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A NOTE ON THE SOLUTION FOR A CIRCULAR GAS JET

William Walters

September 1977

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USA ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
USA BALLISTIC RESEARCH LABORATORY
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The classical solution for a circular gas jet given by Schlichting is obtained in exact form. An exact form of the solution is also presented for a more general form of the governing equations with boundary conditions such that the constants of integration need not be equal to zero.

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TABLE OF CONTENTS

I. INTRODUCTION

Schlichting[1] presented a solution for a laminar (or turbulent) gas jet exiting from a small circular orifice and mixing with the surrounding fluid. The pressure along and across the jet is assumed constant. The flow is axisymmetric and steady. Boundary layer theory is applicable to this problem and the velocity profiles are considered to be similar.

Under the similarity transform, the governing equation is[1]

$$\frac{FF'}{\eta^2} - \frac{F'^2}{\eta} - \frac{FF''}{\eta} = \frac{d}{d\eta}\left(F'' - \frac{F'}{\eta}\right) \tag{1}$$

where $F = \dfrac{\psi}{\nu x}$,

$\eta = \dfrac{y}{x}$,

prime (') denotes differentiation with respect to η,

ψ is the stream function,

ν is the kinematic viscosity,

x is the axial coordinate, and

y is the radial coordinate.

Equation (1) is a third order, ordinary, nonlinear differential equation. The boundary conditions are[1] at $\eta = 0$, $F = 0$ and $F' = 0$.

An exact form of Schlichting's solution is obtained for the given boundary conditions and an exact solution is obtained for a more general class of problems where the boundary conditions are not restricted to those specified for a freely expanding jet.

II. EXACT SOLUTION OF THE CIRCULAR GAS JET EQUATION

Equation (1) is integrable since

$$-\frac{d}{d\eta}\left(\frac{FF'}{\eta}\right) = -\frac{FF''}{\eta} - \frac{F'^2}{\eta} + \frac{FF'}{\eta^2},$$

and direct integration of Equation (1) yields

[1] *H. Schlichting,* *__Boundary Layer Theory__*, *McGraw-Hill Book Co., 6th ed., N.Y., 1968, p. 219-221.*

$$-\frac{FF'}{\eta} = F'' - \frac{F'}{\eta} + C_1, \qquad (2)$$

or

$$FF' = F' - \eta F'', \qquad (3)$$

for the given boundary conditions. Equation (3) is a second order, nonlinear, ordinary differential equation.

Now, Equation (3) may be integrated since

$$\frac{d}{d\eta}(\eta F') - 2F' = \eta F'' - F',$$

and

$$\frac{1}{2}\frac{dF^2}{d\eta} = FF',$$

or Equation (3) becomes

$$2\eta F' = 2C_2 + 4F - F^2. \qquad (4)$$

It is noteworthy that a solution to Equation (3) is given by Kamke[2] by transforming the independent variable. Since direct integration is possible, such transformations are unnecessary.

It is possible to separate the variables in Equation (4) or $\frac{2dF}{2C_2 + 4F - F^2} = \frac{d\eta}{\eta}$ and integration yields,

$$-\frac{1}{x}\tanh^{-1}\frac{2-F}{2x} = \ell n\,\eta + C_3,$$

where

$$x = \pm\frac{\sqrt{2C_2 + 4}}{2}.$$

Then the function F becomes

$$\frac{2-F}{2x} = \tanh\left[\ell n\,(\eta C_3)^{-x}\right]$$

$$= \frac{(\eta C_3)^{-x} - (\eta C_3)^{x}}{(\eta C_3)^{-x} + (\eta C_3)^{x}},$$

---

[2]*E. Kamke, Differentialgleichungen Lösungmethoden und Lösungen, Chelsea Publishing Co., 1948, p. 563.*

or

$$F = \frac{2x\ (\eta C_3)^{2x} - 2x + 2\ (\eta C_3)^{2x} + 2}{1 + (\eta C_3)^{2x}} ,$$

is an exact solution of Equation (3).

For the freely expanding jet, at $\eta = 0$; $F = 0$ and $F' = 0$ or from Equation (4), $C_2 = 0$ or $x = \pm 1$.* Then,

$$F = \frac{4\ (\eta C_3)^2}{1 + (\eta C_3)^2}$$

represents an exact solution to Equation (3), where C_3 is an arbitrary constant. This solution satisfies the boundary condition that $F = 0$ and $F' = 0$ at $\eta = 0$, for any C_3. Note that for $C_3 = \frac{\gamma}{2}$, where γ is a constant, and for $\xi = \gamma\eta$, F transforms to Schlichting's[1] solution or

$$F = \frac{\xi^2}{1 + \frac{1}{4}\ \xi^2} .$$

However, any multiple of C_3 will yield a similar expression for F which satisfies the posed boundary condition. The general solution is given by

$$F = \frac{4\ (\eta C_3)^2}{1 + (\eta C_3)^2} ,$$

and the second constant of integration, C_3, may be determined directly, from the given value of the momentum in this case.

III. EXACT SOLUTION OF THE GENERAL EQUATION

The general expression, Equation (2) may also be solved exactly when it is not required that C_1 be zero. Equation (2) can be written as

$$FF' + \eta F'' - F' - \eta\ C_1 = 0,$$

and integrated to yield

$$F^2 + 2\eta F' - 4F = \eta^2 C_1 + 2C_2. \tag{5}$$

* $x = +1$ and $x = -1$ yield solutions of exactly the same form.

Now $F^2 - 4F \equiv (F-2)^2 - 4$, and Equation (5) becomes

$$(F-2)^2 + 2\eta F' = \eta^2 C_1 + 2C_2 + 4. \tag{6}$$

The left hand side (LHS) of Equation (6) is simplified by letting $\alpha(\eta) = F(\eta) - 2$, and Equation (6) becomes

$$\alpha^2 + 2\eta\alpha' = \eta^2 C_1 + 2C_2 + 4. \tag{7}$$

The term η is removed from the LHS of (7) by transforming the independent variable η such that $\alpha(\eta) = \bar{\alpha}(\xi)$ and $\xi = \ell n\ |\eta|$, or

$$\bar{\alpha}^2 + 2\bar{\alpha}'(\xi) = 2C_2 + 4 + C_1 e^{2\xi}. \tag{8}$$

The constant coefficients are rearranged by letting $y(\xi) = \frac{\bar{\alpha}(\xi)}{2}$, then Equation (8) becomes,

$$y^2 + y' = \frac{e^{2\xi} C_1}{4} + \frac{C_2}{2} + 1. \tag{9}$$

Equation (9) is of the Riccati type and may be transformed to a linear equation. This is accomplished by the Riccati transformation

$$u'(\xi) = y(\xi)\, u(\xi),$$

and Equation (9) becomes

$$u'' = u\left[\frac{e^{2\xi}C_1}{4} + \frac{C_2}{2} + 1\right]. \tag{10}$$

Now, the inverse transformation on the independent variable ξ, where

$$u(\xi) = \bar{u}(z) \qquad , \qquad z = \sqrt{\frac{C_1}{4}}\, e^{\xi}$$

yields,

$$z^2\, \bar{u}'' + z\, \bar{u}' - \left(z^2 + \frac{C_2}{2} + 1\right) \bar{u} = 0. \tag{11}$$

Equation (11) is the modified Bessel equation for $\nu^2 = \frac{C_2}{2} + 1$. The solutions are $I_\nu(z)$ and $K_\nu(z)$ which are modified Bessel functions of the first and second kind respectively[3], or

$$\bar{u} = A\, I_{\pm\nu}(z) + B\, K_{\nu}(z), \tag{12}$$

where A and B are arbitrary constants and I_{ν} and $I_{-\nu}$ are linearly dependent if ν is an integer. Also $I_{\nu}(z)$ and $K_{\nu}(z)$ are real and positive for $z > 0$ and $\nu > -1$.

Thus, Equation (2) may be solved by transforming it into a Riccati equation and then into a modified Bessel equation. Once boundary conditions are stipulated, the integration constants can be determined and the solution for $\bar{u}$ can be transformed back to F. This is not carried out for the general case, since the order of the modified Bessel functions (ν) depends on the value of C_2. I.e., it is preferable to know whether ν is an integer or non-integer before proceeding with the inverse transformations.

Solutions of this type are applicable to equations such as those [illegible] a steady state boundary layer, where similarity transforms can yield [illegible]rate solutions.

IV. SOLUTION FOR F

Utilizing the boundary condition that when the radial coordinate is zero, the radial velocity is zero and the axial velocity represents some mean value implies[1]

$$\text{at } \eta = 0;\ F = 0 \text{ and } F' = 0,$$

or from Equation (5), $C_2 = 0$.

In this case $\nu = \pm 1$, $I_1(z)$ and $I_{-1}(z)$ are linearly dependent (in fact, equal) and the general solution (12) becomes

$$\bar{u} = AI_1(z) + BK_1(z). \tag{13}$$

Then,

$$u(\xi) = AI_1\left(\sqrt{\frac{C_1}{4}}\, e^{\xi}\right) + BK_1\left(\sqrt{\frac{C_1}{4}}\, e^{\xi}\right),$$

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[3] *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, Edited by M. Abramowitz and I. Stegun, National Bureau of Standards, Applied Mathematics Series, 55, June, 1964, p. 374.*

$$y(\xi) = \frac{\sqrt{C_1}}{2} e^{\xi} \left[\frac{AI_1'\left(\frac{\sqrt{C_1}}{2} e^{\xi}\right) + BK_1'\left(\frac{\sqrt{C_1}}{2} e^{\xi}\right)}{AI_1\left(\frac{\sqrt{C_1}}{2} e^{\xi}\right) + BK_1\left(\frac{\sqrt{C_1}}{2} e^{\xi}\right)} \right],$$

$$\alpha(\eta) = \sqrt{C_1}\,\eta \left[\frac{AI_1'\left(\frac{\sqrt{C_1}}{2} \eta\right) + BK_1'\left(\frac{\sqrt{C_1}}{2} \eta\right)}{AI_1\left(\frac{\sqrt{C_1}}{2} \eta\right) + BK_1'\left(\frac{\sqrt{C_1}}{2} \eta\right)} \right]$$

and finally,

$$F = 2 + \sqrt{C_1}\,\eta \left[\frac{AI_1'\left(\frac{\sqrt{C_1}}{2} \eta\right) + BK_1'\left(\frac{\sqrt{C_1}}{2} \eta\right)}{AI_1\left(\frac{\sqrt{C_1}}{2} \eta\right) + BK_1\left(\frac{\sqrt{C_1}}{2} \eta\right)} \right]$$

results. Note that the numerator and denominator of the expression in brackets may be divided by A or B, so A and B are not independent. One constant can always be eliminated. This results from the use of the Riccati transformation used to obtain Equation (10). This transformation converted a first order Equation, (9), into a second order Equation, (10), thereby introducing an additional constant. The two constants of integration associated with Equation (2) are C_1 and A or B. The constant C_1 appears in Equation (2) and C_2 has already been evaluated.

Finally, the expression for F may be written as

$$F = 2 - \sqrt{C_1} + \frac{C_1 \eta}{2} \left(\frac{AI_o\left(\frac{\sqrt{C_1}}{2} \eta\right) - BK_o\left(\frac{\sqrt{C_1}}{2} \eta\right)}{AI_1\left(\frac{\sqrt{C_1}}{2} \eta\right) + BK_1\left(\frac{\sqrt{C_1}}{2} \eta\right)} \right).$$

From the expression for F, the radial and axial velocities and their derivatives may be obtained, in a manner analogous to Schlichting's treatment. For this general formulation, boundary conditions must be specified to determine C_1 and A or B.

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