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A SURVEY OF MULTIATTRIBUTE/MULTICRITERION EVALUATION THEORIES.(U)
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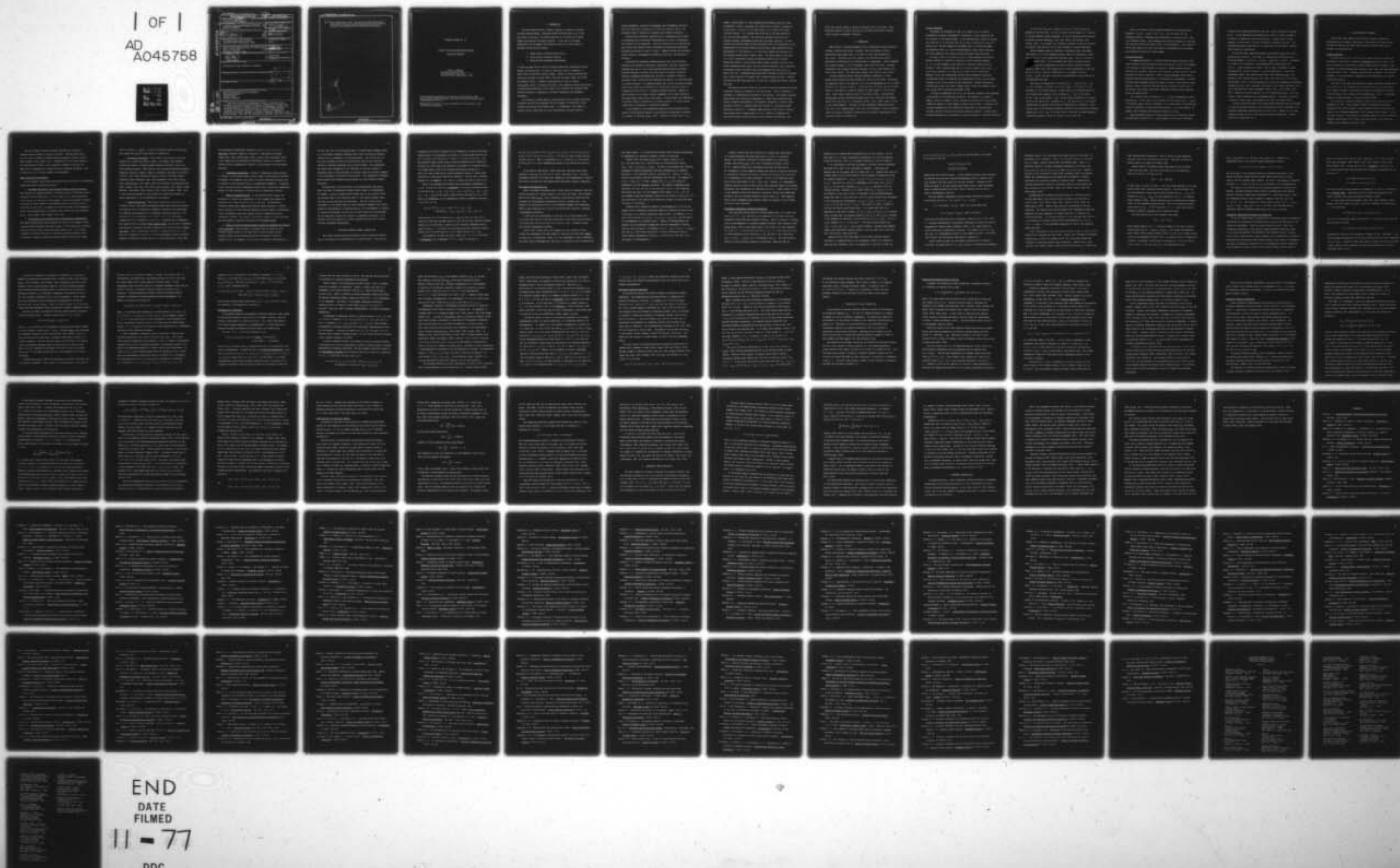

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER 14 TR- Technical Report No. 26	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) 6 A SURVEY OF MULTIATTRIBUTE/MULTICRITERION EVALUATION THEORIES,	5. TYPE OF REPORT & PERIOD COVERED 9 Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Peter C. Fishburn	8. CONTRACT OR GRANT NUMBER(s) 15 N00014-75-C-0857
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Pennsylvania State University University Park, Pennsylvania 16802	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 121407
11. CONTROLLING OFFICE NAME AND ADDRESS (code 613B: SJW) Office of Naval Research, Dept. of the Navy 800 N. Quincy Street Arlington, Virginia 22217	12. REPORT DATE 11 September 1977 13. NUMBER OF PAGES 77
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 80p.	15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this document is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Multiattribute/multicriterion decisions
Evaluation theories
Utility theory
Dominance analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report provides a comprehensive survey of theories for the evaluation of decision alternatives and/or consequences of decision that are characterized by a number of attributes or performance criteria. The evaluation theories are classified under a certainty/risk/uncertainty trichotomy and include varieties of utility theory, noncompensatory preference structures, theories of stochastic dominance, theories of risk, and many others. More than 300 references are provided, about half of which

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20. have appeared after 1970. The survey also discusses various choice models for multiattribute/multicriterion situations and includes a concluding section on assessment methodology.

TECHNICAL REPORT NO. 26

A SURVEY OF MULTIATTRIBUTE/MULTICRITERION EVALUATION THEORIES

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This research was sponsored by the Office of Naval Research, ONR Contract Number N00014-75-C-0857, ONR Contract Authority identification number NR047-112/11-24-76.

1. INTRODUCTION

The past decade has seen a dramatic increase in research on multiple criteria decision making. Substantial gains have been made in all of the main areas of the topic. My purpose here will be to review one of these, namely multicriterion evaluation theories. To set this in the larger perspective, let me suggest the following as three of the main areas of multiple criteria decision making:

a. Formal models of multicriterion choice.

b. Multicriterion evaluation theories.

c. Multicriterion assessment methodologies.

To these one might add the areas of problem formulation, multiperson decision making, and applications. To avoid an additional layer of complexity, I shall focus on individual decision making. Aspects of social evaluation and choice are discussed in Arrow (1963), Arrow and Scitovsky (1969), Sen (1970) and Fishburn (1973a). A number of applications are cited in Zeleny's bibliography (1976a), and Keeney and Raiffa (1976) summarize several applications of multicriterion expected utility theory that illustrate the interplay among problem formulation, redefinition, evaluation assumptions and assessment procedures.

The category of formal models of multicriterion choice includes algorithms, procedures and selection paradigms that are designed to choose good or best decision alternatives from feasible sets. It encompasses a vast number of specific topics, among which are vector maximization, multiple objective linear programming, interactive programming, goal programming, portfolio selection algorithms, maximization of subjective expected utility, and procedures based on varieties of dominance and outranking relations.

Multicriterion evaluation theories focus on assumptions about values or preferences and on structured representations of values or preferences that follow from the assumptions. They are concerned with the presumptions about values that underlie choice procedures and optimization algorithms, including the meaning of "good" or "best". Examples include multiattribute utility theories, noncompensatory preference structures, and theories of dominance.

Multicriterion assessment methodologies deal with the elicitation, estimation and scaling of individuals' preferences, utilities, subjective probabilities, and so forth in multiattribute/multicriterion situations. Although assessment procedures are often guided by specific evaluation theories, assessment methodology may also help to identify a theory of evaluation and/or choice that best describes the behavior of a decision maker in a given situation. The viewpoint adopted here is that evaluation precedes choice even though the determination of an applicable evaluation model or paradigm may follow choice in a revealed preference approach. However, I shall proceed as if at least a partial assessment of the terms involved in the choice model occurs prior to the choice. In other words, the presentation is slanted towards a prescriptive or normative approach to decision making rather than to a predictive or descriptive approach.

A good example of the distinctions among the three preceding areas is found in the interactive approach to multicriterion optimization described, for example, by Geoffrion *et al.* (1972). Suppose the feasible set X is a compact, convex subset of a finite-dimensional Euclidean space and we wish to maximize a concave, increasing real valued utility function u defined on $\{(f_1(x), f_2(x), \ldots, f_n(x)) : x \in X\}$, where each f_i is a concave real valued criterion function. It is assumed that X and the f_i are known explicitly but u has not been assessed. The interactive approach treats this as a standard nonlinear programming problem with one notable exception. At each iteration the decision maker provides information about his preferences in the neighborhood of the current feasible solution which, when translated into an approximation of the gradient of u at that point, guides the selection of a new feasible solution with a higher u value. The choice model in this case is a utility maximizing nonlinear programming procedure with the noted exceptional feature. The evaluation theory behind the model consists of the several assumptions made about u and the criterion functions, which themselves are marginal utility functions, plus the presumption that more utility is better than less. Assessment methodology enters through the specific procedures used at each iteration to estimate local tradeoffs between criteria. Related procedures may have been used prior to this step to assess the several criterion functions.

The general formulation used for the survey of multiattribute/multicriterion evaluation theories is presented in the next section. This is followed by a discussion of classificatory attributes of the theories. The primary attribute around which later sections are organized is a three-valued attribute whose values correspond approximately to the familiar categories of decision under certainty (section 4), decision under risk (section 5), and decision under uncertainty (section 6). Throughout these sections I shall refer to various choice models when it seems helpful to do so. However, to emphasize the distinction between evaluation theories and assessment methodologies, the latter will receive almost no mention in the main body of the survey. This deliberate omission is partly rectified in the final section which presents a brief review of assessment literature.

2. FORMULATION

Many writers, including MacCrimmon (1973), differentiate among attributes, criteria, objectives and goals. Although I shall not adhere to precise distinctions among these terms, it is useful to note some differences in their usage. Attributes are often thought of as differentiating aspects, properties or characteristics of alternatives or consequences. Typical examples are a person's height, a car's horsepower, and a firm's net worth. Although "attribute" often carries evaluative overtones, it is not used primarily as a direct value concept. The other terms, however, have obvious value content.

Criteria generally denote evaluative measures, dimensions or scales against which alternatives may be gauged in a value or worth sense. Objectives are sometimes viewed in the same way, as in the interchangeable use of "criterion function" and "objective function." Objectives may also denote specific desired levels of attainment (to climb Mt. Everest) or vague ideals (to live the good life). Goals usually indicate either of the latter notions. Although some writers make a careful distinction between goals (e.g. potentially attainable levels) and objectives (e.g. unattainable ideals), their common usages are more or less interchangeable. The goals in goal programming (Charnes and Cooper, 1961, 1975, 1977; Charnes et al., 1975; Lee, 1972; Kornbluth, 1973) are specific levels of criteria variables or functions that are usually simultaneously unrealizable within the feasible set.

<u>Attribute Mappings</u>

Throughout our discussion we shall let X denote a set of decision alternatives, potential consequences of decisions, or other things that may be of concern to the decision maker. In the multiattribute context we suppose that there are $n \geq 2$ attributes that can be used to differentiate among the objects in X. We shall assume for the moment that n is finite and number the attributes from 1 to n with the understanding, unless stated otherwise, that these numbers do not reflect relative importance among attributes.

For each i from 1 to n there is a set X_i, whose elements are potential specific "values" or "levels" of attribute i, and an attribute mapping $f_i\colon X \to X_i$ that assigns to each object in X a specific level of the ith attribute. The f_i functions map each $x \in X$ into an n-tuple $(f_1(x), f_2(x), \ldots, f_n(x))$ which describes x in terms of its "values" on the n attributes. Needless to say, many interesting practical and philosophical questions--including the choice of attributes, the definitions of the X_i, and the determinations of the f_i—are raised by this formulation. Readers interested in pursuing these issues can consult Chipman (1966), Wilkie and Pessemier (1973), Plott <u>et al</u>. (1975) and Keeney and Raiffa (1976).

It should be emphasized that the X_i can take many different forms depending on the natures of the underlying attributes. The elements in X_i might be numbers, vectors of numbers, colors, qualitative descriptors of various kinds, probability measures on some algebra of events, and so forth. In addition, the attribute mappings should be understood as descriptive or identification functions that may or may not have direct evaluative content. In other words, attribute mappings are not necessarily criterion functions or objective functions.

To illustrate this point, suppose X is a set of simple probability measures on the real line. If $f_i(x)$ is the ith central moment of x, then the f_i are well defined attribute mappings. If, as in the mean-variance approach in portfolio theory (Markowitz, 1952, 1959; Tobin, 1965; Sharpe, 1964; Lintner, 1965), it is assumed that preference increases in mean and decreases in variance, then the first two f_i can be viewed as criterion functions.

Another example arises in traditional consumption theory (Houthakker, 1961) where X is a set of commodity bundles--vectors of quantities of goods and services in a finite-dimensional Euclidean space. Here X is already in a multiattribute form. If it is assumed that utility increases in each dimension then the x_i components have obvious direct evaluative content. Lancaster (1966, 1971, 1975) argues that it is more appropriate to first map each $x \in X$ into a vector of characteristics of consumption activity $(f_1(x),\ldots,f_n(x))$ and then to talk about an individual's utility function on the characteristic space.

When each x is mapped into an n-tuple $(f_1(x),\ldots,f_n(x))$ in $X_1\times\ldots\times X_n$, it is not uncommon to identify x with this n-tuple or to replace x by the surrogate of its attribute values. In many cases $f_i(x)$ is abbreviated as x_i and we speak about elements in X as n-tuples $(x_1,\ldots,x_n)$ in the product set $X_1\times\ldots\times X_n$, or write $X\subseteq X_1\times\ldots\times X_n$. In most actual situations X is a proper subset of the product set: elements in $X_1\times\ldots\times X_n$ that are not in X represent combinations of attribute values that are unrealizable or infeasible. Nevertheless, many of the axiomatic preference theories for multiattribute situations assume that $X = X_1\times\ldots\times X_n$, or at least that an individual can make meaningful comparisons between all pairs of n-tuples in the product set.

There is of course a value assumption embedded in the multiattribute mapping $x \to (f_1(x),\ldots,f_n(x))$, or $x \to (x_1,\ldots,x_n)$ for short, and the subsequent practice of speaking about preferences or utilities on $X_1 \times \ldots \times X_n$ or its feasible subset. This assumption says that two elements in X that map into the same n-tuple have equal values, or are indifferent. Since aspects of the holistic nature of x can be lost when it is decomposed into attributes, this assumption should not be taken lightly.

Criterion Functions

As suggested previously, a criterion function usually indicates a real valued function on X that directly reflects the worth or value of the elements in X according to some criterion or objective. These functions are also referred to as objective functions, goal functions, scoring functions, ranking functions and utility functions. Unlike attribute mappings, which usually describe objective characteristics of alternatives or consequences, criterion functions often represent subjective values on a more or less arbitrary scale. However, values of criterion functions may have objective content such as net profits, test scores, times until completion, payback periods, expected values and market shares.

In a situation with m criteria ($j = 1,\ldots,m$) and corresponding criteria functions g_j: $X \to Re$, each x in X is mapped into an m-tuple $(g_1(x),\ldots,g_m(x))$ of criterion values, scores or utilities. It is then common to associate some notion of preference or value with these m-tuples. It is often assumed, for example, that preference monotonically increases in each g_j.

Some developments based on criterion functions do not explicitly assume a multiattribute structure for X. A good example of this is the outranking relations choice methods described by Roy (1971, 1974) and Bernard and Besson (1971). In this approach the alternatives in X, which is usually assumed to be finite, are mapped into score vectors $(g_1(x),\ldots,g_m(x))$ which are then compared in various ways to develop outranking or dominance relations. The outranking relations, which need not be transitive, are then used to identify "good" subsets of alternatives.

On the other hand, many multicriterion choice models assume that X has a multiattribute structure. This leads to a composite multiattribute-multicriterion mapping $x \rightarrow (g_1(f_1(x),\ldots,f_n(x)),\ldots,g_m(f_1(x),\ldots,f_n(x)))$. Frequently X is taken to be a subset of a finite-dimensional Euclidean space with $(f_1(x),\ldots,f_n(x))$ replaced by x itself. This is done, for example, in goal programming (see earlier references), in various approaches to interactive programming (Saska, 1968; Benayoun and Tergny, 1969; Benayoun et al., 1971; Geoffrion, 1970; Geoffrion et al., 1972; Boyd, 1970; Dyer, 1972, 1973; Zionts and Wallenius, 1976), in vector maximization's search for undominated alternatives (DaCunha and Polak, 1967; Geoffrion, 1968; Philip, 1972; Benson and Morin, 1977), and in multiobjective linear programming (Zeleny, 1974; Yu and Zeleny, 1975, 1976), domination structure analysis (Yu, 1974; Bergstresser et al., 1976) and Zeleny's (1976b) parametric goal programming approach. Additional discussions of several of these topics are provided by Roy (1971) and Hirsch (1976).

It may be noted that many of the choice methods mentioned above are designed to avoid the problems involved in the conceptualization and/or assessment of an integrated preference-preserving utility function on X. However, all of these methods are based on more or less definite evaluative assumptions. Some of these will be mentioned later.

3. CLASSIFICATION OF THEORIES

Like so many other things, multicriterion evaluation theories are differentiated by a number of attributes. This section will identify some of these and indicate the rough classification of theories that will be followed in ensuing sections.

A Basic Trichotomy

The main attribute that I shall use to classify evaluation theories is the extent to which risk or uncertainty explicitly enters the theory. This will be treated as a three-valued attribute whose values are similar to the categories of decision making under certainty, under risk, and under uncertainty (Luce and Raiffa, 1957). Although uncertainty may be endemic to all decision problems, it is often absent or disguised in some formulations, and I shall follow this practice.

The first of the three main categories includes evaluation theories that do not explicitly use probabilities or uncertain events in the evaluations. This category includes a large number of theories of preference and utility (Luce and Suppes, 1965; Fishburn, 1970a; Chipman et al., 1971). An excellent example in economic theory is provided by Debreu (1959a).

The second main category encompasses theories that explicitly involve probability or risk. This includes a number of multiattribute theories (Fishburn, 1970a, 1977a; Farquhar, 1976b; Keeney and Raiffa, 1976) that are based on the von Neumann-Morgenstern (1947) expected utility theory. It also includes a variety of other models for comparing gambles or risky alternatives (Rapoport and Wallsten, 1972; Payne, 1973; Slovic et al., 1977; Libby and Fishburn, 1977; Fishburn and Vickson, 1977).

The third category involves evaluation theories that explicitly consider uncertain events or states of the world. The best known theory for this case is probably the Ramsey-Savage personalistic expected utility theory (Ramsey, 1931; Savage, 1954). Although this is not always thought of as a multiattribute theory, it can certainly be viewed as such with the event set and consequence set comprising the two principal attributes. Each of these may in turn involve a number of subattributes.

<u>Other Classificatory Attributes</u>

We now consider briefly seven other aspects that can be used to differentiate among multicriterion evaluation theories.

a. <u>The number and nature of the attribute and/or criterion functions</u>. Some theories are designed primarily for specific types of attribute/criterion structures, such as when each $X_i = \{0,1\}$ or each X_i is a finite qualitative set or each X_i is a continuum of real numbers. Most of the theories discussed later apply to finite numbers of attributes/criteria, but infinite sets of attributes are also used. An example of the latter arises in the denumerable-period time preference theory of Koopmans (1960, 1972b) and others (Koopmans <u>et al</u>., 1964; Diamond, 1965; Burness, 1973, 1976).

b. <u>The structure of the feasible set of alternatives or consequences</u>. This aspect differentiates among feasible sets that are, for example, convex/connected/compact/separable topological spaces or finite-dimensional Euclidean spaces and various types of less structured feasible sets. It is closely connected in certain obvious ways to the basic trichotomy presented above. In addition, it makes a distinction between axiomatic theories that assume that X or $\{(g_1(x),\ldots,g_m(x)): x \in X\}$ is a Cartesian product set and those that assume only that X is some subset of a product set.

c. The basis of evaluation. This refers to the nature of the value construct(s) on which the theory is based. For example, many axiomatic theories are based on a holistic binary preference relation on the set of objects being evaluated. Other theories use a quaternary preference-intensity comparison relation or employ a family of preference relations for different criteria. Still other theories are based on choices, including revealed preference theory (Samuelson, 1938, 1948; Houthakker, 1950, 1961; Richter, 1966; Chipman *et al.*, 1971; Shafer, 1975) and "stochastic" preference/utility theory (Quandt, 1956; Luce, 1958, 1959; Luce and Suppes, 1965; Chipman, 1960a; Marschak, 1960; Marley, 1968; Tversky, 1972a, 1972b; Fishburn, 1973b). Most of the theories discussed in later sections are either based directly on binary comparisons or can be interpreted in this manner.

d. Ordering assumptions. When binary relations are involved in the evaluative theory, this aspect distinguishes among these relations according to properties such as transitivity, asymmetry, reflexivity and completeness. Two commonly used assumptions for an asymmetric preference relation $\succ$ ("is preferred to") are transitivity ($x \succ y$ and $y \succ z \Rightarrow x \succ z$) and negative transitivity ($x \succ z \Rightarrow$ either $x \succ y$ or $y \succ z$). A relation that is asymmetric and transitive will be called a strict partial order, and one that is asymmetric and negatively transitive (and hence transitive also) will be called a strict weak order. When an indifference relation $\sim$ is defined from $\succ$ by $x \sim y$ if and only if neither $x \succ y$ nor $y \succ x$, it is an equivalence relation (reflexive, symmetric, transitive) provided that $\succ$ is a strict weak order; in this case the preference-or-indifference relation $\succsim$ ($x \succsim y \Leftrightarrow x \succ y$ or $x \sim y$) is a weak order (reflexive, complete, transitive). Some writers, including Aumann (1962, 1964a, 1964b), Kannai (1963), and Roy (1973) and Hirsch (1976), do not assume that the preference-or-indifference relation is complete and therefore add an incomparability relation to the preference and indifference relations.

e. Independence assumptions. Notions of independence among attributes or criteria in an evaluative sense are very common in multicriterion theories. For example, the assumption that global or holistic preference increases with an increase in any criterion value is an independence assumption. In expected utility theories the basic independence axioms refer to evaluative independence between the risk (probability) attribute and the consequences attribute.

f. Degree of compensatoriness. In the Euclidean space context, the attributes or criteria are compensatory if local changes that preserve indifference can be made around any point in the space. Noncompensatory preferences obtain when compensating tradeoffs among attributes or criteria are not possible, in which case the preference structure might be lexicographic. Various intermediate cases arise between the fully compensatory and noncompensatory extremes. This aspect of evaluation theories is often associated with the presence or absence of continuity or Archimedean axioms.

g. Extent to which the decision maker's subjective judgments are involved in the evaluation. This attribute is concerned with the extent to which different decision makers in the same type of situation using the same evaluative model may have different evaluative realizations (Libby and Fishburn, 1977). For example, if X is a set of probability distributions on the real line, and if the evaluative model is a mean-variance dominance model or a first-degree stochastic dominance model, then the resultant dominance relation will be independent of the decision maker. This is true also for the vector dominance relation in multicriterion cases if each criterion function is ordinally equivalent across decision makers. Goal programming may require more information of the individual in the form of goals or acceptable levels on each attribute or criterion along with relative judgments of the seriousness of deviations from the goals. Most compensatory preference models presume that different decision makers will have different tradeoff structures.

The importance of this last aspect for differentiating among choice models and their corresponding evaluation models cannot be overemphasized. For example, a desire to develop choice models that do not actively involve the decision maker in the evaluative phase, or that require minimal inputs from him, has motivated many of these models. The lack of more active involvement of the decision maker is often defended by arguments that revolve around his inaccessibility or unidentifiability, his unwillingness or inability to reveal his preferences, and his lack of clarity about his own preferences and the subsequent problems this implies for assessment procedures. It is therefore clear that aspect g is closely connected with the topic of multicriterion assessment methodologies.

4. EVALUATION THEORIES WITHOUT PROBABILITIES

This section reviews multiattribute/multicriterion evaluation theories that do not explicitly use probabilities or uncertain events. The other two categories in our basic trichotomy will be examined in the next two sections. Taking account of the discussion in section 2, we shall assume here that the set of objects to be evaluated is a subset X of a product set $X_1 \times X_2 \times \ldots \times X_n$. The evaluation might concern either a global preference/utility function or structure on X with each X_i an attribute or the range of a criterion function, or it might refer to one of the criterion functions g_j defined on X. I shall let $\succ$ denote some form of strict preference relation or "better than" relation on X, which may or may not be transitive. The only basic condition imposed on $\succ$ is asymmetry: $x \succ y$ and $y \succ x$ cannot both hold for any $x,y \in X$.

Much of our discussion will center around independence assumptions for $\succ$ on X. We shall say that X_i is <u>independent</u> of the other attributes/criteria if and only if $(a_1,\ldots,x_i,\ldots,a_n) \succ (a_1,\ldots,y_i,\ldots,a_n) \Rightarrow (b_1,\ldots,x_i,\ldots,b_n) \succ (b_1,\ldots,y_i,\ldots,b_n)$ for all cases in which the four n-tuples are in X. Given that X_i is independent, we can unambiguously define an asymmetric relation $\succ_i$ on X_i from $\succ$ on X by

$$x_i \succ_i y_i \text{ iff } (a_1,\ldots,x_i,\ldots,a_n) \succ (a_1,\ldots,y_i,\ldots,a_n)$$
$$\text{for some } (a_1,\ldots,x_i,\ldots,a_n),(a_1,\ldots,y_i,\ldots,a_n) \in X.$$

Note here that if X is so sparse in $X_1 \times \ldots \times X_n$ that there are never two n-tuples in X that have the same values of X_i for all but one i and have different values of X_i for the other i, then all X_i are trivially independent with $\succ_i$ empty for each i. It is partly for this reason that many evaluative theories assume that X is either equal to or is a "large" subset of $X_1 \times \ldots \times X_n$.

More generally, we shall say that a subset $\{X_i\colon i \in I\}$, or I for short, is <u>independent</u> of its complement $I^c = \{1,\ldots,n\}\backslash I$, if and only if

(x_i for $i \in I$, a_i for $i \in I^c$) $\succ$ (y_i for $i \in I$, a_i for $i \in I^c$) $\Rightarrow$ (x_i for $i \in I$, b_i for $i \in I^c$) $\succ$ (y_i for $i \in I$, b_i for $i \in I^c$) for all cases in which the four n-tuples are in X. When I is independent of I^c, a relation $\succ_I$ on the product of the X_i for $i \in I$ can be unambiguously defined in the obvious way from $\succ$ on X.

In the rest of this section I shall first say a few words about interdependent preferences and then look at various independent cases. The basic independent discussion is divided into compensatory and noncompensatory theories. The section concludes with some remarks about preference intensity comparisons.

The General Interdependent Case

Apart from general discussions about various types of preference orders and utility functions for a binary relation $\succ$ on X (Luce and Suppes, 1965; Fishburn, 1970a, 1973c; Krantz et al., 1971), relatively little specific theory has been developed for interdependent preferences/utilities on product sets. This perhaps is not surprising in view of the widespread judgment that independence holds in many situations and in view of the difficulties involved in assessing interdependent preferences.

I shall note four developments in this area, all of which assume that $\succ$ is a strict weak order on X and that there exists a real valued utility function u on X such that, for all $x,y \in X$, $x \succ y$ iff $u(x) > u(y)$. Several of these have definite independence overtones.

Debreu (1960), among others, has suggested for the consumption theory context that independence for some subsets of goods may fail when other subsets of goods (e.g. clothing goods, foods, etc.) are independent of their complements. The latter subset independence may then lead to an additive utility representation over these subsets. In a related article Gorman (1968) discusses implications of independence for families of subsets of goods or attributes.

Roskies (1965) and Krantz et al. (1971) present axioms for $\succ$ on $X_1 \times \ldots \times X_n$ that imply that u can be written in a multiplicative form as $u(x_1,\ldots,x_n) = u_1(x_1)u_2(x_2)\ldots u_n(x_n)$. If every u_i has constant sign (positive everywhere or negative everywhere) then this representation is ordinally equivalent to the independent additive representation. However, if some u_i does not have constant sign then independence does not hold. For example, if $u_1(x_1) > 0$ and $u_1(y_1) < 0$ then a change from x_1 to y_1 will reverse all preferences over the remaining attributes. The simplest independence-type assumption that must hold for the multiplicative case is a sign dependence axiom which says that, for each nonempty proper subset I of $\{1,\ldots,n\}$, any two nonempty conditional preference orders over the product of the X_i for $i \in I$ with the values of the other X_i held fixed must be equal or else be the duals (reverses) of one another.

Fishburn (1972) defines the degree of interdependence of $\succ$ on X as the highest order of preference interaction among the attributes that must be used in writing u in an ordinally equivalent additive form. For example, if $n = 3$ and u can be written as $u_{12}(x_1,x_2) + u_{13}(x_1,x_3) + u_{23}(x_2,x_3)$, then the degree of interdependence is no greater than 2. Degrees of interdependence that exceed 1 are not necessarily incompatible with the independence of each X_i from the other attributes. For example, if $u(x_1,\ldots,x_n) = v(u_1(x_1),\ldots,u_n(x_n))$ with each u_i a real valued function and v strictly increasing in each u_i (which implies independence for each X_i), it may still be true that $\succ$ on X has degree of interdependence n.

Finally, consider the case in which X is a compact and convex subset of a finite-dimensional Euclidean space with $n \geq 2$ and u is continuous (Debreu, 1964; Fishburn, 1970a) with a unique maximum at an ideal point (Coombs, 1964; Davis et al., 1970, 1972; Srinivasan and Shocker, 1973a) $x^* \in X$. Suppose further that u decreases along every ray away from x^*. If u decreases in a fully symmetric fashion away from x^*, as when the isoutility or indifference contours are circles or spheres with centers at x^*, then each X_i is independent of the other attributes. In fact, in the spherical case u can be written additively as $u(x) = -\Sigma(x_i - x_i^*)^2$. However, if a nonsymmetric distance function is used to scale utility (with different weights for different dimensions) then independence will generally not hold. A simple example of this in two dimensions arises with elliptical isoutility contours with common axes through x^* that are oriented differently than the axes of X. Other examples arise in goal programming when different weights are applied to the different criterion or goal functions.

Independent Compensatory Transitive Preferences

The traditional theory under this heading assumes that $\succ$ is a strict weak order on $X = X_1 \times \ldots \times X_n$, that each X_i is independent of the others, and that compensatory trade-offs exist between attributes. When X is infinite, it assumes also an order-denseness or continuity assumption (Debreu, 1954; Fishburn, 1970a; Krantz et al., 1971) so that there exists a real valued u on X that preserves the order of $\succ$. It then follows that each $\succ_i$ on X_i is a strict weak order with an order-preserving utility function u_i on X_i, and that u can be written as $u(x) = v(u_1(x_1),\ldots,u_n(x_n))$ with v increasing in each u_i. This form should be familiar to multicriterion optimization researchers, especially when we interpret x_i as x viewed from the perspective of the ith criterion. On the other hand, if $\succ$ is itself interpreted as preference on X from the viewpoint of the ith criterion, then u is a criterion function for the ith criterion.

It is interesting to note what can happen to the neat situation described above when one or more of its hypotheses is weakened. I shall give two examples that use the simple case of X finite and $n = 2$. Suppose first that X is a proper subset of $X_1 \times X_2$ that consists of the following six ordered pairs that are linearly ordered by $\succ$: $(x_1,a_2) \succ (y_1,a_2) \succ (y_1,b_2) \succ (z_1,b_2) \succ (z_1,c_2) \succ (x_1,c_2)$. Then each of X_1 and X_2 is independent of the other, but $\succ_1$ on $X_1 = \{x_1,y_1,z_1\}$ is the cyclic relation $x_1 \succ_1 y_1 \succ_1 z_1 \succ_1 x_1$ and hence is not an ordering relation. Although $\succ_2$ is a linear order on $X_2 = \{a_2,b_2,c_2\}$, it is not possible in this case to form v as described in the preceding paragraph. Suppose next that $X = X_1 \times X_2$ and that $\succ$ on X is a strict partial order consisting of the following four relationships: $(x_1,a_2) \succ (y_1,b_2)$, $(x_1,b_2) \succ (y_1,a_2)$, $(y_1,c_2) \succ (x_1,d_2)$ and $(y_1,d_2) \succ (x_1,c_2)$. Then each of $\succ_1$ and $\succ_2$ is empty so that both are trivial strict weak orders. It is easily seen that it is impossible in this case to define u_1 on $\{x_1,y_1\}$ and u_2 on $\{a_2,b_2,c_2,d_2\}$ (without being concerned about the relation between $\succ_i$ and u_i) so that $v(u_1,u_2)$ increases in each argument and preserves $\succ$ in the sense that $v(u_1(z_1), u_2(z_2)) > v(u_1(w_1), u_2(w_2))$ whenever $z \succ w$. If this could be done then we would require both $u_1(x_1) > u_1(y_1)$ and $u_1(y_1) > u_1(x_1)$, which is absurd. Although these examples suggest some interesting research problems, little has been done in this area apart from specific cases noted below.

Despite the fact that there seems to be no widely accepted rigorous definition of compensatoriness in the independence context, as a minimum we might say that attributes i and j are compensatory if and only if there are

$x_i \succ_i y_i$, $x_i' \succ_i y_i'$, $x_j \succ_j y_j$, $x_j' \succ_j y_j'$ and a,b in the product of the other n-2 attributes such that

$$(x_i, y_j, a) \succsim (y_i, x_j, a) \text{ and}$$

$$(y_i', x_j', b) \succsim (x_i', y_j', b),$$

perhaps with one or both $\succsim$ being $\succ$. In well behaved Euclidean space situations this implies that there are connected indifference curves or regions in the $X_i \times X_j$ subspace with fixed values of the other variables. Although all the models discussed in the next several pages usually have at least the minimal sense of compensatoriness noted above, some of them will also be seen to exhibit noncompensatory features.

The most familiar independent evaluative model is probably the additive utility model that has u_i: $X_i \to Re$ for $i = 1,\ldots,n$ with

$$x \succ y \text{ iff } \Sigma u_i(x_i) > \Sigma u_i(y_i) \quad \text{when } \succ \text{ is a strict weak order,}$$

and

$$x \succ y \Rightarrow \Sigma u_i(x_i) > \Sigma u_i(y_i) \quad \text{when } \succ \text{ is acyclic.}$$

Acyclicity holds when there are no preference cycles such as $x^1 \succ x^2 \succ \ldots \succ x^N \succ x^1$. Although noncompensatory lexicographic preferences on finite sets can be represented by additive models (Fishburn, 1970a, p. 49), these models are usually discussed in compensatory situations. For example, if $X = X_1 \times \ldots \times X_n$ and $u_i(X_i)$ is a nondegenerate interval of real numbers for each i in the weak order context, then the additive model must be compensatory.

Debreu (1960) provided the first general axiomatization of additive utilities. He assumed that $X = X_1 \times \ldots \times X_n$, each X_i is a connected and separable topological space, $\succ$ is a continuous strict weak order on X, and every I is independent of its complement. When $n \geq 3$ and every attribute is essential (no $\succ_i$ is empty), Debreu's axioms imply the additive model. When $n = 2$, additivity requires a stronger independence assumption to the effect that $(x_1,a_2) \succsim (y_1,b_2)$ and $(y_1,c_2) \succsim (z_1,a_2)$ imply $(x_1,c_2) \succsim (z_1,b_2)$. Debreu's approach is also discussed by Gorman (1968), Koopmans (1972a), and Fishburn (1970a) and a generalization of his method has been applied to ordinal preferences over uncertain lifetimes by Fishburn (1978). Algebraically-oriented alternatives to Debreu's topological additive utility theory have been developed by Luce and Tukey (1964), Luce (1966), Krantz (1964) and Krantz _et al_. (1971).

Axiomatizations for additive utilities when X is a finite set with $\succ$ a strict weak order, strict partial order, or acyclic, can be found in Tversky (1964), Scott (1964), Adams (1965), Fishburn (1970a, 1970b) and Krantz _et al_. (1971). Several other cases will be mentioned in the next subsection. The finite-X case requires higher-order independence axioms that generalize the basic assumption in a manner like the $n = 2$ axiom in the preceding paragraph. The theories mentioned in that paragraph imply that the u_i are unique up to similar positive affine transformations $au_i + b_i$ for $i = 1,\ldots,n$ and $a > 0$; the uniqueness properties for finite X are generally weaker than this.

Other contributions to the basic additive model are made by Jaffray (1974), Narens (1974) and Narens and Luce (1976). Sayeki (1972) discusses the weighted form $\Sigma w_i u_i(x_i)$ in which the weights but not the u_i functions change under revisions of the decision maker's goal orientation. Because the w_i can be arbitrary real numbers, Sayeki includes an axiom that allows w_i to change sign under different goal orientations. This is related to sign dependence mentioned earlier for the multiplicative form. Additional discussion of Sayeki's model is in Sayeki and Vesper (1973).

Special forms of the additive model arise when the X_i are similar. I shall note two cases. The first occurs when all X_i are essentially identical except for the index and has

$$u(x_1,\ldots,x_n) = \Sigma w_i \rho(x_i)$$

so that $u_i(x_i) = w_i \rho(x_i)$ for each i. This form arises naturally in the time-period context with i denoting different periods. The equal-weights case (no time preference) arises from the additive model when $(x_1,\ldots,x_n)$ is indifferent to $(x_{\sigma(1)},\ldots,x_{\sigma(n)})$ for any permutation σ on $\{1,\ldots,n\}$ (Debreu, 1959b; Fishburn, 1970a). Another specialization is axiomatized by Koopmans (1960) for the denumerable-period setting. This is the constant discount factor model $u(x_1,x_2,\ldots) = \Sigma\alpha^{i-1}\rho(x_i)$ with $0 < \alpha < 1$. Other cases based on preference intensity comparisons will be mentioned later.

The other special form is the weighted linear model

$$u(x_1,\ldots,x_n) = \Sigma w_i x_i,$$

which assumes $X_i \subseteq Re$ for all i. A specific example is the linear criterion function model $u(g_1(x),\ldots,g_m(x)) = \Sigma w_j g_j(x)$. With integer programming in mind, Aumann (1964b) presents axioms in which X is the set of integer lattice points in the nonnegative orthant of Re^n. He assumes that $\succsim$ is reflexive and transitive (a preorder or quasi-order) and defines $x \succ y$ iff $x \succsim y$ and not $(y \succsim x)$, and $x \sim y$ iff $x \succsim y$ and $y \succsim x$. His representation has

$\Sigma w_i x_i > \Sigma w_i y_i$ when $x \succ y$, and $\Sigma w_i x_i = \Sigma w_i y_i$ when $x \sim y$. Aumann's key independence axiom is the two-part linear independence condition

$$x \succ y \Rightarrow x + z \succ y + z, \text{ and } x \sim y \Rightarrow x + z \sim y + z.$$

The second part of this condition implies the weighted linear model in the context of Debreu's (1960) additive utility theory when each X_i is a real interval with the relative usual topology. The proof of this employs a bisection procedure and uses the fact that the u_i are continuous in his representation.

The weights of the theories in the preceding paragraph are arbitrary real numbers. Williams and Nassar (1966) present an axiomatization that implies positive decreasing weights and is interpreted in a cash flows context. Their key independence axiom, which is similar to Aumann's, says that $x \succsim y$ iff $x - y \succsim (0,\ldots,0)$. They axiomatize a general model in which $u(x) = x_1 + \alpha_2 x_2 + \alpha_2 \alpha_3 x_3 + \ldots + (\alpha_2 \alpha_3 \ldots \alpha_n) x_n$ with $0 < \alpha_i < 1$ for each i, and then show that an additional temporal consistency axiom implies that all α_i are equal.

Independent Compensatory Nontransitive Preferences

Of the two common forms of intransitivity in preference theory--nontransitive indifference and nontransitive preference--more attention has been given to nontransitive indifference. There are several reasons for this. First, nontransitive indifference is well suited to single-attribute situations, where it can arise from imperfect discriminability and threshold effects. Second, strict partial orders provide a basic framework for the combination of transitive preference and nontransitive indifference (defined as the absence of preference). And third, nontransitive indifference can be accommodated in several appealing utility models for special types of strict partial orders, the best known of which are semiorders and interval orders (Armstrong, 1948, 1950; Luce 1956, 1973; Scott and Suppes, 1958; Roberts, 1970, 1971; Fishburn, 1970b, 1970c, 1973c; Mirkin, 1972). We say that $\succ$ on X is a <u>semiorder</u> if and only if it is a strict partial order that satisfies the following two conditions for all $x,y,z,w \in X$:

$$x \succ z \text{ and } y \succ w \Rightarrow x \succ w \text{ or } y \succ z,$$

$$x \succ z \text{ and } z \succ w \Rightarrow x \succ y \text{ or } y \succ w.$$

If only the first of these conditions is assumed for $\succ$, then it is referred to as an <u>interval order</u>. Semiorders were first defined and examined by Luce (1956); interval orders were introduced by Fishburn (1970c).

It can be shown (Scott and Suppes, 1958; Scott, 1964; Fishburn, 1970c; Mirkin, 1972) that, when X is finite, $\succ$ on X is a semiorder iff there exists $u\colon X \to Re$ such that

$$x \succ y \text{ iff } u(x) > u(y) + 1, \text{ for all } x,y \in X;$$

and $\succ$ on X is an interval order iff there are $u\colon X \to Re$ and φ from X into the positive reals such that

$$x \succ y \text{ iff } u(x) > u(y) + \varphi(y), \text{ for all } x,y \in X.$$

Extensions for infinite X are discussed by Fishburn (1973c). When $X \subseteq X_1 \times \ldots \times X_n$, the preceding representations can be given an additive utility form by replacing $u(x)$ by $\Sigma u_i(x_i)$. Axioms for these cases when X is finite are noted by Fishburn (1970b), and Luce (1973) presents an infinite-X axiomatization of additive semiorders when $X = X_1 \times X_2$.

In contrast to examples of nontransitive indifference, all defensible examples of nontransitive preferences that I am aware of (May, 1954; Davidson et al., 1955; Weinstein, 1968; Tversky, 1969; Lichtenstein and Slovic, 1971; Schwartz, 1972) are multiattribute examples. The typical example suggests that preferences between different pairs of alternatives can be governed by different attributes, or by different "weightings" of attributes, in such a way that successive comparisons lead to cyclic preferences. Several models that allow preference cycles are essentially noncompensatory and will be mentioned under the next heading. Under the present heading I will note three models that allow preference cycles and which have definite compensatory aspects.

The first of these is an additive difference model proposed by Morrison (1962) and Tversky (1969). Tversky's version takes $X = X_1 \times \ldots \times X_n$ with

$$x \succsim y \text{ iff } \Sigma h_i(u_i(x_i) - u_i(y_i)) \geq 0,$$

where u_i: $X_i \rightarrow Re$ and h_i is an increasing and continuous real valued function on a real interval for which $h_i(-t) = -h_i(t)$. Tversky suggests that this model can represent situations in which the individual first compares x and y on each attribute and then adds these n difference comparisons to arrive at a holistic comparison. He notes that the additive utility model is the special case of the additive difference model in which each h_i is linear and that, when $n \geq 3$, $\succsim$ is transitive in his model if and only if all h_i are linear. Since $h_i(0) = 0$, the model requires each X_i to be independent of the other attributes.

Although Beals et al. (1968) have axiomatized an additive difference model for similarity judgments, I am not aware of an axiomatization of the additive difference model for preference judgments. However, the present author, in correspondence with Duncan Luce, has developed axioms for the $X = X_1 \times X_2$ case that yield a model that is quite similar to Tversky's model. The key independence axiom in the new system is an extended independence axiom that has definite overtones of preference intensity comparisons between the two attributes. The model generalizes Tversky's additive difference model by allowing a discontinuity at which preferences become lexicographic. For example, one version of the model has

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } h_1(u_1(x_1) - u_1(y_1)) + u_2(x_2) - u_2(y_2) \geq 0,$$

where h_1 is continuous and increasing on a closed interval $[-\lambda, \lambda]$ with $\lambda \geq 0$, $h_1(t) = \infty$ for $t > \lambda$, and $h_1(-t) = -h_1(t)$ for all t in the symmetric domain of h_1. If $\lambda = 0$ then this version is fully lexicographic (and noncompensatory) with X_1 the dominant attribute; and if h_1 is linear then the additive model obtains. If h_1 is convex on $[0, \lambda]$ then the relative importance of X_1 differences accelerates as the difference increases.

A third independent, compensatory and not necessarily transitive model has been axiomatized by Luce (1977) for the $X = X_1 \times X_2$ case. Luce's model allows additive compensatory action between X_1 and X_2 to change to lexicographic dominance by X_1 as the X_1 difference increases. The displayed version of the model in the preceding paragraph is a special case of Luce's model when h_1 is linear on $[-\lambda, \lambda]$, but his general model cannot be expressed in the h_1 format since the lexicographic discontinuity need not be uniform in X_1 differences. The lexicographic part of X_1 in Luce's model is described by a semiorder $\gg$ on X_1 defined by $x_1 \gg y_1$ iff $(x_1, x_2) \succ (y_1, y_2)$ for all $x_2, y_2 \in X_2$. The compensatory part is described by the symmetric complement C of >> on X_1, where $x_1 C y_1$ iff there are $a_2, b_2, c_2, d_2 \in X_2$ for which $(x_1, a_2) \succsim (y_1, b_2)$ and $(y_1, c_2) \succsim (x_1, d_2)$. With $\delta(x_1) = \sup\{u_1(z_1) - u_1(x_1)\colon\ z_1 C x_1\}$ for all $x_1 \in X_1$, Luce's representation has

$$(x_1, x_2) \succsim (y_1, y_2) \text{ iff } u_1(x_1) > u_1(y_1) + \delta(y_1) \text{ or } [-\delta(x_1) \leq u_1(x_1) - u_1(y_1) \leq \delta(y_1) \text{ and } u_1(x_1) + u_2(x_2) \geq u_1(y_1) + u_2(y_2)].$$

Thus the basic additive model applies when $u_1(x_1) - u_1(y_1) \in [-\delta(x_1), \delta(y_1)]$, but otherwise X_1 lexicographically dominates X_2.

<u>Noncompensatory Preferences</u>

In discussing primarily noncompensatory evaluation theories I shall assume for expositional simplicity that the set X of potential things to which the relation $\succ$ might apply is a product set $X_1 \times X_2 \times \ldots \times X_n$. Since there seems to be no widely accepted definition of noncompensatory $\succ$, I shall begin with a definition proposed in Fishburn (1976a) and explore its ramifications.

For each $i \in \{1, \ldots, n\}$ let $\succ_i^0$ be defined on X_i by

$$x_i \succ_i^0 y_i \text{ iff } (x_i, a) \succ (y_i, a) \text{ for } \underline{\text{every}} \text{ a in the product of the other } n - 1 \text{ attributes.}$$

Note that $\succ_i^0$ is different than $\succ_i$ defined earlier and is not predicated on any notion of independence. We shall say that $\succ$ is <u>strongly noncompensatory</u> if and only if, for all $x, y, z, w \in X$, $[\{i\colon x_i \succ_i^0 y_i\} = \{i\colon z_i \succ_i^0 w_i\}$ and $\{i\colon y_i \succ_i^0 x_i\} = \{i\colon w_i \succ_i^0 z_i\}]$ implies that $x \succ y$ iff $z \succ w$. In other words, $\succ$ is strongly noncompensatory exactly when preference between any pair of n-tuples in X is completely determined by the two disjoint subsets of attributes on which each is better than the other according to the $\succ_i^0$. The question "How much better?" is irrelevant for strongly noncompensatory preferences.

Several aspects of this definition are worth noting. First, it depends in no way on whether $\succ$ is transitive. Second, it implies that each X_i is independent of the other attributes, with $\succ_i^0 = \succ_i$. Hence we can write $\succ_i$ in place of $\succ_i^0$. Third, it implies the strong independence feature that holds for additive compensatory models, namely that every $I \subseteq \{1,\ldots,n\}$ is independent of its complement. And fourth, if the minimal compensatory definition presented earlier is required to have either $(x_i,y_j,a) \succ (y_i,x_j,a)$ or $(y_i',x_j',b) \succ (x_i',y_j',b)$, then a strongly noncompensatory $\succ$ can never be minimally compensatory.

By extending the preference notation to disjoint subsets of $\{1,\ldots,n\}$, with $I \succ J$ iff $x \succ y$ whenever $I = \{i: x_i \succ_i y_i\}$ and $J = \{i: y_i \succ_i x_i\}$, and $I \sim J$ iff $x \sim y$ whenever $I = \{i: x_i \succ_i y_i\}$ and $J = \{i: y_i \succ_i x_i\}$, every strongly noncompensatory preference structure can be efficiently characterized by $\succ$ and $\sim$ on the subsets. A structure for which $\{1\} \succ \{2\}$, $\{1\} \succ \{3\}$ and $\{2,3\} \succ \{1\}$ indicates that attribute 1 dominates either 2 or 3 by itself and that attributes 2 and 3 together dominate 1 by itself.

As shown in Fishburn (1976a), the preference notation on disjoint subsets of attributes can be used to characterize a variety of special types of strongly noncompensatory preference structures. The most commonly discussed of these is the <u>lexicographic structure,</u> which obtains if and only if there is a permutation σ on $\{1,\ldots,n\}$ such that, for all x and y in X,

$$x \succ y \text{ iff not } (x_i \sim_i y_i) \text{ for some } i, \text{ and } x_{\sigma(i)} \succ_{\sigma(i)} y_{\sigma(i)} \text{ for the smallest } i \text{ for which not } (x_{\sigma(i)} \sim_{\sigma(i)} y_{\sigma(i)}).$$

Under this definition, $X_{\sigma(1)}$ is the dominant attribute, $X_{\sigma(2)}$ is the next most important attribute, and $X_{\sigma(n)}$ is the least important attribute. Fishburn (1976a) proves that a strongly noncompensatory $\succ$ is lexicographic if and only if the relation $\succ$ applied to subsets of attributes is acyclic, decisive ($I \cap J = \emptyset$ and $I \cup J \neq \emptyset \Rightarrow I \succ J$ or $J \succ I$) and "superadditive" ($I \succ J$, $I' \succ J'$ and $(I \cup I') \cap (J \cup J') \neq \emptyset \Rightarrow I \cup I' \succ J \cup J'$). Fishburn (1975a) shows that a strongly noncompensatory $\succ$ is lexicographic if $\succ$ on X is a strict weak order and for each i there are x_i, y_i and z_i such that $x_i \succ_i y_i \succ_i z_i$.

Mathematical research on lexicographic preferences derives in large part from Hausdorff's work (1957) on products of ordered sets. Its emergence in economics owes much to Georgescu-Roegen (1954, 1968), Hausner (1954) and Chipman (1960b, 1971). A survey of lexicographic topics is provided by Fishburn (1974a). This survey includes a discussion of nontransitive lexicographic preferences, which can arise when the $\succ_i$ relations are semiorders or interval orders. It also notes variations that occur when strict adherence to the lexicographic idea is relaxed (Davidson et al., 1955; Coombs, 1964; Tversky, 1969). An example of this, which follows ideas of Simon (1955), Georgescu-Roegen (1954), Encarnación (1964a) and Ferguson (1965), defines $\succ$ on X in terms of relations $\succ_i'$ on the X_i for which $x_i \succ_i' y_i$ iff y_i is an unacceptable or unsatisfactory level of X_i and x_i is judged to be better than y_i (x_i might be either satisfactory or unsatisfactory) on the basis of the ith attribute or criterion. The definition takes $x \succ y$ iff $x_i \succ_i' y_i$ for some i and this is true for the smallest i for which either $x_i \succ_i' y_i$ or $y_i \succ_i' x_i$. In this modified scheme, criterion 1 is the most important criterion and criterion n is the least important.

The preceding definition is fully lexicographic in terms of the $\succ_i'$ relations, and $\succ$ as thus defined is a strict weak order on X. Several closely related models, which have been discussed by Coombs (1964), Dawes (1964) and Einhorn (1970), among others, are designed to partition the alternatives in X into an acceptable subset A and an unacceptable subset $X\backslash A$. When each X_i is partitioned into an acceptable subset A_i and its unacceptable complement $X_i\backslash A_i$, the general model under consideration has $x \in A$ iff $\{i: x_i \in A_i\}$ is contained in a specified nonempty family F of nonempty acceptable subsets of $\{1,\ldots,n\}$. If $F = \{\{1,\ldots,n\}\}$ then the model is conjunctive with x acceptable iff every x_i is acceptable. On the other hand, if F contains all nonempty subsets of $\{1,\ldots,n\}$ then the model is said to be disjunctive. From an evaluative viewpoint, each model of this type (one for each F) establishes a strict weak order on X that has at most two indifference classes, namely A and $X\backslash A$.

Although the generic F model of the preceding paragraph is not strongly noncompensatory under our earlier definition, it is sometimes referred to as noncompensatory. To compare it to the strongly noncompensatory situation let $x \succ y$ mean that $x \in A$ and $y \in X\backslash A$, and take $x_i \succ_i y_i$ iff $x_i \in A_i$ and $y_i \in X_i\backslash A_i$. Suppose further, as in the strong noncompensatory case, that x,y,z and w are such that $\{i: x_i \succ_i y_i\} = \{i: z_i \succ_i w_i\}$ and $\{i: y_i \succ_i x_i\} = \{i: w_i \succ_i z_i\}$. Strong noncompensatoriness would then require $x \succ y$ iff $z \succ w$, but this can fail for the generic F model since the conditions of the preceding sentence do not imply that $\{i: z_i \in A_i\} \in F$ and $\{i: w_i \in A\} \notin F$ when $\{i: x_i \in A_i\} \in F$ and $\{i: y_i \in A_i\} \notin F$. However, if F is required to satisfy a reasonable regularity condition—which says that, if I_1, I_2, I_3, I_4 are subsets of $\{1,\ldots,n\}$ that are mutually disjoint except for I_3 and I_4, then $I_1 \cup I_3 \in F$ and $I_2 \cup I_4 \in F$ imply $I_1 \cup I_4 \in F$ or $I_2 \cup I_3 \in F$—then reversals of preference (in the sense that $x \succ y$ and $w \succ z$) are impossible when $\{i: x_i \succ_i y_i\} = \{i: z_i \succ_i w_i\}$ and

$\{i: y_i \succ_i x_i\} = \{i: w_i \succ_i z_i\}$. Hence the conjunctive, disjunctive and other F models have a very definite noncompensatory flavor even though they are not strongly noncompensatory.

Preference Intensity Comparisons

We conclude this section with several remarks on preference intensity comparisons. Such comparisons may be either holistic or conditioned on a particular attribute or criterion. For example, let $\succ^*$ and $\succ_i^*$ be binary relations on $X \times X$. Then $(x,y) \succ^* (z,w)$ could mean that degree of preference for x over y exceeds degree of preference for z over w, and $(x,y) \succ_i^* (z,w)$ could indicate that the difference in preference between x and y on the basis of criterion i exceeds the difference in preference between z and w on the basis of criterion i. Some choice models that identify efficient sets of alternatives from dominance or outranking relations have overtones of the latter type of comparison. The outranking models developed by Roy (1971, 1973, 1974) and others are a case in point. The usages of the scoring functions g_j in these models strongly suggests a degree-of-preference orientation. This seems true also, though to a lesser extent, in some of the goal programming models.

The basic theory of preference-difference comparisons extends from the ordered metric rankings of Coombs (1964), Siegel (1956) and Fishburn (1964) through a number of theories (Frisch, 1926; Alt, 1936; Suppes and Winet, 1955; Suppes and Zinnes, 1963; Pfanzagl, 1959) that imply the existence of a real valued u on X such that

$$(x,y) \succ^* (z,w) \text{ iff } u(x) - u(y) > u(z) - u(w), \text{ for all } x,y,z,w \in X.$$

Inexact or vague degree-of-preference theories are discussed by Adams (1965) and Fishburn (1970d). Several of these theories as well as others are discussed in Fishburn (1970a, Chapter 6) and Krantz et al. (1971, Chapter 4). Their mathematical structures are similar to those in two-attribute additive utility theories; the key difference-comparison axioms are like the independence axioms in the $n = 2$ additivity theories.

When u satisfies the utility difference representation of the preceding paragraph and $X = X_1 \times \ldots \times X_n$, it may be possible to express u in an additive way as $u(x) = \Sigma u_i(x_i)$. Axioms for this case are presented by Krantz et al. (1971, p. 492) and Dyer and Sarin (1977). The latter authors begin from the perspective of an additive representation and ask what must be true so that $u = \Sigma u_i$ can be interpreted in a meaningful way as a function whose differences preserve preference intensities. The opposite approach begins with the preceding difference representation and asks what must be true so that its u can be written in the additive form. It is easily seen (Fishburn, 1970a, p. 93) that this can be done if and only if there is a fixed element $(e_1,\ldots,e_n)$ in X such that, whenever $i \in \{1,\ldots,n\}$ and $x_j = y_j$ and $z_j = w_j$ for all $j \neq i$, $(x,y) \succ^* (z,w)$ iff $((x_i,e_j \text{ for } j \neq i), (y_i,e_j \text{ for } j \neq i)) \succ^* ((z_i,e_j \text{ for } j \neq i), (w_i,e_j \text{ for } j \neq i))$.

The utility difference representation in which u(x) can be written as $\Sigma u_i(x_i)$ can be further specialized when all X_i are the same except for their indices. Fishburn (1970a, Chapter 7) presents axioms based on Debreu's topological approach which implies that u(x) can be written as $\Sigma w_i \rho(x_i)$ with the $w_i > 0$. An additional stationarity axiom, which says that $(x_1,\ldots,x_{n-1},e_0) \succ (y_1,\ldots,y_{n-1},e_0)$ iff $(e_0,x_1,\ldots,x_{n-1}) \succ (e_0,y_1,\ldots,y_{n-1})$ for some fixed e_0, then implies the constant discount rate form in which $u(x) = \Sigma\alpha^{i-1}\rho(x_i)$. In the stationarity axiom (Koopmans, 1960), which is similar to the temporal consistency axiom of Williams and Nassar (1966), $\succ$ is defined from $\succ^*$ by $x \succ y$ iff $(x,y) \succ^* (y,y)$. In general it is customary to define the basic preference relation in this way when $\succ^*$ on $X\times X$ is taken as the primitive relation.

5. COMPARISONS OF RISKY ALTERNATIVES

In this section we discuss evaluation theories in which the alternatives are probability measures p,q,... in a set P of measures defined on an algebra of subsets of a set X of decision consequences. For expositional simplicity the measures in P will usually be referred to as probability distributions or gambles on X. Even when X is not multiattribute, the probabilities and consequences constitute two primary attributes so that the situation is essentially a multiattribute situation. Representations of gambles as probability vectors when X is finite and characterizations of gambles in terms of their moments when $X \subseteq Re$ suggest other multiattribute forms.

The theories of the present section will be divided into three main classes. The first class consists of special theories for a von Neumann-Morgenstern utility function (1947) when X is equal to or a subset of a product set $X_1 \times \ldots \times X_n$. The second class contains a variety of stochastic dominance theories that for the most part assume that $X \subseteq Re$ and that consequence x is preferred to consequence y when $x > y$. The third class involves a number of other theories of comparison when $X \subseteq Re$ and preference increases in x.

Multiattribute Expected Utility Theories

Throughout this subsection we shall assume that a preference relation $\succ$ on P satisfies the expected utility model

$$p \succ q \text{ iff } \int_X u(x)dp(x) > \int_X u(x)dq(x), \text{ for all } p,q \in P,$$

where u is a real valued utility function on X for which $\int udp$ is finite and well defined for all $p \in P$. Axioms for various cases of this model are presented by von Neumann and Morgenstern (1947), Marschak (1950), Herstein and Milnor (1953), Blackwell and Girshick (1954), DeGroot (1970) and Fishburn (1970a, 1975b) among others. A brief review of these and other expected utility theories, including ones based on partial orders and lexicographic utilities, is given by Fishburn (1977b). Also see Fishburn (1974a) for more on lexicographic expected utility.

Within the context of the usual expected utility model we shall consider briefly some special assumptions on preferences between gambles and their effects on u when $X \subseteq X_1 \times \ldots \times X_n$. Additional coverage of this topic can be obtained from the reviews by Farquhar (1976b) and Fishburn (1977a) and the book by Keeney and Raiffa (1976).

The first special form of a von Neumann-Morgenstern utility function on multiattribute consequences that was axiomatized was the additive form $u(x) = \Sigma u_i(x_i)$. This was done independently by Fishburn (1965a) and Pollak (1967) for $X = X_1 \times \ldots \times X_n$. Later Fishburn (1971) proved that, when X is an arbitrary subset of $X_1 \times \ldots \times X_n$, u can be written additively if and only if $p \sim q$ whenever p and q are two gambles in P (the set of simple measures on X) such that the marginal distribution of p on X_i equals the marginal distribution of q on X_i for each i. When $X = X_1 \times \ldots \times X_n$, it suffices to express this condition in terms of simple 50-50 gambles (Fishburn, 1965a; Raiffa, 1969).

A different independence notion that is more similar to the idea of independence in section 4 was introduced by Pollak (1967), Keeney (1968, 1971, 1972), Raiffa (1969) and Meyer (1970). Usually referred to as utility independence, it says that $I \subseteq \{1,\ldots,n\}$ is utility independent of its complement I^c if and only if the preference order over probability distributions on the product of the attributes in I conditioned on fixed values of the attributes in I^c does not depend on the fixed values of these other attributes. When I is utility independent of I^c, and when $X = X_1 \times \ldots \times X_n$ and P includes the simple measures on X, it then follows from the basic expected utility theory that there are real valued functions v_a and v_b on the product of the X_i for $i \in I^c$ with $v_a > 0$ and a real valued function w on the product of the X_i for $i \in I$ such that

$$u(x_1,\ldots,x_n) = v_a(x_i \text{ for } i \in I^c)w(x_i \text{ for } i \in I) + v_b(x_i \text{ for } i \in I^c).$$

If a sufficient number of the $I \subseteq \{1,\ldots,n\}$ are utility independent of their complements (see previous references for details) it follows that u on $X = X_1 \times \ldots \times X_n$ is either additive or else has an essentially multiplicative form $u(x) = u_1(x_1)\ldots u_n(x_n)$ in which each u_i has constant sign. More complex combinatorial forms for u arise when utility independence applies to more restricted I families.

Generalizations of utility independence have been considered for the case in which X is an arbitrary subset of $X_1 \times X_2$ (Fishburn, 1976b) and for $X = X_1 \times X_2 \times \ldots \times X_n$ with complete reversals and empty orders allowed when the fixed values of the attributes in I^c are changed (Fishburn, 1974b; Fishburn and Keeney, 1974, 1975). The latter work ties into the sign dependence condition for multiplicative utilities in the nonprobabilistic context and has the effect of allowing v_a in the preceding paragraph to change sign or equal zero. The former generalization illustrates the difficulties in obtaining the decomposed form of the preceding paragraph when X is an infinite subset of $X_1 \times X_2$.

Utility independence breaks down when the individual's risk attitude towards I (Keeney, 1973; Pollak, 1973) depends on the fixed values of the other attributes. However, more complex independence conditions can accommodate changes in conditional risk attitude. An example is a bilateral independence notion (Fishburn, 1973d, 1974b, 1977a) that uses two sets of fixed values for the attributes in I^c rather than one set as in utility independence. A general system of fractional independence conditions that can use more than two sets of fixed values for the conditioning attributes has been developed by Farquhar (1975, 1976a). Farquhar's theory is presently the most general independence theory for multiattribute expected utility. It gives rise to a great variety of specialized forms for $u(x_1,\ldots,x_n)$ and includes utility independence and bilateral independence as special cases.

In addition to the independence theories mentioned above, we note that a closely related body of theory has been developed specifically for the time-stream context (Fishburn, 1965b, 1970a; Meyer, 1970, 1977; Bell, 1974; Keeney and Raiffa, 1976, Chapter 9). This includes forms for u such as $u(x) = \Sigma\alpha^{i-1}\rho(x_i)$ that are designed for the homogeneous product set context as well as more general additive and multiplicative representations. In a different vein, Kirkwood (1976) has presented a notion of parametrically dependent preferences.

Another area of potential development is the application of approximation theory (Cheney, 1966; Lorentz, 1966) to the estimation of $u(x_1,\ldots,x_n)$ when no independence conditions are presumed. A start in this direction has been made by Fishburn (1977c).

Stochastic Dominance Comparisons

A major problem in using expected utility theory is the difficulty of accurately assessing the decision maker's utility function on the consequence space X. A great deal of attention has therefore been given to comparisons of distributions on X that are based on limited information about u. If $\int udp \geq \int udq$ for every u that satisfies the limited information then p is said to stochastically dominate q with respect to that information. More precisely, suppose that what is known about the decision maker's utility function within the expected utility context can be described by a set U of real valued functions on X such that every $u \in U$ satisfies the given data and every $u \notin U$ violates the data. Suppose further that every $u \in U$ is integrable with respect to p and q. Then we say that p stochastically dominates q with respect to U, and write $p \geq_U q$, if and only if $\int udp \geq \int udq$ for all $u \in U$. And p strictly stochastically dominates q with respect to U, or $p >_U q$, if and only if $p \geq_U q$ and not $(q \geq_U p)$.

In several interesting cases $\geq_U$ can be conveniently stated in terms of the distributions without direct reference to U. Hence some definitions of stochastic dominance relations are based directly on properties of the distributions. Examples will be given momentarily.

The viewpoint on stochastic dominance expressed here follows the general treatment proposed by Brumelle and Vickson (1975) and Fishburn (1975c). A comprehensive introduction to the theoretical side of stochastic dominance is provided by Fishburn and Vickson (1977), and Whitmore and Findlay (1977) includes a number of chapters on applications and implementation. In the rest of this subsection we shall examine three X contexts with respect to stochastic dominance, namely $X \subseteq Re$, $X \subseteq Re^n$ and X arbitrary.

When $X \subseteq Re$, it is convenient to work with cumulative distribution functions F and G rather than with their underlying measures p and q. General definitions of first degree stochastic dominance (FSD, represented by $\geq_1$) and second degree stochastic dominance (SSD, represented by $\geq_2$) in the real line context are provided by

$$F \geq_1 G \text{ iff } F(x) \leq G(x) \text{ for all } x \in Re,$$

$$F \geq_2 G \text{ iff } \int_{-\infty}^{x} F(y)dy \leq \int_{-\infty}^{x} G(y)dy \text{ for all } x \in Re.$$

To avoid certain technical problems (see Tesfatsion, 1976; Fishburn and Vickson, 1977) we shall assume that X is a closed and bounded interval and that X includes the supports of F and G. It can then be shown that $F \geq_U G$ iff $F \geq_1 G$ when U is the class of all strictly increasing functions on X, or the class of all nondecreasing functions on X, or some appropriately rich subset of one of these. The statement "$F \geq_U G$ iff $F \geq_1 G$" is a typical stochastic dominance theorem that relates uniform expected utility comparisons ($\int udF \geq \int udG$ for all $u \in U$) to properties of F and G ($F(x) \leq G(x)$ for all x). Although FSD is often defined in terms of a U class rather than by $\geq_1$, the fact that different U classes (essentially involving nondecreasing preferences over X) give equivalence to $\geq_1$ lends some support to the definition used above. The type of FSD equivalence theorem noted above appears to have been independently arrived at by Lehmann (1955), Quirk and Saposnick (1962) and Fishburn (1964).

Second degree stochastic dominance is associated with nondecreasing concave utility functions on X and corresponds to the notion of risk aversion (Pratt, 1964; Arrow, 1965). A typical SSD theorem says that $F \geq_U G$ iff $F \geq_2 G$ when U is the class of all nondecreasing concave u on X. Here again several authors, including Hardy et al. (1934), Fishburn (1964) and Hadar and Russell (1969) have independently discovered this type of result. Other FSD/SSD references include Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Hadar and Russell (1971). Fishburn (1974c) discusses stochastic dominance between convex combinations of distributions.

Several additional developments for real line SD may be noted. Whitmore (1970) presents a third degree SD definition in terms of thrice continuously differentiable u for which $u' > 0$, $u'' < 0$ and $u''' > 0$ and shows that $\int udF \geq \int udG$ uniformly for this class iff the mean of F is as large as the mean of G and

$$\int_{x=-\infty}^{z} \int_{y=-\infty}^{x} F(y)dy \leq \int_{x=-\infty}^{z} \int_{y=-\infty}^{x} G(y)dy \text{ for all } z.$$

In a related vein, Vickson (1975a, 1975b, 1977) and Bawa (1975) examine stochastic dominance for decreasing absolute risk averse utility functions ($u' > 0$, $u'' < 0$, $r' \leq 0$) where $r(x) = -u''(x)/u'(x)$ is the Arrow-Pratt measure of absolute risk aversion at x, and Meyer (1977) considers the problem of determining what must be true of F and G so that $\int udF \geq \int udG$ for all u for which $r(x) \in [r_1(x), r_2(x)]$ for all x when r_1 and r_2 are extended real valued functions with $r_1 \leq r_2$. Vickson (1976) also develops F vs G comparison rules under FSD and SSD when u has been accurately assessed at a finite number of points. Finally, with $X = [0,b]$, Fishburn (1976c) defines and analyzes a continuum of stochastic dominance relations in which the relation for $\alpha \in [1,\infty)$ is defined in terms of fractional integrals by

$$F \geq_\alpha G \text{ iff } \int_0^x (x - y)^{\alpha-1} dF(y) \leq \int_0^x (x - y)^{\alpha-1} dG(y) \text{ for all } x \in [0,b].$$

This definition is equivalent to the prior definitions of $\geq_1$ and $\geq_2$ when $\alpha = 1$ and $\alpha = 2$. Fishburn proves that every $\geq_\alpha$ is transitive, that every $>_\alpha$ is a strict partial order, and that $\geq_\alpha$ and $>_\alpha$ are respectively proper subsets of $\geq_\beta$ and $>_\beta$ when $\alpha < \beta$. Classes of utility functions that are congruent with these relations are also identified.

Turning next to the multiattribute consequence context, we shall assume that X is a rectangular (product of intervals) subset of Re^n. For the FSD case let U_1 be the set of all nondecreasing Borel measurable functions on X and define $F \succsim_1 G$ iff $\int udF \geq \int udG$ for all $u \in U_1$ where F and G are multivariate distribution functions on Re^n. The most general FSD results for this context appear to be those obtained by Lehmann (1955). His general theorem says in effect that $F \succsim_1 G$ iff $\int_B dF \geq \int_B dG$ for every increasing ($x \in B$, $h \geq 0$, $x + h \in X \Rightarrow x + h \in B$) Borel subset B of X. This result has recently been rediscovered by Levhari et al. (1975). If the distributions are independent in their marginals in the sense that $F(x_1,\ldots,x_n) = F_1(x_1)\ldots F_n(x_n)$, and similarly for G, then Lehmann shows that $F \succsim_1 G$ iff $F_i \geq_1 G_i$ for each i. Some other contributions to the FSD case have been made by Levy and Paroush (1974a, 1974b).

For the multidimensional SSD case let U_2 be the set of all nondecreasing and concave Borel measurable functions on X and define $F \succsim_2 G$ iff $\int udF \geq \int udG$ for all $u \in U_2$. Various results for this case have been obtained by

Sherman (1951), Strassen (1965) and Veinott (see Bessler and Veinott, 1966), and more recently by Levhari *et al*. (1975), Peleg (1975) and Brumelle and Vickson (1975). The basic theorem in this area (Strassen, 1951; Brumelle and Vickson, 1975) says that if X is bounded with $\hat{x}$ and $\hat{y}$ the random vectors for F and G respectively, then $F \succsim_2 G$ if and only if there is a random vector $\hat{z}$ such that $\hat{y}$ is equal in distribution to $\hat{x} + \hat{z}$ and the expected value of $\hat{z}$ given $\hat{x}$ is nonpositive ($\leq (0,\ldots,0)$) with probability 1. In the independence context with the means of F and G finite, we get $F \succsim_2 G$ iff $F_i \geq_2 G_i$ for each i (Fishburn and Vickson, 1977).

Finally, consider the case where X is arbitrary so that there is no natural order (complete or partial) on its elements. Fishburn (1964, 1974d, 1975d) shows how the basic ideas of stochastic dominance can be used when the order of X is taken to be the decision maker's preference order. For example, if $\succsim$ is a weak order on X and p and q are simple distributions with combined support $A \subseteq X$, then $\Sigma u(x)p(x) \geq \Sigma u(x)q(x)$ for all u that preserve $\succsim$ on A iff $p(\{x: a \succsim x\}) \leq q(\{x: a \succsim x\})$ for all $a \in A$. A large variety of similar results for other types of information about u are presented in the aforementioned references. Fishburn (1977d) also proposes a definition of stochastic dominance for the case in which the decision maker's preference relation on the consequences may be intransitive. This definition says that p dominates q if and only if

$$p(\{x: q(\{y: x \succ y\}) \leq \lambda\}) \leq q(\{x: p(\{y: x \succ y\}) \leq \lambda\})$$

and

$$p(\{x: q(\{y: y \succ x\}) \geq \lambda\}) \leq q(\{x: p(\{y: y \succ x\}) \geq \lambda\})$$

for all $\lambda \in [0,1]$. Although this conception is not directly relatable to order preserving utility functions when $\succ$ has cycles, it has a number of appealing properties including equivalence to the previous FSD notion when $\succ$ is a strict weak order on the joint support of p and q.

Other Theories for Univariate Gambles

Our purpose in the rest of this section is to summarize multiattribute theories for the comparison of risky prospects that are not directly based on expected utility. It will be assumed that $X \subseteq Re$ with X a set of potential returns on investment, net profits, or some other variable for which preference increases in x.

The main class of theories under this heading are mean-risk theories that, except for a general *conception of risk* discussed by Coombs (1974) and Coombs and Huang (1968), assume that a larger mean or expected return is preferred to a smaller mean, and a smaller risk is preferred to a larger risk. For distribution F we shall let $\mu(F)$ denote the mean of F and take R(F) as some real valued measure of the risk of F. The models in this class can be differentiated in two main ways: (1) the particular form of R(F) that is used, and (2) whether the model is a dominance model, a completely ordered compensatory model, or a completely ordered lexicographic model in which either the risk measure or the mean dominates.

The most common risk measure discussed in the literature is the variance $\sigma^2(F)$ or standard deviation $\sigma(F)$ of distribution F (Markowitz, 1952, 1959; Tobin, 1965; Sharpe, 1964; Lintner, 1965). This has been modified in two ways by subtraction of the mean. Baumol (1963) proposes $R(F) = K\sigma(F) - \mu(F)$ with $K > 0$ (see also Bickel, 1969 and Agnew et al., 1969), and Pollatsek and

Tversky (1970) axiomatize the measure $R(F) = \theta\sigma^2(F) - (1 - \theta)\mu(F)$ with $0 < \theta < 1$. Other measures of risk focus on low outcomes. With e and t respectively the point of no loss and no gain and a desired target level (t) of return, these measures include two forms of semivariance (Markowitz, 1959; Mao, 1970a, 1970b; Hogan and Warren, 1972, 1974; Porter, 1974), namely the below-mean semivariance

$$R(F) = \int_{-\infty}^{\mu(F)} (\mu(F) - x)^2 dF(x)$$

and the below-target semivariance

$$R(F) = \int_{-\infty}^{t} (t - x)^2 dF(x);$$

Fishburn's (1977e) generalized below-target measure

$$R(F) = \int_{-\infty}^{t} (t - x)^{\alpha} dF(x), \quad \alpha > 0;$$

the probability of ruin, the probability of loss (Markowitz, 1959; Pruitt, 1962), and the weighted loss measure

$$R(F) = \int_{-\infty}^{e} (e - x) dF(x)$$

used by Domar and Musgrave (1944). Stone (1973) presents a general model that includes most of these measures as special cases.

The mean-risk dominance model says that distribution F strictly dominates distribution G if and only if $\mu(F) \geq \mu(G)$, $R(F) \leq R(G)$, and at least one of the inequalities is strict. The dominance relation in this case is a strict partial order. When the dominance model is used, the objective is to identify the efficient (undominated) set of feasible risky prospects. The greatest success in this regard has been with the mean-variance model used by Markowitz and others. This model, in both its dominance and tradeoff forms, has been criticized on certain logical grounds by several writers, including Borch (1963, 1969, 1974), Feldstein (1969), Chipman (1973), Levy (1974) and Fishburn (1975e).

The compensatory mean-risk tradeoff model assumes that there is a real valued utility function u on (μ,R) pairs such that u increases in μ, decreases in R, and has

$$F \succ G \text{ iff } u(\mu(F), R(F)) > u(\mu(G),R(G)).$$

The preceding dominance relation is included in the preference relation $\succ$, which is a strict weak order in the present case. On occasion the utility function is further specialized, as in Van Moeseke's (1963, 1965) model $u(\mu(F),\sigma(F)) = \mu(F) + m\sigma(F)$. Although m must be negative for u to decrease in risk, Van Moeseke includes positive m to denote risk-seeking situations. Questions about $\mu - \sigma$ tradeoff curves and congruence with expected utility for the compensatory $u(\mu,\sigma)$ model are discussed by Samuelson (1967, 1970), Samuelson and Merton (1974), Feldstein (1969), Tsiang (1972, 1974), Chipman (1973), and Levy (1974), among others. Tradeoff models that associate R with low returns have been discussed by Mao (1970a, 1970b), Conrath (1973), Fishburn (1977e) and Libby and Fishburn (1977).

When R(F) denotes the probability of ruin for distribution F, the lexicographic mean-risk model with risk dominant has $F \succ G$ iff $R(F) < R(G)$ or $[R(F) = R(G)$ and $\mu(F) > \mu(G)]$. A related lexicographic model seeks to maximize expected return subject to probability of ruin or failure not exceeding a level specified by the decision maker (Reder, 1947; Roy, 1952; Shubik, 1961; Encarnación, 1964b; Agnew et al., 1969; Machol and Lerner, 1969; Joy and Barron, 1974), and Conrath (1973) recommends a hybrid model that maximizes utility in a mean-risk compensatory model subject to probability of ruin being acceptably small. Although mean-risk lexicographic models in which the mean is dominant do not seem to have been discussed in the literature, Fishburn (1975e) shows that such a model can be logically implied by assumptions that lie behind the mean-variance approach.

In addition to the mean-risk models mentioned above, choices and/or preferences among univariate gambles have been examined on other bases. These include higher order moments in addition to the mean and variance (Lichtenstein, 1965; Alderfer and Bierman, 1970; Tsiang, 1972; Payne, 1973) and a linear model for special types of gambles in which the attractiveness of a gamble is a weighted sum of probability of winning, probability of losing, amount that may be won, and amount that may be lost (Slovic, 1967; Slovic and Lichtenstein, 1968, 1971; Rapoport and Wallsten, 1972; Payne, 1973, 1975).

6. COMPARISONS UNDER UNCERTAINTY

The third category of our basic trichotomy of evaluation theories views each decision alternative or act as a function f that assigns a consequence in X to each state in a set S of exclusive and exhaustive states of the world (Savage, 1954). If $S = \{s_1,\ldots,s_n\}$ then, with $f(s_i) = x_i$ for each i, an act can be thought of as an n-tuple $(x_1,\ldots,x_n)$ in X^n. In any case, f(s) denotes the consequence in X that obtains when s is the true state of the world.

The most widely accepted normative theory for decision making under uncertainty is the Ramsey-Savage personalistic expected utility theory (Ramsey, 1931; Savage, 1954). This theory was first completely axiomatized by Savage within an infinite-states context. A full technical account of Savage's theory is also given in the final chapter of Fishburn (1970a). Savage's axioms imply the existence of a finitely-additive probability measure η on 2^S and a real valued utility function u on X such that, for all f,g: $S \to X$,

$$f \succ g \text{ iff } \int_S u(f(s))d\eta(s) > \int_S u(g(s))d\eta(s).$$

Here η is the individual's personal or subjective probability measure on S and u is his utility function on the consequences. Subsequent axiomatizations of this and related models for subjective expected utility have been presented by Suppes (1956), Davidson and Suppes (1956), Pratt et al. (1964), Jeffrey (1965), Bolker (1966, 1967), Luce and Krantz (1971), Fishburn (1970a, 1973e) and Balch and Fishburn (1974), among others. Several of these (Jeffrey, Bolker, Luce and Krantz, Balch and Fishburn) are conditional models in which the decision maker's probabilities depend on the act selected. In Savage's model, η is independent of the acts.

If the decision maker's probabilities are presumed to be known then the present situation maps into the context of the preceding section since each f induces a probability distribution on X. On the other hand, if the utilities but not the probabilities are assumed to be known precisely, then we encounter a situation that is dual to the stochastic dominance approach of the preceding section. Fishburn (1964, 1965c) and Barron (1973) examine various types of information about η and show what must be true of u so that $\int u(f(s))d\eta(s) \geq \int u(g(s))d\eta(s)$ for all η that satisfy the given information. For example, if $S = \{s_1,\ldots,s_n\}$ and if all that is presumed known about η is $\eta(s_1) \geq \eta(s_2) \geq \ldots \geq \eta(s_n) \geq 0$ with $\Sigma\eta(s_i) = 1$, then we can conclude that $\Sigma u(f(s_i))\eta(s_i) \geq \Sigma u(g(s_i))\eta(s_i)$ for all such η if and only if

$$\sum_{i=1}^{k} u(f(s_i)) \geq \sum_{i=1}^{k} u(g(s_i)) \quad \text{for } k = 1,2,\ldots,n.$$

Fishburn (1964, Chapter 11) also examines the case where $X = X_1 \times \ldots \times X_n$ and u is additive over the attributes in the context of incomplete information on utilities and/or probabilities. It is shown that many of the uniform expected utility comparison problems can be viewed as linear programming problems. Other approaches for comparing subjective expected utilities when the decision maker's probabilities are not fully known are discussed in Fishburn et al. (1968).

A special case of the Ramsey-Savage approach for the finite-S setting is the so-called Laplace criterion (1814) for which the states are regarded as equally likely. This is sometimes defended on the basis of the principle of insufficient reason (see, for example, Fishburn, 1964, pp. 140-143), and axiomatizations of the criterion have been given by Chernoff (1954) and Milnor (1954).

The other primary methods for comparing acts in a finite states formulation are non-probabilistic approaches. Apart from the Laplace criterion, which is sometimes viewed as a non-probabilistic averaging model, the four main approaches under this heading are maximin (Wald, 1950), maximax, Hurwicz-α, and minimax loss (Savage, 1951). Summaries and criticisms of these approaches have been presented by a number of writers, including Goodman (1954), Milnor (1954), Luce and Raiffa (1957), Ackoff (1962), Fishburn (1966) and MacCrimmon (1973). Milnor's article is especially useful in that it differentiates among the approaches axiomatically.

Maximin says that f is better than g if $\min_i u(f(s_i)) > \min_i u(g(s_i))$; maximax says that f is better if $\max_i u(f(s_i)) > \max_i u(g(s_i))$; Hurwicz-$\alpha$ for $0 \leq \alpha \leq 1$ has f over g if $\alpha \max_i u(f(s_i)) + (1 - \alpha) \min_i u(f(s_i)) > \alpha \max_i u(g(s_i)) + (1 - \alpha) \min_i u(g(s_i))$; and minimax loss has f over g within the context of a set A of acts if $\max_i [\sup_A u(a(s_i)) - u(f(s_i))] < \max_i [\sup_A u(a(s_i)) - u(g(s_i))]$. The first two of these are based solely on simple preference comparisons between consequences but the latter two have obvious cardinal utility implications. Although all four can give different orderings of a set A of acts, Savage (1954, p. 170) says that Wald's actual use of maximin is equivalent to his own minimax loss approach. Extensive discussions of this approach in statistical decision theory, including its application to mixed acts or randomized strategies, are given by Wald (1950) and Savage (1951, 1954). It is of course closely related to the minimax solution approach for zero-sum games (von Neumann and Morgenstern, 1947; Luce and Raiffa, 1957).

7. ASSESSMENT METHODOLOGIES

As explained earlier, I have deliberately omitted discussion of assessment methodologies from the preceding sections to avoid confounding this crucial area with evaluation theories *per se*. In this final section I shall indicate briefly some of the main themes of assessment and attempt to provide a balanced introduction to its literature.

Most of the assessment procedures that relate to the evaluative theories surveyed in earlier sections are concerned with the estimation of either subjective probabilities or utilities and/or criteria weights. Major aspects of the assessment of subjective or personal probability are discussed in the early survey by Edwards (1954), the important assessment articles by Winkler (1967a, 1967b), and the more recent surveys by Savage (1971), Slovic and Lichtenstein (1971), and Hogarth (1975). Various difficulties and biases that can affect judgments of personal probabilities are noted in these works and in Winkler and Murphy (1973) and Tversky and Kahneman (1974). The latter review is backed up by a series of interesting studies on specific sources of bias (Tversky and Kahneman, 1973; Kahneman and Tversky, 1972, 1973). Probability assessment is of course intimately concerned with the evaluative theories discussed in sections 5 and 6.

Extensive coverage of multiattribute/multicriterion utility assessment is provided by Fishburn (1967), Raiffa (1969), Slovic and Lichtenstein (1971), MacCrimmon (1973), Green and Wind (1973), Huber (1974), Kneppreth et al. (1974), Johnson and Huber (1976) and Keeney and Raiffa (1976). A rather large proportion of this material is concerned with the additive utility form $u(x_1,\ldots,x_n) = \Sigma u_i(x_i)$ in both the nonprobabilistic and the risky settings, although Raiffa (1969) and Keeney and Raiffa (1976) extensively discuss the multiplicative and other algebraic forms in the risky context of section 5. Assessment procedures that do not necessarily presuppose a decomposed form for a multiattribute utility function include holistic procedures (Slovic and Lichtenstein, 1971; Fischer, 1977), tradeoff methods (Thurstone, 1931; MacCrimmon and Toda, 1969; MacCrimmon and Siu, 1974), and approximate fits to pairwise assessment data

(Smith et al. 1974). Some specialized procedures designed for interactive programming methods are discussed in the references in the penultimate paragraph of section 2.

The additive utility form has been considered in the general Σu_i format as well as in specialized forms that are available when each X_i is a set of real numbers. The latter include the weighted linear model $\Sigma w_i x_i$ (Gulliksen, 1956; Srinivasan and Shocker, 1973b; Srinivasan et al. 1973; Dawes and Corrigan, 1974; Einhorn and Hogarth, 1975) and the weighted Euclidean distance model with ideal point (Srinivasan and Shocker, 1973a; Pekelman and Sen, 1974).

Two general approaches are used to estimate the u_i in the general case. First, each u_i might be assessed separately up to compensating scale weights (Galanter, 1962; Fishburn, 1967; Edwards, 1972; Keeney and Raiffa, 1976; Fischer, 1977). Meyer and Pratt (1968) and Bradley and Frey (1975) discuss u_i fits to limited data for a real variable in the expected utility context. Second, one key u_i might be assessed directly with the others determined from it through some form of tradeoff data (Fishburn, 1967; MacCrimmon and Siu, 1974).

In the special Euclidean models only the criterion or aspect weights may require estimation although functional forms and the location of an ideal point may also be at issue. Many studies have focussed on the weighting problem, including Churchman and Ackoff (1954), Eckenrode (1965), Stimson (1969), Edwards (1972), Srinivasan and Shocker (1973a, 1973b), Srinivasan et al.(1973), Pekelman and Sen (1974) and Fischer (1977). Fischer's study compares the assessments obtained with holistic and additive procedures and observes that the two are often very similar. Dawes and Corrigan (1974), Einhorn and Hogarth (1975) and Wainer (1976) conclude that the weights in the linear model may make little practical difference in certain types of decision situations. Their work also suggests that a linear model of a decision maker's selection process in a repetitive situation may do better against an external criterion of success than the decision maker himself. This has given rise to a man versus model of man controversy. Recent contributions to this issue have been made by Libby (1976a, 1976b) and Goldberg (1976).

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