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CONCAVITY ARGUMENTS AND GROWTH ESTIMATES FOR LINEAR INTEGRODIFF--ETC(U)
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR-77-1217
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:

4. TITLE (and Subtitle): CONCAVITY ARGUMENTS ~~&~~ and GROWTH ESTIMATES FOR LINEAR INTEGRODIFFERENTIAL EQUATIONS IN HILBERT SPACE II. DAMPED EQUATIONS & APPLICATIONS TO A CLASS OF HOLOHERDAL ISOTROPIC DIELECTRICS,
5. TYPE OF REPORT & PERIOD COVERED: Interim rept.,
6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): Frederick Bloom and
8. CONTRACT OR GRANT NUMBER(s): AFOSR [redacted] 77-3396

9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of South Carolina, Department of Mathematics and Computer Science, Columbia, SC
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A4

11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientifc Research/NM, Bolling AFB, DC 20332
12. REPORT DATE: [redacted]77
13. NUMBER OF PAGES: 29

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 31p.
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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AFOSR-TR- 77-1217

Concavity Arguments and Growth Estimates for Linear Integrodifferential Equations in Hilbert Space II. Damped Equations and Applications to a Class of Holohedral Isotropic Dielectrics*

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Abstract

Concavity arguments are employed so as to obtain growth estimates for solutions to two initial-value problems associated with a class of damped integrodifferential equations in Hilbert space; by applying the results obtained in this abstract setting we obtain growth estimates for the gradients of electric displacement fields which occur in a class of holohedral isotropic nonconducting rigid dielectrics.

*This research was supported in part by AFOSR Grant 77-3396

1. Introduction

In [1] we employed a modified version of a concavity argument due to Levine [2] to obtain growth estimates for solutions to a class of initial-value problems associated with an undamped linear integrodifferential equation in Hilbert space; our results were subsequently applied to the derivation of growth estimates for the gradients of electric displacement fields occuring in a class of rigid nonconducting material dielectrics of Maxwell-Hopkinson type. Unlike the stability and growth estimates which were obtained in [3], the results of [1] do not require that the electric displacement fields belong to certain uniformly bounded sets in a given function space.

In the present work we will extend certain of the results of [1] to a class of damped linear integrodifferential equations in Hilbert space; it will be clear from the form of our results, and the specific content of the associated hypotheses, that analogous estimates for the undamped equations can not be recovered from the estimates contained in the present paper by simply setting the damping coefficient equal to zero. The growth estimates derived in §2 are applied, in §3, to the evolution of electric displacement fields which occur in a class of holohedral isotropic rigid dielectrics of the type first studied by Toupin & Rivlin [4]; stability and growth estimates for electric displacement fields which occur in such dielectrics were previously obtained in [5] but, as in [3], these estimates were derived via a logarithmic convexity argument which, by its intrinsic nature, requires that the class of electric displacement fields considered as admissible satisfy an a priori upper bound in the norm of a certain Hilbert space.

2. Growth Estimates for a Damped Abstract Integrodifferential Equation.

As in [1] we will denote by H any real Hilbert space with inner-product $\langle , \rangle$ and norm $||(\cdot)||$. By H_+ we denote a second Hilbert space with inner-product $\langle , \rangle_+$ and norm $||(\cdot)||_+$; we assume that $H_+ \subseteq H$, both algebraically and topologically, and we let $\gamma > 0$ denote the embedding constant for the map $i: H_+ \to H$ (i.e., for all $\underset{\sim}{v} \in H_+$, $||\underset{\sim}{v}|| \le \gamma ||\underset{\sim}{v}||_+$). Finally we define H_- to be the completion of H under the norm $||(\cdot)||_-$ that is given by

$$||w||_- = \sup_{\underset{\sim}{v} \in H_+} (|\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle| / ||\underset{\sim}{v}||_+)$$

Appearing in the statement of the abstract initial-value problems which we will consider are operators $\underset{\sim}{N} \in L_S(H_+, H_-)$ and $\underset{\sim}{K} \in L^2((-\infty,\infty); L_S(H_+,H_-))$ where $L_S(H_+,H_-)$ denotes the space of all bounded linear operators from H_+ into H_-. We assume that $\underset{\sim}{K}_t(t)$, the strong operator derivative of $\underset{\sim}{K}$, exists and that $\underset{\sim}{K}_t \in L^2((-\infty,\infty); L_S(H_+,H_-))$. In addition we will require that

(i) $-\langle \underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v} \rangle \ge 0, \ \forall \underset{\sim}{v} \in H_+$

(ii) $\langle \underset{\sim}{v}, \underset{\sim}{N}\underset{\sim}{v} \rangle \ge 0, \ \forall \underset{\sim}{v} \in H_+$

and

(iii) $\int_0^\infty ||\underset{\sim}{K}(\tau)||_{L_S(H_+,H_-)} d\tau < \infty$ and

$$\int_0^T \int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt < \infty$$

for each $T > 0$. We remark that whereas (ii) was not needed for the logarithmic convexity argument that was employed in [5] we did require in [5] that $\underset{\sim}{K}(0)$ satisfy a hypothesis which is stronger than (i) above, namely,

$$\text{(i')} \quad -\langle \underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v}\rangle \geq \kappa ||\underset{\sim}{v}||_+^2, \quad \forall \underset{\sim}{v} \in H_+$$

$$\text{with } \kappa \geq \gamma T \sup_{[0,\infty)} ||\underset{\sim}{K}_t(t)||_{L_S(H_+,H_-)}$$

Now, let $\Gamma > 0$ be a given real number. The problems of interest to us here assume the following forms:

<u>Problem A</u> For any $\alpha > 0$ denote by $\underset{\sim}{u}^\alpha$ a strong solution(1) to

$$\text{(2.1a)} \quad \underset{\sim}{u}^\alpha_{tt} + \Gamma \underset{\sim}{u}^\alpha_t - \underset{\sim}{N}\underset{\sim}{u}^\alpha + \int_{-\infty}^t \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^\alpha(\tau)d\tau = \underset{\sim}{0}, \quad 0 \leq t < T$$

$$\text{(2.1b)} \quad \underset{\sim}{u}^\alpha(0) = \alpha \underset{\sim}{u}_0, \ \underset{\sim}{u}^\alpha_t(0) = \underset{\sim}{v}_0 \quad (\underset{\sim}{u}_0, \underset{\sim}{v}_0 \in H_+)$$

$$\text{(2.1c)} \quad \underset{\sim}{u}^\alpha(\tau) = \underset{\sim}{U}(\tau), \quad -\infty < \tau < 0$$

where $\underset{\sim}{U}:(-\infty,0) \to H_+$ and satisfies $\int_{-\infty}^0 ||\underset{\sim}{U}(\tau)||_+ d\tau < \infty$.

We seek to derive a lower bound for $\sup_{-\infty<t<T} ||\underset{\sim}{u}^\alpha||_+$ in terms of α, γ, Γ, $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0$, $\underset{\sim}{U}$, the length of the interval $[0,T)$, and the operator norms of $\underset{\sim}{N}$, $\underset{\sim}{K}$, and $\underset{\sim}{K}_t$.

(1) $\underset{\sim}{u}^\alpha \in C^2([0,T); H_+)$, for each $\alpha > 0$, with $\underset{\sim}{u}^\alpha_t \in C^1([0,T); H_+)$ and $\underset{\sim}{u}^\alpha_{tt} \in C([0,T); H_-)$.

<u>Problem B</u> For any $\beta > 0$ denote by $\underset{\sim}{u}^{\beta}$ a strong solution to

$$(2.2a)\qquad \underset{\sim}{u}^{\beta}_{tt} + \Gamma \underset{\sim}{u}^{\beta}_{t} - \underset{\sim}{N}\underset{\sim}{u}^{\beta} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\beta}(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T$$

$$(2.2b)\qquad \underset{\sim}{u}^{\beta}(0) = \underset{\sim}{u}_0,\ \underset{\sim}{u}^{\beta}_t(0) = \underset{\sim}{v}_0 \qquad (\underset{\sim}{u}_0, \underset{\sim}{v}_0 \in H_+)$$

$$(2.2c)\qquad \underset{\sim}{u}^{\beta}(\tau) = g(\beta)\underset{\sim}{U}(\tau), \quad -\infty < \tau < 0$$

where $g(\beta)$ is a monotonically increasing function of β for $0 < \beta < \infty$. We seek a lower bound for $\sup_{-\infty<t<T} ||\underset{\sim}{u}^{\beta}||_{+}$ in terms of the data indicated above.

In each of the two problems stated above we will require, in addition to conditions (i) - (iii) that $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0$ and $\underset{\sim}{U}$ satisfy

(iv) $\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle > 0$

and

(v) $\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau \rangle < 0$

We are now ready to state and prove our first estimate, namely,

<u>Theorem II.1</u> Let $\underset{\sim}{u}^{\alpha}$ be a strong solution to (2.1a) - (2.1c) and suppose that

$$(2.3a)\qquad (a)\quad ||\underset{\sim}{u}_0||^2 \le \frac{2}{\Gamma} \langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle$$

$$(2.3b)\qquad (b)\quad T > \frac{1}{\Gamma} \ln \left[\frac{2\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle}{2\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle - \Gamma ||\underset{\sim}{u}_0||^2} \right]$$

Then for each $\alpha > ||\underset{\sim}{v}_0|| / \langle \underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0 \rangle^{1/2}$

$$\text{(2.4)} \qquad \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\alpha}(t)||_{+} \geq \left[\frac{|<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>|}{\gamma\sum_T}\right]^{\frac{1}{2}} \sqrt{\alpha}$$

where

$$\text{(2.5)} \qquad \sum_T = \tfrac{1}{2}||\underset{\sim}{N}||_{L_S(H_+,H_-)} + \int_0^{\infty}||\underset{\sim}{K}(\tau)||_{L_S(H_+,H_-)}d\tau$$

$$+ \int_0^T \int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)}d\tau dt$$

<u>Proof</u> Suppose that for some $\alpha = \bar{\alpha}$,

$$\bar{\alpha} > ||\underset{\sim}{v}_0||/<\underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0>^{\frac{1}{2}}$$

$$\text{(2.6)} \qquad \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar{\alpha}}||_{+} < \left[\frac{<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>}{\gamma\sum_T}\right]^{\frac{1}{2}} \sqrt{\bar{\alpha}}$$

where T satisfies (2.3b). If, as in [1], we set

$$\text{(2.7)} \qquad F_{\bar{\alpha}}(t) = <\underset{\sim}{u}^{\bar{\alpha}}(t), \underset{\sim}{u}^{\bar{\alpha}}(t)>, \quad 0 \leq t < T$$

then a direct computation yields

$$\text{(2.8)} \qquad F_{\bar{\alpha}}F_{\bar{\alpha}}'' - (\bar{\alpha}+1)F'^2 = 4(\bar{\alpha}+1)S_{\bar{\alpha}}^2$$

$$+ 2F_{\bar{\alpha}}\{<\underset{\sim}{u}^{\bar{\alpha}}, \underset{\sim}{u}_{tt}^{\bar{\alpha}}> - (2\bar{\alpha}+1)<\underset{\sim}{u}_t^{\bar{\alpha}}, \underset{\sim}{u}_t^{\bar{\alpha}}>\}$$

where $S_{\bar{\alpha}}^2 = ||\underset{\sim}{u}^{\bar{\alpha}}||^2\ ||\underset{\sim}{u}_t^{\bar{\alpha}}||^2 - <\underset{\sim}{u}^{\bar{\alpha}}, \underset{\sim}{u}_t^{\bar{\alpha}}> \geq 0$ by virtue of the

Schwartz inequality. Thus

$$(2.9) \qquad F_{\bar\alpha}F_{\bar\alpha}'' - (\bar\alpha+1)F_{\bar\alpha}'^2 \geq 2F_{\bar\alpha}J_{\bar\alpha}, \quad 0 \leq t < T$$

where by (2.8) and (2.1a)

$$(2.10) \qquad J_{\bar\alpha}(t) = \langle \underset{\sim}{u}^{\bar\alpha}, \underset{\sim\sim}{N}\underset{\sim}{u}^{\bar\alpha}\rangle - \langle \underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau\rangle$$

$$- \Gamma\langle \underset{\sim}{u}^{\bar\alpha}, u_t^{\bar\alpha}\rangle - (2\bar\alpha+1)\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim}{u}_t^{\bar\alpha}\rangle$$

We will show that $[e^{\Gamma t}(F_{\bar\alpha}^{-\bar\alpha})']' \leq 0$ for $0 \leq t < T$ by proving that, under hypotheses (i) - (v) above, $J_{\bar\alpha}(t) \geq -(\Gamma/2)F_{\bar\alpha}'(t)$; this, in turn, will lead to a contradiction of (2.6). Directly from (2.10) we have

$$(2.11) \qquad J_{\bar\alpha}'(t) = 2\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim\sim}{N}\underset{\sim}{u}^{\bar\alpha}\rangle - \frac{d}{dt}\langle \underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau\rangle$$

$$- \Gamma\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim}{u}_t^{\bar\alpha}\rangle - \Gamma\langle \underset{\sim}{u}^{\bar\alpha}, \underset{\sim}{u}_{tt}^{\bar\alpha}\rangle - 2(2\bar\alpha+1))\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim}{u}_{tt}^{\bar\alpha}\rangle$$

$$= - 4\bar\alpha\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim\sim}{N}\underset{\sim}{u}^{\bar\alpha}\rangle - \frac{d}{dt}\langle \underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau\rangle$$

$$- \Gamma\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim}{u}_t^{\bar\alpha}\rangle - \Gamma\langle \underset{\sim}{u}^{\bar\alpha}, \underset{\sim}{u}_{tt}^{\bar\alpha}\rangle + 2\Gamma(2\bar\alpha+1)\langle \underset{\sim}{u}_t^{\bar\alpha}, \underset{\sim}{u}_t^{\bar\alpha}\rangle$$

$$+ 2(2\bar\alpha+1)\langle \underset{\sim}{u}_t^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau\rangle$$

By integrating (2.11) from zero to t, using the fact that

$$(2.12) \qquad J_{\bar\alpha}(0) = \bar\alpha^2\langle \underset{\sim}{u}_0, \underset{\sim\sim}{N}\underset{\sim}{u}_0\rangle - \bar\alpha\langle \underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle$$

$$-\Gamma\bar\alpha\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle - (2\bar\alpha+1)||\underset{\sim}{v}_0||^2$$

and dropping the term proportional to $||\mathbf{u}_t^{\bar{\alpha}}||^2$, we obtain

(2.13) $$J_{\bar{\alpha}}(t) \geq J_{\bar{\alpha}}(0) - 2\bar{\alpha}(\langle \mathbf{u}^{\bar{\alpha}}, \mathbf{N}\mathbf{u}^{\bar{\alpha}}\rangle - \bar{\alpha}^2\langle \mathbf{u}_0, \mathbf{N}\mathbf{u}_0\rangle)$$

$$- \langle \mathbf{u}^{\bar{\alpha}}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\mathbf{u}^{\bar{\alpha}}(\tau)d\tau\rangle$$

$$+ \bar{\alpha}\langle \mathbf{u}_0, \int_{-\infty}^{0} \mathbf{K}(-\tau)\mathbf{U}(\tau)d\tau\rangle$$

$$- \Gamma(\langle \mathbf{u}^{\bar{\alpha}}, \mathbf{u}_t^{\bar{\alpha}}\rangle - \bar{\alpha}\langle \mathbf{u}_0, \mathbf{v}_0\rangle)$$

$$+ 2(2\bar{\alpha}+1)\int_0^t \langle \mathbf{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\mathbf{u}^{\bar{\alpha}}(\lambda)d\lambda\rangle d\tau$$

$$= (2\bar{\alpha}+1)[\bar{\alpha}^2\langle \mathbf{u}_0, \mathbf{N}\mathbf{u}_0\rangle - ||\mathbf{v}_0||^2] - 2\bar{\alpha}\langle \mathbf{u}^{\bar{\alpha}}, \mathbf{N}\mathbf{u}^{\bar{\alpha}}\rangle$$

$$+ 2(2\bar{\alpha}+1)\int_0^t \langle \mathbf{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\mathbf{u}^{\bar{\alpha}}(\lambda)d\lambda\rangle d\tau$$

$$- \langle \mathbf{u}^{\bar{\alpha}}, \int_{-\infty}^{t} \mathbf{K}(t-\tau)\mathbf{u}^{\bar{\alpha}}(\tau)d\tau\rangle - \Gamma\langle \mathbf{u}^{\bar{\alpha}}, \mathbf{u}_t^{\bar{\alpha}}\rangle$$

However, in view of hypothesis (i) above

(2.14) $$\int_0^t \langle \mathbf{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\mathbf{u}^{\bar{\alpha}}(\lambda)d\lambda\rangle d\tau \geq$$

$$\int_0^t \frac{d}{dt}\langle \mathbf{u}^{\bar{\alpha}}, \int_{-\infty}^{\tau} \mathbf{K}(\tau-\lambda)\mathbf{u}^{\bar{\alpha}}(\lambda)d\lambda\rangle d\tau$$

$$- \int_0^t \langle \mathbf{u}^{\bar{\alpha}}(\tau), \int_{-\infty}^{\tau} \mathbf{K}_\tau(\tau-\lambda)\mathbf{u}^{\bar{\alpha}}(\lambda)d\lambda\rangle d\tau$$

and, therefore, (2.13) yields

(2.15) $$J_{\bar{\alpha}}(t) \geq (2\bar{\alpha}+1)[\bar{\alpha}^2\langle \mathbf{u}_0, \mathbf{N}\mathbf{u}_0\rangle - ||\mathbf{v}_0||^2$$

$$+ 2\bar{\alpha}\,|\,\langle \mathbf{u}_0, \int_{-\infty}^{0} \mathbf{K}(-\tau)\mathbf{U}(\tau)d\tau\rangle\,|]$$

$$- 2\bar{\alpha} \langle \underset{\sim}{u}^{\bar{\alpha}}, \underset{\sim\sim}{N u}^{\bar{\alpha}} \rangle$$

$$+ (4\bar{\alpha}+1)\langle \underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar{\alpha}}(\tau)d\tau \rangle$$

$$- 2(2\bar{\alpha}+1)\int_{0}^{t}\langle \underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{\tau} \underset{\sim}{K}_{\tau}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda \rangle d\tau$$

$$- \Gamma\langle \underset{\sim}{u}^{\bar{\alpha}}, \underset{\sim}{u}_{t}^{\bar{\alpha}} \rangle$$

where we have made use of hypothesis (v) above. If we set

$$M_{T,\bar{\alpha}} = \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau \rangle|}{\gamma \sum_T}\right]^{\frac{1}{2}} \sqrt{\bar{\alpha}}$$

where $\sum_T$ is given by (2.5) then routine estimates employing the the Schwartz inequality, coupled with the assumption (2.6), yield the lower bounds

(2.16a) $$\langle \underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar{\alpha}}(\tau)d\tau \rangle \geq - \gamma M_{T,\bar{\alpha}}^2 \int_0^{\infty} ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

(2.16b) $$-\int_{0}^{t}\langle \underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{t} \underset{\sim}{K}_{\tau}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda \rangle d\tau$$

$$\geq - \gamma M_{T,\bar{\alpha}}^2 \int_0^T \int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt$$

and

(2.16c) $$- \langle \underset{\sim}{u}^{\bar{\alpha}}, \underset{\sim\sim}{N u}^{\bar{\alpha}} \rangle \geq - \gamma M_{T,\bar{\alpha}}^2 ||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

Combining the above estimates with (2.15), making use of the definition of $\sum_T$ again, and using the assumption that $\bar{\alpha} > ||\underset{\sim}{v}_0||/\langle \underset{\sim}{u}_0, \underset{\sim\sim}{N u}_0 \rangle^{\frac{1}{2}}$ we easily obtain from (2.15)

$$(2.17)\qquad J_{\bar\alpha}(t) \geq -\Gamma\langle \underset{\sim}{u}^{\bar\alpha}, \underset{\sim}{u}^{\bar\alpha}_t\rangle = -\frac{\Gamma}{2} F_{\bar\alpha}'(t), \quad 0 \leq t < T$$

Therefore, by (2.9)

$$(2.18)\qquad F_{\bar\alpha}F_{\bar\alpha}'' - (\bar\alpha+1)F_{\bar\alpha}'^2 \geq -\Gamma F_{\bar\alpha}F_{\bar\alpha}', \quad 0 \leq t < T$$

or

$$(2.19)\qquad [e^{\Gamma t}(F_{\bar\alpha}^{-\bar\alpha})']' \geq 0, \quad 0 \leq t < T$$

Two successive integrations of (2.19) easily yield the lower bound

$$(2.20)\qquad F_{\bar\alpha}^{\bar\alpha}(t) \geq F_{\bar\alpha}^{\bar\alpha}(0)[1 - (1-e^{-\Gamma t})\bar\alpha F_{\bar\alpha}'(0)/\Gamma F_{\bar\alpha}(0)]^{-1}$$

The expression in the brackets above will vanish at

$$(2.21)\qquad t_\infty \equiv \frac{1}{\Gamma}\,\ell n[2\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle/(2\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle - \Gamma||\underset{\sim}{u}_0||^2)]$$

provided that $[\Gamma F_{\bar\alpha}(0)/\bar\alpha F_{\bar\alpha}'(0)] < 1$. But

$$(2.22)\qquad \frac{\Gamma F_{\bar\alpha}(0)}{\bar\alpha\, F_{\bar\alpha}'(0)} = \frac{\Gamma\bar\alpha^2||\underset{\sim}{u}_0||^2}{2\bar\alpha\langle\bar\alpha \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle} < 1$$

if, as per the hypothesis of the theorem, the initial datum is restricted so as to satisfy $||\underset{\sim}{u}_0||^2 < \frac{2}{\Gamma}\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle$. By our other basic hypothesis, i.e., (2.3b), it follows that $t_\infty < T$ and, therefore, $\sup\limits_{0<t<T} ||\underset{\sim}{u}^{\bar\alpha}(t)|| = +\infty$. Thus,

$$(2.23)\qquad +\infty = \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar\alpha}(t)|| \leq \gamma \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar\alpha}(t)||_+$$

which contradicts (2.6) and establishes the growth estimate (2.4).

Q.E.D.

The last theorem admits the following corollary whose proof is immediate (based on our previous computations):

Corollary II.1 Let $f(\alpha)$ be a real-valued function for $0 < \alpha < \infty$ with $\sup\limits_{\alpha<0}\left(\frac{f(\alpha)}{\alpha}\right) < \infty$. For each $\alpha > 0$ let $\underset{\sim}{u}^{\alpha} \in C^2([0,T_\alpha); H_+)$ be a strong solution to (2.1a), on $[0, T_\alpha)$, subject to (2.1c) and the initial conditions

$$(2.24) \qquad \underset{\sim}{u}^{\alpha}(0) = f(\alpha)\underset{\sim}{u}_0, \quad \underset{\sim}{u}^{\alpha}_t(0) = \underset{\sim}{v}_0$$

where
$$\frac{\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle}{||\underset{\sim}{u}_0||^2} > \frac{\Gamma}{2} \sup_{\alpha>0} \left(\frac{f(\alpha)}{\alpha}\right) \text{ and}$$

$$(2.25) \qquad T_\alpha > \frac{1}{\Gamma} \ln \left[2\alpha\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle / 2\alpha\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle - \Gamma f(\alpha)||\underset{\sim}{u}_0||^2\right]$$

Then for each α satisfying

$$(2.26) \qquad f(\alpha) \geq ||\underset{\sim}{v}_0|| / \langle \underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0\rangle^{\frac{1}{2}}$$

it follows that

$$(2.27) \qquad \sup_{-\infty<t<T_\alpha} ||\underset{\sim}{u}^{\alpha}(t)||_+ \geq \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|}{\gamma \Sigma_{T_\alpha}}\right]^{\frac{1}{2}} \sqrt{f(\alpha)}$$

where Σ_{T_α} is given by (2.5) with $T \to T_\alpha$

Having completed our formal discussion of problem A we now turn our attention to the system consisting of (2.2a) - (2.2c) and state

<u>Theorem II.2</u> For each real $\beta > 0$ let $\underset{\sim}{u}^{\beta} \in C^2([0,T); H_*)$ be a strong solution of (2.2a) - (2.2c) and suppose that the initial datum satisfy

$$\langle \underset{\sim}{u}_0, \underset{\sim\sim}{Nu} \rangle \geq ||\underset{\sim}{v}_0||^2 \tag{2.28}$$

Let $\delta > 0$ be any positive constant; then if

$$T > \frac{1}{\Gamma} \ln [2\alpha_\delta \langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle / (2\alpha_\delta \langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle - \Gamma ||\underset{\sim}{u}_0||^2)]$$

where $\alpha_\delta = (\Gamma ||\underset{\sim}{u}_0||^2 / 2\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle) + \delta$,

$$\sup_{-\infty<t<T} ||\underset{\sim}{u}^{\beta}||_+ \geq \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau \rangle|}{\gamma \Sigma_T}\right]^{\frac{1}{2}} \sqrt{g(\beta)} \tag{2.29}$$

for all β, $0 < \beta < \infty$.

<u>Proof</u>. The proof strongly resembles the line of argument followed in the proof of the previous theorem with one important difference. We assume that for some $\beta = \bar{\beta}$, $0 < \bar{\beta} < \infty$

$$\sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar{\beta}}||_+ < N_{T,\bar{\beta}} \tag{2.30}$$

where

$$N_{T,\bar{\beta}} = \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau \rangle|}{\gamma \Sigma_T}\right]^{\frac{1}{2}} \sqrt{g(\bar{\beta})} \tag{2.31}$$

and $$T > \frac{1}{\Gamma} \ln \left[\frac{2\alpha_\delta \langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle}{2\alpha_\delta \langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle - \Gamma ||\underset{\sim}{u}_0||^2}\right]$$

Defining $F_{\bar{\beta}}(t) = \langle \underset{\sim}{u}^{\bar{\beta}}(t), \underset{\sim}{u}^{\bar{\beta}}(t)$, $0 \leq t < T$, we now compute that for any $\alpha > 0$

$$F_{\bar\beta}F''_{\bar\beta} - (\alpha+1)F'^2_{\bar\beta} \geq 2\, F_{\bar\beta}\, L_{\alpha,\bar\beta}\ , \quad 0 \leq t < T \tag{2.32}$$

where

$$L_{\alpha,\bar\beta}(t) = <\underset{\sim}{u}^{\bar\beta}, \underset{\sim}{N}\underset{\sim}{u}^{\bar\beta}> - <\underset{\sim}{u}^{\bar\beta}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\beta}(\tau)d\tau> - \Gamma<\underset{\sim}{u}_t^{\bar\beta}, \underset{\sim}{u}^{\bar\beta}> - (2\alpha+1)<\underset{\sim}{u}_t^{\bar\beta}, \underset{\sim}{u}_t^{\bar\beta}> \tag{2.33}$$

A direct computation analogous to that used in passing from (2.10) to (2.15), then yields

$$\begin{aligned} L_{\alpha,\bar\beta}(t) \geq\ & (2\alpha+1)[<\underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0> - ||\underset{\sim}{v}_0||^2] - \Gamma<\underset{\sim}{u}^{\bar\beta}, \underset{\sim}{u}_t^{\bar\beta}> \\ & + 2g(\bar\beta)(2\alpha+1)|<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>| \\ & - 2\alpha<\underset{\sim}{u}^{\bar\beta}, \underset{\sim}{N}\underset{\sim}{u}^{\bar\beta}> \\ & + (4\alpha+1)<\underset{\sim}{u}^{\bar\beta}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\beta}(\tau)d\tau> \\ & - 2(2\alpha+1)\int_0^t<\underset{\sim}{u}^{\bar\beta}, \int_{-\infty}^{\tau}\underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\bar\beta}(\lambda)d\lambda>d\tau \end{aligned} \tag{2.34}$$

If we now make use of (2.28), so as to drop the first expression on the right-hand side of (2.34), and then employ the assumption embodied in (2.30) to bound, from below, the last three expressions in the above estimate we easily obtain

$$\begin{aligned} L_{\alpha,\bar\beta}(t) \geq\ & -\frac{\Gamma}{2}\, F'_{\bar\beta}(t) + 2(2\alpha+1)g(\bar\beta)|<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>| \\ & - (2\alpha+1)\gamma N^2_{T,\bar\beta}\left[(\frac{2\alpha}{2\alpha+1})||\underset{\sim}{N}||_{L_S(H_+,H_-)}\right. \end{aligned} \tag{2.35}$$

$$+ \left(\frac{4\alpha+1}{2\alpha+1}\right)\int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

$$+ 2\int_0^T \int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt\Big]$$

or

$$L_{\alpha,\bar\beta}(t) \geq -\frac{\Gamma}{2}F'_{\bar\beta}(t) + 2(2\alpha+1)\Big[g(\bar\beta)|<\underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>| \tag{2.36}$$

$$- \gamma N^2_{T,\bar\beta}\Big(\tfrac{1}{2}||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

$$+ \int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho + \int_0^T\int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt\Big)\Big]$$

$$= -\frac{\Gamma}{2}F'_{\bar\beta}(t)$$

in view of the definitions of $\sum_T$ and $N_{T,\bar\beta}$. Thus, by (2.32) and (2.36_2) it follows that

$$F_{\bar\beta}F''_{\bar\beta} - (\alpha+1)F'^2_{\bar\beta} \geq -\Gamma F_{\bar\beta}F'_{\bar\beta}, \quad 0 \leq t < T \tag{2.37}$$

for any $\alpha > 0$. Integrating (2.37) we find the estimate

$$F_{\bar\beta}(t) \geq F_{\bar\beta}(0)[1 - (1-e^{-\Gamma t})\alpha F'_{\bar\beta}(0)/\Gamma F_{\bar\beta}(0)]^{-1} \tag{2.38}$$

However, (2.38) implies that $F_{\bar\beta}(t)$ tends to $+\infty$ as

$$t \to t_\infty = \frac{1}{\Gamma}\ell n\left[\frac{2\alpha<\underset{\sim}{u}_0, \underset{\sim}{v}_0>}{2\alpha<\underset{\sim}{u}_0, \underset{\sim}{v}_0> - \Gamma||\underset{\sim}{u}_0||^2}\right]$$

(provided we now choose $\alpha = \alpha_\delta$,

$$\alpha_\delta \equiv (\Gamma||\underset{\sim}{u}_0||^2/2<\underset{\sim}{u}_0, \underset{\sim}{v}_0>) + \delta \tag{2.39}$$

so that t_∞ exists). By virtue of the hypotheses of the theorem $T > t_\infty$ and thus $\sup_{[0,T)} ||\underset{\sim}{u}^{\beta}|| = +\infty$. The contradiction to (2.30) now follows via the same type of argument that was eemployed in the last theorem. Q.E.D.

3. Growth Estimates for Holohedral Isotropic Dielectrics

Let $\Omega \subseteq R^3$ be a bounded region which is filled with a non-conducting material dielectric; we assume that $\partial\Omega$, the boundary of Ω, is smooth enough to admit application of the divergence theorem. Let $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{P}$, and $\underset{\sim}{D}$ denote, respectively, the electric field vector, the magnetic flux density, the polarization vector, and the electric displacement vector in Ω; the fields $\underset{\sim}{E}$ and $\underset{\sim}{D}$ are related by

$$\underset{\sim}{D} = \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P}$$

where $\epsilon_0 > 0$ is a physical constant. If we define, in the usual manner, the magnetic intensity $\underset{\sim}{H}$ via

$$\underset{\sim}{H} = \mu_0^{-1} \underset{\sim}{B}, \quad \epsilon_0 \mu_0 = c^{-2}$$

where c is the speed of light in a vacuum, then in a Lorentz reference frame (x^i, t), $i = 1,2,3$, Maxwell's equations have the form

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \text{curl } \underset{\sim}{E} = \underset{\sim}{0}, \quad \text{div } \underset{\sim}{B} = 0 \tag{3.1a}$$

$$\text{curl } \underset{\sim}{H} - \frac{\partial \underset{\sim}{D}}{\partial t} = \underset{\sim}{0}, \quad \text{div } \underset{\sim}{D} = 0 \tag{3.1b}$$

provided the density of free current, the magnetization, and the density of free charge all vanish in Ω and the dielectric medium is non-deformable. In order to obtain a determinate system of equations for the fields appearing in (3.1a) and (3.1b) we must specify a set of constitutive relations, e.g., in a vacuum $\underset{\sim}{P} = \underset{\sim}{0}$ and thus

$$(3.2) \qquad \underset{\sim}{D} = \epsilon_0 \underset{\sim}{E}, \quad \underset{\sim}{H} = \mu_0^{-1} \underset{\sim}{B}$$

while in a rigid, linear, stationary nonconducting dielectric these relations assume the form

$$(3.3) \qquad \underset{\sim}{D} = \underset{\sim}{\epsilon} \cdot \underset{\sim}{E}, \; \underset{\sim}{B} = \underset{\sim}{\mu} \cdot \underset{\sim}{H}$$

with $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ constant second order tensors; these latter relations were first put forth by Maxwell [6] in 1873. A more general theory was introduced by Volterra [7] in 1912 to treat the case where the dielectric is anisotropic, nonlinear, and magnetized; his constitutive equations had the general form

$$(3.4a) \qquad \underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{\epsilon} \cdot \underset{\sim}{E}(\underset{\sim}{x},t) + \underset{-\infty}{\overset{t}{\mathcal{D}}}(\underset{\sim}{E}(\underset{\sim}{x},\tau)), \; \underset{\sim}{x} \in \Omega$$

$$(3.4b) \qquad \underset{\sim}{B}(\underset{\sim}{x},t) = \underset{\sim}{\mu} \cdot \underset{\sim}{H}(\underset{\sim}{x},t) + \underset{-\infty}{\overset{t}{\mathcal{B}}}(\underset{\sim}{H}(\underset{\sim}{x},\tau), \; \underset{\sim}{x} \in \Omega,$$

If the functional $\underset{\sim}{\mathcal{D}}$ is linear and isotropic, and the body satisfies various restrictions that follow from material symmetry considerations, then (3.4a) can be shown to reduce to an equation of the form

$$(3.5) \qquad \underset{\sim}{D}(\underset{\sim}{x},t) = \epsilon \underset{\sim}{E}(x,t) + \int_{-\infty}^{t} \phi(t-\tau) \underset{\sim}{E}(\underset{\sim}{x},\tau) d\tau, \; \underset{\sim}{x} \in \Omega,$$

which embodies the earlier constitutive hypothesis of Hopkinson [8]. The relations (3.4a), (3.4b) were critically examined by Toupin and Rivlin [4] who showed that the a priori separation of electric and magnetic effects which is hypothesized in these equations is inadequate with regard to predicting such phenonmena as the Faraday effect in dielectrics. Toupin and Rivlin [4] thus proposed an extension of the theory embodied in (3.4a) and (3.4b) to one specified by constitutive relations of the form

$$\text{(3.6a)} \qquad \underset{\sim}{D}(\underset{\sim}{x},t) = \sum_{j=0}^{n} \underset{\sim}{a}_j \cdot \underset{\sim}{E}^{(j)}(\underset{\sim}{x},t) + \sum_{j=0}^{n} \underset{\sim}{c}_j \cdot \underset{\sim}{B}^{(j)}(\underset{\sim}{x},t)$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\phi}_1(t,\tau) \cdot \underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau + \int_{-\infty}^{t} \underset{\sim}{\phi}_2(t,\tau) \cdot \underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

$$\text{(3.6b)} \qquad \underset{\sim}{H}(\underset{\sim}{x},t) = \sum_{j=0}^{n} \underset{\sim}{b}_j \cdot \underset{\sim}{E}^{(j)}(\underset{\sim}{x},t) + \sum_{j=0}^{n} \underset{\sim}{d}_j \cdot \underset{\sim}{B}^{(j)}(\underset{\sim}{x},t)$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\psi}_1(t,\tau) \cdot \underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau + \int_{-\infty}^{t} \underset{\sim}{\psi}_2(t,\tau) \cdot \underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

where $\underset{\sim}{E}^{(j)}(\underset{\sim}{x},t) = \dfrac{\partial^j \underset{\sim}{E}(\underset{\sim}{x},t)}{\partial t^j}$, etc., $\underset{\sim}{a}_j \ldots \underset{\sim}{d}_j$ are constant tensors and the kernels $\underset{\sim}{\phi}_i$, $\underset{\sim}{\psi}_i$, $i = 1,2$, are continuous tensor functions of t and τ which are assumed to satisfy growth conditions of the form

$$\text{(3.6c)} \qquad \phi_1(t,\tau) < \underset{\sim}{C}/(t-\tau)^{1+\rho}, \ \rho > 0$$

It can be shown that if the dielectric does not exhibit aging then $\underset{\sim}{D}$ and $\underset{\sim}{H}$ are periodic functions of t whenever $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are.

By combining this result with the assumed growth conditions on the kernel functions $\underset{\sim}{\phi}_i$, ψ_i and employing early results of Volterra on the theory of functionals [7] Toupin and Rivlin [4] conclude that $\underset{\sim}{\phi}_i$ and $\underset{\sim}{\psi}_i$ depend on t and τ only through the difference $t-\tau$; they then prove that if the dielectric has holohedral symmetry (admits the full othogonal group as its group of material symmetry transformations) the consititutive relations (3.6a) and (3.6b) can be reduced to an uncoupled set of the form

$$\text{(3.7a)} \qquad \underset{\sim}{D}(\underset{\sim}{x},t) = \sum_{j=0}^{n} a_j \underset{\sim}{E}^{(j)}(\underset{\sim}{x},t) + \int_{-\infty}^{t} \phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau$$

$$\text{(3.7b)} \qquad \underset{\sim}{H}(\underset{\sim}{x},t) = \sum_{j=0}^{n} b_j \underset{\sim}{B}^{(j)}(\underset{\sim}{x},t) + \int_{-\infty}^{t} \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

where ϕ,ψ are now scalar functions.

In this section we will examine the implications of the growth estimates obtained in §2 for the special case of (3.7a), (3.7b) which corresponds to the simplifying assumptions

$$\text{(3.8)} \qquad a_j = 0,\ b_j = 0,\quad j \ge 1$$

$$\text{(3.9a)} \qquad \underset{\sim}{E}(\underset{\sim}{x},t) = \begin{cases} 0, & -\infty < t < -t_h \\ \underset{\sim}{E}_h(x,t), & -t_h \le t < 0 \end{cases}$$

$$\text{(3.9b)} \qquad \underset{\sim}{B}(\underset{\sim}{x},t) = \begin{cases} 0, & -\infty < t < -t_h \\ B_h(x,t), & -t_h \le t < 0 \end{cases}$$

where $t_h > 0$ is a given positive constant and where $\underset{\sim}{E}_h$, is assumed to satisfy

(3.10) $$\lim_{t\to -t_h^+} \int_\Omega (\underset{\sim}{E}_h(\underset{\sim}{x},t))_i (\underset{\sim}{E}_h(\underset{\sim}{x},t))_i d\underset{\sim}{x} = 0$$

with a similar hypothesis applying to $\underset{\sim}{B}_h$ (in [5] it was assumed that $t_h = 0$). Therefore, the constitutive relations (3.7a), (3.7b) reduce to

(3.11a) $$\underset{\sim}{D}(\underset{\sim}{x},t) = a_0 \underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau, \quad \underset{\sim}{x} \in \Omega$$

and

(3.11b) $$\underset{\sim}{H}(\underset{\sim}{x},t) = b_0 \underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau, \quad \underset{\sim}{x} \in \Omega$$

where we assume that ϕ,ψ are monotonically decreasing functions which are (at least) twice continuously differentiable on $(0,\infty)$. If we now set

(3.12) $$\Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t), \quad t \ge 0$$

$$\phi^1(t) = \frac{1}{a_0}\phi(t), \quad \phi^n(t) = \int_{-t_h}^{t} \phi^1(t-\tau)\phi^{n-1}(\tau)d\tau, \quad n \ge 2$$

and define $\Psi(t)$ in terms of $\psi(t)$ in an analogous fashion, then it is easy to show that the technique of successive approximations, when applied to (3.11a) and (3.11b), yields, respectively,

(3.13a) $$\underset{\sim}{E}(\underset{\sim}{x},t) = a_0^{-1}\underset{\sim}{D}(\underset{\sim}{x},t) + a_0^{-1}\int_{-t_h}^{t} \Phi(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau, \quad \underset{\sim}{x} \in \Omega$$

and

(3.13b) $$\underset{\sim}{B}(\underset{\sim}{x},t) = b_0^{-1}\underset{\sim}{H}(\underset{\sim}{x},t) + b_0^{-1}\int_{-t_h}^{t} \Psi(t-\tau)\underset{\sim}{H}(\underset{\sim}{x},\tau)d\tau, \quad \underset{\sim}{x} \in \Omega,$$

As a direct consequence of (3.9a) and (3.11a) we have

$$\underset{\sim}{D}(x,t) = \begin{cases} 0, & -\infty < t < -t_h \\ \underset{\sim}{D}_h(\underset{\sim}{x},t), & -t_h \le t < 0 \end{cases}$$

and, in view of (3.10),

$$(3.15) \qquad \lim_{t\to -t_h^+} \int_\Omega (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i \, d\underset{\sim}{x} = 0$$

We shall, however, require that the past history $\underset{\sim}{D}_h$ of the electric displacement field satisfy a slightly stronger condition than (3.15), namely, that

$$(3.16) \qquad \lim_{t\to -t_h^+} \underset{\sim}{D}_h(\underset{\sim}{x},t) = \underset{\sim}{0}, \quad \text{uniformly in } \Omega$$

If, in addition, $\underset{\sim}{D}_h(\underset{\sim}{x},t)$ is continuous in t for all t < 0 then for all $\underset{\sim}{x} \in \Omega$, $\underset{\sim}{D}_h(\underset{\sim}{x}, -t_h) = \underset{\sim}{0}$ (for our purposes it is sufficient to assume that $\underset{\sim}{D}_h(\underset{\sim}{x},t)$ is continuous in t for all t in some neighborhood of $-t_h$ of the form $[-t_h,\sigma]$, $\sigma < 0$).

The inverted constitutive relations (3.13a) and (3.13b), when coupled with Maxwell's equations, our hypotheses relative to the past history $\underset{\sim}{D}_h$, and the vector identity

$$(3.17) \qquad \underset{\sim}{\Delta}\underset{\sim}{V}(\underset{\sim}{x}) = \text{grad}(\text{div}\, \underset{\sim}{V}(\underset{\sim}{x})) - \text{curl curl}\, \underset{\sim}{V}(\underset{\sim}{x}), \quad \underset{\sim}{x} \in \Omega$$

now yield the following result regarding the evolution of the electric displacement field $\underset{\sim}{D}$ in a holohedral isotropic dielectric of the type specified by (3.11a) and (3.11b):

<u>Lemma</u> The evolution of the electric displacement field $\underset{\sim}{D}(\underset{\sim}{x},t)$, in any holohedral isotropic nonconducting dielectric (which conforms to the constitutive hypotheses (3.11a) and (3.11b)) is governed by the following system of damped integrodifferential equations in Ω

$$(3.18)\qquad \frac{\partial^2 D_i}{\partial t^2} + \Psi(0)\frac{\partial D_i}{\partial t} + \dot{\Psi}(0)[D_i - c_o\delta_{ik}\delta_{j\ell}\frac{\partial^2 D_k}{\partial x_j \partial x_\ell}]$$

$$+ \int_{-t_n}^{t} (\ddot{\Psi}(t-\tau)D_i(\tau) - (\frac{b_o}{a_o})\phi(t-\tau)\delta_{ik}\delta_{j\ell}\frac{\partial^2 D_k(\tau)}{\partial x_j \partial x_\ell})d\tau$$

$$= 0;\ i = 1,2,3,\ c_0 \equiv b_0/a_0\dot{\Psi}(0)$$

provided the past history $\underset{\sim}{D}_h$ satisfies $\underset{\sim}{D}_h(\underset{\sim}{x},-t_h) = \underset{\sim}{0}$ for all $\underset{\sim}{x} \in \Omega$ and $\dot{\Psi}(0) \neq 0$.

<u>Proof</u> By (3.13a) and the second Maxwell equation in (3.16) it follows that div $\underset{\sim}{E} = \underset{\sim}{0}$. Thus, by (3.17)

$$(3.19)\qquad \underset{\sim}{\Delta}\underset{\sim}{E}(\underset{\sim}{x},t) = -\text{ curl curl }\underset{\sim}{E}(\underset{\sim}{x},t)$$

However, by the first Maxwell relation in (3.1a) and (3.14b) we have

$$(3.20)\qquad \text{curl }\underset{\sim}{E}(\underset{\sim}{x},t) = -\frac{\partial}{\partial t}\underset{\sim}{B}(\underset{\sim}{x},t)$$

$$= -\frac{1}{b_0}\frac{\partial}{\partial t}\underset{\sim}{H}(\underset{\sim}{x},t) - \frac{1}{b_0}\Psi(0)\underset{\sim}{H}(\underset{\sim}{x},t)$$

$$- \frac{1}{b_0}\int_{-t_h}^{t}\Psi_t(t-\tau)\underset{\sim}{H}(\underset{\sim}{x},\tau)d\tau$$

Therefore,

$$\text{(3.21)} \qquad \underset{\sim\sim}{\Delta E}(\underset{\sim}{x},t) = \frac{1}{b_0}\frac{\partial}{\partial t}[\text{curl } \underset{\sim}{H}(\underset{\sim}{x},t)] + \frac{1}{b_0}\Psi(0)\text{curl } \underset{\sim}{H}(\underset{\sim}{x},t)$$

$$+ \frac{1}{b_0}\int_{-t_h}^{t} \Psi_t(t-\tau)\text{curl } \underset{\sim}{H}(\underset{\sim}{x},\tau)d\tau$$

$$= \frac{1}{b_0}\underset{\sim}{D}_{tt}(\underset{\sim}{x},t) + \frac{1}{b_0}\Psi(0)\underset{\sim}{D}_t(x,t)$$

$$+ \frac{1}{b_0}\int_{-t_h}^{t} \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\underset{\sim}{x},\tau)d\tau$$

where we have employed the first Maxwell equation in (3.1b). However,

$$\text{(3.22)} \qquad \int_{-t_h}^{t} \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\underset{\sim}{x},\tau)d\tau = -\int_{-t_h}^{t} \Psi_\tau(t-\tau)\underset{\sim}{D}_\tau(\underset{\sim}{x},\tau)d\tau$$

$$= -\Psi_\tau(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)\Big|_{-t_h}^{t}$$

$$+ \int_{-t_h}^{t} \Psi_{\tau\tau}(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau$$

$$= \Psi_t(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)\Big|_{-t_h}^{t}$$

$$+ \int_{-t_h}^{t} \Psi_{tt}(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau$$

and so (3.21) may be rewritten in the form

$$\text{(3.23)} \qquad \underset{\sim\sim}{\Delta E}(\underset{\sim}{x},t) = b_0^{-1}(\underset{\sim}{D}_{tt}(\underset{\sim}{x},t) + \Psi(0)\underset{\sim}{D}_t(\underset{\sim}{x},t)$$

$$+ \dot{\Psi}(0)\underset{\sim}{D}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \Psi_{tt}(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau)$$

if we use the assumption that $\underset{\sim}{D}_h(\underset{\sim}{x}, -t_h) = \underset{\sim}{0}$ for all $x \in \Omega$. The result now follows if we substitute for $\underset{\sim}{E}$ on the left-hand side of (3.23) from (3.13a) and rearrange terms. Q.E.D.

In conjunction with the system of integrodifferential equations (3.18), for the components of the electric displacement vector $\underset{\sim}{D}$, we will consider initial and boundary data of the form

(3.24a) $\underset{\sim}{D}(\underset{\sim}{x},0) = \underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_t(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x})$, $\underset{\sim}{x} \in \bar{\Omega}$ (2)

(3.24b) $\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0}$, $(\underset{\sim}{x},t) \in \partial\Omega \times [-t_h, T)$.

It will be clear from the analysis presented below that our results will also hold for more general boundary conditions than the homogeneous condition specified by (3.24b). In order to correlate the initial-boundary value problem (3.18), (3.24a), (3.24b) with the abstract initial value problems considered in §2 we introduce the same spaces which were employed in the analysis presented in [1], i.e., we let $C_0^\infty(\Omega)$ denote the set of three dimensional vector fields with compact support in Ω whose components are in $C^\infty(\Omega)$ and we take $H = L_2(\Omega)$, the completion of $C_0^\infty(\Omega)$ under the norm induced by the inner-product

(3.25a) $\langle \underset{\sim}{v},\underset{\sim}{w}\rangle_{L_2} = \int_\Omega v_i w_i \, d\underset{\sim}{x}$

while $H_+ = H_0^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ under the norm induced

(2) $\underset{\sim}{D}_0$ and $\underset{\sim}{D}_1$ are assumed to be continuous on $\bar{\Omega}$.

by the inner - product

$$(3.25b) \qquad \langle \underset{\sim}{v},\underset{\sim}{w}\rangle_{H_0^1} = \int_\Omega \frac{\partial v_i}{\partial x_j}\,\frac{\partial w_i}{\partial x_j}\,d\underset{\sim}{x}$$

Finally $H_- = H^{-1}(\Omega)$, the completion of $C_0^\infty(\Omega)$ under the norm

$$(3.25c) \qquad ||\underset{\sim}{v}||_{H^{-1}} = \sup_{\underset{\sim}{w}\in H_0^1}\left[|\int_\Omega v_i w_i d\underset{\sim}{x}|/(\int_\Omega \frac{\partial w_i}{\partial x_j}\,\frac{\partial w_i}{\partial x_j}\,d\underset{\sim}{x})^{\frac{1}{2}}\right]$$

It is well known that $H_0^1(\Omega) \subseteq L_2(\Omega)$, both topologically and algebraically, and that $H_0^1(\Omega)$ is dense in $L_2(\Omega)$; we denote the embedding constant for the inclusion map i: $H_0^1(\Omega) \rightarrow L_2(\Omega)$ by γ and define operators $\hat{\underset{\sim}{N}} \in \mathcal{L}_S(H_0^1(\Omega);\ H^{-1}(\Omega))$ and $\hat{\underset{\sim}{K}} \in L^2((-\infty,\infty);\ \mathcal{L}_S(H_0^1(\Omega);\ H^{-1}(\Omega)))$ as follows: for any $\underset{\sim}{v} \in H_0^1(\Omega)$

$$(3.26a) \qquad (\hat{\underset{\sim}{N}}\underset{\sim}{v})_i \equiv \dot{\Psi}(0)[c_0\delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j \partial x_\ell} - v_i],\ c_0 \equiv b_0/a_0\dot{\Psi}(0)$$

$$(3.26b) \qquad (\hat{\underset{\sim}{K}}(t)\underset{\sim}{v})_i \equiv \ddot{\Psi}(t)v_i - (\frac{b_0}{a_0})\Phi(t)\delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j \partial x_\ell}$$

To verify the formal symmetry of the operator $\hat{\underset{\sim}{N}}$ we compute[3]

$$(3.27) \qquad \langle \underset{\sim}{v},\hat{\underset{\sim}{N}}\underset{\sim}{v}\rangle_{L_2} = \int_\Omega v_i(\hat{\underset{\sim}{N}}\underset{\sim}{v})_i\,d\underset{\sim}{x}$$

$$= \dot{\Psi}(0)\left[\int_\Omega c_0\delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j\partial x_\ell}v_i d\underset{\sim}{x} - \int_\Omega v_i v_i d\underset{\sim}{x}\right]$$

$$= \dot{\Psi}(0)c_0\left[\int_{\partial\Omega} v_i\,\frac{\partial v_i}{\partial x_j}\,n_j\,d\underset{\sim}{x} - \int_\Omega \frac{\partial v_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,d\underset{\sim}{x}\right]$$

$$- \dot{\Psi}(0)||\underset{\sim}{v}||^2_{L_2}$$

(3) Any $v \in H_0^1(\Omega)$ satisfies $\underset{\sim}{v} = 0$ on $\partial\Omega$ by virtue of a standard trace theorem.

$$= -\dot{\Psi}(0)[c_0||\underset{\sim}{v}||^2_{H^1_0} + ||\underset{\sim}{v}||^2_{L_2}] = \langle \hat{\underset{\sim}{N}}\underset{\sim}{v}, \underset{\sim}{v}\rangle_{L_2}$$

with a similar result for $\underset{\sim}{K}(t)$, $-\infty < t < \infty$. From (3.27) and the definition of c_0 it is clear that condition (ii) of §2, i.e., $\langle \underset{\sim}{v}, \hat{\underset{\sim}{N}}\underset{\sim}{v}\rangle_{L_2} \geq 0$, $\forall \underset{\sim}{v} \in H^1_0(\Omega)$ will be satisfied if $\dot{\Psi}(0) < 0$ and $(b_0/a_0) < 0$. Also, for any $\underset{\sim}{v} \in H^1_0(\Omega)$

$$\text{(3.28)} \qquad \langle \underset{\sim}{v}, \hat{\underset{\sim}{K}}(0)\underset{\sim}{v}\rangle_{L_2} = \int_\Omega v_i[\hat{\underset{\sim}{K}}(0)\underset{\sim}{v}]_i \, d\underset{\sim}{x}$$

$$= \ddot{\Psi}(0)\int_\Omega v_i v_i d\underset{\sim}{x}$$

$$- \left(\frac{b_0}{a_0}\right)\Phi(0)\int_\Omega \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} v_i d\underset{\sim}{x}$$

$$= \ddot{\Psi}(0)||\underset{\sim}{v}||^2_{L_2} - \left(\frac{b_0}{a_0}\right)\Phi(0)[\int_{\partial\Omega} v_i \frac{\partial v_i}{\partial x_j} n_j \, d\underset{\sim}{x}$$

$$- \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} d\underset{\sim}{x}]$$

$$= \ddot{\Psi}(0)||\underset{\sim}{v}||^2_{L_2} + \left(\frac{b_0}{a_0}\right)\Phi(0)||\underset{\sim}{v}||^2_{H^1_0}$$

and, therefore, if $(a_0/b_0) < 0$ then

$$\text{(3.29)} \qquad -\langle \underset{\sim}{v}, \hat{\underset{\sim}{K}}(0)\underset{\sim}{v}\rangle_{L_2} \geq 0, \ \forall \underset{\sim}{v} \in H^1_0(\Omega)$$

provided $\Phi(0) \geq 0$ and $\ddot{\Psi}(0) \leq 0$; with these latter conditions statisfied, and $(a_0/b_0) < 0$, it is clear that the operator $\hat{\underset{\sim}{K}}(0)$ will satisfy condition (i) of §2. Clearly, in the present situation,

$\Gamma = \Psi(0)$ and so we must require that $\Psi(0) > 0$. We now want to delineate the forms assumed by the hypotheses in (iii) of §2; to this end we must first compute

(3.30) $$||\hat{\underset{\sim}{K}}(t)||_{L_S(H_0^1(\Omega);\, H^{-1}(\Omega))} = \sup_{\underset{\sim}{v}\in H_0^1(\Omega)} \frac{|\langle \underset{\sim}{v}, \hat{\underset{\sim}{K}}(t)\underset{\sim}{v}\rangle_{L_2}|}{||\underset{\sim}{v}||^2_{H_0^1}}$$

$$= \sup_{\underset{\sim}{v}\in H_0^1(\Omega)} \frac{|\int_\Omega v_i[\hat{\underset{\sim}{K}}(t)\underset{\sim}{v}]_i d\underset{\sim}{x}|}{||\underset{\sim}{v}||^2_{H_0^1}}$$

$$= \sup_{\underset{\sim}{v}\in H_0^1(\Omega)} \frac{|\ddot{\Psi}(t)||\underset{\sim}{v}||^2_{L_2} + (\frac{b_0}{a_0})\Phi(t)||\underset{\sim}{v}||^2_{H_0^1}|}{||\underset{\sim}{v}||^2_{H_0^1}}$$

$$\le |\ddot{\Psi}(t)| \sup_{\underset{\sim}{v}\in H_0^1(\Omega)} \left(\frac{||\underset{\sim}{v}||^2_{L_2}}{||\underset{\sim}{v}||^2_{H_0^1}}\right) + \left|\frac{b_0}{a_0}\right| |\Phi(t)|$$

$$\le \gamma^2|T(t)| + \left|\frac{b_0}{a_0}\right| |\Phi(t)|,$$

$T(t) \equiv \ddot{\Psi}(t)$, and in a similar fashion

(3.31) $$||\hat{\underset{\sim}{K}}_t(t)||_{L_S(H_0^1(\Omega),\, H^{-1}(\Omega))} \le \gamma^2|\dot{T}(t)| + \left|\frac{b_0}{a_0}\right| |\dot{\Phi}(t)|$$

Therefore, the conditions embodied in (iii) of §2 will be satified by $\hat{\underset{\sim}{K}}(t)$, as defined by (3.26b), provided

(3.32a) $$\gamma^2 \int_0^\infty |T(t)|dt + \left|\frac{b_0}{a_0}\right| \int_0^\infty |\Phi(t)|dt < \infty$$

and

(3.33b) $$\gamma^2 \int_0^T\int_{-\infty}^t |\dot{T}(t-\tau)|d\tau dt + \left|\frac{b_0}{a_0}\right| \int_0^T\int_{-\infty}^t |\dot{\Phi}(t-\tau)|d\tau dt < \infty$$

for each $T < \infty$. Collecting our results we can state the following

Lemma The operators $\hat{\underset{\sim}{N}} \in L_S(H_0^1(\Omega); H^{-1}(\Omega))$ and $\hat{\underset{\sim}{K}} \in L^2((-\infty,\infty); L_S(H_0^1(\Omega); H^{-1}(\Omega))$, which are defined by (3.26a) and (3.26b), respectively, satisfy conditions (i), (ii), and (iii) of §2 provided

(i') $(b_0/a_0) < 0$

(ii') $\dot{\Psi}(0) < 0$, $\ddot{\Psi}(0) \leq 0$

(iii') $\Phi(0) \geq 0$

and $T(t)$, $\Phi(t)$ satisfy (3.32a) and (3.32b) for each $T < \infty$. (Additionally, we require that $\Psi(0) > 0$ so that the damping coefficient in the system (3.18) is positive).

As an example of the way in which the results of §2 apply to the situation at hand we will consider the following initial-value problem for $\underset{\sim}{D}^\beta \in C^2([0,T); H_0^1(\Omega))$, $\beta > 0$:

$$\underset{\sim}{D}^\beta_{tt} + \Psi(0)\underset{\sim}{D}^\beta_t - \hat{\underset{\sim}{N}}\underset{\sim}{D}^\beta + \int_{-t_h}^t \hat{\underset{\sim}{K}}(t-\tau)\underset{\sim}{D}^\beta(\tau)d\tau = \underset{\sim}{0}, \quad 0 \leq t < T \tag{3.34}$$

$$\underset{\sim}{D}^\beta(0) = \underset{\sim}{D}_0,\ \underset{\sim}{D}^\beta_t(0) = \underset{\sim}{D}_1\ (\underset{\sim}{D}_0, \underset{\sim}{D}_1 \in H_0^1(\Omega)) \tag{3.35}$$

$$\underset{\sim}{D}^\beta(t) = \begin{cases} 0, & -\infty < t < -t_h \\ g(\beta)\underset{\sim}{D}_h(t), & -t_h \leq t < 0 \end{cases} \tag{3.36}$$

where $g(\beta)$ is a monotonically increasing function of β for $0 < \beta < \infty$ and $\underset{\sim}{D}_h$ is continuous in the $||\cdot||_{L_2}$ norm with $||\underset{\sim}{D}_h(t)||_{L_2} \to 0$ as $t \to -t_h^+$. We assume that the conditions delineated in the above lemma are satisfied so that $\hat{\underset{\sim}{N}}$, $\hat{\underset{\sim}{K}}(t)$, satisfy (i), (ii), and (iii) of §2; then the following result is a direct consequence of theorem II.2: Let $\underset{\sim}{D}^{\beta} \in C^2([0,T);\, H_0^1(\Omega))$, $\beta > 0$, be a strong solution of (3.34) - (3.36), with $\Psi(0) > 0$. Suppose that

(3.37a)
$$\left|\frac{b_0}{a_0}\right| \, ||\underset{\sim}{D}_0||^2_{H_0^1} + |\dot{\Psi}(0)| \, ||\underset{\sim}{D}_0||^2_{L_2} \geq ||\underset{\sim}{D}_1||^2_{L_2}$$

(3.37b) $\langle \underset{\sim}{D}_0, \underset{\sim}{D}_1 \rangle_{L_2} > 0$ (the form assumed by condition (iv) of §2)

(3.37c)
$$\int_{-t_h}^{0} T(-\tau)\langle \underset{\sim}{D}_0, \underset{\sim}{D}_h(\tau)\rangle_{L_2} d\tau < \left|\frac{b_0}{a_0}\right| \int_{-t_h}^{t} \Phi(-\tau)\langle \underset{\sim}{D}_0, \underset{\sim}{D}_h(\tau)\rangle_{H_0^1} d\tau$$

(the form assumed by condition (v) of §2)

and that

(3.37d)
$$T > \frac{1}{\Psi(0)} \ln[2\pi_\delta \langle \underset{\sim}{D}_0, \underset{\sim}{D}_1\rangle_{L_2} / 2\pi_\delta \langle \underset{\sim}{D}_0, \underset{\sim}{D}_1\rangle_{L_2} - \Psi(0)||\underset{\sim}{D}_0||^2_{L_2}]$$

for some $\delta > 0$, where

(3.38)
$$\pi_\delta = (\Psi(0)||D_0||^2_{L_2} / 2\langle \underset{\sim}{D}_0, \underset{\sim}{D}_1\rangle_{L_2}) + \delta$$

then

(3.39)
$$\sup_{-\infty<t<T} ||\underset{\sim}{D}^{\beta}(t)||_{H_0^1} \geq \frac{\sqrt{g(\beta)}}{\sqrt{\gamma \tilde{\Sigma}_T}} \times$$
$$\left(\left|\frac{b_0}{a_0}\right| \int_{-t_h}^{0} \Phi(-\tau)\langle \underset{\sim}{D}_0, \underset{\sim}{D}_h(\tau)\rangle_{H_0^1} d\tau - \int_{-t_h}^{0} T(-\tau)\langle \underset{\sim}{D}_0, \underset{\sim}{D}_h(\tau)\rangle_{L_2} d\tau\right)^{\frac{1}{2}}$$

where

(3.40)
$$\tilde{\Sigma}_T = \left|\frac{b_0}{a_0}\right| \left(\frac{1}{2} + \int_0^{\infty} |\Phi(t)| dt + \int_0^T \int_{-\infty}^{t} |\dot{\Phi}(t-\tau)| d\tau dt\right)$$
$$+ \gamma^2 \left(\frac{1}{2}|\dot{\Psi}(0)| + \int_0^{\infty} |T(t)| dt + \int_0^T \int_{-\infty}^{t} |\dot{T}(t-\tau)| d\tau dt\right)$$

$$\geq \hat{\Sigma}_T = \tfrac{1}{2}||\hat{\underset{\sim}{N}}||_{\mathcal{L}_S(H_0^1(\Omega);\ H^{-1}(\Omega))}$$

$$+ \int_0^\infty ||\hat{\underset{\sim}{K}}(t)||_{\mathcal{L}_S(H_0^1(\Omega);\ H^{-1}(\Omega))} dt$$

$$+ \int_0^T \int_{-\infty}^t ||\hat{\underset{\sim}{K}}(t-\tau)||_{\mathcal{L}_S(H_0^1(\Omega);\ H^{-1}(\Omega))} d\tau dt$$

Once $g(\beta)$, the initial data $\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_1(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$, and the past history $\underset{\sim}{D}_h(\underset{\sim}{x},t)$, $-t_h < t < 0$, have been specified, along with the constitutive constants a_0 and b_0 and the memory functions $\Phi(t)$ and $\Psi(t)$, $-t_h < t < T$, all of the quantities on the right-hand side of (3.39) are computable. If only the form of either $\Phi(t)$ or $\Psi(t)$ (or both) are known, e.g., $\Phi(t) = \exp(-\lambda_1 t)$, $\Phi(t) = \exp(-\lambda_2 t)$, with $\lambda_1 > 0$, $\lambda_2 > 0$ not specified or known a priori, then (3.39) could be used as the basis for a series of experimental tests to try to determine (or simply bound) λ_1 and λ_2, i.e., we might hold the initial data, the past history $\underset{\sim}{D}_h$, the time interval $[-t_h, T)$, etc., fixed and measure $\sup\limits_{-t_h<t<T} ||\underset{\sim}{D}^\beta(t)||_{H_0^1}$ as β is varied continously; in principle such a practice may be difficult to carry out if we can not first verify that the hypotheses of the above lemma, as well as (3.37a) - (3.37d), are satisfied. Other experimental tests could be based on estimates which are analogous to (3.39) and which follow by applying Theorem II.1 and its Corollary to the present situation; the delineation of the precise forms assumed by these estimates is a simple exercise, completely analogous to that which resulted in obtaining (3.39) from Theorem II.2 and is, therefore, left to the reader.

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