AD-A046 313 STANFORD RESEARCH INST MENLO PARK CALIF F/G 12/1

NUMERICAL SOLUTION OF NONLINEAR PARABOLIC EQUATIONS.(U)

JUN 77 S SCHECHTER DAHC04-72-C-0030

UNCLASSIFIED NL

1 OF 1

AD A046313

END

DATE FILMED

12-77

DDC

ARO 10485.3-M

12

B.S.

AD A046313

Final Report June 1977

NUMERICAL SOLUTION OF NONLINEAR PARABOLIC EQUATIONS

DDC
RECEIVED
NOV 7 1977
F

By: SAMUEL SCHECHTER

Prepared for:

U.S. ARMY RESEARCH OFFICE
P.O. BOX 12211
RESEARCH TRIANGLE PARK, NORTH CAROLINA 27709
Attn: Dr. Paul T. Boggs, Mathematics Division

Contract No. DAHC04-72-C-0030

STANFORD RESEARCH INSTITUTE
Menlo Park, California 94025 · U.S.A.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) Numerical Solution of Nonlinear Parabolic Equations		5. TYPE OF REPORT & PERIOD COVERED (9) Final Report. 17 April 1972 – ~~to~~ 30 May 1977
		6. PERFORMING ORG. REPORT NUMBER Final Report, SRI Project 1867
7. AUTHOR(s) (10) Samuel Schechter		8. CONTRACT OR GRANT NUMBER(s) (15) DAHC04-72-C-0030
9. PERFORMING ORGANIZATION NAME AND ADDRESS Stanford Research Institute, 333 Ravenswood Avenue, Menlo Park, California 94025		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (12) 27 p.
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709		12. REPORT DATE (11) June 1977 — 13. NO. OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (if diff. from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear Equations, Parabolic Differential Equations, Relaxation methods, Quadratic Systems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A method of solution is obtained for a class of nonlinear parabolic partial differential equations. An analysis is made of the existence and uniqueness of a solution to a special class of semilinear systems arising → next page

DD FORM 1 JAN 73 **1473**

EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 332 500

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

mt

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19. KEY WORDS (Continued)

20 ABSTRACT (Continued)

from various discretisations of the differential equation. A numerical procedure for solving singular problems is given. A method of approximate block relaxation is shown to converge globally, and an application to a quadratic system is presented.

DD FORM 1 JAN 73 1473 (BACK)

EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

CONTENTS

NUMERICAL SOLUTION OF NONLINEAR PARABOLIC EQUATIONS

Samuel Schechter*

1. Introduction

We will consider the parabolic differential equation

$$\frac{\partial g(\theta)}{\partial t} = \frac{\partial^2 f(\theta)}{\partial x^2} \tag{1.1}$$

for the unknown function $\theta(x,t)$ over the set $0 < x < 1$, $0 < t \le T$ and given boundary and initial conditions

$$\theta(x,0) = \nu_0(x) \tag{1.2}$$

$$\theta(i,t) = \mu_i(t), \quad i = 0,1.$$

We assume that

$$g'(\theta) > 0 \tag{1.3}$$

$$f'(\theta) > 0$$

for all θ in a set S.

Typical examples are found where

$$g(\theta) = \theta, \; f(\theta) = \theta^m, \; m > 0. \tag{1.4}$$

For m = 2 we get the equation for flow of gas in a pipe [9], while Richtmyer [4] examines the case m = 5 for a running wave. Examples

*This work was supported by the U.S. Army Research Office-Durham, under contract DAHCO4-72-C-0030.

arise in flows in porous media [10]. Other examples may be found in Ames [12].

We approximate (1.1) by using a weighted form of Euler's summation formula (or second order Padé approximation) [13],

$$g_i^1 - g_i^o = \tfrac{1}{2}k(\alpha \dot{g}_i^o + \beta \dot{g}_i^1) + (k^2/12)(\gamma \ddot{g}_i^o - \delta \ddot{g}_i^1) \tag{1.5}$$

where

$$g_i^j = g(u_i^j),\ u_i^j = \theta(x_i, t + jk),\ \dot{g} = g_t$$

$$i = 1,2,\ldots,\ n;\ j = 0,1$$

$$nh = 1, k = \Delta t,\ \alpha + \beta > o\ ,$$

and α, β, γ, δ are nonnegative constants. If these constants are specialized to all equal 1, then we obtain the usual formula, with remainder of order k^5. Other choices give a lower order accuracy, usually $O(k^3)$.

We will find it convenient to drop the superscript 1 in later formulas. To obtain an approximation for (1.1), we replace $\dot{g}$ by $f_{\bar{xx}}$ in (1.5), where $f_{\bar{xx}} = (f_{i-1} - 2f_i + f_{i+1})/h^2$. This will yield an implicit scheme, with a nonlinear system in n unknowns to be solved at each time step.

If we take the special case (1.4) and set $\alpha = \gamma = \delta = 0$, $\beta = 2$, we get the backward Euler system

$$u_i - u_i^o = k(u_{i-1}^m - 2u_i^m + u_{i+1}^m/h^2\ , \tag{1.6}$$

where m is here an exponent, not a superscript.

To simplify our notation we define, for the vectors u and v in R^n, the Schur product $uv = u \circ v = vu$ as the vector with components $u_i v_i$. If $f(x)$ is a function of one variable, we define the vector $f(u)$ by

$$f(u) = (f(u_1),\ldots,f(u_n))^T$$

also called a diagonal mapping [8].

In particular, u^m will denote the vector whose components are raised to the power m.

For each vector u we may correspond a diagonal matrix, which we denote by its upper case, $U = \text{diag}\ (u_1,\ldots,u_n) = \text{diag}\ (u)$ and conversely. It then follows that $uv = Uv$, where the right side is ordinary matrix multiplication.

If $f'(x)$ is the derivative of $f(x)$ then $F'(u) = \text{diag}\ (f'(u))$ is the Jacobian of $f(u)$. If u depends on t, then $f(u)_t = f'(u)\dot{u} = F'(u)\dot{u}$.

With this notation in hand, we may write (1.6) as

$$u - u^o = -2\lambda K u^m,\ 2\lambda = k/h^2 \quad , \qquad (1.6)$$

where K is the usual tridiagonal matrix of order n

$$K = [-1, 2, -1].$$

Alternatively, we may write (1.6) in quasilinear form as

$$u - u^o = -2\lambda A(u)u \quad ,$$

where $A(u) = KU^{m-1}$.

Returning to the more general case (1.1) we get

$$\dot{g}(u) = -h^{-2} K f(u)$$
$$\ddot{g}(u) = -h^{-2} K F' (u)\dot{u} \quad . \tag{1.7}$$

Since $\dot{g}(u) = G'(u)\dot{u}$,

we get, after solving for $\dot{u}$,

$$g(u) = h^{-4} L(u)\, f(u) \quad ,$$

where we set

$$L(u) = K F'(u)\, G'(u)^{-1} K \quad .$$

Inserting these into (1.5) and collecting terms, we get

$$g(u) + \lambda\beta K f(u) + \tfrac{1}{3}\lambda^2 \delta\, L(u) f(u) = R_o \tag{1.8}$$

$$R_o = g^o - \lambda\alpha K f^o + \tfrac{1}{3}\lambda^2 \gamma\, L^o f^o \quad ,$$

where the zero superscript indicates u is replaced by u^o. We seek a solution to (1.8) and a method to construct it.

Since the latter calculations used only the fact that K was independent of u and t, the same result is obtainable for a general linear operator T so that $g_t = T(f)$.

We now assume that $L(\bar{u})$ is replaced by $L_o = L(\bar{u})$ where $\bar{u}$ is some approximation to u,* a solution of (1.8). If we denote the left side of (1.8) by p(u), then (1.8) becomes

$$r(u) := p(u) - R_o = 0. \tag{1.8}$$

Lemma: Let $S = \{\theta : f'(\theta) > o, \; g'(\theta) > o\}$. If S is convex (that is an interval), then (1.8) has at most one solution u^* with components in S.

Proof: Assume there are two solutions, u and v. Then

$$(u - v, \; p(u) - p(v)) = o \quad .$$

From the hypotheses we get

$$(u - v, \; g(u) - g(v)) = (u - v, \; G'(z)(u-v)) > 0$$

for z in the interval $[u,v]$, and since K is positive definite,

$$(u-v, \; K(f(u) - f(v)) = (u-v, \; KF'(w)(u-v)) > 0 \quad .$$

The same applies to $L_o f$. This gives a contradiction and proves the Lemma.

If we assume that f(0) and g(0) have positive derivatives for all 0, then we obtain the existence of a solution. From the previous lemma it will be unique. Thus, we assume that

$$f'(0) > 0, \; g'(0) > 0 \text{ for all real } \theta \quad . \qquad (1.9)$$

We may then assume that $f(0) \equiv 0$, since by inverting $f(u) = v$ we get a new equation in v with $g(u) = g(f^{-1}(v))$. With $f(u) = u$, (1.8) becomes a semilinear problem, with g(u) a diagonal isotone mapping.

If $\delta = 0$, then p(u) becomes an M-function [8] so that existence is obtained in this case by a theorem of Rheinboldt [5]. In the case where $\delta > o$, we obtain a gradient map. Since the Jacobian of p satisfies

$$p'(u) \geq \lambda\beta K \geq \lambda\beta\lambda_{min}(K) \cdot I \quad ,$$

the existence and uniqueness follow from results in Schechter [7]. Both of the above theorem also provide methods of construction of the solution. In particular we may use a variety of relaxation methods [8]. We have thus proved:

<u>Theorem</u>. If f and g satisfy (1.9), then (1.8) has a unique solution that may be obtained by a relaxation method.

It should be noted that the solution method does not require a priori estimates for this special class of problems.

2. Existence and Estimates

Although existence and uniqueness are obtained as stated in the previous section the actual methods available to compute the solution are usually dictated by the nature of the problem. If $\delta=0$, we may resort to nonlinear SOR with overrelaxation parameter ω in $(0,1]$ or use a Jacobi iteration [8]. Since these methods tend, in general, to be slowly convergent and since we wish to cover the more general case $\delta > 0$, we consider the use of relaxation or SOR-Newton methods [6], [7], [14].

For this purpose we find it necessary to require that f and g have continuous derivations and at times continuous second derivatives. This will allow us to choose relaxation parameters, at times, outside the unit interval.

Before proceeding, we note some estimates of the solution for the case $\delta=0$. We may assume as above that f is the identity map and $g(0)=0$. Otherwise, we may subtract a constant from g.

If u^* is the solution, it then follows from Taylor's expansion that

$$(D + A)u^* = R_o \tag{2.1}$$

where we set $A = \lambda\beta K$ and for some z in $[0,u^*]$, $D = G'(z)$. Since D + A is a monotone matrix, its inverse is nonnegative and we get

$$u^* = (D + A)^{-1} R_o \ , \tag{2.2}$$

and if $R_o > 0$, then $u^* > 0$.

In practice we may achieve $R_o > 0$ by choosing either a small α or small time step, if $g^o > 0$. In this case we also obtain an upper bound on u^*, since $D > 0$,

$$0 < u^* \leq A^{-1} R_o \tag{2.3}$$

We note in passing that (2.3) can be used as a basis for the usual fixed point iteration process. Since this is usually slow we do not pursue it.

A similar upper bound is available if $\delta \geq 0$ since L_o is positive definite and monotone. If we set $B = \frac{1}{3}\lambda\delta L_o$, then

$$(D + A + B)u^* = R_o \tag{2.4}$$

and if $u^* > 0$, we get

$$u^* \leq (A + B)^{-1} R_o.$$

To achieve the positivity of the solution we may choose δ or k small enough so that $R_o - Bu^* > o$. We may of course, without this restriction, obtain a bound on (u^*,u^*) by using the energy function associated with the symmetric problem. This bound yields

$$\left|R_o\right| \geq \frac{1}{2} \lambda_{\min} (A) \cdot \left|u^*\right| \quad .$$

For some special choices of β and δ where δ is not small this positivity may again be obtained. If $\beta = 2$ and $\delta = 3$, then, absorbing λ into K, the operator on u^* may be written as $D + 2K + KL_oK = (I + KL_o)(D + K) + K(I - L_oD)$. Note that the first term on the right is monotone since the factors are M-matrices. If z and $\bar{u}$ are close, then L_oD is near the identity. If the time step is small, this will prevail with smooth functions. If the second term is combined with R_o, an argument similar to the previous one may be applied to achieve the result.

For the example (1.4) cited above we see that the derivative of f does not appear to be positive for all θ. In many applications, it is usual to have the solution nonnegative so that if f is restricted to this set, we do have a positive derivative. If a solution appears in that set, we showed previously that it will be unique.

In order to achieve existence we may extend the definitions of f or g so that the positivity conditions are valid. If a positive solution was available before the extension it should not be lost.

Let us consider the example $m = 2$ since it is the simplest and most common case. Other cases may be treated in a similar fashion. By inverting f we get $g(\theta) = \theta^{\frac{1}{2}}$. We now extend g by defining

$$g(\theta) = |\theta|^{\frac{1}{2}} \operatorname{sgn}(\theta) \tag{2.5}$$

This function has positive derivatives except at $\theta = 0$ where it is infinite. The function is monotone so that uniqueness is available as indicated above. To get existence we make use of the potential function $E(u)$ of $r(u)$. This potential is strictly convex and $E \to \infty$ as $u \to \infty$. Thus its level sets are bounded, yielding a global minimum and the existence of a solution.

If we perturb g by a small amount we can get a more constructive method of solution and allow for a large variety of solution methods. For example, we may change g to be

$$g(\epsilon,\theta) = ((|\theta| + \epsilon)^{\frac{1}{2}} - \epsilon^{\frac{1}{2}}) \operatorname{sgn} \theta \tag{2.6}$$

How does this change the solution? We assume, in general, that for all $\theta > 0$,

$$0 < g(\epsilon_1,\theta) - g(\epsilon_2,\theta) \le c(\epsilon_2 - \epsilon_1) \text{ for } \epsilon_1, < \epsilon_2 \tag{2.7}$$

for some positive constant c depending on θ. Consider first the case of $\delta = 0$ and let u and v be the positive solutions to the equations (1.8) for the parameters ϵ_1 and ϵ_2, respectively. Thus,

$$p(\epsilon_1,u) \equiv g(\epsilon_1,u) + Au = g(\epsilon_2,v) + Av \equiv p(\epsilon_2,v) = R_o$$

and

$$p(\epsilon_1,v) > p(\epsilon_2,v) = p(\epsilon_1,u)$$

so that $v > u$. This follows from the fact that p is inverse isotone. Since the solutions are bounded from below by zero we get a decreasing sequence of solutions as ϵ goes to zero. This again verifies the existence.

From the hypotheses there is a positive diagonal matrix C such that

$C(\epsilon_2 - \epsilon_1) \geq g(\epsilon_1, v) - g(\epsilon_2, v) = g(\epsilon_1, v) - g(\epsilon_1, u) + A(v-u) \geq A(v-u)$

$$A^{-1}C(\epsilon_2 - \epsilon_1) \geq v-u > 0.$$

Thus, if we let ϵ_1 go to zero,

$$A^{-1}C\epsilon_2 \geq v-u^* > 0 ,$$

which estimates the perturbed solution, with possibly a new positive C.

If we choose $\epsilon = O(k^m)$ for a suitable m then we stay within the error of the equation.

The example (2.6) given above is differentiable for all θ and satisfies (2.7).

Consider the case of $\delta \geq 0$ and let $E(\epsilon, u)$ be the potential function of r corresponding to $g(\epsilon, u)$. Thus, r is the gradient of E. We assume that for all θ

$$|g(\epsilon_2, \theta) - g(\epsilon_1, \theta)| \leq c|\epsilon_2 - \epsilon_1| ,$$

for some positive c. It follows that the integral $\bar{g}$ of g as a function of θ also satisfies such an estimate.

Let u_1 and u_2 be solutions corresponding to ϵ_1 and ϵ_2 and set $E(\epsilon_i, u_j) = E_{ij}$. Then, since the first solution minimizes $E(\epsilon_1, u)$,

$$\begin{aligned} E_{12} - E_{11} &= \tfrac{1}{2}(u_2 - u_1)^T E''(z)(u_2 - u_1) \\ &\geq \tfrac{1}{2} \lambda_{min}(A) |u_2 - u_1|^2 \end{aligned} .$$

We will show that the left side is of order $\epsilon_2 - \epsilon_1 = e$. First, since $E_{ij} - E_{jj}$ depends only on $\bar{g}$, this difference is $O(e)$. Since $E_{jj} \leq E_{ji}$,

$$E_{12} - E_{11} = E_{12} - E_{22} + E_{22} - E_{21} + E_{21} - E_{11}$$
$$\leq E_{12} - E_{22} + E_{21} - E_{11} \leq C\,|e|\ .$$

This yields the estimate

$$|u_2 - u_1| \leq C\,|\epsilon_2 - \epsilon_1|^{\frac{1}{2}}\quad .$$

If $\epsilon_2 \to 0$, it follows readily from the uniqueness that $u_2 \to u^*$. This gives the estimate desired,

$$|u^* - u_1| \leq c\,\epsilon_1^{\frac{1}{2}}\quad .$$

3. Quadratic System

We consider the specific example (1.4) with $m = 2$. We then get a quadratic system to solve for given $R_o > 0$

$$\rho(v) = v + Av^2 - R_o \tag{3.1}$$

with A positive definite and monotone. The Jacobian of ρ is

$$\rho' = I + 2AV = M$$

and if we have $V > 0$, then ρ' is nonsingular. It will be so, even if some but not all entries in V vanish. Assume that v^o is positive and set $\rho'(u^o) = M$. We may try to use a modified Newton iteration or a block relaxation method to find the positive solution.

The iteration appears as

$$M(v^1 - v^o) = -\omega\rho(v^o) \tag{3.2}$$

for a given choice of parameter ω. We assume that ω is chosen sufficiently small so that $v^1 > 0$. *If $\rho < 0$, then since M is monotone we do not need to* restrict ω in this case.

Since M is not symmetric ρ is not a gradient mapping and it is not apparent that this iteration will converge. If we transform the variable to $u = v^2$, then a variety of choices are available for ω to yield convergence.

Let us examine the Newton step for the transformed problem for $u^o > 0$

$$r(u) = u^{\frac{1}{2}} + Au - R_o = \rho(v)\,. \tag{3.3}$$

Let $J = r'$ so that

$$J(u^o)(u^1 - u^o) = -\omega r^o = -\omega\rho^o$$

$$J = \tfrac{1}{2}U^{-\frac{1}{2}} + A = \tfrac{1}{2}(I + 2AV)V^{-1} = \tfrac{1}{2}MV^{-1}\,.$$

If we replace u by v^2, then

$$\tfrac{1}{2}M(V^o)^{-1}(V^1 + V^o)(v^1 - v^o) = -\omega\rho^o\,.$$

Thus we see that this iteration, which can be guaranteed to converge for the proper choice of ω since r is now a gradient mapping, is quite close to the previous one. The extra factor in J is the matrix $\frac{1}{2}(I + (V^o)^{-1}V^1)$, which if the iterates are close enough will be near the identity.

If we assume that the iterates in (3.2) satisfy

$$V^1(V^o)^{-1} \leq 1 + \tau \qquad \tau > 0 ,$$

then the conditions for using the approximate relaxation methods of the Appendix to this report are satisfied. In particular, the condition (3.3) of that section is satisfied with $\beta = 2/(2 + \tau)$. We may therefore consider (3.2) as an approximate relaxation method to (3.3). Since (3.2) contains no square roots, it may seem less expensive to use.

APPENDIX A

AN APPROXIMATE BLOCK RELAXATION

Samuel Schechter*

1. Introduction

In the iterative method of nonlinear relaxation it is often desirable to avoid derivative computation or to use other approximations for the evaluation of the iterates. In [3] it was shown that such approximations are valid if coordinate directions are used at each step. In this note we indicate an extension of this method to block relaxation which would include a modified Newton method for a restricted class of problems.

An example is provided to show that positivity of the Hessian of a smooth convex function does not imply the same property for the finite difference approximation in the large, for more than one dimension.

2. Assumptions and Estimates

We will use the notation of [2] and [3] and propose to minimize a real valued smooth function $g(u) \in C^3(R^n)$ by iteration. We assume $u^o \in R^n$ is a given guess to solve $r(u) = g'(u) = o$. Let $r^o = r(u^o)$, $A = g''(u)$ be the Hessian matrix, and m a multiindex taken from the set $(1,\ldots,n) = Z$. Let A_m denote the principal submatrix of $A(u^o)$ defined by m. We henceforth assume that A_m is positive definite.

Given a nonsingular matrix K of the same order as A_m, we define an approximate block relaxation step by

$$
\begin{aligned}
& u^1 = u^o + \omega d^o, \\
(2.1)\qquad & d^o_m \equiv d = -K^{-1} r^o_m, \\
& d^o_{m'} = o
\end{aligned}
$$

*This work was supported by the U.S. Army Research Office, Durham, under Contract DAHCO4-72-C-0030.

Where m' is the complementary multiindex to m and ω, $0 < \omega < 2$, is some relaxation parameter. Since we wish $g(u^1)$ to decrease, we require that $(r^o, d^o) < 0$ so that, for $r_m^o \neq 0$,

$$(r_m^o, K^{-1} r_m^o) > 0.$$

We do not require K to be symmetric.

To estimate the change in g or

$$\Delta g = g(u^1) - g(u^o)$$

we use Taylor's expansion. Using the notation $B_j(u) = \partial_j A(u)$ and $B_{jm}(u)$ for its corresponding principal minor we get

$$\Delta g = (g'(u^o),\ u^1 - u^o) + \frac{1}{2}(u^1 - u^o,\ A(u^o)(u^1 - u^o))$$

$$+ \frac{1}{6}\sum_j (u^1 - u^o,\ B_j(z)(u^1 - u^o)(u^1 - u^o)_j$$

$$= \omega(r_m^o, d) + \frac{1}{2}\omega^2(d,\ A_m d) + \frac{1}{6}\omega^3 \sum_{j\in m} (d,\ B_{jm}(z)d)d_j$$

for a suitable z between u^o and u^1.

If we let

$$\alpha = (d,\ A_m d)/(d, Kd) > 0$$

then

(2.2) $$-\Delta g = \frac{1}{2}\,\omega(d, Kd)[2 - \alpha\omega - \frac{1}{3}\omega^2 \sum_{j\in m} (d, B_{jm}\, d)d_j/(d, Kd)].$$

We assume, as in [3], that for W on the line segment I joining u^o to $u^o + 2d^o$,

$$\Big(\sum_{j\in m} |B_{jm}(w)|^2\Big)^{\frac{1}{2}} \leq \mu$$

where the spectral norm is used in the sum.

Let

$$\Phi_K = (d, Kd)/(d,d)$$

$$\psi_K = \mu|d|/\Phi_K$$

$$\alpha_o = \max(\alpha, 1)$$

then if we set

$$\gamma_o = \frac{2}{\alpha_o + \sqrt{(\alpha_o^2 + \frac{8}{3}\psi_K)}}$$

it follows that $\gamma_o \leq 1$. If we then choose ω in

$$0 < \omega < 2\gamma_o$$

we get, as in [3], that

$$-\Delta g \geq \frac{1}{2}\omega\,(d, Kd)(2 - \hat{\omega}) > 0$$

where $\hat{\omega} = \omega/\gamma_o$. This guarantees that g will decrease with each iterate whose active residual is not zero. To get the next iterate, a new m, K and ω are to be chosen. They may of course be the same as the previous choice in certain instances.

3. Convergence

In order to obtain convergence of the iterates we need further restrictions on the choice of the matrices K. A method for obtaining this is to get an estimate of the form

$$(3.1) \qquad -\Delta g \geq C|r_m^o|^2$$

where C is a constant independent of the iterates. This requires uniform upper and lower bounds on the K matrices and a lower bound on γ_o.

This may be achieved by assuming that K is a reasonable approximation to A_m and that the spectral norm of K is uniformly bounded. To this end we assume that there exist positive constants, β and C_o independent of the iterates such that

$$(3.2) \qquad |K| < C_o$$

$$(3.3) \qquad \beta(d, A_m d) \le (d, Kd).$$

These inequalities readily yield

$$(3.4) \qquad (d, Kd) \ge C_1 |r_m^o|^2 .$$

The lower bound of γ_o is obtained from a lower bound on Φ_K. Since A_m is positive definite it follows as in [2] that

$$A_m \ge \lambda > 0$$

$$\Phi_K \ge \lambda\beta > 0$$

from a priori estimates on the level set containing the iterates. This is attained by the assumption that the original level set is bounded.

Under the further assumptions of theorem 4.1 of [3] we obtain global convergence of this approximate block relaxation process. We may state:

Theorem 3.1. Assume that the level set

$$D = \{u | g(u) \le g(u^o)\}$$

is bounded and that the sequence of multi indeces $\{m_p\}$ is a cyclic ordering covering Z infinitely often. Assume that the sequence of matrices $\{A_{m_p}\}$ are positive definite and that the matrices $\{K_p\}$ are given to satisfy (3.2) and (3.3) then a sequence $\{\omega_p\}$ may be chosen so that for the process (2.1), $r(u^p) \to 0$. If the solution is unique, the iterates $\{u^p\}$ converge to the solution.

We note that convexity of g is not required, only convexity in the subspaces defined by $\{m_p\}$.

The proof, once the estimate (3.4) is known, follows along the lines of Theorem 8.1 of [2].

4. Remarks

i) It is clear from (2.2) that when the matrices B_{jm}, $j \in m$ are all positive definite and if the $d_j \le 0$, we obtain a decrease in g for

$$0 < \alpha\,\omega < 2$$

and γ_o does not enter. For the usual choice of $\alpha = 1$, the full range of ω is then available. This was noted previously [3], in the scalar case. An example of this is given by a class of semilinear elliptic equations of the form $\Delta\Phi = f(\Phi)$ where $f''(\Phi) \ge 0$. For the usual finite difference approximation to such problems the B_{jm} take the form $B_{jm} = f''(\Phi_j)(e_j e_j^T)_m$, where e_j is the jth unit vector in R^n, and are nonnegative definite.

If $r_m^o \ge 0$ and K is monotone we will get $d \le 0$. The choice of $K = A_m$ in the cited example will yield this property.

ii) A common choice of K, in general, might be a finite difference approximation to A_m

$$a_{ij}(u) = \frac{\Delta_j r_i}{h_j} + O(h_j)$$

$$\Delta_j r_i = r_i(u + h_j e_j) - r_i(u).$$

Thus

$$K = A_m + BH = (\Delta_j r_i / h_j)$$

where H is a diagonal matrix with entries h_j, $j \in m$ and B contains entries in g'''. If H is sufficiently small, we may estimate

$$|(d, BHd)| \le \epsilon(d,d)$$

where ϵ depends only on g. We then get

$$(d, Kd) \geq (d, A_m d)(1 - \epsilon/\Phi_A)$$

$$\geq \beta \; (d, A_m d)$$

with $\beta = 1 - \epsilon/\lambda$, which is positive for $\epsilon < \lambda$.

iii) If we wish to choose $K = I$, the identity, then we may choose $\beta = 1/\Lambda$ where Λ is the largest of the eigenvalues of A_m in a suitably large but bounded domain.

iv) Since the method described includes the case of m being the full index set $(1, 2, \ldots, n)$, we may consider solving an elliptic system

$$r(u) \equiv Lu - f = 0$$

by a sequence of problems

$$K(u^1 - u^o) = - \omega r(u^o)$$

and the above results will apply. For example the choice of $K = - \Delta_h$, the discrete Laplacian, has been proposed by Concus and Golub [1] for solving a linear elliptic problem. This has the virtue that K may be a much simpler operator to use than L. Our results indicate that such methods are feasible in the nonlinear case as well.

5. A Counterexample

For a smooth convex function of one variable, it is well known that the difference quotient of its derivative is nonnegative:

$$(g'(u+h) - g'(u))/h \geq 0$$

for all u and u + h in the domain of definition. Furthermore if $g''(u) \geq \lambda > 0$ for all u, then its difference quotients have the same lower bound for all u and h.

The question now arises about the corresponding statement in higher dimensions. That is, if we define the matrix

$$D = (\Delta_{ij})$$

by $\Delta_{ij} = \Delta_j r_i / h_j$ can we conclude that $(w, Dw) \geq 0$ for all u and h_j, w

if $A(u) \geq 0$ for all u?

Furthermore if $A(u) \geq \lambda > 0$ globally can we conclude that there is a $\lambda_o > 0$ such that

$$(w, Dw) \geq \lambda_o \ (w,w)$$

for all w, h_j ?

It follows from the previous section that for sufficiently smooth g the statements are true for all u but for sufficiently small h_j. We will show, by a counterexample, that the statements are not valid for all h_j.

The example we will first use is $g = (x^2 + y^2)^{\frac{1}{2}}$ in two dimensions. Thus g is convex and $r_1 = x/g$, $r_2 = y/g$. We choose the point $(x_o, y_o) = (1, 1)$ and set $h_1 = h$, $h_2 = 0$ so that $\Delta_{12} = a_{12}$, $\Delta_{22} = a_{22}$

$$D = \begin{pmatrix} \Delta_{11} & a_{12} \\ \Delta_{21} & a_{22} \end{pmatrix}$$

$$g_o = g(1,1) = \sqrt{2}\ , \quad g_1 = g(1+h,1), \quad x_1 = 1+h$$

$$\Delta_{11} = (\frac{x_1}{g_1} - \frac{1}{g_o})/h\ , \quad \Delta_{21} = (\frac{1}{g_1} - \frac{1}{g_o})/h\ ,$$

$$a_{12} = -\,xyg^{-3} \quad , \quad a_{22} = x^2\, g^{-3}.$$

If it were true that $(w, D\,w) \geq 0$, then, since the eigenvalues of D would be nonnegative, the determinant is likewise. Thus we show that $\det(D) < 0$ and get a contradiction. Since at (1,1)

$$a_{22} = g_o^{-3} = -\,a_{12}$$

$$\det(D) = \frac{1}{g_o^3}(\Delta_{11} + \Delta_{21}) = \frac{1}{hg_o^3}\left(\frac{2+h}{g_1} - \sqrt{2}\right).$$

But $\left(\frac{2+h}{g_1}\right)^2 = \frac{(1 + x_1)^2}{1 + x_1^2} = 1 + 2\frac{x_1}{1 + x_1^2} < 2$

for $h \neq 0$, so that the det (D) < 0 and we get a contradiction.

To obtain an example for a smooth, in fact C^∞, convex function we choose a perterbation of g:

$$g_\epsilon = (x^2 + y^2 + \epsilon)^{\frac{1}{2}} \quad \text{for } \epsilon > 0.$$

Let D_ϵ be the corresponding matrix at (1,1). It follows from continuity considerations that the $\det(D_\epsilon)$ is negative for small enough ϵ. In fact a direct check for $\epsilon \leq 1/2$ and $h > 20\epsilon$ shows that $\det(D_\epsilon) < 0$. Since the Hessian of g_ϵ is uniformly bounded from below, both conjectures are false.

It follows from the remark (ii) that if we choose $K = D + E$, where E is positive definite and of order $|HB|$, we will satisfy the positivity.

APPENDIX B

PUBLICATIONS SUPPORTED BY THIS CONTRACT

Schechter, S., "On the Choice of Relaxation Parameters for Nonlinear Problems" in: Numerical Solution of Systems of Nonlinear Algebraic Equations, eds. G. D. Byrne and C. A. Hall, Academic Press, New York, pp. 349-372, 1973.

Schechter S., "An Approximate Block Relaxation" to appear in SIAM J. Numer. Anal.

Ablow, C. M. and S. Schechter, "Compylotropic Coordinates", submitted for publication, Abstract in Notices of the American Mathematical Society, Vol. 24, No. 3, No. 746-C3, 1977.

REFERENCES

1. P. Concus and G. H. Golub, Use of fast direct methods for the efficient numerical solution of nonseparable elliptic equations, SIAM J. Numer. Anal. 10 (1973), pp. 1103-1120.

2. S. Schechter, Minimization of a convex function by relaxation, Chap. 7 in Integer and Nonlinear Programming, J. Abadie ed., North Holland, Amsterdam, 1970, pp. 177-189.

3. S. Schechter, On the choice of relaxation parameters for nonlinear problems, Numerical Solution of Systems of Nonlinear Algebraic Equations, G. D. Byrne and C. A. Hall eds., Academic Press, New York, 1973, pp. 349-372.

4. Richtmyer, R. D. and K. W. Morton, Difference Methods for Initial Value Problems, 2nd edition, Interscience Publishers, New York, 1967.

5. Rheinboldt, W., "On M-Functions and Their Application to Nonlinear Gauss-Seidel Iterations and to Network Problems," J. Math. Anal. Applic., Vol. 32, pp. 274-307 (1970).

6. Schechter, S., "Iteration Methods for Nonlinear Problems," Trans. Am. Math. Soc. 104, pp. 179-189 (1962).

7. Schechter, S., "Relaxation Methods for Convex Problems." SIAM J. Numer. Anal., Vol. 5, pp. 601-612, (1968).

8. Ortega, J., and W. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

9. Saul'yev, V. K., Intergration of Equations of Parabolic Type by the Method of Nets, Pergamon Press, New York, 1964.

10. Scheidegger, A. E., The Physics of Flow Through Porous Media, Macmillan Publishing Company, New York, 1960.

11. Douglas, J., "A Survey of Numerical Methods for Parabolic Differential Equations," Advances in Computers, Vol. 2, Academic Press, New York, pp. 1-52, 1961.

12. W. F. Ames, Nonlinear Partial Differential Equations in Engineering, Academic Press, New York, 1965.

13. Liniger, W. and R. A. Willoughby, "Efficient Integration Methods for Stiff Systems of Ordinary Differential Equations," SIAM J. Numer. Anal. 7 pp. 47-66 (1970).

14. S. Schechter, "An Approximate Block Relaxation," to appear.