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Concavity Arguments and Growth Estimates for Linear Integrodifferential Equations in Hilbert Space I. Undamped Equations and Applications to Maxwell-Hopkinson Dielectrics*

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Abstract

Employing a modified version of a concavity argument for abstract differential equations, we obtain growth estimates for solutions to a class of initial-value problems associated with an undamped linear integrodifferential equation in Hilbert space; our results are applied to the derivation of growth estimates for the gradients of electric displacement fields occurring in rigid nonconducting material dielectrics of Maxwell-Hopkinson type.

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1. Introduction

In two recent papers [1], [2] this author considered the problem of deriving stability and growth estimates for electric displacement fields in rigid nonconducting material dielectrics; in [1] we employed the constitutive theory of Maxwell-Hopkinson [3] while in [2] a special class of isotropic holohedral dielectrics, of the type first studied by Toupin and Rivlin [4], was considered. In both [1] and [2] the initial-boundary value problems which govern the evolution of the electric displacement field in the dielectric lead one, in a natural way, to study the evolution of solutions to certain initial-value problems associated with abstract linear integrodifferential equations in Hilbert space.

The analysis of the abstract initial-value problems appearing in both [1] and [2] are based on logarithmic convexity arguments and a basic ingredient in any such argument is the a priori restriction to solutions which lie in certain uniformly bounded classes; the desire to remove this a priori restriction is the basic motivation for the current work. As we emphasize below the growth estimates derived in this paper are based on a simple concavity argument due to Levine and Payne; while concavity arguments have not previously been used to study the growth behavior of solutions to integrodifferential equations they have been employed, with some success, to prove nonexistence and instability theorems for initial-boundary value problems associated with nonlinear partial differential equations of both hyperbolic and parabolic type.[5] - [8]; concavity arguments have also been used to derive growth estimates for solutions to initial-boundary value problems arising in nonlinear elastodynamics [9].

In the present work we employ the same basic abstract setting that was previously used both in [1] and [2], namely, we take H to be any real Hilbert space with inner-product $<,>$ and let $H_+ \subseteq H$ (algebraically and topologically) be a second Hilbert space with inner-product denoted by $<,>_+$; we then define H_- to be the completion of H under the norm

$$||\underset{\sim}{w}||_- = \sup_{\underset{\sim}{v}\in H_+} \frac{|<\underset{\sim}{v},\underset{\sim}{w}>|}{||\underset{\sim}{v}||_+}$$

By $L_S(H_+,H_-)$ we denote the space of all symmetric bounded linear operators from H_+ into H_-. The abstract initial-value problems to be considered in this paper are of the form

$$\text{(1.1)} \qquad \underset{\sim}{u}_{tt} - \underset{\sim}{N}\underset{\sim}{u} + \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \underset{\sim}{0},\ 0 \le t < T$$

$$\text{(1.2)} \qquad \underset{\sim}{u}(0) = \underset{\sim}{u}_o,\ \underset{\sim}{u}_t(0) = \underset{\sim}{v}_o$$

$$\text{(1.3)} \qquad \underset{\sim}{u}(\tau) = \underset{\sim}{U}(\tau),\ -\infty < \tau < 0$$

where $\underset{\sim}{u} \in C^2([0,T);H_+)$, such that $\underset{\sim}{u}_t \in C^1([0,T);H_+)$ and $\underset{\sim}{u}_{tt} \in C([0,T);H_+)$, and $\underset{\sim}{u}_o$, $\underset{\sim}{v}_o \in H_+$. Also,

(i) $\underset{\sim}{N} \in L_S(H_+,H_-)$

(ii) $\underset{\sim}{K}(t),\ \underset{\sim}{K}_t(t) \in L^2((-\infty,\infty);\ L_S(H_+,H_-))$,

where $\underset{\sim}{K}_t$ denotes the strong operator derivative; the past history $\underset{\sim}{U}$ (taken to be identically zero in both [1] and [2]) is required to satisfy only $\int_{-\infty}^{0}||\underset{\sim}{U}(\tau)||d\tau < \infty$ so that, in particular, we do not require either that $\lim_{t\to 0^-} ||\underset{\sim}{U}(t)-\underset{\sim}{u}_0|| = 0$ or that $\lim_{t\to 0^-} ||\underset{\sim}{U}_t(t)-\underset{\sim}{v}_0|| = 0$.

In [1] and [2] the intrinsic structure of the logarithmic convexity arguments employed required us to restrict our attention to solutions of (1.1) - (1.3) which lie in uniformly bounded classes of the form

$$(1.4) \qquad N = \{\underset{\sim}{v} \in C^2([0,T);H_+) \mid \sup_{[0,T)} ||\underset{\sim}{v}(t)||_+ < N^2\}$$

for some real number N. In addition to (i) and (ii) above the operator $\underset{\sim}{K}(t)$ was required to satisfy

$$\text{(iii)} \quad -\langle \underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v}\rangle \geq \kappa ||\underset{\sim}{v}||_+^2, \ \forall \underset{\sim}{v} \in H_+ \text{ with}$$
$$\kappa \geq \gamma T \sup_{[0,\infty)} ||\underset{\sim}{K}_t(t)||_{L_S(H_+,H_-)}$$

where γ is the imbedding constant for the map i: $H_+ \to H$ (i.e., $||\underset{\sim}{v}|| \leq \gamma ||\underset{\sim}{v}||_+$, $\underset{\sim}{v} \in H_+$ and some $\gamma > 0$); no definiteness condition was imposed on $\underset{\sim}{N}$, however, in either [1] or [2]. In the present work we drop the a priori restriction that our solutions lie in uniformly bounded classes of the type prescribed by (1.4); furthermore, we may weaken (iii) and shall require that

$$\text{(iii')} \quad -\langle \underset{\sim}{v}, K(0)v\rangle \geq 0, \ \forall \underset{\sim}{v} \in H_+.$$

However, in addition to (i), (ii), and (iii') we now require that $\underset{\sim}{N}$ satisfy

$$\text{(iv)} \quad \langle \underset{\sim}{v}, N\underset{\sim}{v}\rangle \geq 0, \ \forall \underset{\sim}{v} \in H_+$$

and that

(v) $\int_0^\infty ||\underset{\sim}{K}(\rho)||_{\mathcal{L}_S(H_+,H_-)} d\rho < \infty$ and

$$\int_0^T \int_{-\infty}^t ||\underset{\sim}{K}_t(t-\rho)||_{\mathcal{L}_S(H_+,H_-)} d\rho dt < \infty$$

for each $T < \infty$. Finally we restrict our choice of initial datum $(\underset{\sim}{u}_0, \underset{\sim}{v}_0)$ so that

(vi) $<\underset{\sim}{u}_0, \underset{\sim}{v}_0> \; > 0$ and $<\underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau> \; < 0$,

i.e., in both of the problems considered in the next section it will be assumed that the past history $\underset{\sim}{U}$ and the initial data $\underset{\sim}{u}_0$ and $\underset{\sim}{v}_0$ have been chosen so as to satisfy condition (vi) above.

2. Growth Estimates for an Undamped Abstract Integrodifferential Equation

We begin by considering two problems which are special cases of (1.1) - (1.3), namely,

Problem A For any $\alpha > 0$ we denote by $\underset{\sim}{u}^\alpha \in C^2([0,T);H_+)$ a strong solution of

(2.1a) $\underset{\sim}{u}^\alpha_{tt} - \underset{\sim}{N}\underset{\sim}{u}^\alpha + \int_{-\infty}^t \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^\alpha(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T$

(2.1b) $\underset{\sim}{u}^\alpha(0) = \alpha\underset{\sim}{u}_0, \quad \underset{\sim}{u}^\alpha_t(0) = \underset{\sim}{v}_0$

(2.1c) $\underset{\sim}{u}^\alpha(\tau) = \underset{\sim}{U}(\tau), \quad -\infty < \tau < 0$

We seek a lower bound for $\sup\limits_{-\infty<t<T} ||\underset{\sim}{u}^\alpha||_+$ in terms of α, the initial data $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0$, the past history $\underset{\sim}{U}$, the length T of the interval

$[0,T)$, the imbedding constant γ, and the operator norms $||\underset{\sim}{N}||_{L_S(H_+,H_-)}$, $||\underset{\sim}{K}||_{L_S(H_+,H_-)}$, $||\underset{\sim}{K}_t||_{L_S(H_+,H_-)}$.

Problem B For any $\beta > 0$ we denote by $u^{\beta} \in C^2([0,T);H_+)$ a strong solution of

$$\text{(2.2a)} \quad \underset{\sim}{u}^{\beta}_{tt} - \underset{\sim}{N}\underset{\sim}{u}^{\beta} + \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\beta}(\tau)d\tau = \underset{\sim}{0}, \; 0 \le t < T$$

$$\text{(2.2b)} \quad \underset{\sim}{u}^{\beta}(0) = \underset{\sim}{u}_0, \; \underset{\sim}{u}^{\beta}_t(0) = \underset{\sim}{v}_0$$

$$\text{(2.2c)} \quad \underset{\sim}{u}^{\beta}(\tau) = g(\beta)U(\tau), \; -\infty < \tau < 0$$

where $g(\beta) > 0$ is a monotonically increasing real-valued function of β, $0 \le \beta < \infty$. We seek a lower bound for $\sup_{-\infty<t<T} ||\underset{\sim}{u}^{\beta}||_+$ in terms of $g(\beta)$, the initial data $\underset{\sim}{u}_0$, $\underset{\sim}{v}_0$, the past history $\underset{\sim}{U}$, the length T of the interval $[0,T)$, the imbedding constant γ, and the operator norms $||\underset{\sim}{N}||_{L_S(H_+,H_-)}$, $||\underset{\sim}{K}||_{L_S(H_+,H_-)}$, and $||\underset{\sim}{K}_t||_{L_S(H_+,H_-)}$.

Before proceeding with the statements and proofs of the growth estimates which apply to solutions of Problems A and B, respectively, we first need the following

Lemma If $\underset{\sim}{K}(t)$ satisfies (ii) and (v) of §1 and $\underset{\sim}{u}:(-\infty,T) \to H_+$ is such that $\sup_{-\infty<t<T} ||\underset{\sim}{u}||_+ \le M_T < \infty$ then for all t, $0 \le t < T$,

$$\text{(2.3)} \quad |\langle \underset{\sim}{u}(t), \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau \rangle| \le \gamma M_T^2 \int_0^{\infty} ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

and

$$\text{(2.4)} \quad \int_0^t \langle \underset{\sim}{u}, \int_{-\infty}^{\tau}\underset{\sim}{K}_{\tau}(t-\lambda)\underset{\sim}{u}(\lambda)d\lambda \rangle d\tau \le \gamma M_T^2 \int_0^T \int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt$$

<u>Proof</u> To prove (2.3) note that

$$(2.5)\quad |\langle \underset{\sim}{u}(t), \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle|$$

$$= |\langle \underset{\sim}{u}(t), \int_0^{\infty} \underset{\sim}{K}(\rho)\underset{\sim}{u}(t-\rho)d\rho\rangle|$$

$$\le ||\underset{\sim}{u}(t)|| \int_0^{\infty} ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} ||\underset{\sim}{u}(t-\rho)||_+ d\rho$$

$$\le \gamma(\sup_{-\infty<t<T} ||\underset{\sim}{u}(t)||_+)^2 \int_0^{\infty} ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

$$= \gamma M_T^2 \int_0^{\infty} ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

where we have employed the simple change of variable $\rho = t-\tau$, the Schwartz inequality, and the definition of the embedding constant γ. In order to establish the estimate (2.4) we again employ the Schwartz inequality and the hypothesis that $\sup_{-\infty<t<T} ||\underset{\sim}{u}||_+ \le M_t < \infty$ so as to obtain

$$(2.6)\quad \int_0^{t} \langle \underset{\sim}{u}, \int_{-\infty}^{\tau} \underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}(\lambda)d\lambda\rangle d\tau$$

$$\le \int_0^{t} |\langle \underset{\sim}{u}, \int_{-\infty}^{\tau} \underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}(\lambda)d\lambda\rangle| d\tau$$

$$\le \int_0^{t} ||\underset{\sim}{u}(\tau)|| \int_{-\infty}^{\tau} ||\underset{\sim}{K}_\tau(\tau-\lambda)||_{L_S(H_+,H_-)} ||\underset{\sim}{u}(\lambda)||_+ d\lambda d\tau$$

$$\le \gamma \sup_{-\infty<t<T} ||\underset{\sim}{u}||_+ \int_0^{t}\int_{-\infty}^{\tau} ||\underset{\sim}{K}_\tau(\tau-\lambda)||_{L_S(H_+,H_-)} ||\underset{\sim}{u}(\lambda)||_+ d\lambda d\tau$$

$$\le \gamma(\sup_{-\infty<t<T} ||\underset{\sim}{u}||_+)^2 \int_0^{t}\int_{-\infty}^{\tau} ||\underset{\sim}{K}_\tau(\tau-\lambda)||_{L_S(H_+,H_-)} d\lambda d\tau$$

$$\le \gamma M_T^2 \int_0^{T}\int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt$$

Q.E.D.

Remark For future reference we also note here the simple estimate

$$|\langle u, Nu\rangle| \le \gamma\|u\|_+\|Nu\| \le \gamma M_T^2\|N\|_{L_S(H_+,H_-)}$$

valid for any $u:(-\infty,T) \to H_+$ such that $\sup_{-\infty<t<T}\|u\|_+ \le M_T < \infty$.

We are now in a position to state and prove the basic growth estimates which apply to the solutions of Problems A and B cited above:

Theorem II.1 For each real $\alpha > 0$ let $u^\alpha \in C^2([0,T);H_+)$ be a strong solution of (2.1a) - (2.1c). If $T > \|u_0\|^2/2\langle u_0,v_0\rangle$ then for each $\alpha \ge \alpha_0$ $\equiv \|v_0\|^2/\langle u_0,Nu_0\rangle$

$$(2.7) \qquad \sup_{-\infty<t<T}\|u^\alpha(t)\|_+ \ge \left[\frac{|\langle u_0,\int_{-\infty}^0 K(-\tau)U(\tau)d\tau\rangle|}{\gamma\Psi_T}\right]^{1/2}\sqrt{\alpha}$$

where

$$(2.8) \qquad \Psi_T = \frac{1}{2}\|N\|_{L_S(H_+,H_-)} + \int_0^\infty\|K(\rho)\|_{L_S(H_+,H_-)}d\rho + \int_0^T\int_{-\infty}^t\|K_t(t-\tau)\|_{L_S(H_+,H_-)}d\tau\, dt$$

Proof Let T be chosen so as to satisfy $T > \|u_0\|^2/2\langle u_0,v_0\rangle$ and assume that for some $\alpha = \bar\alpha \ge \alpha_0$

$$(2.9) \qquad \sup_{-\infty<t<T}\|u^{\bar\alpha}\|_+ < \left[\frac{|\langle u_0,\int_{-\infty}^0 K(-\tau)U(\tau)d\tau\rangle|}{\gamma\Psi_T}\right]^{1/2}\sqrt{\bar\alpha}$$

For each t, $0 \le t < T$, we define the real-valued function $F_{\bar\alpha}(t) = \langle u^{\bar\alpha}(t), u^{\bar\alpha}(t)\rangle$. Then

$$(2.10)\qquad F_{\bar\alpha}'(t) = 2<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}^{\bar\alpha}>,\quad F_{\bar\alpha}''(t) = 2<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}> + 2<\underset{\sim}{u}_{tt}^{\bar\alpha},\underset{\sim}{u}^{\bar\alpha}>$$

Direct computation (compare, Levine [5], § 2) now yields

$$(2.11)\qquad F_{\bar\alpha}F_{\bar\alpha}'' - (\bar\alpha+1)F_{\bar\alpha}'^{2} = 4(\bar\alpha+1)S_{\bar\alpha}^{2} + 2F_{\bar\alpha}\{<\underset{\sim}{u}^{\bar\alpha},\underset{\sim}{u}_{tt}^{\bar\alpha}> - (2\bar\alpha+1)<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}>\}$$

where $S_{\bar\alpha}^{2} \equiv <\underset{\sim}{u}^{\bar\alpha},\underset{\sim}{u}^{\bar\alpha}><\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}> - <\underset{\sim}{u}^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}>^{2} \geq 0$ by the Schwartz inequality. Therefore

$$(2.12)\qquad F_{\bar\alpha}F_{\bar\alpha}'' - (\bar\alpha+1)F_{\bar\alpha}'^{2} \geq 2F_{\bar\alpha}G_{\bar\alpha},\qquad 0 \leq t < T$$

where

$$(2.13)\qquad G_{\bar\alpha}(t) \equiv <\underset{\sim}{u}^{\bar\alpha},\underset{\sim}{u}_{tt}^{\bar\alpha}> - (2\bar\alpha+1)<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}>.$$

We will show that provided (2.9) obtains, $G_{\bar\alpha}(t) \geq 0$, $0 \leq t < T$. First of all, by (2.1a)

$$(2.14)\qquad G_{\bar\alpha}(t) = <\underset{\sim}{u}^{\bar\alpha},\underset{\sim}{N}\underset{\sim}{u}^{\bar\alpha}> - <\underset{\sim}{u}^{\bar\alpha},\int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau> - (2\bar\alpha+1)<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_t^{\bar\alpha}>$$

so that

$$(2.15)\qquad G_{\bar\alpha}'(t) = 2<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{N}\underset{\sim}{u}^{\bar\alpha}> - \frac{d}{dt}<\underset{\sim}{u}^{\bar\alpha},\int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau> - 2(2\bar\alpha+1)<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{u}_{tt}^{\bar\alpha}>$$

$$= -4\bar\alpha<\underset{\sim}{u}_t^{\bar\alpha},\underset{\sim}{N}\underset{\sim}{u}^{\bar\alpha}> - \frac{d}{dt}<\underset{\sim}{u}^{\bar\alpha},\int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau> + 2(2\bar\alpha+1)<\underset{\sim}{u}_t^{\bar\alpha},\int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau>$$

where we have used the fact that $\underset{\sim}{N} \in L_S(H_+,H_-)$ and (again) (2.1a). By combining (2.14) with (2.1b) we easily obtain

$$(2.16)\quad G_{\bar{\alpha}}(0) = \bar{\alpha}^{-2}<\underset{\sim}{u}_0,\underset{\sim}{N}\underset{\sim}{u}_0> - (2\bar{\alpha}+1)||\underset{\sim}{v}_0||^2$$
$$- \bar{\alpha}<\underset{\sim}{u}_0,\int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>$$

Therefore, if we integrate (2.15) from zero to t, $(0 < t < T)$ we obtain

$$(2.17)\quad G_{\bar{\alpha}}(t) = G_{\bar{\alpha}}(0) - 2\bar{\alpha}[<\underset{\sim}{u}^{\bar{\alpha}},\underset{\sim}{N}\underset{\sim}{u}^{\bar{\alpha}}> - \bar{\alpha}^2<\underset{\sim}{u}_0,\underset{\sim}{N}\underset{\sim}{u}_0>]$$
$$+ 2(2\bar{\alpha}+1)\int_0^t<\underset{\sim}{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau}\underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda>d\tau$$
$$-[<\underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar{\alpha}}(\tau)d\tau> - \bar{\alpha}<\underset{\sim}{u}_0,\int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>]$$
$$= (2\bar{\alpha}+1)[\bar{\alpha}^{-2}<\underset{\sim}{u}_0,\underset{\sim}{N}\underset{\sim}{u}_0> - ||\underset{\sim}{v}_0||^2]$$
$$-2\bar{\alpha}<\underset{\sim}{u}^{\bar{\alpha}},\underset{\sim}{N}\underset{\sim}{u}^{\bar{\alpha}}> - <\underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar{\alpha}}(\tau)d\tau>$$
$$+ 2(2\bar{\alpha}+1)\int_0^t<\underset{\sim}{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau}\underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda>d\tau$$

However,

$$(2.18)\quad \int_0^t <\underset{\sim}{u}_\tau^{\bar{\alpha}}, \int_{-\infty}^{\tau}\underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda>d\tau$$
$$= \int_0^t \frac{d}{d\tau}<\underset{\sim}{u}^{\bar{\alpha}}, \int_{-\infty}^{\tau}\underset{\sim}{K}(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda>d\tau$$
$$- \int_0^t<\underset{\sim}{u}^{\bar{\alpha}}(\tau), \underset{\sim}{K}(0)\underset{\sim}{u}^{\bar{\alpha}}(\tau)>d\tau$$
$$- \int_0^t<\underset{\sim}{u}^{\bar{\alpha}}(\tau), \int_{-\infty}^{\tau}\underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\bar{\alpha}}(\lambda)d\lambda>d\tau$$

Substituting for the last expression in (2.17_2) from (2.18) and simplifying we obtain

$$(2.19)\quad G_{\bar\alpha}(t) = (2\bar\alpha+1)[\bar\alpha^2<\underset{\sim}{u}_0,\underset{\approx}{N}\underset{\sim}{u}_0> - ||\underset{\sim}{v}_0||^2]$$

$$- 2\bar\alpha<\underset{\sim}{u}^{\bar\alpha},\underset{\approx}{N}\underset{\sim}{u}^{\bar\alpha}> + (4\bar\alpha+1)<\underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau>$$

$$- 2\bar\alpha(2\bar\alpha+1)<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>$$

$$- 2(2\bar\alpha+1)\int_0^t<\underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{\tau}\underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\bar\alpha}(\lambda)d\lambda>d\tau$$

$$- 2(2\bar\alpha+1)\int_0^t<\underset{\sim}{u}^{\bar\alpha}(\tau), \underset{\sim}{K}(0)\underset{\sim}{u}^{\bar\alpha}(\tau)>d\tau$$

However, by (iii') of §1, $-\int_0^t<\underset{\sim}{u}^{\bar\alpha}, \underset{\sim}{K}(0)\underset{\sim}{u}^{\bar\alpha}>d\tau \geq 0$ and, therefore, (2.19) yields

$$(2.20)\quad G_{\bar\alpha}(t) \geq (2\bar\alpha+1)[\bar\alpha^2<\underset{\sim}{u}_0,\underset{\approx}{N}\underset{\sim}{u}_0> - ||\underset{\sim}{v}_0||^2$$

$$+ 2\bar\alpha|<\underset{\sim}{u}_0, \int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau>|] - 2\bar\alpha<\underset{\sim}{u}^{\bar\alpha},\underset{\approx}{N}\underset{\sim}{u}^{\bar\alpha}>$$

$$+ (4\bar\alpha+1)<\underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau>$$

$$- 2(2\bar\alpha+1)\int_0^t<\underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{\tau}\underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\bar\alpha}(\lambda)d\lambda>d\tau$$

where we have used the assumption that $<\underset{\sim}{u}_0,\int_{-\infty}^{0}\underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau> < 0$. For the sake of convenience we now set

$$M_{T,\bar\alpha} = \left[\frac{|<u_0, \int_{-\infty}^{0}K(-\tau)U(\tau)d\tau>|}{\gamma\Psi_T}\right]^{1/2}\sqrt{\bar\alpha}$$

where Ψ_T is given by (2.8). Then by the Lemma of §1 and the assumed inequality (2.9)

$$(2.21a)\quad <\underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{t}\underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\bar\alpha}(\tau)d\tau> \geq -\gamma M_{T,\bar\alpha}^2\int_0^\infty||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)}d\rho$$

and

$$(2.21b)\qquad -\int_0^t \langle \underset{\sim}{u}^{\bar\alpha}, \int_{-\infty}^{\tau} \underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\bar\alpha}(\lambda)d\lambda\rangle d\tau$$

$$\geq -\gamma M_{T,\bar\alpha}^2 \int_0^T\int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt$$

Also, by the Remark which follows the lemma of §1 we have

$$(2.21c)\qquad -\langle \underset{\sim}{u}^{\bar\alpha}, \underset{\sim}{N}\underset{\sim}{u}^{\bar\alpha}\rangle \geq -\gamma M_{T,\bar\alpha}^2 ||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

Combining the estimates (2.21a), (2.21b), and (2.21c) with (2.20) and making use of the fact that

$$\bar\alpha \geq ||\underset{\sim}{v}_0|| \Big/ \langle \underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0\rangle^{\frac{1}{2}}$$

we obtain

$$(2.22)\qquad G_{\bar\alpha}(t) \geq (2\bar\alpha+1)\{2\bar\alpha|\langle \underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|$$

$$-\gamma M_{T,\bar\alpha}^2\Big[\left(\frac{2\bar\alpha}{2\bar\alpha+1}\right)||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

$$+\left(\frac{4\bar\alpha+1}{2\bar\alpha+1}\right)\int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

$$+ 2\int_0^T\int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt\Big]\}$$

$$\geq (2\bar\alpha+1)\{2\bar\alpha|\langle \underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|$$

$$-\gamma M_{T,\bar\alpha}^2\Big[||\underset{\sim}{N}||_{L_S(H_+,H_-)} + 2\int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)} d\rho$$

$$+ 2\int_0^T\int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)} d\tau dt\Big]\}$$

$$= 2(2\bar\alpha+1)\{\bar\alpha|\langle \underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle| - \gamma\Psi_T M_{T,\bar\alpha}^2\} = 0$$

in view of the definition of $M_{T,\bar\alpha}$. Therefore, if (2.9) obtains then

(2.23) $$G_{\bar{\alpha}}(t) \geq 0, \quad 0 \leq t < T$$

and thus, from (2.12) and the fact that $F_{\bar{\alpha}} \geq 0$, $0 \leq t < T$, it follows that

(2.24) $$F_{\bar{\alpha}}F_{\bar{\alpha}}'' - (\bar{\alpha}+1)F_{\bar{\alpha}}'^{2} \geq 0, \quad 0 \leq t < T$$

However,

(2.25) $$(F_{\bar{\alpha}}^{-\bar{\alpha}})'' = -\bar{\alpha}F_{\bar{\alpha}}^{-\bar{\alpha}-2}(F_{\bar{\alpha}}F_{\bar{\alpha}}'' - (\bar{\alpha}+1)F_{\bar{\alpha}}'^{2}) \leq 0$$

by (2.24). Integrating this last inequality we obtain

(2.26) $$F_{\bar{\alpha}}^{\bar{\alpha}}(t) \geq F_{\bar{\alpha}}^{\bar{\alpha}+1}(0)[F_{\bar{\alpha}}(0) - \bar{\alpha}tF_{\bar{\alpha}}'(0)]^{-1}, \quad 0 \leq t < T$$

Clearly, the right hand-side of (2.26) tends to $+\infty$ as $t \to t_{\infty} = F_{\bar{\alpha}}(0)/\bar{\alpha}F_{\bar{\alpha}}'(0)$. But from the definition of $F_{\bar{\alpha}}(t)$, (2.1b), and (2.10_1)

(2.27) $$\frac{F_{\bar{\alpha}}(0)}{\bar{\alpha}F_{\bar{\alpha}}'(0)} = \frac{\langle\bar{\alpha}\underset{\sim}{u}_0,\bar{\alpha}\underset{\sim}{u}_0\rangle}{2\bar{\alpha}\langle\bar{\alpha}\underset{\sim}{u}_0,\underset{\sim}{v}_0\rangle} < T$$

by virtue of our hypothesis relating the length of the interval $[0,T)$ and the initial datum. Thus $0 < t_{\infty} < T$ and $\sup_{[0,T)} ||\underset{\sim}{u}^{\bar{\alpha}}(t)|| = +\infty$. However

(2.28) $$\sup_{0\leq t<T} ||\underset{\sim}{u}^{\bar{\alpha}}|| \leq \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar{\alpha}}|| \leq \gamma \sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar{\alpha}}||_{+}$$

and thus it follows that $\sup_{-\infty<t<T} ||\underset{\sim}{u}^{\bar{\alpha}}||_{+} = +\infty$; this, in turn, contradicts the assumption (2.9) and establishes the growth estimate (2.7). Q.E.D.

Theorem II.1 has the following extension the proof of which follows directly from the previous computation.

Corollary II.1 For each $\alpha > 0$ let $\underset{\sim}{u}^\alpha \in C^2([0,T_\alpha); H_+)$ be a strong solution to (2.1a), on $[0,T_\alpha)$, subject to (2.1c) and the initial conditions

$$\underset{\sim}{u}^\alpha(0) = f(\alpha)\underset{\sim}{u}_0, \quad \underset{\sim}{u}^\alpha_t(0) = \underset{\sim}{v}_0, \tag{2.29}$$

where $f(\alpha) > 0$ is a real-valued monotonically increasing function of α, $0 \le \alpha < \infty$, and $T_\alpha > \left(\frac{f(\alpha)}{2\alpha}\right) \frac{||\underset{\sim}{u}_0||^2}{\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0\rangle}$. Then for each

$$\alpha \ge \bar{\alpha}_0 \equiv \inf_\alpha \{f(\alpha) \ge ||\underset{\sim}{v}_0|| / \langle \underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0\rangle^{1/2}\}$$

$$\sup_{-\infty<t<T_\alpha} ||\underset{\sim}{u}^\alpha(t)||_+ \ge \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|}{\gamma\Psi_{T_\alpha}}\right]^{1/2} \sqrt{f(\alpha)} \tag{2.30}$$

where

$$\Psi_{T_\alpha} = \frac{1}{2}||\underset{\sim}{N}||_{L_S(H_+,H_-)} + \int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)}d\rho$$

$$+ \int_0^{T_\alpha}\int_{-\infty}^t ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)}d\tau dt$$

We now turn our attention to Problem B and state

Theorem II.2 For each real $\beta > 0$ let $\underset{\sim}{u}^\beta \in C^2([0,T);H_+)$ be a strong solution of (2.2a)-(2.2c). If $\langle \underset{\sim}{u}_0, \underset{\sim}{N}\underset{\sim}{u}_0\rangle \ge ||\underset{\sim}{v}_0||^2$ then for each $T > 0$

$$\sup_{-\infty<t<T} ||\underset{\sim}{u}^\beta||_+ \ge \left[\frac{|\langle \underset{\sim}{u}_0, \int_{-\infty}^0 \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|}{\gamma\Psi_T}\right]^{1/2} \sqrt{g(\beta)} \tag{2.31}$$

for all β, $0 < \beta < \infty$.

Proof Suppose that for some $\beta = \bar{\beta}$, $0 < \bar{\beta} < \infty$,

$$(2.32)\qquad \sup_{-\infty<t<T} ||u^{\bar{\beta}}||_+ < \left[\frac{|<u_0, \int_{-\infty}^{0} K(-\tau)U(\tau)d\tau>|}{\gamma\Psi_T}\right]^{1/2} \sqrt{g(\bar{\beta})} \equiv L_{\bar{\beta},T}$$

Define $F_{\bar{\beta}}(t) = <u^{\bar{\beta}}(t), u^{\bar{\beta}}(t)>$, $0 \le t < T$. Then

$$(2.33)\qquad F_{\bar{\beta}}F_{\bar{\beta}}'' - (\alpha+1)F_{\bar{\beta}}'^2 \ge 2F_{\bar{\beta}}H_{\alpha,\bar{\beta}}, \quad 0 \le t < T,$$

for any $\alpha > 0$, where

$$(2.34)\qquad H_{\alpha,\bar{\beta}}(t) = <u^{\bar{\beta}}, Nu^{\bar{\beta}}> - <u^{\bar{\beta}}, \int_{-\infty}^{t} K(t-\tau)u^{\bar{\beta}}(\tau)d\tau> - (2\alpha+1)<u_t^{\bar{\beta}}, u_t^{\bar{\beta}}>$$

A direct computation, similar to that employed in (2.15)-(2.17_2), yields

$$(2.35)\qquad H_{\alpha,\bar{\beta}}(t) = (2\alpha+1)[<u_0, Nu_0> - ||v_0||^2] - 2\alpha<u^{\bar{\beta}}, u^{\bar{\beta}}> - <u^{\bar{\beta}}, \int_{-\infty}^{t} K(t-\tau)u^{\bar{\beta}}(\tau)d\tau> + 2(2\alpha+1)\int_0^t <u_\tau^{\bar{\beta}}, \int_{-\infty}^{\tau} K(\tau-\lambda)u^{\bar{\beta}}(\lambda)d\lambda>d\tau$$

By making use of the hypothesis that $<u_0, Nu_0> \ge ||v_0||^2$, the decomposition (2.18) with $u^{\bar{\alpha}} \to u^{\bar{\beta}}$, the condition (2.2c) with $\beta = \bar{\beta}$, and the fact that $-\int_0^t <u^{\bar{\beta}}, K(0)u^{\bar{\beta}}>d\tau \ge 0$, we obtain the estimate

$$(2.36)\qquad H_{\alpha,\bar{\beta}}(t) \ge 2(2\alpha+1)g(\bar{\beta})|<u_0, \int_{-\infty}^{0} K(-\tau)U(\tau)d\tau>| - 2\alpha<u^{\bar{\beta}}, Nu^{\bar{\beta}}> + (4\alpha+1)<u^{\bar{\beta}}, \int_{-\infty}^{t} K(t-\tau)u^{\bar{\beta}}(\tau)d\tau> - 2(2\alpha+1)\int_0^t <u^{\bar{\beta}}, \int_{-\infty}^{\tau} K_\tau(\tau-\lambda)u^{\bar{\beta}}(\lambda)d\lambda>d\tau$$

valid for all t, $0 \le t < T$, and all $\alpha > 0$. In view of the lemma of §1 and our assumption (2.32) on $\underset{\sim}{u}^{\overline{\beta}}$ we have the lower bounds

$$(2.37a)\qquad \langle \underset{\sim}{u}^{\overline{\beta}}, \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}^{\overline{\beta}}(\tau)d\tau\rangle \ge -\gamma L^2_{\overline{\beta};T}\int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)}d\rho$$

and

$$(2.37b)\qquad -\int_0^t \langle \underset{\sim}{u}^{\overline{\beta}}, \int_{-\infty}^{\tau} \underset{\sim}{K}_\tau(\tau-\lambda)\underset{\sim}{u}^{\overline{\beta}}(\lambda)d\lambda\rangle d\tau$$

$$\ge -\gamma L^2_{\overline{\beta},T}\int_0^T\int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)}d\tau dt$$

while by the Remark following the Lemma of §1,

$$(2.37c)\qquad -\langle \underset{\sim}{u}^{\overline{\beta}}, \underset{\sim}{N}\underset{\sim}{u}^{\overline{\beta}}\rangle \ge -\gamma L^2_{\overline{\beta},T}||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

Combining (2.37a) - (2.37c) with (2.36) we have

$$(2.38)\qquad H_{\alpha,\overline{\beta}}(t) \ge (2\alpha+1)[2g(\overline{\beta})|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|$$

$$-\gamma L^2_{\overline{\beta},T}\{(\tfrac{2\alpha}{2\alpha+1})||\underset{\sim}{N}||_{L_S(H_+,H_-)}$$

$$+(\tfrac{4\alpha+1}{2\alpha+1})\int_0^\infty ||\underset{\sim}{K}(\rho)||_{L_S(H_+,H_-)}d\rho$$

$$+2\int_0^T\int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)}d\tau dt\}]$$

$$\ge (2\alpha+1)[2g(\overline{\beta})|\langle \underset{\sim}{u}_0, \int_{-\infty}^{0} \underset{\sim}{K}(-\tau)\underset{\sim}{U}(\tau)d\tau\rangle|$$

$$-\gamma L^2_{\overline{\beta},T}\{||\underset{\sim}{N}||_{L_S(H_+,H_-)} + 2\int_0^\infty ||K(\rho)||_{L_S(H_+,H_-)}d\rho$$

$$+2\int_0^T\int_{-\infty}^{t} ||\underset{\sim}{K}_t(t-\tau)||_{L_S(H_+,H_-)}d\tau dt\}]$$

$$= 0$$

in view of (2.32) and the definition of Ψ_T. By combining (2.33) with (2.38) we now obtain for any $\alpha > 0$

$$F_{\bar{\beta}} F_{\bar{\beta}}'' - (\alpha+1) F_{\bar{\beta}}'^2 \geq 0, \quad 0 < t < T \tag{2.39}$$

from which it follows that

$$F_{\bar{\beta}}^{\alpha} \geq F_{\bar{\beta}}^{\alpha+1}(0)[F_{\bar{\beta}}(0) - \alpha t F_{\bar{\beta}}'(0)]^{-1}, \quad 0 \leq t < T \tag{2.40}$$

However, the right-hand side of (2.40) tends to $+\infty$ as $t \to \bar{t}_\infty = F_{\bar{\beta}}(0)/\alpha F_\beta'(0)$. From the definition of $F_{\bar{\beta}}$ we have

$$\bar{t}_\infty = ||\underset{\sim}{u}_0||^2/2\alpha\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle \tag{2.41}$$

and thus $\bar{t}_\infty < T$ provided we choose

$$\alpha \geq \alpha_T = \left(\frac{1}{2T}\right) \frac{||\underset{\sim}{u}_0||^2}{\langle \underset{\sim}{u}_0, \underset{\sim}{v}_0 \rangle} \tag{2.42}$$

Having chosen α so as to satisfy (2.42) it follows from (2.40) that $\sup\limits_{[0,T)} ||\underset{\sim}{u}^{\bar{\beta}}|| = +\infty$ and thus

$$+\infty = \sup_{[0,T)} ||\underset{\sim}{u}^{\bar{\beta}}|| \leq \sup_{(-\infty,T)} ||\underset{\sim}{u}^{\bar{\beta}}|| \leq \gamma \sup_{(-\infty,T)} ||\underset{\sim}{u}^{\bar{\beta}}||_+ \tag{2.43}$$

contradicting (2.32)

Q.E.D.

3. Growth Estimates for the Electric Displacement Field in a Class of Maxwell-Hopkinson Dielectrics

As in [1] we let (x^i, t), $i = 1,2,3$, denote a Lorentz reference frame with t being the time parameter and the x^i rectangular

Cartesian coordinates. If $\underset{\sim}{B}$, $\underset{\sim}{E}$, $\underset{\sim}{H}$, and $\underset{\sim}{D}$ denote, respectively, the magnetic flux density, the electric field, the magnetic intensity, and the electric displacement, then in a rigid nonconducting dielectric Maxwell's equations have the form

$$\text{(3.1a)} \qquad \frac{\partial \underset{\sim}{B}}{\partial t} + \operatorname{curl} \underset{\sim}{E} = \underset{\sim}{0}, \quad \operatorname{div} \underset{\sim}{B} = \underset{\sim}{0}$$

$$\text{(3.1b)} \qquad \operatorname{curl} \underset{\sim}{H} - \frac{\partial \underset{\sim}{D}}{\partial t} = \underset{\sim}{0}, \quad \operatorname{div} \underset{\sim}{D} = \underset{\sim}{0}$$

provided that the density of free current, the magnetization, and the density of free charge all vanish; in (3.1a), (3.1b)

$$\text{(3.1c)} \qquad \underset{\sim}{D} \equiv \epsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \quad \text{and} \quad \underset{\sim}{H} = \mu_0^{-1} \underset{\sim}{B}$$

where $\epsilon_0 > 0$, $\mu_0 > 0$ are physical constants satisfying $\epsilon_0 \mu_0 = c^{-2}$ (c being the speed of light in a vacuum) and $\underset{\sim}{P}$ is the polarization vector. So as to obtain a determinate system of equations for the electromagnetic field in the dielectric we must append a constitutive equation which relates the polarization vector to the fields which appear in (3.1a) and (3.1b). Let $\Omega \subseteq R^3$ be a bounded domain with smooth boundary $\partial\Omega$; then for $(\underset{\sim}{x},t) \in \Omega \times (-\infty,T)$ we take

$$\text{(3.2)} \qquad \underset{\sim}{P}(\underset{\sim}{x},t) = \tilde{\epsilon} \underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-\infty}^{t} \phi(t-\tau) \underset{\sim}{E}(\underset{\sim}{x},\tau) d\tau$$

where $\tilde{\epsilon} < 0$ is assumed to satisfy $|\tilde{\epsilon}| > \epsilon_0$ and ϕ is a twice continuously differentiable function which is monotonically decreasing on $[0,\infty)$. Combining (3.2) with the first relation in (3.1c) we obtain

(3.3) $\quad \underset{\sim}{D}(\underset{\sim}{x},t) = \epsilon\underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-\infty}^{t}\phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau,$

for $(\underset{\sim}{x},t) \in \Omega \times (-\infty,T)$, where $\epsilon < 0$.

In [1] we obtained (via a logarithmic convexity argument) growth estimates for electric displacement fields which occur in Maxwell-Hopkinson dielectrics that are governed by constitutive relations of the form (3.3) with $\epsilon > 0$ and $\underset{\sim}{E}(\underset{\sim}{x},t) = 0$, $(x,t) \in \Omega \times (-\infty,0)$. In order to proceed with the derivation of the integrodifferential equation which governs the evolution of the electric displacement field in the dielectric which is specified by (3.3), with $\epsilon < 0$, we will make the simplifying assumption that there exists $t_h > 0$ such that the past history of $\underset{\sim}{E}$ has the form

$$\underset{\sim}{E}(\underset{\sim}{x},t) = \begin{cases} \underset{\sim}{0}, & t < -t_h \\ \underset{\sim}{E}_h(\underset{\sim}{x},t), & -t_h \le t < 0 \end{cases}$$

with $\lim_{t\to -t_h^+} \int_\Omega (\underset{\sim}{E}_h)_i(\underset{\sim}{E}_h)_i\, d\underset{\sim}{x} = 0$; in this case it is clear that (3.3) reduces to

(3.3') $\quad \underset{\sim}{D}(\underset{\sim}{x},t) = \epsilon\underset{\sim}{E}(\underset{\sim}{x},t) + \int_{-t_h}^{t}\phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau,\ (\underset{\sim}{x},t) \in \Omega \times (-\infty,T)$

We now invert (3.3') by employing the usual technique of successive approximations and obtain

(3.4) $\quad \underset{\sim}{E}(\underset{\sim}{x},t) = \epsilon^{-1}\underset{\sim}{D}(\underset{\sim}{x},t) + \epsilon^{-1}\int_{-t_h}^{t}\Phi(t-\tau)\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau$

where $(x,t) \in \Omega \times (-\infty,T)$ and

(3.5) $$\Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t)$$

$$\phi^1(t) = \epsilon^{-1}\phi(t)$$

$$\phi^n(t) = \int_{-t_h}^{t} \phi^1(t-\tau)\phi^{n-1}(\tau)d\tau, \qquad n \geq 2$$

Because of the assumed smoothness of $\phi(t)$, $\Phi(t)$ will be continuously differentiable on $[0,\infty)$ if the series in (3.5), and the associated series obtained by term by term differentiation of (3.5) are uniformly convergent. The required integrodifferential equation for $\underset{\sim}{D}(\underset{\sim}{x},t)$ is now obtained by employing the vector identity

(3.6) $$\Delta\underset{\sim}{V}(\underset{\sim}{x}) = \text{grad}(\text{div } \underset{\sim}{V}(\underset{\sim}{x})) - \text{curl curl } \underset{\sim}{V}(\underset{\sim}{x})$$

in conjunction with Maxwell's equations and the constitutive relations (3.3') and (3.4). If fact, by (3.4) and the vanishing of div $\underset{\sim}{D}$, it follows that div $\underset{\sim}{E} = \underset{\sim}{0}$; thus

(3.7) $$\Delta\underset{\sim}{E} = -\text{ curl curl } \underset{\sim}{E} = \text{curl } \underset{\sim}{B}_t = \mu_0(\text{curl } \underset{\sim}{H})_t$$

However, curl $\underset{\sim}{H} = \underset{\sim}{D}_t$ and so by (3.7_4) and (3.4)

(3.8a) $$\epsilon\mu_0\underset{\sim}{D}_{tt}(\underset{\sim}{x},t) = \Delta\underset{\sim}{D}(\underset{\sim}{x},t) + \int_{-t_h}^{t} \Phi(t-\tau)\Delta\underset{\sim}{D}(\underset{\sim}{x},\tau)d\tau$$

for $(\underset{\sim}{x},t) \in \Omega \times (-\infty,T)$; to (3.8a) we append boundary and initial data of the form

(3.8b) $$\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0}, \quad (\underset{\sim}{x},t) \in \partial\Omega \times (-\infty,T)$$

(3.8c) $$\underset{\sim}{D}(\underset{\sim}{x},0) = \underset{\sim}{D}_0(x), \; \underset{\sim}{D}_t(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x}), \; \underset{\sim}{x} \in \Omega$$

where $\underset{\sim}{D}_0$, $\underset{\sim}{D}_1$ are of class C^1 on $\bar{\Omega}$ and vanish on $\partial\Omega$. Also, by (3.3') and our assumption relative to the past history $\underset{\sim}{E}(\underset{\sim}{x},t)$, $-\infty < t < 0$, it follows that

$$\underset{\sim}{D}(\underset{\sim}{x},t) = \begin{cases} 0, & t < -t_h \\ \underset{\sim}{D}_h(\underset{\sim}{x},t), & -t_h \le t < 0 \end{cases} \tag{3.8d}$$

with $\lim_{t\to -t_h^+} \int_\Omega (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i d\underset{\sim}{x} = \underset{\sim}{0}$. We note here that the analysis presented below can be easily modified to accomodate boundary data of the form

$$\text{grad } \underset{\sim}{D}_k \cdot \underset{\sim}{n} = 0,\ k = 1,2,3 \text{ or } \underset{\sim}{D} \cdot \underset{\sim}{n} = 0 \tag{3.8b'}$$

where $\underset{\sim}{n}$ is the exterior unit normal to $\partial\Omega$.

As in [1] we now let $C_0^\infty(\Omega)$ denote the set of three dimensional vector fields with compact support in Ω whose components are in $C^\infty(\Omega)$ and we take $H = L_2(\Omega)$, the completion of $C_0^\infty(\Omega)$ under the norm induced by the inner product $\langle \underset{\sim}{v},\underset{\sim}{w}\rangle = \int_\Omega \underset{\sim}{v}_i \underset{\sim}{w}_i d\underset{\sim}{x}$; we also take $H_+ = H_0^1(\Omega)$, the completion of $C_0^\infty(\Omega)$ under the norm induced by the inner-product $\langle \underset{\sim}{v},\underset{\sim}{w}\rangle_+ = \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} d\underset{\sim}{x}$. Finally, for H_- we take $H^{-1}(\Omega)$, the completion of $C_0^\infty(\Omega)$ with respect to the norm $||\underset{\sim}{v}||_{H^{-1}(\Omega)} = \sup_{\underset{\sim}{w}\in H_0^1(\Omega)} [\,|\langle \underset{\sim}{v},\underset{\sim}{w}\rangle| / ||\underset{\sim}{w}||_{H_0^1(\Omega)}]$.

For the operators $\underset{\sim}{N} \in L_S(H_0^1(\Omega);H^{-1}(\Omega))$ and $\underset{\sim}{K}(t) \in L^2((-\infty,\infty);$ $L_S(H_0^1(\Omega);H^{-1}(\Omega))$ we have

$$(\underset{\sim}{N}\underset{\sim}{v})_i \equiv \frac{1}{\epsilon\mu_0} \delta_{ik}\delta_{jl} \frac{\partial^2 v_k}{\partial x_j \partial x_l}, \quad \forall \underset{\sim}{v} \in H_0^1(\Omega) \tag{3.9}$$

$$(K(t)\underset{\sim}{v})_i \equiv -\Phi(t) N_{ik} v_k, \quad \forall \underset{\sim}{v} \in H_0^1(\Omega) \tag{3.10}$$

and with these definitions of $\underset{\sim}{N}$ and $\underset{\sim}{K}(t)$ the initial-boundary value problem (3.8a) - (3.8d) now assumes the form

$$(3.11a) \qquad \underset{\sim}{D}_{tt} - \underset{\sim}{N}\underset{\sim}{D} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{D}(\tau)d\tau = 0, \quad 0 \le t < T$$

$$(3.11b) \qquad \underset{\sim}{D}(0) = \underset{\sim}{D}_0, \quad \underset{\sim}{D}_t(0) = \underset{\sim}{D}_1; \qquad \underset{\sim}{D}_0, \underset{\sim}{D}_1 \in H_0^1(\Omega)$$

$$(3.11c) \qquad \underset{\sim}{D}(t) = \begin{cases} \underset{\sim}{0}, & t < -t_h \\ \underset{\sim}{D}_h(t), & -t_h \le t < 0 \end{cases}$$

where $\underset{\sim}{D} \in C^2([0,T); H_0^1(\Omega))$ and $\underset{\sim}{D}_h(t)$, $-t_h \le t < 0$ is prescribed a priori and satisfies $\lim_{t \to -t_h^+} ||\underset{\sim}{D}_h(t)||_{L_2(\Omega)} = 0$.

In order to apply the results of the previous section to the situation at hand we must first consider the implications of conditions (i) - (v) of §1. Conditions (i) and (ii) on $\underset{\sim}{N}$, $\underset{\sim}{K}(t)$, and $\underset{\sim}{K}_t(t)$ are trivially satisfied in view of our smoothness assumptions on ϕ and the fact that $\underset{\sim}{v} = \underset{\sim}{0}$ on $\partial\Omega$ for all $\underset{\sim}{v} \in H_0^1(\Omega)$ by virtue of a standard trace theorem; thus the definitions (3.9) and (3.10) and integration by parts yield, respectively

$$(3.12a) \qquad <\underset{\sim}{w}, \underset{\sim}{N}\underset{\sim}{v}> = <\underset{\sim}{N}\underset{\sim}{w}, \underset{\sim}{v}>, \quad \forall \underset{\sim}{v}, \underset{\sim}{w} \in H_0^1(\Omega)$$

and

$$(3.12b) \qquad <\underset{\sim}{w}\underset{\sim}{K}(t)\underset{\sim}{v}> = <\underset{\sim}{K}(t)\underset{\sim}{w}, \underset{\sim}{v}>, \quad \forall \underset{\sim}{v}, \underset{\sim}{w} \in H_0^1(\Omega), \ t \quad (-\infty, \infty),$$

with a similar result for $\underset{\sim}{K}_t$.

Condition (iii') assumes the form

$$(3.13)\qquad -\langle \underset{\sim}{v}, \underset{\sim}{K}(0)\underset{\sim}{v}\rangle = -\int_\Omega \underset{\sim}{v}_i [\underset{\sim}{K}(0)\underset{\sim}{v}]_i \, d\underset{\sim}{x}$$

$$= \frac{\Phi(0)}{\epsilon\mu_0}\int_\Omega \delta_{ik}\delta_{jl} v_i \frac{\partial^2 v_k}{\partial x_j \partial x_l} \, d\underset{\sim}{x}$$

$$= \frac{\Phi(0)}{\epsilon\mu_0}\left[\int_{\partial\Omega} \underset{\sim}{v}_k \frac{\partial v_k}{\partial x_j} n_j \, dS_{\underset{\sim}{x}} - \int_\Omega \frac{\partial v_k}{\partial x_j}\frac{\partial v_k}{\partial x_j} \, d\underset{\sim}{x}\right]$$

$$= -\frac{\Phi(0)}{\epsilon\mu_0} \, ||\underset{\sim}{v}||^2_{H^1_0(\Omega)} \geq 0$$

for all $\underset{\sim}{v} \in H^1_0(\Omega)$. As $\epsilon < 0$, $\mu_0 > 0$, condition (iii') is equivalent to $\Phi(0) \geq 0$. (By using the definition of Φ, i.e. (3.5), it is not difficult to show that

$$(3.14)\qquad \Phi(t) + \frac{1}{\epsilon}\,\phi(t) = -\frac{1}{\epsilon}\int_{t_h}^{t} \phi(t-\tau)\Phi(\tau)\,d\tau$$

and, thus, $\Phi(0) = -\frac{1}{\epsilon}\,\phi(0) + \int_{-t_h}^{0} \phi(-\tau)\Phi(\tau)\,d\tau$).

As for condition (iv) of §1 we have, by a similar computation

$$(3.15)\qquad \langle \underset{\sim}{v}, \underset{\sim}{N}\underset{\sim}{v}\rangle = \frac{1}{\epsilon\mu_0}\int_\Omega \delta_{ik}\delta_{jl} v_i \frac{\partial^2 v_k}{\partial x_j \partial x_l}\, d\underset{\sim}{x}$$

$$= -\frac{1}{\epsilon\mu_0}\, ||\underset{\sim}{v}||^2_{H^1_0(\Omega)} \geq 0$$

for all $\underset{\sim}{v} \in H^1_0(\Omega)$. Turning to condition (v) of §1 we note that for any $t \in (-\infty,\infty)$

$$(3.16)\qquad ||\underset{\sim}{K}(t)||_{L_S(H^1_0(\Omega);H^{-1}(\Omega))} = \sup_{\underset{\sim}{v}\in H^1_0(\Omega)} \frac{|\int_\Omega \underset{\sim}{v}_i [\underset{\sim}{K}(t)\underset{\sim}{v}]_i \, dx|}{||\underset{\sim}{v}||^2_{H^1_0(\Omega)}}$$

$$= \sup_{\underset{\sim}{v} \in H_0^1(\Omega)} \frac{\frac{|\Phi(t)|}{|\epsilon|\mu_0} \left| \int_\Omega \delta_{ij}\delta_{kl} v_i \frac{\partial^2 v_j}{\partial x_k \partial x_l} dx \right|}{\|\underset{\sim}{v}\|^2_{H_0^1(\Omega)}}$$

$$= |\Phi(t)|/|\epsilon|\mu_0$$

In a similar manner we have, for any $t \in (-\infty,\infty)$

(3.17) $$\|\underset{\sim}{K}_t(t)\|_{L_S(H_0^1(\Omega);H^{-1}(\Omega))} = |\dot{\Phi}(t)|/|\epsilon|\mu_0$$

and therefore, the conditions represented by (v) of §1 will be satisfied provided

(3.18) $$\int_0^\infty |\Phi(t)|dt < \infty \text{ and } \int_0^T \int_{-\infty}^t |\dot{\Phi}(t-\tau)|d\tau dt < \infty$$

for each $T < \infty$. Finally the conditions repreented by (vi) of §1 will be satisfied if

(3.19) $$\int_\Omega (D_0(\underset{\sim}{x}))_i (D_1(\underset{\sim}{x}))_i dx > 0$$

and

(3.20) $$\int_{-t_h}^0 \Phi(-\tau) \int_\Omega \frac{\partial}{\partial x_j} (\underset{\sim}{D}_0(\underset{\sim}{x}))_k \frac{\partial}{\partial x_j} (\underset{\sim}{D}_h(\underset{\sim}{x},\tau))_k d\underset{\sim}{x} d\tau > 0.$$

In all that follows we will assume that $\Phi(t)$, as given by (3.5), satisfies (3.18), that $\Phi(0) \geq 0$, and that $\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_1(\underset{\sim}{x})$, and $\underset{\sim}{D}_h(\underset{\sim}{x},t)$, $-t_h \leq t < 0$, satisfy (3.19) and (3.20) as well as the condition that $\lim_{t \to -t_h^+} \int_\Omega (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i (\underset{\sim}{D}_h(\underset{\sim}{x},t))_i d\underset{\sim}{x} = 0$. Our first growth estimate for $\underset{\sim}{D}(\underset{\sim}{x},t)$ is then a direct consequence of theorem II.1, namely

<u>Theorem III.1</u> For each real $\alpha > 0$, let $\underset{\sim}{D}^{\alpha} \in C^2([0,T);H_0^1(\Omega))$ be a solution of (3.8a) subject to the initial conditions $\underset{\sim}{D}^{\alpha}(\underset{\sim}{x},0) = \alpha\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_t^{\alpha}(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$, and the specification of the past history which is given by (3.8d). If

$$(3.21) \qquad T > \int_\Omega (\underset{\sim}{D}_0(\underset{\sim}{x}))_i(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x} / 2\int_\Omega (\underset{\sim}{D}_0(\underset{\sim}{x}))_i(\underset{\sim}{D}_1(\underset{\sim}{x}))_i d\underset{\sim}{x}$$

then for each $\alpha \geq \alpha_0$,

$$(3.22) \qquad \alpha_0 \equiv (|\epsilon|\mu_0\int_\Omega (\underset{\sim}{D}_1(\underset{\sim}{x}))_i(\underset{\sim}{D}_1(\underset{\sim}{x}))_i d\underset{\sim}{x} / \int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x})^{1/2}$$

it follows that

$$(3.23) \qquad \sup_{-\infty<t<T} (\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}^{\alpha}(\underset{\sim}{x},t))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}^{\alpha}(\underset{\sim}{x},t))_i d\underset{\sim}{x})^{1/2}$$

$$\geq \frac{\sqrt{\alpha}}{\Phi_T} (\int_{-t_h}^{0} \Phi(-\tau)\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_k \frac{\partial}{\partial x_j}(\underset{\sim}{D}_h(\underset{\sim}{x},\tau))_k d\underset{\sim}{x}d\tau)^{1/2}$$

where Φ_T is the positive square root of

$$(3.24) \qquad \Phi_T^2 = \frac{\gamma}{2} + \gamma|\epsilon|\mu_0(\int_0^\infty |\Phi(t)|dt + \int_0^T\int_{-\infty}^{t} |\dot{\Phi}(t-\tau)|d\tau dt)$$

In addition to the theorem above we also have the following extension (a direct consequence of Corollary II.1 of §2):

<u>Corollary III.1</u> For each real $\alpha > 0$, let $\underset{\sim}{D}^{\alpha} \in C^2([0,T_\alpha);H_0^1(\Omega))$ be a solution of (3.8a), on $[0,T_\alpha)$, subject to the initial conditions $\underset{\sim}{D}^{\alpha}(\underset{\sim}{x},0) = f(\alpha)\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_t^{\alpha}(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$ (and the specification of the past history that is given by (3.8d) where $f(\alpha) > 0$ is a real-valued monotonically increasing function of α, $0 \leq \alpha < \infty$ and

$$(3.25)\qquad T_\alpha > \left(\frac{f(\alpha)}{2\alpha}\right)\left[\int_\Omega (\underset{\sim}{D}_0(\underset{\sim}{x}))_i(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x} \Big/ \int_\Omega (\underset{\sim}{D}_0(\underset{\sim}{x}))_i(\underset{\sim}{D}_1(\underset{\sim}{x}))_i d\underset{\sim}{x}\right]$$

Then for each $\alpha \geq \bar{\alpha}_0$,

$$(3.26)\qquad \bar{\alpha}_0 = \inf_\alpha\left\{f(\alpha) \geq \left[\frac{|\epsilon|\mu_0\int_\Omega (\underset{\sim}{D}_1(\underset{\sim}{x}))_i(\underset{\sim}{D}_1(\underset{\sim}{x}))_i d\underset{\sim}{x}}{\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x}}\right]^{1/2}\right\}$$

if follows

$$(3.27)\qquad \sup_{-\infty<t<T_\alpha}\left(\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}^\alpha(\underset{\sim}{x},t))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}^\alpha(\underset{\sim}{x},t))_i d\underset{\sim}{x}\right)^{1/2}$$

$$\geq \frac{\sqrt{f(\alpha)}}{\Phi_{T_\alpha}}\left(\int_{-t_h}^0 \Phi(-\tau)\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_k \frac{\partial}{\partial x_j}(\underset{\sim}{D}_h(\underset{\sim}{x},\tau))_k d\underset{\sim}{x}d\tau\right)^{1/2}$$

where Φ_{T_α} is given by (3.24) with $T \to T_\alpha$. Our last growth estimate for the electric displacement field corresponds to theorem II.2 of §2 and assumes the following form:

<u>Theorem III.2</u> For each real $\beta > 0$, let $\underset{\sim}{D}^\beta \in C^2([0,T);H_0^1(\Omega))$ be a solution of (3.8a) subject to the initial conditions (3.8c) and a past history of the form

$$(3.28)\qquad \underset{\sim}{D}^\beta(\underset{\sim}{x},t) = \begin{cases} \underset{\sim}{0}, & t < -t_h \\ g(\beta)\underset{\sim}{D}_h(\underset{\sim}{x},t), & -t_h \leq t < 0 \end{cases}$$

where $g(\beta) > 0$ is a monotonically increasing real-valued function of β, $0 \leq \beta < \infty$. If

$$(3.29)\qquad \frac{1}{|\epsilon|\mu_0}\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x}$$

$$\geq \int_\Omega (\underset{\sim}{D}_1(\underset{\sim}{x}))_i(\underset{\sim}{D}_1(\underset{\sim}{x}))_i d\underset{\sim}{x}$$

then for each $T > 0$

$$(3.30)\qquad \sup_{-\infty<t<T} \int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}^\beta(\underset{\sim}{x},t))_i \frac{\partial}{\partial x_j}(\underset{\sim}{D}^\beta(\underset{\sim}{x},t))_i d\underset{\sim}{x}$$

$$\geq \frac{\sqrt{g(\beta)}}{\Phi_T}\left(\int_{-t_h}^0 \Phi(-\tau)\int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_k \frac{\partial}{\partial x_j}(\underset{\sim}{D}_h(\underset{\sim}{x},\tau))_k d\underset{\sim}{x}d\tau\right)^{1/2}$$

where Φ_T is determined by (3.24).

We conclude with some preliminary observations concerning the applicability of the growth estimates represented by (3.23), (3.27) and (3.30) and for convenience sake we will concentrate our remarks on the last estimate in this set. Suppose that a material dielectric occupies some region $\Omega \subseteq R^3$ and that it has already been determined that the electric field and the electric displacement field in Ω are related by a constitutive equation of the form (3.4) where $\epsilon < 0$, $\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0}$, $(\underset{\sim}{x},t) \in \Omega \times (-\infty,-t_h)$ with t_h some positive constant, and $\Phi(t) = e^{-\lambda t}$; however, the rate at which Φ decays exponentially, governed by $\lambda > 0$, has not yet been determined. Consider the initial-boundary value problem (3.8a) - (3.8d) which governs the evolution of the electric displacement field in Ω; in the course of an experiment all of the quantities appearing in (3.8a) - (3.8d) are either known or controllable with the exception of the as yet undetermined decay rate λ, i.e., the quantities T, t_h, $\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{D}_1(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$ and $\underset{\sim}{D}_h(\underset{\sim}{x},t)$, $(\underset{\sim}{x},t) \in \Omega \times (-t_h,0)$ are controllable in the experimental sense while the constants μ_0, γ, and $|\epsilon|$ are either known a priori, determined by Ω, or determinable via simple experiments (to determine $|\epsilon|$ prescribe $\underset{\sim}{D}_0$, take $\underset{\sim}{D}(\underset{\sim}{x},t) = 0$, $(\underset{\sim}{x},t) \in \Omega \times (-\infty,0)$, and measure E_0; then (3.4) determines $|\epsilon|$ as

$$|\epsilon| = (\int_\Omega (\underset{\sim}{D}_0(\underset{\sim}{x}))_i(\underset{\sim}{D}_0(\underset{\sim}{x}))_i d\underset{\sim}{x})^{1/2}/(\int_\Omega (\underset{\sim}{E}_0(\underset{\sim}{x}))_i(\underset{\sim}{E}_0(\underset{\sim}{x}))_i d\underset{\sim}{x})^{1/2})$$

Suppose that we now carry out a series of experiments holding t_h, T, $\underset{\sim}{D}_0$, $\underset{\sim}{D}_1$ and $\underset{\sim}{D}_h(\underset{\sim}{x},t)$, $(\underset{\sim}{x},t) \in \Omega \times (-t_h,0)$ all fixed but modifying the past history as per (3.28) by continuously varing β; as we vary β we compute $\sup_{-\infty<t<T} ||\underset{\sim}{D}^\beta(t)||_{H_0^1(\Omega)}$. Set

$$(3.31) \qquad \mathcal{D}(t) = \int_\Omega \frac{\partial}{\partial x_j}(\underset{\sim}{D}_0(\underset{\sim}{x}))_k \frac{\partial}{\partial x_j}(\underset{\sim}{D}_h(\underset{\sim}{x},t))_k d\underset{\sim}{x}, \quad -t_h < t < 0$$

and assume $\underset{\sim}{D}_0(\underset{\sim}{x})$, $\underset{\sim}{x} \in \Omega$, and $\underset{\sim}{D}_h(\underset{\sim}{x},t)$, $(\underset{\sim}{x},t) \in \Omega \times (-t_h,0)$ chosen so that $\mathcal{D}(t) > 0$, $-t_h < t < 0$. From (3.24), with $\Phi(t) = e^{-\lambda t}$, we have

$$(3.32) \qquad \Phi_T^2 = \gamma/2 + \gamma|\epsilon|\mu_0(\int_0^\infty e^{-\lambda t}dt + \lambda\int_0^T\int_{-\infty}^t e^{-\lambda(t-\tau)}d\tau dt)$$

$$= \gamma/2 + \gamma|\epsilon|\mu_0(1/\lambda + T)$$

$$\equiv \Phi_{\lambda,T}$$

and, therefore, from (3.30) we have

$$(3.33) \qquad (\sup_{-\infty<t<T} ||\underset{\sim}{D}^\beta(t)||_{H_0^1(\Omega)})^2 \geq \frac{g(\beta)}{\Phi_{\lambda,T}} (\int_{-t_h}^0 e^{\lambda\tau}\mathcal{D}(\tau)d\tau)$$

$$\geq \frac{g(\beta)e^{-\lambda t_h}}{\Phi_{\lambda,T}} (\int_{-t_h}^0 \mathcal{D}(\tau)d\tau)$$

or

$$(3.34) \qquad e^{\lambda t_h}\Phi_{\lambda,T} \geq \int_{-t_h}^0 \mathcal{D}(\tau)d\tau[\frac{g(\beta)}{(\sup_{-\infty<t<T} ||\underset{\sim}{D}^\beta(t)||_{H_0^1(\Omega)})^2}]$$

From (3.34) we see that the quantity

$$\sqrt{g(\beta)}/(\sup_{-\infty<t<T} ||\underset{\sim}{D}^{\beta}(t)||_{H_0^1(\Omega)}) \equiv G(\beta)$$

is bounded for all $\beta > 0$ and thus

(3.35) $$e^{\lambda t_h}\Phi_{\lambda,T} \geq (\sup_{\beta>0} G^2(\beta))\int_{-t_h}^{0} \mathcal{D}(t)dt$$

providing a bound on the exponential decay rate λ.

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