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PERIODIC INTERPOLATING SPLINES AND THEIR LIMITS.(U)
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PERIODIC INTERPOLATING SPLINES AND THEIR LIMITS

A. S. Cavaretta, Jr. and D. J. Newman

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ABSTRACT

Let (x_ν, y_ν), $\nu = 1,\dots,k$ be points of interpolation with $0 < x_1 < \cdots < x_k \leq 2\pi$ and let $1 < p \leq \infty$. We consider sequences $\{S_m\}$ of 2π-periodic functions which interpolate the data optimally in the sense that

$$\|S_m^{(m)}\|_p = \text{minimum} .$$

The main results, which depend on the parity of k, concern the asymptotic behavior of $S_m(x)$ as m tends to infinity.

1. $k = 2n + 1$. Let $T(x)$ be the unique polynomial

$$T(x) = \frac{a_0}{2} + \sum_{q=1}^{n} (a_q \cos qx + b_q \sin qx) \tag{1}$$

that interpolates the data. Then

$$\lim_{m\to\infty} S_m(x) = T(x) \quad \text{uniformly in } x . \tag{2}$$

2. $k = 2n$. Among all interpolants (1) let $T(x)$ be that polynomial such that

$$a_n^2 + b_n^2 = \text{minimum} .$$

For this $T(x)$ again (2) holds.

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SIGNIFICANCE AND EXPLANATION

We consider points in the plane which recur in a wave-like pattern. We determine infinitely many curves passing through these points and indexed by a natural number m. The index m indicates the smoothness of the curve; and as m becomes larger the curves take on a limiting shape which is explicitly determined in the report.

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PERIODIC INTERPOLATING SPLINES AND THEIR LIMITS

A. S. Cavaretta, Jr. and D. J. Newman

In 1940 J. Favard wrote "Sur l'interpolation," a paper which has attracted some attention during the past few years [2], [3], [8]. Favard's work raises many interesting questions (e.g. [3]) and we would now like to address one which appears in the very final paragraph of his paper: to wit, how does one handle interpolation of periodic data by smooth m-times differentiable functions and what is the asymptotic behavior of these smooth interpolating functions as m tends to infinity. The present paper is accordingly divided into two sections. In the first section we approach the interpolation problem as a constrained minimization and obtain a sequence S_m of spline interpolants. The asymptotic behavior of this sequence is determined in the second section.

We deal always with uniform convergence on the period, while the constrained minimization is posed in each of the L_p norms, $1 < p \leq \infty$. However the limit of the sequence S_m does not depend on p, but rather is completely determined by the given data. The case when $p = 2$ has been settled earlier by M. V. Golitschek [6] and also by I. J. Schoenberg [9]. The new ideas necessary to handle the case of $p \neq 2$, and in particular $p = \infty$, consistute the novelty of the present paper.

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§1. Fix (x_ν, y_ν), $\nu = 1,\ldots,k$ as the points of interpolation and assume $0 \le x_1 < x_2 < \cdots < x_k < 2\pi$ and $k \ge 2$. We denote by W_p^m the space of 2π-periodic functions whose $(m-1)$th derivative is absolutely continuous and whose m^{th}-derivative is in the Lebesgue space $L_p(0,2\pi)$. Put

$$F_p = \{f \in W_p^m \mid f(x_\nu) = y_\nu,\ \nu = 1,\ldots,k\} .$$

Our problem is to determine solutions of

$$\min_{f \in F_p} \|f^{(m)}\|_p . \tag{1.1}$$

For nonperiodic data the results for this interpolation problem are well known [4], [2], and we merely adapt the effective approach of de Boor and others to the present case of periodic data. When $p = \infty$ the result for the perodic case reads as

Theorem 1. Problem (1.1) has a solution S which is a periodic perfect spline function of degree m with at most $2[\frac{k}{2}]$ knots. More precisely

$$|S^{(m)}(x)| = c$$

except at the knots and

$$c = \min_{f \in F_\infty} \|f^{(m)}\|_\infty .$$

Moreover if $m \ge k$ the solution S is unique.

Remark: Except for the uniqueness statement, this fact has been observed by others, notably B. D. Bojanov [1] and W. Forst [5].

Proof: We first convert our interpolation conditions into moment conditions. For every integer j and $\nu = 1,\ldots,k$ put

$$x_{\nu+jk} = x_\nu + 2\pi j$$
$$y_{\nu+jk} = y_\nu .$$

Based on the nodes $\{x_\nu\}_{\nu=-\infty}^{\infty}$, construct the B-splines M_ν of degree $m-1$ defined as the Peano kernels of divided differences of order m. Thus for any $f \in F_\infty$ and any integer ν

$$[y_\nu,\ldots,y_{\nu+m}] = \int_{x_\nu}^{x_{\nu+m}} M_\nu(t) f^{(m)}(t)dt \ .$$

Now form $\varphi_\nu(t) = \sum_{j=-\infty}^{\infty} M_{\nu+jk}(t)$, $\nu = 1,\ldots,k$. Clearly the φ_ν are 2π-periodic and linearly independent.

Now for any $f \in F_\infty$ set

$$c_\nu = \int_0^{2\pi} \varphi_\nu(t) f^{(m)}(t)dt, \quad \nu = 1,\ldots,k \ . \tag{1.2}$$

We now solve the constrained minimization problem

$$\min \|g\|_\infty \text{ subject to } \int_0^{2\pi} \varphi_\nu(t)g(t)dt = c_\nu, \quad \nu = 1,\ldots,k \ . \tag{1.3}$$

Let M denote the linear span of $\{\varphi_\nu\}_{\nu=1}^k$. We view M as a k-dimensional subspace of $L_1(0,2\pi)$, and on M we define a linear functional λ by setting

$$\lambda\varphi_\nu = c_\nu, \quad \nu = 1,\ldots,k \ .$$

It follows from the Hahn-Banach theorem and the identification of L_1^* with L_∞ that there exists a solution $g_* \in L_\infty(0,2\pi)$ of the above minimum problem (1.3). Clearly $\|g_*\|_\infty = \|\lambda\|$.

To determine the structure of the function $g_*(t)$, let us suppose that λ achieves its norm for the function $s \in M$. Thus $\|s\|_1 = 1$ and $\lambda s = \|\lambda\|$. Suppose for the moment that the periodic spline s vanishes on no interval. Then from Rolle's theorem we conclude that $s(t)$ has no more than $2[\frac{k}{2}]$ zeros on its period. Observe also

$$\|g_*\|_\infty = \|\lambda\| = \lambda s = \int_0^{2\pi} s(t)g_*(t)dt \le \|g_*\|_\infty \|s\|_1 = \|g_*\|_\infty \ . \tag{1.4}$$

So equality must hold throughout and so

$$g_*(t) = \|g_*\| \text{ sign } s(t) \ ,$$

except at $2[\frac{k}{2}]$ points or less where $s(t) = 0$.

Now if $m \ge k$ it follows from the minimum support property of the B-splines $M_\nu(t)$ that s cannot vanish on an interval. So $g_*(t)$ is in this case uniquely determined

by sign $s(t)$. If $m < k$ and $s(t)$ should actually vanish on an interval, we need some additional technical details in order to see that there is some solution $g_*(t)$ with constant absolute value and at most $2[\frac{k}{2}]$ sign changes. For these details, we refer the reader to [2].

To conclude our proof of Theorem 1 observe first

$$\int_0^{2\pi} g(t)dt = \int_0^{2\pi} f^{(m)}(t)dt = 0 .$$

So g has an m-fold periodic integral, defined uniquely up to an additive constant. Let $S_1(t)$ be one such integral and put $p(t) = S_1(t) - f(t)$. Then

$$0 = \int_0^{2\pi} p^{(m)}(t)\varphi_\nu(t)dt = \int_0^{2\pi} p^{(m)}(t)\Big(\sum_{j=-\infty}^{\infty} M_\nu(t + 2\pi j)\Big)dt$$

$$= \sum_{j=-\infty}^{\infty} \int_0^{2\pi} p^{(m)}(t)M_\nu(t + 2\pi j)dt$$

$$= \sum_{j=-\infty}^{\infty} \int_{2\pi j}^{2\pi(j+1)} p^{(m)}(t - 2\pi j)M_\nu(t)dt$$

$$= \sum_{j=-\infty}^{\infty} \int_{2\pi j}^{2\pi(j+1)} p^{(m)}(t)M_\nu(t)dt$$

$$= \int_{-\infty}^{\infty} p^{(m)}(t)M_\nu(t)dt .$$

Thus the divided differences of $p(t)$ vanish; hence on the nodes $p(t)$ must reduce to a constant p_0. So $S(t) = S_1(t) - p_0$ interpolates the data and yields the desired solution of Theorem 1.

For arbitrary p, $1 < p < \infty$, we obtain by the same method as above the following proposition. In case $p = 2$, the interpolating spline S is structurally characterized as the interpolating spline with knots at the x_ν, see [8]; for other p the structure of S is less transparent, but has been discussed to some extent by Golomb [7], also by de Boor [3].

Proposition: For each p, $1 < p < \infty$, there is a unique $S \in F_p$ minimizing $\|s^{(m)}\|_p$. This S is characterized by the relation

$$s^{(m)}(x) = |s(x)|^{q-1} \operatorname{sign} s(x)$$

where q is conjugate to p and s is a spline in M.

Before leaving questions of interpolation, let us recall some results concerning interpolation by trigonometric polynomials. For k odd, say $k = 2n + 1$, there is a unique trigonometric polynomial $T(x)$ of order n satisfying

$$T(x_\nu) = y_\nu, \quad \nu = 1,\ldots,2n + 1 . \tag{1.5}$$

It is convenient at times to write

$$T(x) = \frac{a_0}{2} + \sum_{q=1}^{n} (a_q \cos qx + b_q \sin qx) \tag{1.6}$$

or

$$T(x) = \sum_{\nu=1}^{2n+1} y_\nu \ell_\nu(x) \tag{1.7}$$

where $\ell_\nu(x)$ are the Lagrange functions for the nodes $x_1,\ldots,x_{2n+1}$.

For k even, $k = 2n$, there is a one parameter family of interpolating n^{th} order trigonometric polynomials with

$$T(x_\nu) = y_\nu, \quad \nu = 1,\ldots,2n \tag{1.8}$$

and Schoenberg [8] has suggested singling out the one with least amplitude for the terms of highest frequency:

Definition: Among the $T(x)$ of the form (1.6) and satisfying the interpolation conditions (1.8) we determine the unique $T(x)$ which satisfies the condition that

$$a_n^2 + b_n^2 = \text{minimum} .$$

This T is called the proximal interpolant.

Consider the function $\psi(x) = \prod_{\nu=1}^{2n} \sin \frac{x - x_\nu}{2}$. Evidently ψ vanishes precisely at the x_ν and so if $T_1(x)$ satisfies the interpolation conditions (1.8) so does $T_1(x) + \lambda\psi(x)$ for any value λ. Using this observation one can easily establish the following lemma [9]:

Lemma 1: The proximal interpolant $T(x)$ is uniquely characterized as the interpolant of the form

$$T(x) = \frac{1}{2} a_0 + \sum_{q=1}^{n-1} (a_q \cos qx + b_q \sin qx) + \alpha \sin(nx - \frac{1}{2} \sum_{\nu=1}^{2n} x_\nu) \,. \tag{1.9}$$

§2. For the data (x_ν, y_ν), $\nu = 1,\ldots,k$, we solve the problem in Theorem 1 for every positive integer m. This yields a sequence $\{S_m(x)\}_{m=1}^{\infty}$ of periodic perfect splines, all of which interpolate the same data. Alternatively, obtain as in the Proposition a similar sequence $\{S_m(x)\}_{m=1}^{\infty}$ which minimizes $\|S^{(m)}\|_p$, $1 < p < \infty$. In any case, the following theorem describes the asymptotic behavior of these sequences.

Theorem 2: *Depending on whether* k *is odd or even, let* $T(x)$ *be the unique interpolating trigonometric polynomial as determined in* §1 *by* (1.6) *for* $k = 2n + 1$ *and by* (1.9) *for* $k = 2n$. *Then for every* $j = 0,1,\ldots,$ *we have*

$$\lim_{m\to\infty} S_m^{(j)}(x) = T^{(j)}(x) \quad \text{uniformly in } x .$$

§2.1. *The number of data is odd:* $k = 2n + 1$.

We need to write the Fourier series for our functions. Since we will differentiate the resulting series, it is convenient to use the complex form even though all functions concerned are real. Accordingly, let

$$S_m(x) = \sum_{-\infty}^{\infty} c_{s,m} e^{isx} = U_m(x) + R_m(x) ; \tag{2.1}$$

here we have set

$$U_m(x) = \sum_{-n}^{n} c_{s,m} e^{isx}, \quad R_m(x) = \sum_{|s|>n} c_{s,m} e^{isx} .$$

Lemma 2: *For* $j = 0,1,2,\ldots,$ $\lim_{m\to\infty} R_m^{(j)}(x) = 0$ *uniformly in* x.

Proof: Observe that

$$c_{s,m} = \frac{1}{2\pi} \int_0^{2\pi} S_m(t) e^{-ist} dt = \frac{1}{2\pi} \int_0^{2\pi} S_m^{(m)}(t) \frac{e^{-ist}}{(-is)^m} dt ,$$

so by Hölder's inequality

$$|c_{s,m}| \le \frac{1}{|s|^m} \|S_m^{(m)}\|_p .$$

Since T also interpolates the given data, we have from the optimality property of S_m that

(2.2) $$\|S_m^{(m)}\|_p \le \|T^{(m)}\|_p \le n^m \|T\|_p ,$$

where the last inequality is that of Bernstein-Zygmund. Hence

(2.3) $$|c_{s,m}| \le \left|\frac{n}{s}\right|^m \|T\|_p ,$$

so

(2.4) $$|R_m(x)| \le \sum_{|s|>n} |c_\nu| \le \|T\|_p \sum_{|s|>n} \left|\frac{n}{s}\right|^m .$$

Clearly as m tends to infinity, the right hand side of (2.4) tends to 0, and thus the lemma is proved for $j = 0$. For arbitrary j we find in exactly the same way

(2.5) $$|R_m^{(j)}(x)| \le n^j \|T\|_p \sum_{|s|>n} \left|\frac{n}{s}\right|^{m-j} , \quad m > j + 1 ,$$

which completes the proof.

Using Lemma 2, the case of $k = 2n + 1$ of Theorem 2 follows quite easily. Indeed

$$S_m(x) = U_m(x) + R_m(x)$$

and from

$$\lim_{m\to\infty} R_m(x) = 0$$

it follows

$$\lim_{m\to\infty} U_m(x_\nu) = y_\nu, \qquad \nu = 1,\ldots,2n + 1 .$$

But as $y_\nu = T(x_\nu)$ and each U_m is a trigonometric polynomial of order n, we conclude

$$\lim_{m\to\infty} U_m(x) = T(x)$$

uniformly in x. Finally observing that the operation of differentiation is continuous when restricted to trigonometric polynomials of order n, we have

$$\lim_{m\to\infty} S_m^{(j)}(x) = \lim_{m\to\infty} U_m^{(j)}(x) + \lim_{m\to\infty} R_m^{(j)}(x) = \lim_{m\to\infty} U_m^{(j)}(x) = T^{(j)}(x)$$

uniformly in x.

§2.2. The number of data is even: $k = 2n$.

The even case is more subtle and to prove the result we need the precise value of

$$\lim_{m\to\infty} \|n^{-m} S_m^{(m)}\|_p , \quad 1 < p \le \infty$$

which is itself a result of independent interest. For convenience in what follows, we take the norm to be

$$\|f\|_p = \begin{cases} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right\}^{\frac{1}{p}} & 1 \le p < \infty \\ \text{ess sup } |f(x)| & p = \infty \end{cases} .$$

As a preliminary observation in the case $k = 2n$ we first pick a point $x_0 \neq x_\nu \bmod 2\pi$, $\nu = 1,\ldots,2n$. We then prove

Lemma 3: *Fix* $1 < p \le \infty$ *and let* $\{S_m\}$ *be the sequence of interpolating splines as determined in* §1. *Then*

(2.6) $$S_m(x_0) = O(1) .$$

Proof: Recall that $S_m(x) = U_m(x) + R_m(x)$ and from Lemma 2 (which still applies as the parity of the number of data points plays no role)

$$\lim_{m\to\infty} R_m(x) = 0 .$$

Thus it is sufficient to show

(2.7) $$\eta_m \equiv U_m(x_0) = O(1) .$$

Using the Lagrange interpolation formula, we have

$$U_m(x) = \sum_{\nu=1}^{2n} (y_\nu + \varepsilon_\nu^m) \ell_\nu(x) + \eta_m \ell_0(x)$$

where again from Lemma 2 $\varepsilon_\nu^m = 0(1)$, $\nu = 1,\ldots,2n$. Now for some $\alpha \neq 0$ and ξ,

$$\ell_0(x) = \alpha \sin(nx + \xi) + \text{lower terms.}$$

Then the same argument which produced (2.3) yields

(2.8) $$\int_0^{2\pi} S_m(x) \sin(nx + \xi) dx = O(1) .$$

On the other hand

$$\int_0^{2\pi} S_m(x)\sin(nx + \xi)dx = \int_0^{2\pi} (U_m(x) + R_m(x))\sin(nx + \xi)dx$$

$$= \int_0^{2\pi} U_m(x)\sin(nx + \xi)dx \tag{2.9}$$

$$= \int_0^{2\pi} \Big(\sum_{\nu=1}^{2n} (y_\nu + \varepsilon_\nu^m)\ell_\nu(x)\Big)\sin(nx + \xi)dx + \eta_m \int_0^{2\pi} \ell_0(x)\sin(nx + \xi)dx$$

$$= O(1) + \eta_m \cdot \alpha\pi .$$

Together (2.8) and (2.9) yield (2.7).

Taken in conjunction Lemmas 2 and 3 guarantee that subsequential limits of the sequence $S_m(x)$ exist uniformly in x. In particular from Lemma 3 we have that the trigonometric polynomials U_m are bounded at $2n + 1$ distinct points and so the sequence U_m has subsequential limits. But from Lemma 2 it follows that any such limit (which must be a trigonometric polynomial) is also a limit of the corresponding subsequence of S_m. We finish the proof of Theorem 2 by showing that the only such limiting trigonometric polynomial is the unique proximal interpolant T singled out in §1. This is proved by the following two lemmas, where we argue for subsequences without indicating this in the notation.

Lemma 4: Let $\{S_m\}$ denote any subsequence and suppose

$$\lim_{m\to\infty} S_m(x) = \tilde{T}(x) \equiv \alpha \sin(nx + \omega) + \text{lower order terms} .$$

Then

$$\liminf_{m\to\infty} \|n^{-m} S_m^{(m)}(x)\|_p \geq \frac{|\alpha|}{2\|\sin x\|_q} , \qquad 1 < p \leq \infty, \ \frac{1}{p} + \frac{1}{q} = 1 . \tag{2.10}$$

Proof: We argue for m even; m odd goes analogously. Clearly

$$\lim_{m\to\infty} \int_0^{2\pi} S_m(x)\sin(nx + \omega)dx = \int_0^{2\pi} \tilde{T}(x)\sin(nx + \omega)dx = \pi\alpha .$$

On integrating by parts m times we get

$$\lim_{m\to\infty} (-1)^{\frac{m}{2}} \frac{1}{n^m} \int_0^{2\pi} S_m^{(m)}(x)\sin(nx + \omega)dx = \pi\alpha ;$$

and since

$$\left|\frac{1}{2\pi}\int_0^{2\pi} S_m^{(m)}(x)\sin(nx+\omega)dx\right| \le \|S_m^{(m)}(x)\|_p \|\sin(nx+\omega)\|_q = \|S_m^{(m)}(x)\|_p \|\sin x\|_q ,$$

the result follows.

Lemma 5: *Suppose for some* β *and* τ

$$U(x) = \beta \sin(nx+\tau) + V(x)$$

satisfies $U(x_\nu) = y_\nu$, $\nu = 1,\dots,2n$; $V(x)$ *a trigonometric polynomial of order* $n-1$. *The*

$$\limsup_{m\to\infty} \|n^{-m} S_m^{(m)}(x)\|_p \le \frac{|\beta|}{2\|\sin x\|_q}, \qquad 1 < p \le \infty . \tag{2.11}$$

Proof: By a translation we can assume $\tau = 0$. Because of the minimal nature of $S_m(x)$, we need only produce for any given $\varepsilon > 0$ an $F(x)$ which interpolates and for which

$$\|F^{(m)}(x)\|_p \le \frac{n^m}{2\|\sin x\|_q} (|\beta| + \varepsilon) .$$

Also as seen in §1 $\|S_m^{(m)}(x)\|_p$ depends continuously on the data and hence the F we produce need only interpolate the data (x_ν, y_ν), $\nu = 1,\dots,2n$ within ε and not necessarily exactly.

For q conjugate to p, we consider the function

$$\lambda \frac{|\sin x|^q}{\sin x}, \qquad \lambda = \frac{1}{2(\|\sin x\|_q)^q}$$

with its Fourier series given by

$$\sum_{s=1}^{\infty} c_s \sin sx .$$

Form

$$F(x) = \sum_{s=1}^{\infty} c_s \frac{\rho^s}{s^m} U(sx) \tag{2.12}$$

for ρ near $1-$ and U as in the statement of the lemma. Since

$$c_1 = \frac{1}{\pi}\int_0^{2\pi} \lambda \frac{|\sin x|^q}{\sin x} \sin x\, dx = 1$$

we have

$$F(x) = \rho U(x) + O\left(\frac{1}{2^m}\right) .$$

Thus $F(x)$ satisifes the interpolation conditions within an error of $O(1 - \rho) + O\left(\frac{1}{2^m}\right)$, which is less than ε for ρ close to 1- and m large, as they will be chosen.

Next observe from the uniform convergence of (2.12) when $\rho < 1$, we have for m even (a similar analysis will hold if m odd)

$$(2.13)\quad F^{(m)}(x) = \sum_{s=1}^{\infty} c_s \rho^s U^{(m)}(sx) = (-1)^{\frac{m}{2}} \beta n^m \sum_{s=1}^{\infty} c_s \rho^s \sin nsx + \sum_{s=1}^{\infty} c_s \rho^s V^{(m)}(sx) .$$

Now observe that the series of the first term is the Abel sum for

$$\sum_{s=1}^{\infty} c_s \sin snx = \lambda \frac{|\sin nx|^q}{\sin nx} .$$

For the series of the second term, we have

$$V^{(m)} = O(n - 1)^m \quad \text{and} \quad c_s = O(1) ,$$

and so the series is

$$O\left(\frac{1}{1 - \rho} \cdot (n - 1)^m\right) .$$

Thus (2.13) is estimated by

$$(2.14)\qquad F^{(m)}(x) = (-1)^{\frac{m}{2}} \beta n^m \lambda \frac{|\sin nx|^q}{\sin nx} + O(1 - \rho) n^m + O\left(\frac{1}{1 - \rho} (n - 1)^m\right) .$$

We make the error in (2.14) less than εn^m by first choosing ρ sufficiently close to 1- and then choosing m sufficiently large. Taking the L_p norm we then have

$$(2.15)\qquad \left\| n^{-m} F^{(m)}(x) \right\|_p \leq |\beta| \lambda \left\| \frac{|\sin nx|^q}{\sin nx} \right\|_p + \varepsilon .$$

But as

$$\lambda \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|\sin nx|^{pq}}{|\sin nx|^p} \right)^{\frac{1}{p}} = \lambda \left(\frac{1}{2\pi} \int_0^{2\pi} |\sin x|^q \right)^{\frac{1}{p}} = \lambda \|\sin x\|_q^{\frac{q}{p}} = \frac{1}{2\|\sin x\|_q} ,$$

(2.15) implies (2.11).

Now Theorem 2 follows immediately from Lemmas 4 and 5 and the previous remarks. The proximal interpolant T, by its very definition, provides a U for Lemma 5 with the

smallest possible β. Thus (2.11) holds with this β. So by Lemma 4 in conjunction with (2.11) no subsequence of S_m can converge to any other trigonometric polynomial other than the proximal interpolant. It follows that the full sequence S_m converges uniformly to T. We then can argue just as in §2.1 that for $j = 0,1,\ldots$

$$\lim_{m\to\infty} S_m^{(j)}(x) = T^{(j)}(x) \quad \text{uniformly in } x .$$

We will close with a corollary concerning the perfect splines of Theorem 1, the case $p = \infty$.

Corollary: Let T be the interpolant of our Theorem 2 and let α denote the amplitude of the terms of frequency n in T. Then

$$\lim_{m\to\infty} n^{-m} \|S_m^{(m)}\|_\infty = \frac{\pi}{4}\alpha .$$

Proof: This follows immediately from Lemmas 4 and 5 and the observation that when $q = 1$, $\lambda = \frac{\pi}{4}$.

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Let (x_ν, y_ν), $\nu = 1,\dots,k$ be points of interpolation with $0 < x_1 < \cdots < x_k \leq 2\pi$ and let $1 < p \leq \infty$. We consider sequences $\{S_m\}$ of 2π-periodic functions which interpolate the data optimally in the sense that

$$\|S_m^{(m)}\|_p = \text{minimum} .$$

The main results, which depend on the parity of k, concern the asymptotic

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20. ABSTRACT - Cont'd.

behavior of $S_m(x)$ as m tends to infinity.

1. $k = 2n + 1$. Let $T(x)$ be the unique polynomial

$$T(x) = \frac{a_0}{2} + \sum_{q=1}^{n} (a_q \cos qx + b_q \sin qx) \tag{1}$$

that interpolates the data. Then

$$\lim_{m \to \infty} S_m(x) = T(x) \quad \text{uniformly in } x \,. \tag{2}$$

2. $k = 2n$. Among all interpolants (1) let $T(x)$ be that polynomial such that

$$a_n^2 + b_n^2 = \text{minimum} \,.$$

For this $T(x)$ again (2) holds.