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STOCHASTIC BILINEAR MODELS

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Abstract

In this paper we consider the behavior for large t of the solutions of Ito equations of the form

$$dx = Axdt+\Sigma B_i xdw_i+\Sigma g_i dw_i$$

In particular we investigate properties of the invariant measures of such systems and describe a certain isotropic characteristic of the invariant measure.

Introduction

The steady stream of papers on bilinear systems in recent years is due in part to the very general nature of this class of systems and in part to the rather satisfying degree to which they can be analyzed. For example there are now two different kinds of approximation theorems [1,2] and [3,4] which illustrate the approximation of quite general nonlinear systems with bilinear ones and there is a general isomorphism theorem for bilinear systems [5,6,7] which uses controllability and observability to prove a uniqueness theorem for bilinear realizations. The Volterra series is a perfectly satisfactory description of the input-output map defined by bilinear systems [8] and the realization problem for bilinear systems is now reasonably well understood [9,10,11]. In our earlier papers [12,13,14] we have undertaken a study of bilinear stochastic models. In the stochastic case the problems seem to be more difficult and, with the exception of those results which have been obtained by analysis of the moments, rather little is known.

In this paper we continue our earlier study of the limiting probability distribution associated with Ito equations of the form

$$dx = Axdt + \sum_{i=1}^{m} B_i xdw_i + \sum_{i=m+1}^{r} g_i dw_i \qquad (*)$$

Our main results concern rather precise estimates on the asymptotic decay (for large $||x||$) of the invariant measures. We introduce a new mixing condition which is stronger than controllability (as used in [14]) but analogous to irreducibility in Markov chain theory, and which guarantees that the invariant measure has a certain isotropic property for $||x||$ large.

2. Motivation

We recall here a few examples from [14] and some comments. For the scalar Ito equation

$$dx = axdt+bxdw_1+cdw_2$$

there is an invariant measure given by

$$\rho(x) = \frac{K(\alpha,\beta)}{(\alpha x^2+1)^\beta} \; ; \quad \alpha = b^2/c^2; \quad \beta = 1-a^2/b^2$$

provided $c \neq 0$ and $a < 0$. We see in this case that only a finite number of moments exist for the limiting distribution, regardless of how many exist initially. By juxtaposing two or more such equations with different values of β we can construct invariant measures for bilinear systems which have the form

$$\rho(x) = \frac{K}{(\alpha_1 x_1^2+1)^{\beta_1}(\alpha_2 x_2^2+1)^{\beta_2}\cdots(\alpha_n x_n^2+1)^{\beta_n}}$$

These systems are "fully elliptic" in that there are as many (in fact twice as many) independent Wiener processes as dim x.

On the other hand, we show in [14] that the bilinear model (*) can have a gaussian invariant measure. In particular, for the Ito equation

$$dx = (S-bb')xdt+\Omega xdw_1+bdw_2$$

with $S-\frac{1}{2}\Omega^2$ and Ω both skew symmetric the gaussian measure

$$\rho(x) = \frac{1}{\sqrt{(2\pi)^n}} e^{-x'x/2}$$

is invariant. This construction is valid for x of any dimension, assuming controllability, and hence may describe a highly degenerate elliptic situation.

This then leads to questions about the "typical" asymptotic behavior of bilinear stochastic equations. In particular, we are interested in the following question. Is it to be expected that the existence of moments is isotropic, i.e. independent of the direction in x-space, as it is for the second example, or is it to be expected that a different number of moments will exist depending on the direction as in the first example? Our results here show that the first alternative is the rule

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and the second is in some sense exceptional.

3. Moment Observability

The main tools available so far to study bilinear stochastic equations are the associated moment equations and certain ideas from controllability and hypoellipticity. We recall from [14] that if we introduce

$$x^{[p]} = \begin{bmatrix} x_1^p \\ \alpha_2 x_1^{p-1} x_2 \\ \vdots \\ x_n^p \end{bmatrix} \in \mathbb{R}^{\binom{n+p-1}{p}}$$

then

$$\frac{d}{dt}\mathcal{E}x^{[p]} = ((A-\Sigma \tfrac{1}{2}B_i^2)_{[p]}+\Sigma \tfrac{1}{2}(B_{i[p]})^2)\mathcal{E}x^{[p]} + G_p\mathcal{E}x^{[p-2]} \qquad (\dagger)$$

where the meaning of [p] is to be derived from expressing the derivative of $x^{[p]}$ along solutions of $\dot{x} = Fx$ as

$$\dot{x} = Fx \iff \frac{d}{dt}x^{[p]} = F_{[p]}x^{[p]}$$

(See [11-14] for more details.) The matrix G_p is constructed in [14].

We recall that for A a square matrix and c a vector, the pair (A,c) is called observable if c is a cyclic vector for A'. This is necessary and sufficient for the initial state of the free dynamical system

$$\dot{x} = Ax; \quad y = \langle c,y\rangle$$

to be determinable from the observation of y and it is sufficient to insure that $y(\cdot)$ is bounded on $[0,\infty)$ if and only if $e^{At}x(0)$ is bounded on $[0,\infty)$.

In two dimensions it happens that for some matrices A, any $c \neq 0$ gives an observable pair. More precisely, in this case if A has complex eigenvalues and c is nonzero then (A,c) is an observable pair (we are assuming A and c are defined over the real field). However, for dimensions greater than 2 there is for each A, vectors c(A) such that c(A) is not zero and yet (A,c(A)) is not an observable pair. We make these comments by way of contrast to what follows.

Consider the Ito equation (*) and the associated moment equation ($\dagger$). We focus attention on the homogeneous part of the moment equation. The Ito equation (*) will be called p-moment observable if for every $c \neq 0$ the pair

$$((A-\tfrac{1}{2}\Sigma B_i^2)_{[p]} + \tfrac{1}{2}\Sigma(B_{i[p]})^2,\ c^{[p]})$$

is observable. What this definition means is this. Note that (see [12])

$$(\langle c,x\rangle)^p = \langle c^{[p]},x^{[p]}\rangle$$

so that

$$\mathcal{E}\langle c,x\rangle^p = \langle c^{[p]},\mathcal{E}x^{[p]}\rangle$$

thus we are isolating the condition under which we can determine any pth moment from the time history of any other pth moment. Only observational vectors of the form $c^{[p]}$, a rather thin subset of all $\binom{n+p-1}{p}$ vectors, have meaning in this context.

Recall that a set of square matrices $\{B_i\}$ is said to act irreducibly if there is no change of basis such that they all take the form

$$PB_iP^{-1} = \begin{bmatrix} B_{11}^i & B_{12}^i \\ 0 & B_{22}^i \end{bmatrix}$$

It should be clear that if $\{B_i\}$ is reducible then $\{B_{i[p]}\}$ is also, as is the whole associative algebra generated by the B_i's or the $B_{i[p]}$'s. Thus we see that the following lemma is valid.

Lemma 1: A necessary condition for p-moment observability of (*) is that $\{A,B_1,B_2\ldots B_m\}$ should act irreducibly.

In order to develop additional intuition about the question of p-moment observability we point out that for (A,b,c) of appropriate dimensions

$$(\langle c,e^{At}b\rangle)^p = \langle c^{[p]},e^{A_{[p]}t}b^{[p]}\rangle$$

but that if

$$\langle c,e^{At}b\rangle = (\sum_{i=1}^{n}\alpha_i e^{\lambda_i t})$$

then

$$\langle c^{[p]},e^{A_{[p]}t}b^{[p]}\rangle = (\Sigma\alpha_i e^{\lambda_i t})^p = \Sigma\beta_{(i_1,i_2,\ldots i_p)}e^{(\lambda_{i1}+\lambda_{i2}+\ldots+\lambda_{ip})t}$$

Thus if the $\binom{n+p-1}{p}$ in number p-fold sums of the eigenvalues are distinct then the triple $(A_{[p]},b^{[p]},c^{[p]})$ is controllable and observable by well known linear system theoretic arguments. This leads to the following conclusion.

Lemma 2: If A has distinct eigenvalues say $\lambda_1,\lambda_2,\ldots,\lambda_n$ and if the p-fold sums of these eigenvalues are distinct then b is a cyclic vector for A if and only if $b^{[p]}$ is a cyclic vector for $A_{[p]}$. Likewise c is a cyclic vector for A' if and only if $c^{[p]}$ is a cyclic vector for $A'_{[p]}$.

We now address the question of existence of p-moment observable systems.

Theorem 1: The stochastic equation

$$dx = Axdt+Bxdw_1+gdw_2 \qquad (**)$$

is generically p-moment observable.

Remark: What we mean by this is that there is an open and dense subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ such that for A and B in this subset we have p-moment observability.

Proof: The set of A's which have no repeated eigenvalues is an open and dense set. If the eigenvalues of A are unrepeated we can put it in diagonal form. In this case $A_{[p]}$ is also diagonal.

The invariant subspaces of A in this case are just the subspaces characterized by vectors with a certain number of zero components. There are, of course 2^n such subspaces corresponding to all possible ways of fixing zero elements. Now consider the moment equations

$$\frac{d}{dt}\mathcal{E}x^{[p]} = [(A-\tfrac{1}{2}B^2)_{[p]} + \tfrac{1}{2}(B_{[p]})^2]\,\mathcal{E}x^{[p]} + G_p\, x^{[p-2]}$$

We let A have unrepeated eigenvalues and be in diagonal form. We are interested in the set of vectors

$$[(A-\tfrac{1}{2}B^2)_{[p]} + \tfrac{1}{2}(B_{[p]})^2]'^k c^{[p]}; \quad k=0,1,\ldots\binom{n+p-1}{p}$$

Now the determinant of matrix formed by these columns can, for each value of A and c, be regarded as a polynomial in the entries of B. We want to show that this polynomial does not vanish identically unless c is zero. From our previous remarks we see that the zeroth degree term in the polynomial vanishes only if c has one or more zero entries.

By reordering the variables we can arrange matters so that a given invariant subspace of A consists of all vectors of the form

$$S(A) = \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{matrix} \}\text{r-components} \\ \}\text{(n-r)-components} \end{matrix}$$

$$S(A_{[p]}) = \begin{bmatrix} 0 \\ * \end{bmatrix} \begin{matrix} \}\binom{n+p-1}{p}-\binom{n-r-p-1}{p}\text{-components} \\ \}\binom{n-r-p-1}{p}\text{-components} \end{matrix}$$

Now for this subspace to remain invariant when we add $\frac{1}{2}B^2{}_{[p]} + \frac{1}{2}(B_{[p]})^2$ it is necessary that

$$\tfrac{1}{2}(B^2)_{[p]} + \tfrac{1}{2}(B_{[p]})^2 \quad \left[\begin{array}{c|c} \sim & 0 \\ \hline \sim & \sim \end{array}\right]$$

But for generic B no such zero block exists and so generically we have p-moment observability.

4. Asymptotic Properties of Invariant Measures

We now apply the previous results to the study of the possible forms of invariant measures achieved as the limiting distribution of stochastic differential equations of the form here. The main thrust of the following theorem together with Theorem 1, is to show that typically when limiting distributions exist they are isotropic in the sense that their rate of decay along any line in x-space is roughly the same.

Theorem 2: If the moments of (*) up to and including order p-1 are asymptotically stable and if the moment equation of order p is unstable then for no c does

$$\lim_{t\to\infty} \mathcal{E}(\langle c, x(t)\rangle)^p$$

exist, provided (*) is p-moment observable.

Proof: This is an easy consequence of the earlier remarks on observability and the definition of p-moment observability.

One reason this result seems interesting is that it points out more precisely what forms the invariant measures for bilinear stochastic systems can take. In particular it can be used as a guide in attempting to find series expansions for the invariant measures.

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