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DENSE FAMILIES OF LOW-COMPLEXITY ATTAINABLE SETS OF MARKETS

by

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Abstract

Given the attainable set of utility outcomes for a market (with finitely many traders), its complexity is defined to be the least number of commodities needed for any market giving the same set. This notion is investigated both in the case of quasiconcave and concave utility functions. It is shown that, in either case, there is a dense collection of attainable sets having complexity at most $n(n-1)/2$.

1. Introduction. The question of which n-person cooperative games can arise from economic markets has generated much recent interest. One aspect of this question concerns representations of "attainable sets" of markets. In particular, representations are sought which associate with an attainable set a market of low "complexity" (that is, a market involving as small a number of commodities as possible).

Two classes of markets have been considered in some detail. The broader class consists of markets in which all of the traders' utility functions are upper-semicontinuous and quasiconcave. We shall show that all n-dimensional attainable sets of two recently-studied types arising from such markets can be represented by n-trader markets involving at most $n(n-1)/2$ commodities. A consequence of this is that there is a collection of attainable sets, each of complexity at most $n(n-1)/2$, which is dense in the collection of all n-dimensional attainable sets.

A natural subclass of markets consists of those in which the traders' utility functions are continuous and concave. It is known that the n-dimensional attainable sets of such markets are precisely the convex, compactly generated sets in R^n. An upper bound on the complexity of these sets, due to Kalai and Smorodinsky [3], is $(n-1)^2 - (n-2)$. In this case also, we show that a dense collection of such n-dimensional attainable sets has complexity bounded by $n(n-1)/2$. This has been conjectured to be the upper bound over all n-dimensional attainable sets.

Consider a market consisting of a set of traders $N = \{1,2,\ldots,n\}$, and an m-dimensional commodity space $I^m = \{(y_1,\ldots,y_m): 0 \le y_i \le 1$ for all $i\}$. (We take $I^0 = R^0 = \{0\}$.) For any collection $\{u_i\}_{i=1}^n$ of utility functions of the traders (real-valued functions on I^m), the attainable set of the market is

$$A(u_1,\ldots,u_n) = \{x\epsilon R^n\colon x \leq (u_1(y^1),\ldots,u_n(y^n)), \text{ where}$$
$$\text{each } y^i \epsilon I^m \text{ and } \sum y^i = (1,\ldots,1)\}.$$

This is the set of all utility outcomes which can be achieved by some distribution of the available commodities among the traders.

A set X in R^n is the comprehensive hull of another set Y if $X = \{x\epsilon R^n\colon x \leq y$ for some $y\epsilon Y\}$; in this case, we say that X is (comprehensively) generated by Y. It is not difficult to show that if $u_1,\ldots,u_n$ are upper-semicontinuous and bounded, then $A(u_1,\ldots,u_n)$ is compactly generated (generated by a compact set).

Let U_1 be the collection of all upper-semicontinuous, quasiconcave utility functions, and let U_2 be the subcollection of all continuous, concave utility functions. For $k = 1,2$, let $A_k(n)$ be the collection of all n-dimensional attainable sets of markets in which the traders' utility functions are in U_k. The extent of $A_1(n)$ was investigated in [4], [5], [6] and [7]; in [1], the sets in $A_2(n)$ were characterized as all sets which are generated by convex, compact sets.

Let V be an attainable set in $A_k(n)$. If $u_1,\ldots,u_n$ are functions in U_k defined on I^m (for some fixed m), such that $V = A(u_1,\ldots,u_n)$, then $\{u_i\}_{i=1}^n$ will be called a k-representation for V over I^m. The k-complexity of V is the least $m \geq 0$ such that there exists a k-representation for V over I^m. We use comp V to denote the k-complexity of V; the context will make it clear whether k is 1 or 2.

2. Quasiconcave markets. In this section, we consider the attainable sets of markets in which all traders' utility functions are upper-semicontinuous and quasiconcave. Therefore, we direct our attention to 1-representations and 1-complexity. Without loss of generality, we assume that all n-dimensional attainable sets under consideration are comprehensively generated by compact sets lying in the interior of the unit n-cube I^n. Let D_k be the "corner" generated by the point $(1,\dots,1,0)\epsilon R^k$. In [6] and [7], n-dimensional attainable sets were represented by utility functions obtained from constructions which first represented (over I^{n-1}) the unions of these sets with D_n. This makes the following result of value.

Theorem 2.1: Let V be generated by a nonempty compact set in I^n. For $1 \leq k \leq n$, let $V_k' = \{x\epsilon R^k: (x;0)\epsilon V\}$ and let $V_k = V_k' \cup D_k$. If each V_k is an attainable set (in $A_1(k)$), then V is an attainable set (in $A_1(n)$), and comp $V \leq \sum_{k=1}^{n}$ comp V_k.

Proof: Let c_k = comp V_k, and $c = \sum c_k$. Assume that $V_k = A(u_1^k,\dots,u_k^k)$, where all u_i^k $(1 \leq i \leq k)$ are defined on I^{c_k}. Represent any $x\epsilon I^c$ as $x = ({}^1x,\dots,{}^nx)$, where ${}^kx\epsilon I^{c_k}$. Define $u_i(x) = \min (u_i^i({}^ix),\dots,u_i^n({}^nx))$. We shall verify that $V = A(u_1,\dots,u_n)$, from which the upper bound on the complexity of V follows immediately.

Consider any $z = (z_1,\dots,z_n) \epsilon A(u_1,\dots,u_n)$. There is some allocation $(x^1,\dots,x^n)$, with each $x^i\epsilon I^c$ and $\sum x^i = (1,\dots,1)$, such that $u_i(x^i) \geq z_i$ for all $1 \leq i \leq n$. Therefore, by definition, $u_i^j({}^jx^i) \geq z_i$ for all $j \geq i$. Hence, $z^{(k)} = (z_1,\dots,z_k)\epsilon V_k$ for all $1 \leq k \leq n$. If $z^{(k)} \not\epsilon V_k'$, then $z^{(k)} \leq (1,\dots,1,0)$ and therefore $z_k \leq 0$. Let $\overline{k} = \max (0, \{k: z^{(k)}\epsilon V_k'\})$. If $\overline{k} = 0$, then $z \leq (0,\dots,0)\epsilon V$. If $\overline{k} > 0$, then $z^{(\overline{k})}\epsilon V_{\overline{k}}'$ and $z \leq (z^{(\overline{k})};0)\epsilon V$. In either case, we conclude that $z\epsilon V$ (and , indeed, $\overline{k} = n$) and therefore $V \supset A(u_1,\dots,u_n)$.

On the other hand, given any $z \epsilon V$, it follows that each $z^{(k)} = (z_1,\ldots,z_k) \epsilon V_k' \subset V_k$. For each $1 \leq k \leq n$, let $({}^k x^1,\ldots,{}^k x^k)$ be an allocation, with each ${}^k x^i \epsilon I^{c_k}$ and $\sum_i {}^k x^i = (1,\ldots,1)$, such that $z^{(k)} \leqq (u_1^k({}^k x^1),\ldots,u_k^k({}^k x^k)) \epsilon V_k$. Define $x^i = (0;{}^i x^i,\ldots,{}^n x^i) \epsilon I^c$ for each $1 \leq i \leq n$. Then $z \leq (u_1(x^1),\ldots,u_n(x^n))$, and therefore $V \subset A(u_1,\ldots,u_n)$.

Finally, it may be observed that the construction of each u_i from the u_i^k $(k \geq i)$ preserves both upper-semicontinuity and quasiconcavity. This completes the proof of the theorem.

A set $V \subset R^n$ is finitely generated if it is the comprehensive hull of a finite set (that is, if V is the union of a finite number of corners); V is convexifiable if it is compactly generated and there exists a collection $\{g_1,\ldots,g_n\}$ of strictly increasing, continuous functions such that

$$V_{\{g_1,\ldots,g_n\}} = \{x \epsilon R^n: x \leq (g_1(y_1),\ldots,g_n(y_n)) \text{ for some } y \epsilon V\}$$

is convex.

Let V be generated by a set in I^n. It has been shown that if V is finitely generated (see [6]) or convexifiable (see [7]), then V_n is an attainable set of complexity at most $n-1$. Since these properties, of being finitely generated or of being convexifiable, are inherited by all $V_k'(1 \leq k \leq n)$, an application of the theorem yields the following result.

Corollary 2.2: If $V \subset R^n$ is finitely generated or convexifiable, then V is an attainable set (in $A_1(n)$), and comp $V \leq n(n-1)/2$.

Since the finitely generated sets are (Hausdorff) dense in $A_1(n)$ ([2, Theorem 2]), we can state the following.

Corollary 2.3: $A_1(n)$ has a dense subset consisting of attainable sets of (1-) complexity no greater than $n(n-1)/2$.

3. Concave markets. We now turn our attention to the attainable sets of markets in which all traders' utility functions are continuous and concave, and in consequence we consider 2-representations and 2-complexity. Without loss of generality, we continue to assume that all n-dimensional attainable sets under consideration are generated by (convex, compact) sets lying in the interior of I^n. For any $h \geq 0$, let $D_k(h)$ be the corner generated by the point $(1,\ldots,1,-h) \in R^k$. In [1], n-dimensional attainable sets were represented by utility functions obtained from contructions which first represented (over I^{n-1}) the convex hulls of the unions of these sets with sets $D_n(h)$, for large values of h. Therefore, a variation of the theorem of the previous section will be broadly applicable.

Let V be generated by a convex compact set in I^n, and, as before, let $V_k' = \{x \in R^k : (x;0) \in V\}$. For $2 \leq k \leq n$, let $V_{k-1}' \times R^1 = \{x \in R^k : (x_1,\ldots,x_{k-1}) \in V'_{k-1}\}$ be a cylinder with cross-section V_{k-1}'. We define $V_0' = \{0\}$ and $V_0' \times R^1 = R^1$. For any $h \geq 0$, define $V_k(h)$ to be the (comprehensive) convex hull of $V_k' \cup D_k(h)$. Each $V_k(h)$ is generated by a convex compact set, and is therefore an attainable set (in $A_2(k)$). We say V is resolved by a sequence of non-negative numbers $h_1,\ldots,h_n$ if for all $1 \leq k \leq n$, $V_k(h_k) \cap (V_{k-1}' \times R^1) = V_k'$; in this case, we say V is resolvable.

Theorem 3.1: Let V be generated by a convex compact set in I^n. Assume that V is resolved by $h_1,\ldots,h_n$. Then $\text{comp } V \leq \sum_{k=1}^{n} \text{comp } V_k(h_k) \leq n(n-1)/2$.

Proof. We begin as in the proof of Theorem 2.1. Let $c_k = \text{comp } V_k(h_k)$, and $c = \sum c_k$. Assume that $V_k(h_k) = A(u_1^k,\ldots,u_k^k)$, where all u_i^k $(1 \leq i \leq k)$ are continuous concave functions defined on I^{c_k}. Write $x \in I^c$ as

$x = ({}^1x,\ldots,{}^nx)$, where ${}^kx \in I^{c_k}$. Define $u_i(x) = \min(u_i^i({}^ix),\ldots,u_i^n({}^nx))$. The construction of the functions $u_1,\ldots,u_n$ yields continuous concave functions on I^c. Therefore, it will suffice to verify that $V = A(u_1,\ldots,u_n)$.

Consider any $z = (z_1,\ldots,z_n) \in A(u_1,\ldots,u_n)$. Then each $z^{(k)} = (z_1,\ldots,z_k) \in V_k(h_k)$. Assume that, for some $2 \le k \le n$, $z^{(k)} \notin V_k'$. Since V is resolved by $h_1,\ldots,h_n$, $V_k(h_k) \cap (V_{k-1}' \times R^1) = V_k'$; therefore $z^{(k)} \notin V_{k-1}' \times R^1$, and hence $z^{(k-1)} \notin V_{k-1}'$. Now, assume that $z \notin V = V_n'$. From the result just established, it follows inductively that $z^{(k)} \notin V_k'$ for all $1 \le k \le n$. But $z^{(1)} \in V_1(h_1) = V_1'$. This contradiction affirms that $z \in V$. Therefore, $V \supset A(u_1,\ldots,u_n)$.

On the other hand, it follows as in the proof of Theorem 2.1 that $V \subset A(u_1,\ldots,u_n)$. Therefore $V = A(u_1,\ldots,u_n)$, and the complexity of V is no greater than $c = \sum_{k=1}^{n} \text{comp } V_k(h_k)$. The second inequality follows from the observation that for any $h \geqq 0$, comp $V_k(h) \leqq (k-1)$; see [1].

In the remainder of this section, we will show that there is a dense (in the sense of the Hausdorff distance) collection of resolvable n-dimensional attainable sets. This class consists of those attainable sets having a (uniformly) positive normal at every nonnegative boundary points; these will be referred to as positively supported sets. Specifically, an n-dimensional attainable set V is said to be <u>(uniformly) positively supported</u> if there exists a closed set $Q \subset \{q \in R^n \mid \sum_{i=1}^{m} q_i = 1,\ q > 0\}$ so that for each $x \in \partial V \cap R_+^n$, there is a $q \in Q$ for which $\langle x,q\rangle \geqq \langle y,q\rangle$ for all $y \in V$. Here ∂V denotes the boundary of V, R_+^n, the nonnegative orthant in R^n, and $\langle \cdot,\cdot\rangle$, the usual inner product on R^n.)

<u>Lemma 3.2</u>: If V is positively supported then V is resolvable.

<u>Proof</u>: Let Q be the set of normals specified in the definition. Let $h_1 = 0$ and, for $2 \leq k \leq n$, let $h_k = \max\{\sum_{i<k} q_i/q_k \mid q \in Q\} > 0$. We shall show that V is resolved by the sequence $h_1, h_2, \ldots, h_n$. Since, for $1 \leq k \leq n$, we always have

$$V_k(h_k) \cap (V'_{k-1} \times R^1) \supset V'_k \qquad (3.2.1)$$

and, for $k = 1$, equality is clear, it is enough to show the other inclusion for $k \geq 2$.

Suppose $x \in R^k$ is an element of the left-hand side of (3.2.1). If $x_k \leqq 0$, then since $(x_1, \ldots, x_{k-1}) \in V'_{k-1}$, we must have $(x_1, \ldots, x_{k-1}, 0) \in V'_k$ and so $x \in V'_k$. If $x_k > 0$, then since $x \in V_k(h_k)$, we must have $x \leqq \bar{x}$ where

$$0 \leqq \bar{x} = \lambda y + (1-\lambda)\,(1,1,\ldots,1,-h_k) \in V_k(h_k), \qquad (3.2.2)$$

$y \in V'_k \cap R^k_+$ and $0 < \lambda \leqq 1$. We show $\bar{x} \in V'_k$, and so $x \in V'_k$ by comprehensiveness. If $\bar{x} \notin V'_k$ then there exists $v \in \partial V'_k$, $0 \leqq v \leqq \bar{x}$, $v \neq \bar{x}$. Since $(v;0) \in \partial V \cap R^n_+$, there is a $q \in Q$ so that

$$\langle \bar{x}, q^{(k)} \rangle > \langle v, q^{(k)} \rangle \geq \langle z, q^{(k)} \rangle \qquad (3.2.3)$$

for all $z \in V'_k$, where $q^{(k)} = (q_1, \ldots, q_k) > 0$. Further,

$$\langle v, q^{(k)} \rangle \geqq 0 \geqq \sum_{i<k} q_i - h_k q_k = \langle (1,1,\ldots,1,-h_k), q^{(k)} \rangle$$

so $\langle v, q^{(k)} \rangle \geq \langle z, q^{(k)} \rangle$ for all $z \in V_k(h_k)$. This contradicts (3.2.2) and

(3.2.2), establishing $\bar{x}\epsilon V_k'$. In either case, we have $x\epsilon V_k'$ and thus have shown (3.2.1) to be an equality, completing the proof.

It seems intuitively clear that the positively supported attainable sets are dense among all such sets. We outline a proof here for completeness. We denote by d the Hausdorff distance induced by the norm $\|x\| = \max\limits_{1\le i\le m} |x_i|$.

Suppose V is an n-dimensional attainable set (generated by a compact convex subset of I^n). We can write

$$V = \{x\epsilon R^n \mid \langle x,p\rangle \leqq \alpha_p \quad \text{for} \quad p\epsilon S\}$$

where $S = \{p\epsilon R^n_+ \mid \sum_{i=1}^n p_i = 1\}$ and $\alpha_p = \sup\limits_{x\epsilon V} \langle x,p\rangle$. Note that α_p, the support function of V, varies continuously for $p\epsilon S$. For $0 < \epsilon < 1/n$, define

$$V^\epsilon = \{x\epsilon R^n_+ \mid \langle x,p\rangle \leqq \alpha_p \quad \text{for} \quad p\epsilon Q^\epsilon\} - R^n_+$$

where $Q^\epsilon = \{q\epsilon S \mid q_i \geqq \epsilon \text{ for all } i\}$.

<u>Lemma 3.3</u>: The sets V^ϵ are positively supported and $d(V,V^\epsilon) \to 0$ as $\epsilon \to 0$.

<u>Proof</u>: V^ϵ is positively supported since for $x\epsilon\partial V^\epsilon \cap R^n_+$, there is a $p\epsilon Q^\epsilon$ such that $\langle x,p\rangle = \alpha_p \geqq \langle y,p\rangle$ for all $y\epsilon V^\epsilon$ (otherwise x would be in the interior of V^ϵ).

Since $d(V,V^\epsilon) \leqq d(V \cap R^n_+, V^\epsilon \cap R^n_+)$ and $V^\epsilon \supset V$, it is enough show that, given $\eta > 0$, there is $\bar{\epsilon} > 0$ so that for $\epsilon < \bar{\epsilon}$ and

$x \epsilon V^{\epsilon} \backslash V$, $x \geqq 0$, there is $y \epsilon V$, $y \geqq 0$ so that $||x - y|| \leqq \eta$.

Let ϵ^o · 1/2n. The function $f(z,p) = \langle z,p \rangle - \alpha_p$ is uniformly continuous over $(V^{\epsilon^o} \cap R^n_+) \times S$ and so there is an $\bar{\epsilon} < \epsilon^o$ so that for $z \epsilon V^{\epsilon^o}$, $z \geqq 0$, and $p,q \epsilon S$, we have $||q - p|| \leq \bar{\epsilon}$ implies $|f(z,q) - f(z,p)| \leq \eta$.

Take $\epsilon < \bar{\epsilon}$ and $x \epsilon V^{\epsilon} \backslash V$, $x \geqq 0$. Let $q \epsilon S$ maximize $f(x,p)$ over S. Since $x \epsilon V^{\epsilon} \backslash V$, $\delta = f(x,q) > 0$ but $f(x,p) \leqq 0$ for $p \epsilon Q^{\epsilon}$. Now let $z = x - (\delta,\delta,\ldots,\delta)$. Then $z \epsilon V$ and $y = z^+ \epsilon V \cap R^n_+$ where $(z^+)_i = \max(z_i,0)$. Further $z \leqq y \leqq x$, so

$$||x - y|| \leqq ||x - z|| = \delta \leqq \eta,$$

proving the lemma.

Combining (3.2) and (3.3), we have our final result.

<u>Theorem 3.4</u>: $A_2(n)$ has a dense subset consisting of attainable sets of (2-) complexity no greater than $n(n-1)/2$.

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Given the attainable set of utility outcomes for a market (with finitely many traders), its complexity is defined to be the least number of commodities needed for any market giving the same set. This notion is investigated both in the case of quasiconcave and concave utility functions. It is shown that, in either case, there is a dense collection of attainable sets having complexity at most $n(n-1)/2$.