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A NUMERICAL MODEL OF THE MOON'S ROTATION

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The differential equations for the Moon's rotation are derived in terms of Euler angles referenced to an inertial coordinate system. Also, the variational equations of the rotation with respect to six Euler angle initial conditions, the lunar J_2, the moment-of-inertia ratios β and γ, and the coefficients of the third and higher-degree gravity harmonics, are presented in detail. The equations of motion ~~have been~~ were integrated numerically within the framework of MIT's Planetary Ephemeris Program, and ~~we have adjusted~~ the initial

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cont. conditions were adjusted to fit the libration angles given by the numerical "LLB-5" model of J.G. Williams and others at JPL. The postfit rms orientation difference, after removal of a three-axis rotation to correct for lunar orbital ephemeris differences, was about 0.03 arcsec (selenocentric) over a six-year span. Neglected effects and anticipated improvements in our model are also discussed.

INTRODUCTION

The differential equations of motion of the Moon about its center of mass have been known since the time of Euler (Moutsoulas, 1971), but due to the complexity of the driving terms (torques from Earth, Sun, etc.), they can only be solved approximately. Formulations in which the orbital motions are approximated by functions containing terms secular and periodic in time are amenable to exact solution, and the theories of motion derived are called analytic theories. Modern analytic theories, such as those of Eckhardt (1970) and Migus (1976) are invaluable for their concise description of different modes of physical libration, and for the possible detection of free lunar librations, but are constrained in accuracy by their dependence upon (relatively inaccurate) analytic orbit theories.

In order to obtain the improved accuracy necessary for the interpretation of modern lunar laser ranging and very-long-baseline interferometry observations, Williams (1975) and others at the Jet Propulsion Laboratory developed a series of lunar libration models, the latest being called LLB-5, based upon direct numerical integration of the equations of motion for the libration. The LLB-5 integration is done in non-inertial coordinates, so that the equations of motion contain "inertial-force" terms. Also, no variational equations are integrated in LLB-5 and the partial derivatives needed for adjustment of the parameters of the model are obtained by finite-differencing.

Our lunar rotation is based upon the direct numerical integration of Euler's differential equations for the rotation of the moon, with the rotation being described by Euler angles referenced to an inertial coordinate system. The variational equations for the partial derivatives of the Euler angles and their time-derivatives with respect to all of the parameters in the equations of motion, and with respect to the initial conditions of motion, are integrated in parallel. The detailed formulation of our model, and results of comparing it with LLB-5, are described in the following sections.

EQUATIONS OF MOTION

The state of rotation of a rigid body with respect to an arbitrary coordinate system can be expressed by six quantities: three angles defining a rotation of axes, and their temporal derivatives. We choose to describe the lunar rotation in terms of Euler angles and their derivatives as defined by an inertial coordinate system referred to the mean equinox and equator of 1950.0. The ξ_3 axis is perpendicular to the mean equator of 1950.0 and points northward, the ξ_1 axis is the intersection of the mean equator and ecliptic of 1950.0 and points towards the constellation Aries, and the ξ_2 axis completes the right-handed system. Letting the x_i $(i = 1,3)$ axes be a right-handed system coinciding with the Moon's principal axes of inertia, where the moment about the x_1 axis is least and the moment about x_3 greatest, the Euler angles are defined as indicated in Figure 1.

Euler's equations governing the motion of a rigid body about its center of mass are

$$\begin{aligned} A\dot{\omega}_1 + (C - B)\omega_2\omega_3 &= T_1 \\ B\dot{\omega}_2 - (C - A)\omega_1\omega_3 &= T_2 \\ C\dot{\omega}_3 + (B - A)\omega_1\omega_2 &= T_3 \end{aligned} \tag{1}$$

In this representation the ω_i $(i = 1,3)$ are the components of the lunar angular velocity vector $\vec{\omega}$ along the three principal (body-fixed) lunar axes, x_i; A, B, and C are the moments of inertia about the principal axes; and the T_i are the components of the total torque vector about the corresponding axes.

Since we desire an ultimate accuracy of .01" for our model, we should examine all torques which might result in displacements of this magnitude. The largest libration term is due to the regression of the node of the lunar equator on the ecliptic and is of roughly 5000" amplitude. Thus, so long as we are not driving near a resonant frequency, we should expect that all torques that are no more than 2×10^{-6} times the dominant term (central-body term for the Earth) should be negligible. The far-field torque exerted on the Moon by an external body is directly proportional to the body's mass, and inversely proportional to the cube of its distance.

A calculation of $M/r^3_{minimum}$ for the bodies of the solar system demonstrates that the torque on the Moon may be approximated as the sum of the Earth and Sun induced torques to an accuracy of 1 part in 10^6, with the solar contribution only 1/200 that of the Earth. The torque due to the oblateness of the Earth is not included in the model described in this paper, but we intend to incorporate it in the near future. Introducing the lunar moment of inertia ratios $\alpha = \frac{C-B}{A}$, $\beta = \frac{C-A}{B}$, and $\gamma = \frac{B-A}{C}$, we then have:

$$\begin{aligned}
\dot{\omega}_1 &= -\alpha\omega_2\omega_3 + (T_{\oplus 1} + T_{\odot 1})/A \\
\dot{\omega}_2 &= \beta\omega_1\omega_3 + (T_{\oplus 2} + T_{\odot 2})/B \\
\dot{\omega}_3 &= -\gamma\omega_1\omega_2 + (T_{\oplus 3} + T_{\odot 3})/C \qquad (2)
\end{aligned}$$

where $T_{\oplus i}$ and $T_{\odot i}$ represent the components of the torque due to Earth and Sun, respectively.

We desire the equations of motion in terms of the inertially-referenced Euler angles. The relation between the body-fixed angular rates and the inertial Euler angles is well known (Goldstein, 1950):

$$\begin{aligned}
\omega_1 &= \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\sin\theta \\
\omega_2 &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\sin\theta \\
\omega_3 &= \dot{\psi}\cos\theta + \dot{\phi} \qquad (3)
\end{aligned}$$

Differentiating the equations (3) with respect to time results in a system of equations linear in $\ddot{\psi}$, $\ddot{\theta}$, and $\ddot{\phi}$, the solution of which forms our Euler angle equations of motion:

$$
\begin{aligned}
\ddot{\psi} &= \csc\theta(\dot{\omega}_1 \sin\phi + \dot{\omega}_2 \cos\phi) + \dot{\theta}\dot{\phi}\csc\theta - \dot{\psi}\dot{\theta}\cot\theta \equiv F_1 \\
\ddot{\theta} &= \dot{\omega}_1 \cos\phi - \dot{\omega}_2 \sin\phi - \dot{\psi}\dot{\phi}\sin\theta \equiv F_2 \\
\ddot{\phi} &= \dot{\omega}_3 - F_1 \cos\theta + \dot{\psi}\dot{\theta}\sin\theta \equiv F_3 \qquad (4)
\end{aligned}
$$

The $\dot{\omega}_i$ are found in terms of Euler angles by equations (2) and (3) and it only remains to evaluate the body-axis torques due to Earth and Sun. The potential due to the lunar gravitational field may be approximated as a finite expansion in spherical harmonics as follows (Kaula, 1966):

$$
U(r,L,\lambda) = \frac{-GM_{☾}}{r}\left\{1 - \sum_{n=2}^{20} J_n \left(\frac{a}{r}\right)^n P_n(\sin L) + \sum_{n=2}^{10}\sum_{m=1}^{n}\left(\frac{a}{r}\right)^n [C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\, P_{nm}(\sin L)\right\} \qquad (5)
$$

where
- G = Gaussian gravitational constant
- $M_{☾}$ = mass of the moon
- a = lunar radius
- r = distance from the lunar center of mass
- L = selenocentric latitude
- λ = selenocentric East longitude
- P_n are the Legendre polynomials
- P_{nm} are the associated Legendre functions
- J_n, C_{nm}, and S_{nm} are dimensionless coefficients

In terms of the rectangular coordinates x_i it is easily shown that:

$$r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \tag{6a}$$

$$\sin L = \frac{x_3}{r} \tag{6b}$$

$$\cos \lambda = \frac{x_1}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \tag{6c}$$

$$\sin \lambda = \frac{x_2}{(x_1^2 + x_2^2)^{\frac{1}{2}}} \tag{6d}$$

In our model the Earth and Sun are treated as point-masses of mass M_b ($b = \oplus, \odot$) and therefore experience the force

$$\vec{F}_b = -M_b \nabla_b U \tag{7}$$

where $\nabla_b U$ is the gradient of U evaluated at the position of body b. The resultant torque vector acting on the moon is

$$\begin{aligned} \vec{T}_b &= \vec{r} \times (-\vec{F}) \\ &= M_b (\vec{r} \times \nabla_b U) \end{aligned} \tag{8}$$

with $\vec{r}$ the vector from the center of mass of the Moon to the body. If the selenocentric components of $\vec{r}$ are (x_1, x_2, x_3) then the selenocentric torque components for body b are:

$$T_1 = M_b \left(x_2 \frac{\partial U}{\partial x_3} - x_3 \frac{\partial U}{\partial x_2} \right)$$

$$T_2 = M_b \left(x_3 \frac{\partial U}{\partial x_1} - x_1 \frac{\partial U}{\partial x_3} \right)$$

$$T_3 = M_b\left(x_1 \frac{\partial U}{\partial x_2} - x_2 \frac{\partial U}{\partial x_1}\right) \qquad (9)$$

In the evaluation of $\partial U/\partial x_k$ we note that we may neglect the central force $(1/r^2)$ term since it produces no net torque about the center of mass. In fact any term in $\partial U/\partial x_k$ of the form $g \cdot x_k$, where g is any function not explicitly containing the index k, will cancel in the torque cross product. We obtain for either Earth or Sun:

$$\frac{\partial U}{\partial x_k} = GM_{\text{ℓ}} \sum_{n=2}^{20} \left(\frac{a}{r}\right)^n \frac{J_n}{r} P'_n (\sin L) \frac{\partial \sin L}{\partial x_k} - GM_{\text{ℓ}} \sum_{n=2}^{10} \sum_{m=1}^{n} \left(\frac{a}{r}\right)^n \frac{1}{r}$$

$$\cdot\{[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda]\, P'_{nm} (\sin L) \frac{\partial \sin L}{\partial x_k}$$

$$+ m[-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda]\, P_{nm} (\sin L) \frac{\partial \lambda}{\partial x_k} \}$$

$$+ \text{ terms which do not contribute to the torque} \qquad (10)$$

From equations (6a) and (6b) we derive

$$\frac{\partial \sin L}{\partial x_k} = \frac{\delta_{3k}}{r} - \frac{x_3 x_k}{r^3} \qquad (11a)$$

where δ_{ij} is the Kronecker delta defined by $\delta_{ij} = 1$ for $i = j$ and 0 otherwise. If we differentiate (6d) with respect to x_k we get

$$\cos\lambda \frac{\partial \lambda}{\partial x_k} = (x_1^2 + x_2^2)^{-3/2} (x_1^2\, \delta_{2k} - x_1 x_2\, \delta_{1k})$$

which by (6c) gives:

$$\frac{\partial\lambda}{\partial x_k} = \frac{x_1\ \delta_{2k} - x_2\ \delta_{1k}}{x_1^2 + x_2^2} \tag{11b}$$

The three second-order equations of motion are now rewritten as a set of six first-order differential equations for ease of integration method. Defining

$$\begin{aligned} y_1 &\equiv \psi & \qquad y_4 &\equiv \dot{\psi} \\ y_2 &\equiv \theta & \qquad y_5 &\equiv \dot{\theta} \\ y_3 &\equiv \phi & \qquad y_6 &\equiv \dot{\phi} \end{aligned}$$

we obtain:

$$\begin{aligned} \dot{y}_1 &= y_4 & \qquad \dot{y}_4 &= F_1 \\ \dot{y}_2 &= y_5 & \qquad \dot{y}_5 &= F_2 \\ \dot{y}_3 &= y_6 & \qquad \dot{y}_6 &= F_3 \end{aligned} \tag{12}$$

where the F_i are given by equations (4). The six initial conditions of the state vector $\overline{y}$ are the three Euler angles and their rates at some initial epoch.

--

VARIATIONAL EQUATIONS

Our primary reason for modelling the lunar physical librations is to enable us to process accurate data and consequently arrive at improved estimates for the parameters affecting libration. Often this is accomplished through the iterative use of a linear least-squares estimator;

thus, in addition to the model for motion, it is desirable to supply partial first-derivatives of the state vector with respect to the libration parameters at all times. Referring back to the defining equations (4) for the driving terms F_k we note that in general the

$$F_k \equiv F_k(t, \overline{p}, \overline{y})$$

are explicit functions of time, the set of adjustable parameters $\overline{p}$, and the Eulerian state vector $\overline{y}$. We differentiate the equations of motion (12) with respect to any specific time-independent parameter p_i and interchange the order of differentiation to obtain the variational equations:

$$\frac{d}{dt}\left(\frac{\partial y_k}{\partial p_i}\right) = \frac{\partial y_{k+3}}{\partial p_i} \qquad k = 1,3 \qquad (13)$$

$$\frac{d}{dt}\left(\frac{\partial y_{k+3}}{\partial p_i}\right) = \left.\frac{\partial F_k}{\partial p_i}\right|_{t,\overline{p}\neq i}$$

with $\overline{p} \neq i$ denoting the parameter set $\overline{p}$ exclusive of p_i. The initial conditions for these equations are all zero except when p_i is one of the state vector initial conditions, then

$$\left.\frac{\partial y_k}{\partial p_i}\right|_{t=t_o} = \delta_{k\ell} \text{ for } p_i \equiv \left. y_\ell \right|_{t=t_o}$$

In general a change in p_i will affect the F_k both through an explicit dependence of F_k upon p_i, and also implicitly through a change evoked in the state vector $\overline{y}$ by the integrated effect of the p_i perturbation.

Namely,

$$\left.\frac{\partial F_k}{\partial p_i}\right|_{t,\bar{p}\neq i} = \left.\frac{\partial F_k}{\partial p_i}\right|_{t,\bar{p}\neq i,\bar{y}} + \sum_{\ell=1}^{6} \left.\frac{\partial F_k}{\partial y_\ell}\right|_{t,\bar{p},\bar{y}\neq \ell} \cdot \frac{\partial y_\ell}{\partial p_i} \qquad k = 1,3 \qquad (14)$$

The vector $\partial F_k/\partial p_i\big|_{t,\bar{y},\bar{p}\neq i}$ is found through differentiation of the explicit dependence of the F_k upon p_i, therefore when p_i is one of the libration initial conditions this term vanishes. From equations (4) we have

$$\frac{\partial F_1}{\partial p_i} = \csc\theta\left(\sin\phi\,\frac{\partial\dot{\omega}_1}{\partial p_i} + \cos\phi\,\frac{\partial\dot{\omega}_2}{\partial p_i}\right)$$

$$\frac{\partial F_2}{\partial p_i} = \cos\phi\,\frac{\partial\dot{\omega}_1}{\partial p_i} - \sin\phi\,\frac{\partial\dot{\omega}_2}{\partial p_i}$$

$$\frac{\partial F_3}{\partial p_i} = \frac{\partial\dot{\omega}_3}{\partial p_i} - \cos\theta\,\frac{\partial F_1}{\partial p_i} \qquad (15)$$

To evaluate the quantities $\partial\dot{\omega}_k/\partial p_i$ we may substitute equations (9) into equations (2) and then differentiate:

$$\frac{\partial\dot{\omega}_1}{\partial p_i} = -\omega_2\omega_3\,\frac{\partial\alpha}{\partial p_i} + \sum_{b=\oplus,\odot}\left\{\frac{M_b}{A}\left[x_2\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_3}\right) - x_3\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_2}\right)\right] + T_{b1}\,\frac{\partial}{\partial p_i}\left(\frac{1}{A}\right)\right\}$$

$$\frac{\partial\dot{\omega}_2}{\partial p_i} = \omega_1\omega_3\,\frac{\partial\beta}{\partial p_i} + \sum_{b=\oplus,\odot}\left\{\frac{M_b}{B}\left[x_3\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_1}\right) - x_1\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_3}\right)\right] + T_{b2}\,\frac{\partial}{\partial p_i}\left(\frac{1}{B}\right)\right\}$$

$$\frac{\partial\dot{\omega}_3}{\partial p_i} = -\omega_1\omega_2\,\frac{\partial\gamma}{\partial p_i} + \sum_{b=\oplus,\odot}\left\{\frac{M_b}{C}\left[x_1\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_2}\right) - x_2\,\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_1}\right)\right] + T_{b3}\,\frac{\partial}{\partial p_i}\left(\frac{1}{C}\right)\right\} \qquad (16)$$

The partial derivatives with respect to specific parameters in equations (16) are simple and can be found in Appendix A.1.

The only quantity in (14) which remains to be formulated is the 3×6 "feed-back" matrix $\partial F_k/\partial y_\ell$; it is this term which supplies the initial condition partials. First we differentiate the defining equations (4) for the F_k with respect to each component of the state vector:

$$\frac{\partial F_1}{\partial\psi} = \csc\theta\left[\sin\phi\,\frac{\partial\dot\omega_1}{\partial\psi} + \cos\phi\,\frac{\partial\dot\omega_2}{\partial\psi}\right]$$

$$\frac{\partial F_2}{\partial\psi} = \cos\phi\,\frac{\partial\dot\omega_1}{\partial\psi} - \sin\phi\,\frac{\partial\dot\omega_2}{\partial\psi}$$

$$\frac{\partial F_3}{\partial\psi} = \frac{\partial\dot\omega_3}{\partial\psi} - \cos\theta\,\frac{\partial F_1}{\partial\psi}$$

$$\frac{\partial F_1}{\partial\theta} = \csc\theta\left[\sin\phi\,\frac{\partial\dot\omega_1}{\partial\theta} + \cos\phi\,\frac{\partial\dot\omega_2}{\partial\theta}\right] - \cot\theta\,\csc\theta\left[\dot\omega_1\sin\phi + \dot\omega_2\cos\phi\right]$$
$$- \dot\theta\dot\phi\cot\theta\csc\theta + \dot\psi\dot\theta\csc^2\theta$$

$$\frac{\partial F_2}{\partial\theta} = \cos\phi\,\frac{\partial\dot\omega_1}{\partial\theta} - \sin\phi\,\frac{\partial\dot\omega_2}{\partial\theta} - \dot\psi\dot\phi\cos\theta$$

$$\frac{\partial F_3}{\partial\theta} = \frac{\partial\dot\omega_3}{\partial\theta} - \cos\theta\,\frac{\partial F_1}{\partial\theta} + \sin\theta\,F_1 + \dot\psi\dot\theta\cos\theta$$

$$\frac{\partial F_1}{\partial\phi} = \csc\theta\left[\dot\omega_1\cos\phi + \sin\phi\,\frac{\partial\dot\omega_1}{\partial\phi} - \dot\omega_2\sin\phi + \cos\phi\,\frac{\partial\dot\omega_2}{\partial\phi}\right]$$

$$\frac{\partial F_2}{\partial\phi} = \cos\phi\,\frac{\partial\dot\omega_1}{\partial\phi} - \sin\phi\,\dot\omega_1 - \cos\phi\,\dot\omega_2 - \sin\phi\,\frac{\partial\dot\omega_2}{\partial\phi}$$

$$\frac{\partial F_3}{\partial\phi} = \frac{\partial\dot\omega_3}{\partial\phi} - \cos\theta\,\frac{\partial F_1}{\partial\phi}$$

$$\frac{\partial F_1}{\partial \dot\psi} = \csc\theta \left[\sin\phi \frac{\partial \dot\omega_1}{\partial \dot\psi} + \cos\phi \frac{\partial \dot\omega_2}{\partial \dot\psi}\right] - \dot\theta \cot\theta$$

$$\frac{\partial F_2}{\partial \dot\psi} = \cos\phi \frac{\partial \dot\omega_1}{\partial \dot\psi} - \sin\phi \frac{\partial \dot\omega_2}{\partial \dot\psi} - \dot\phi \sin\theta$$

$$\frac{\partial F_3}{\partial \dot\psi} = \frac{\partial \dot\omega_3}{\partial \dot\psi} - \cos\theta \frac{\partial F_1}{\partial \dot\psi} + \dot\theta \sin\theta$$

$$\frac{\partial F_1}{\partial \dot\theta} = \csc\theta \left[\sin\phi \frac{\partial \dot\omega_1}{\partial \dot\theta} + \cos\phi \frac{\partial \dot\omega_2}{\partial \dot\theta}\right] + \dot\phi \csc\theta - \dot\psi \cot\theta$$

$$\frac{\partial F_2}{\partial \dot\theta} = \cos\phi \frac{\partial \dot\omega_1}{\partial \dot\theta} - \sin\phi \frac{\partial \dot\omega_2}{\partial \dot\theta}$$

$$\frac{\partial F_3}{\partial \dot\theta} = \frac{\partial \dot\omega_3}{\partial \dot\theta} - \cos\theta \frac{\partial F_1}{\partial \dot\theta} + \dot\psi \sin\theta$$

$$\frac{\partial F_1}{\partial \dot\phi} = \csc\theta \left[\sin\phi \frac{\partial \dot\omega_1}{\partial \dot\phi} + \cos\phi \frac{\partial \dot\omega_2}{\partial \dot\phi}\right] + \dot\theta \csc\theta$$

$$\frac{\partial F_2}{\partial \dot\phi} = \cos\phi \frac{\partial \dot\omega_1}{\partial \dot\phi} - \sin\phi \frac{\partial \dot\omega_2}{\partial \dot\phi} - \dot\psi \sin\theta$$

$$\frac{\partial F_3}{\partial \dot\phi} = \frac{\partial \dot\omega_3}{\partial \dot\phi} - \cos\theta \frac{\partial F_1}{\partial \dot\phi} \tag{17}$$

To obtain the quantities $\partial\dot\omega_i/\partial y_\ell$ we may again substitute equations (9) into equations (2) and differentiate, this time with respect to the Euler angles:

$$\frac{\partial \dot\omega_1}{\partial y_\ell} = -\alpha\left(\omega_2 \frac{\partial \omega_3}{\partial y_\ell} + \omega_3 \frac{\partial \omega_2}{\partial y_\ell}\right)$$
$$+ \sum_{b=\oplus,\odot} \frac{M_b}{A}\left\{x_2 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_3}\right) + \frac{\partial x_2}{\partial y_\ell}\frac{\partial U_b}{\partial x_3} - x_3 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_2}\right) - \frac{\partial x_3}{\partial y_\ell}\frac{\partial U_b}{\partial x_2}\right\} \tag{18}$$

$$\frac{\partial \dot{\omega}_2}{\partial y_\ell} = \beta \cdot \left(\omega_1 \frac{\partial \omega_3}{\partial y_\ell} + \omega_3 \frac{\partial \omega_1}{\partial y_\ell} \right)$$

$$+ \sum_{b=\oplus,\odot} \frac{M_b}{B} \left\{ x_3 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_1}\right) + \frac{\partial x_3}{\partial y_\ell}\frac{\partial U_b}{\partial x_1} - x_1 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_3}\right) - \frac{\partial x_1}{\partial y_\ell}\frac{\partial U_b}{\partial x_3} \right\}$$

$$\frac{\partial \dot{\omega}_3}{\partial y_\ell} = -\gamma \left(\omega_1 \frac{\partial \omega_2}{\partial y_\ell} + \omega_2 \frac{\partial \omega_1}{\partial y_\ell} \right)$$

$$+ \sum_{b=\oplus,\odot} \frac{M_b}{C} \left\{ x_1 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_2}\right) + \frac{\partial x_1}{\partial y_\ell}\frac{\partial U_b}{\partial x_2} - x_2 \frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_1}\right) - \frac{\partial x_2}{\partial y_\ell}\frac{\partial U_b}{\partial x_1} \right\}$$

Notice that the summation terms are zero for $\ell = 4$, 5, or 6 since the torques are not dependent upon the angular velocity of the Moon. The terms multiplying the moment of inertia ratios are kinematic and can be derived from equations (3):

$$\begin{array}{lll}
\dfrac{\partial \omega_1}{\partial \psi} = 0 & \dfrac{\partial \omega_2}{\partial \psi} = 0 & \dfrac{\partial \omega_3}{\partial \psi} = 0 \\[2ex]
\dfrac{\partial \omega_1}{\partial \theta} = \dot{\psi} \sin\phi \cos\theta & \dfrac{\partial \omega_2}{\partial \theta} = \dot{\psi} \cos\phi \cos\theta & \dfrac{\partial \omega_3}{\partial \theta} = -\dot{\psi} \sin\theta \\[2ex]
\dfrac{\partial \omega_1}{\partial \phi} = -\dot{\theta} \sin\phi + \dot{\psi} \sin\theta \cos\phi & \dfrac{\partial \omega_2}{\partial \phi} = -\dot{\theta} \cos\phi - \dot{\psi} \sin\phi \sin\theta & \dfrac{\partial \omega_3}{\partial \phi} = 0 \\[2ex]
\dfrac{\partial \omega_1}{\partial \dot{\psi}} = \sin\phi \sin\theta & \dfrac{\partial \omega_2}{\partial \dot{\psi}} = \cos\phi \sin\theta & \dfrac{\partial \omega_3}{\partial \dot{\psi}} = \cos\theta \\[2ex]
\dfrac{\partial \omega_1}{\partial \dot{\theta}} = \cos\phi & \dfrac{\partial \omega_2}{\partial \dot{\theta}} = -\sin\phi & \dfrac{\partial \omega_3}{\partial \dot{\theta}} = 0 \\[2ex]
\dfrac{\partial \omega_1}{\partial \dot{\phi}} = 0 & \dfrac{\partial \omega_2}{\partial \dot{\phi}} = 0 & \dfrac{\partial \omega_3}{\partial \dot{\phi}} = 1
\end{array} \qquad (19)$$

Inside the summation we have the terms $\partial x_i/\partial y_\ell$ which represent the change in the selenocentric coordinates of a perturbing body with respect to changes in the Euler angles. The selenocentric coordinates are transformed from the inertial system by a rotation matrix $\overline{\overline{R}}$ defined by

$$\overline{x} = \overline{\overline{R}}\,(\psi,\ \theta,\ \phi)\ \overline{\xi}$$

and given by Goldstein (1950) as:

$$\overline{\overline{R}}(\psi,\theta,\phi) = \begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & \cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ -\sin\phi\cos\psi - \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & \cos\phi\sin\theta \\ \sin\theta\sin\psi & -\sin\theta\cos\psi & \cos\theta \end{bmatrix} \quad (20)$$

Now $\overline{\xi}$ does not depend on the orientation of the selenocentric coordinate system, so:

$$\frac{\partial x_1}{\partial \psi} = (-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi)\ \xi_1 + (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi)\ \xi_2$$

$$\frac{\partial x_1}{\partial \theta} = \sin\phi\sin\theta\sin\psi\,\xi_1 - \sin\phi\sin\theta\cos\psi\,\xi_2 + \sin\phi\cos\theta\,\xi_3$$

$$\frac{\partial x_1}{\partial \phi} = (-\sin\phi\cos\psi - \cos\phi\cos\theta\sin\phi)\ \xi_1 + (-\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi)\ \xi_2 + \cos\phi\sin\theta\,\xi_3$$

$$\frac{\partial x_2}{\partial \psi} = (\sin\phi\sin\psi - \cos\phi\cos\theta\cos\psi)\xi_1 + (-\sin\phi\cos\psi - \cos\phi\cos\theta\sin\psi)\xi_3$$

$$\frac{\partial x_2}{\partial \theta} = \cos\phi\sin\theta\sin\psi\,\xi_1 - \cos\phi\sin\theta\cos\psi\,\xi_2 + \cos\phi\cos\theta\,\xi_3$$

$$\frac{\partial x_2}{\partial \phi} = (-\cos\phi\cos\psi + \sin\phi\cos\theta\sin\psi)\xi_1 + (-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi)\xi_2$$

$$-\sin\phi\sin\theta\,\xi_3$$

$$\frac{\partial x_3}{\partial \psi} = \sin\phi\cos\psi\,\xi_1 + \sin\theta\sin\psi\,\xi_2$$

$$\frac{\partial x_3}{\partial \theta} = \cos\theta\sin\psi\,\xi_1 - \cos\theta\cos\psi\,\xi_2 - \sin\theta\,\xi_3$$

$$\frac{\partial x_3}{\partial \phi} = 0 \tag{21}$$

The quantities $\partial/\partial y_\ell(\partial U_b/\partial x_k)$ can be found through application of the chain rule:

$$\frac{\partial}{\partial y_\ell}\left(\frac{\partial U_b}{\partial x_k}\right) = \sum_{i=1}^{3} \frac{\partial x_i}{\partial y_\ell}\,\frac{\partial}{\partial x_i}\left(\frac{\partial U_b}{\partial x_k}\right) \tag{22}$$

The terms $\partial x_i/\partial y_\ell$ were evaluated in equations (21), and differentiation of equation (10) yields:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial U_b}{\partial x_k}\right) = GM_{☾} \sum_{n=2}^{20} \left(\frac{a}{r}\right)^n \frac{J_n}{r^2} \left\{ P'_n(\sin L)\, r\, \frac{\partial^2 \sin L}{\partial x_k\, \partial x_i} + P''_n(\sin L)\, r\, \frac{\partial \sin L}{\partial x_k}\,\frac{\partial \sin L}{\partial x_i} \right.$$

$$\left. - (n+1)\, P'_n(\sin L)\, \frac{\partial \sin L}{\partial x_k}\,\frac{x_i}{r} \right\}$$

$$+ GM_{☾} \sum_{n=2}^{10} \sum_{m=1}^{n} \left(\frac{a}{r}\right)^n \frac{1}{r^2} \left\{ -(n+1)\,\frac{x_i}{r} \left[(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda) P'_{nm}(\sin L)\, \frac{\partial \sin L}{\partial x_k} \right. \right.$$

$$\left. + m\,(-C_{nm}\sin m\lambda + S_{nm}\cos m\lambda) P_{nm}(\sin L)\, \frac{\partial \lambda}{\partial x_k} \right]$$

$$- r \left[(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \left[P'_{nm}(\sin L) \frac{\partial^2 \sin L}{\partial x_k \, \partial x_i} \right. \right.$$

$$\left. + P''_{nm}(\sin L) \frac{\partial \sin L}{\partial x_k} \frac{\partial \sin L}{\partial x_i} \right]$$

$$+ m(-C_{nm} \sin m\lambda + S_{nm} \cos m\lambda) \left[P'_{nm}(\sin L) \frac{\partial \sin L}{\partial x_k} \frac{\partial \lambda}{\partial x_i} \right.$$

$$\left. + P_{nm}(\sin L) \frac{\partial^2 \lambda}{\partial x_k \, \partial x_i} + P'_{nm}(\sin L) \frac{\partial \lambda}{\partial x_k} \frac{\partial \sin L}{\partial x_i} \right]$$

$$\left. \left. - m^2 (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin L) \frac{\partial \lambda}{\partial x_k} \frac{\partial \lambda}{\partial x_i} \right] \right\} \qquad (23)$$

The second derivatives are found by differentiating equations (11) as indicated, and simplifying:

$$\frac{\partial^2 \sin L}{\partial x_k \, \partial x_i} = \frac{3 x_3 x_k x_i}{r^5} - \frac{\delta_{3k} x_i + \delta_{3i} x_k + \delta_{ik} x_3}{r^3}$$

$$\frac{\partial^2 \lambda}{\partial x_k \, \partial x_i} = \frac{(x_1 \delta_{1k} + x_2 \delta_{2k})(x_2 \delta_{1i} - x_1 \delta_{2i}) + (x_1 \delta_{1i} + x_2 \delta_{2i})(x_2 \delta_{1k} - x_1 \delta_{2k})}{(x_1^2 + x_2^2)^2}$$

This completes the derivation of the variational equations for our libration model. If there are n parameters for which we desire derivatives we can integrate the 6n first-order variational equations in parallel with the 6 equations of motion by any suitable numerical method.

VERIFICATION

The equations of motion and variational equations have been integrated using an Adams-Moulton predictor-corrector method (Smith, 1968) at 8 steps/day within the framework of the M.I.T. Planetary Ephemeris Program (Ash, 1965). The orbital motions of the Moon and Earth were read from ephemeris tapes which were integrated from initial conditions obtained in a least-squares fit to data: 5 years of laser ranging data of the Moon (King et al., 1976), and many years of optical and radar observations of the planets (Ash et al., 1971). The variational equations have proven to be consistent with the equations of motion to better than .01% by finite differencing of the latter; the fact that one may check variational equations in this manner is another strong incentive (in addition to the operational convenience of the parallel integration of the partial derivatives) for their use.

In order to verify the accuracy of our integration of the equations of motion, we have compared the integrated Euler angles with LLB-5. The LLB-5 Cassini angle initial conditions were transformed to the corresponding Euler angle initial conditions referenced to the inertial 1950.0 mean equator-equinox system. The lunar gravitational harmonic coefficients, moment of inertia ratios, and the mass of the Earth were also set to the values used to generate LLB-5. The result of the integration is compared with LLB-5 in Fig. 2, where differences in the Cassini angles (which must be transformed back from the Euler angles of PEP's output at each tabular point) are plotted as functions of time. The greatest discrepancy is in τ, and has the form of a sinusoid with a 2.9 year period and an amplitude of 0.29". This period is that of the free mode of libration in longitude, corresponding to a homogeneous solution of Euler's equations. This mode has been stimulated by a longitude offset of 0.29" between the lunar ephemeris used to create LLB-5 and that of King et al. The cause of this orbital offset is unknown, but under investigation.

In order to minimize the effects of the JPL-MIT mean lunar orbit difference for our comparison of the two lunar libration integrations, we introduced three ad hoc small angles to describe an arbitrary orientation difference between coordinate systems; specifically, we modelled the angles as constant biases in each of the three Cassini angles. Since the Cassini angles describe the difference between the true lunar libration and the mean lunar orbit (vis-à-vis Cassini's laws), a bias in a Cassini angle follows directly from a corresponding offset in the orientation of the mean lunar orbit. These biases were simultaneously fit with six initial conditions of rotation to minimize the sum-squared differences of Cassini angles between the PEP integration and LLB-5. The adjusted initial conditions were then integrated and the new Cassini angle differences from LLB-5 are plotted with biases included in Figure 3; the estimated biases and post-fit rms Cassini angle differences (about the biases) are displayed in Table 1.

The largest differences between the PEP post-fit integration and LLB-5 appear about equally in the Cassini angles ρ and $I\sigma$, and correspond to a lunar surface displacement of about 16 cm from each angle, or about 4 cm in range to the retro-reflectors farthest from the sub-Earth point. The sinusoidal signatures are predominantly those of the free libration modes of ρ and $I\sigma$ — a combination of 27.2 day and 24 year components in both. These free librations are of a magnitude consistent with the hypothesis that they are being stimulated by the known differences in the lunar ephemerides used in the integrations.

In order to ascertain which libration model more closely represents the true motion of points on the lunar surface about the center of mass we have performed parallel fits to laser ranging observations of the

Apollo retro-reflectors. The maximum difference between the Euler model solution and a corresponding solution using LLB-5 was 0.2 ns (3 cm in range) rms for a series of range observations on a specific retro-reflector over a period of 2 or 3 years, with the rms residual for the Euler model equal or lower than LLB-5 for all such series. It should be emphasized, however, that the lunar orbital emphemeris employed in these data analyses was the same ephemeris which was used to generate the Euler angle libration model and perhaps the lower residuals merely reflect the consistency of orbital and rotational models.

In our development of the Euler angle model we have made many approximations, some of which we will continue to ignore and others we will correct in the future. The lunar orbital and rotational motions are cross-coupled and we are in the process of modifying our computer algorithms to integrate them simultaneously. The Earth's oblateness appears to affect libration at above the .01" level, so we plan to include the low-order gravity field of the Earth in our model. Finally, the Moon is not a rigid body, and is subject to solid body tides raised mainly by the Earth, resulting in an inertia tensor that is non-diagonal and time-variable; also there is undoubtably some tidal dissipation in the Moon. The magnitudes of these effects are difficult to estimate, depending as they do upon unknown quantities such as the lunar Love numbers, and Q for the Moon, though a rough calculation places their effect at about the level of the other errors in our model, so we plan to incorporate a flabby Moon into our theory as well. Ultimately, we hope to reduce the total libration errors to below the anticipated future lunar laser range uncertainties of 3 cm, so that the full benefit of the quality of the laser ranging data can be realized.

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APPENDIX A.1

PARTIAL DERIVATIVES WITH RESPECT TO SPECIFIC LIBRATION PARAMETERS

In order to derive the partial derivatives indicated in equation (16) we must first determine which members to include in our set of adjustable libration parameters. In principle any unknown parameter affecting the libration should be modelled as such and have derivatives generated for it; practically, we need only model the small set of parameters whose uncertainties influence the libration to a measurable extent (defined, for the purpose of this paper, as greater than 1 part in 10^5 of the whole libration). We are left with the following (significant) parameters.

1) six initial conditions of the state vector

2) J_2

3) β and γ

4) third and higher order harmonic coefficients

Note that we have arbitrarily chosen J_2, β, and γ as our independent second-degree coefficients. C_{22} is a combination of all three, while α depends only on β and γ, and since we choose the lunar principal axes of inertia to define our selenocentric coordinate system, C_{21}, S_{21}, and S_{22} are identically zero.

Now we will derive the derivative terms in equation (16). $\partial\beta/\partial p_i$ and $\partial\gamma/\partial p_i$ are non-zero only when $p_i = \beta$ or γ respectively. $\partial\alpha/\partial p_i$, $\partial A^{-1}/\partial p_i$, $\partial B^{-1}/\partial p_i$, and $\partial C^{-1}/\partial p_i$ are all zero for p_i other than the second degree parameters β, γ, and J_2, when we get the following:

$$\alpha = \frac{\beta - \gamma}{1 - \beta\gamma}$$

$$\frac{\partial\alpha}{\partial J_2} = 0$$

$$\frac{\partial\alpha}{\partial\beta} = \frac{1 - \gamma^2}{(1 - \beta\gamma)^2}$$

(A-1)

$$\frac{\partial\alpha}{\partial\gamma} = \frac{\beta^2 - 1}{(1 - \beta\gamma)^2}$$

$$C^{-1} = \frac{1}{M_{\mathbb{C}} a^2} \, \frac{2\beta - \gamma + \beta\gamma}{2J_2(1 + \beta)}$$

$$\frac{\partial C^{-1}}{\partial J_2} = \frac{-1}{J_2} C^{-1}$$

$$\frac{\partial C^{-1}}{\partial\beta} = \frac{1 + \gamma}{M_{\mathbb{C}} a^2 J_2 (1 + \beta)^2}$$

$$\frac{\partial C^{-1}}{\partial\gamma} = \frac{\beta - 1}{2M_{\mathbb{C}} a^2 J_2 (1 + \beta)}$$

$$A^{-1} = C^{-1} \frac{1 + \beta}{1 - \beta\gamma}$$

$$\frac{\partial A^{-1}}{\partial J_2} = \frac{1 + \beta}{1 - \beta\gamma} \, \frac{\partial C^{-1}}{\partial J_2}$$

$$\frac{\partial A^{-1}}{\partial\beta} = \frac{1 + \beta}{1 - \beta\gamma} \, \frac{\partial C^{-1}}{\partial\beta} + \frac{1 + \gamma}{(1 - \beta\gamma)^2} C^{-1}$$

$$\alpha = \frac{\beta - \gamma}{1-\beta\gamma}$$

$$\frac{\partial\alpha}{\partial J_2} = 0$$

$$\frac{\partial\alpha}{\partial\beta} = \frac{1-\gamma^2}{(1-\beta\gamma)^2}$$

$$\frac{\partial\alpha}{\partial\gamma} = \frac{\beta^2-1}{(1-\beta\gamma)^2} \tag{A-1}$$

$$C^{-1} = \frac{1}{M_{\mathbb{C}}a^2}\,\frac{2\beta - \gamma + \beta\gamma}{2J_2(1+\beta)}$$

$$\frac{\partial C^{-1}}{\partial J_2} = \frac{-1}{J_2}\,C^{-1}$$

$$\frac{\partial C^{-1}}{\partial\beta} = \frac{1+\gamma}{M_{\mathbb{C}}a^2 J_2(1+\beta)^2}$$

$$\frac{\partial C^{-1}}{\partial\gamma} = \frac{\beta - 1}{2M_{\mathbb{C}}a^2 J_2(1+\beta)}$$

$$A^{-1} = C^{-1}\,\frac{1+\beta}{1-\beta\gamma}$$

$$\frac{\partial A^{-1}}{\partial J_2} = \frac{1+\beta}{1-\beta\gamma}\,\frac{\partial C^{-1}}{\partial J_2}$$

$$\frac{\partial A^{-1}}{\partial\beta} = \frac{1+\beta}{1-\beta\gamma}\,\frac{\partial C^{-1}}{\partial\beta} + \frac{1+\gamma}{(1-\beta\gamma)^2}\,C^{-1}$$

$$\frac{\partial A^{-1}}{\partial \gamma} = \frac{1+\beta}{1-\beta\gamma}\frac{\partial C^{-1}}{\partial \gamma} + \frac{\beta(1+\beta)}{(1-\beta\gamma)^2} C^{-1}$$

$$B^{-1} = \frac{1+\beta}{1+\gamma} C^{-1}$$

$$\frac{\partial B^{-1}}{\partial J_2} = \frac{1+\beta}{1+\gamma}\frac{\partial C^{-1}}{\partial J_2}$$

$$\frac{\partial B^{-1}}{\partial \beta} = \frac{1+\beta}{1+\gamma}\frac{\partial C^{-1}}{\partial \beta} + \frac{1}{1+\gamma} C^{-1}$$

$$\frac{\partial B^{-1}}{\partial \gamma} = \frac{1+\beta}{1+\gamma}\frac{\partial C^{-1}}{\partial \gamma} - \frac{1+\beta}{(1+\gamma)^2} C^{-1}$$

The remaining chain-rule terms in equations (16) are the $\partial/\partial p_i\,(\partial U_b/\partial x_k)$. The harmonic coefficient partials are merely the terms of the linear combination which the coefficients multiply:

$$\frac{\partial}{\partial J_n}\left(\frac{\partial U_b}{\partial x_k}\right) = \frac{GM_{☾}}{r^2}\left(\frac{a}{r}\right)^n P_n'\,(\sin L)\,[\delta_{3k} - \frac{x_k}{r}\sin L] \qquad \text{(A-2a)}$$

$$\frac{\partial}{\partial C_{nm}}\left(\frac{\partial U_b}{\partial x_k}\right) = \frac{-GM_{☾}}{r^2}\left(\frac{a}{r}\right)^n \Big\{\cos m\lambda\ P_{nm}'(\sin L)\,[\delta_{3k} - \frac{x_k}{r}\sin L]$$

$$- m\sin m\lambda\ P_{nm}\ (\sin L)\ \frac{1}{\cos L}\,[\delta_{2k}\ \cos\lambda - \delta_{1k}\ \sin\lambda\,]\Big\} \qquad \text{(-2b)}$$

$$\frac{\partial}{\partial S_{nm}}\left(\frac{\partial U_b}{\partial x_k}\right) = \frac{-GM_{☾}}{r^2}\left(\frac{a}{r}\right)^n \left\{\sin m\lambda \, P'_{nm}(\sin L)\,[\delta_{3k} - \frac{x_k}{r}\sin L]\right.$$

$$\left. + \, m \cos m\lambda \, P_{nm}(\sin L)\,\frac{1}{\cos L}\,[\delta_{2k}\cos\lambda - \delta_{1k}\sin\lambda]\right\} \qquad \text{(A-2c)}$$

For p_i one of β, γ, or J_2 we apply the chain rule to find the (additional, in the case of J_2) change in the potential gradient due to the change induced in C_{22}:

$$\frac{\partial}{\partial p_i}\left(\frac{\partial U_b}{\partial x_k}\right) = \frac{\partial C_{22}}{\partial p_i}\,\frac{\partial}{\partial C_{22}}\left(\frac{\partial U_b}{\partial x_k}\right) \qquad \text{(A-3)}$$

Since

$$C_{22} = J_2\,\frac{\gamma}{2}\,\frac{1+\beta}{2\beta - \gamma + \beta\gamma}$$

we have

$$\frac{\partial C_{22}}{\partial J_2} = \frac{C_{22}}{J_2}$$

$$\frac{\partial C_{22}}{\partial \beta} = -J_2\,\frac{\gamma(1+\gamma)}{(2\beta - \gamma + \beta\gamma)^2}$$

$$\frac{\partial C_{22}}{\partial \gamma} = J_2\,\frac{\beta(1+\beta)}{(2\beta - \gamma + \beta\gamma)^2} \qquad \text{(A-4)}$$

The second term in the RHS of equation (A-3) is given by (A-2b) with $n = m = 2$.

Table 1. Comparison of Best-fit PEP Euler Integration with LLB-5

	bias	post-fit rms difference
τ	.287"	.009"
ρ	.085"	.021"
$I\sigma$	.066"	.019"

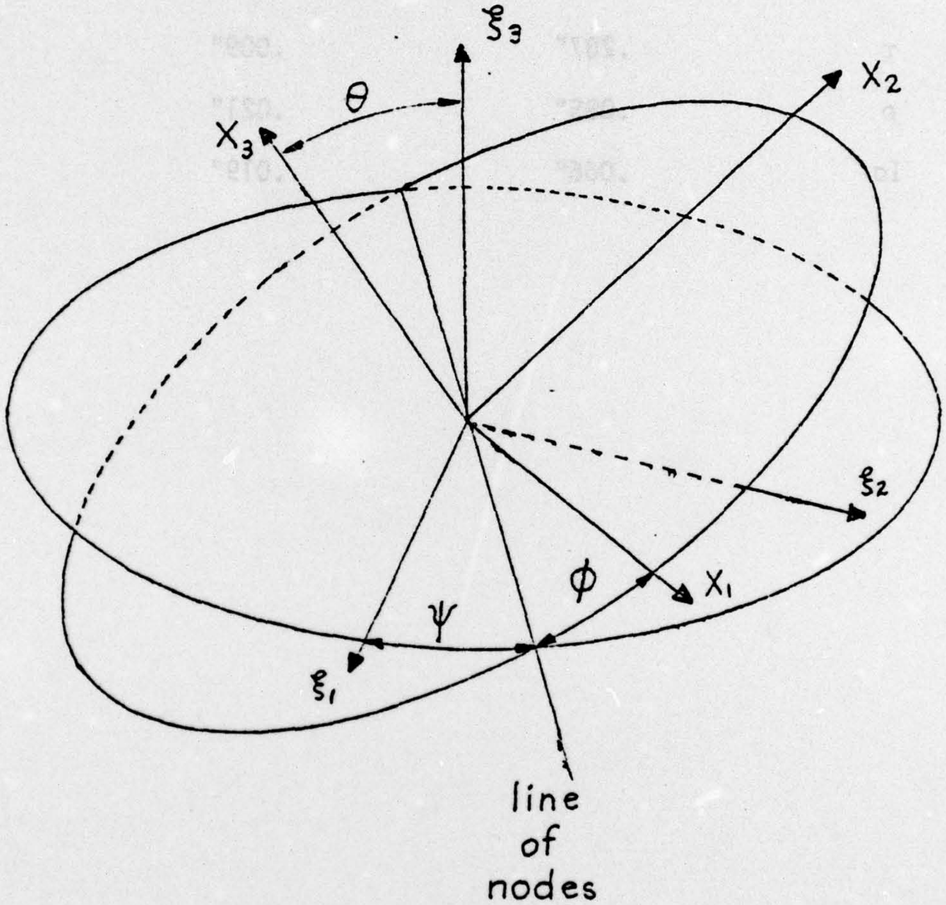

FIG. 1 EULER ANGLES

LLB5 - EULER

INTEGRATION OF TRANSFORMED LLB5 ICS

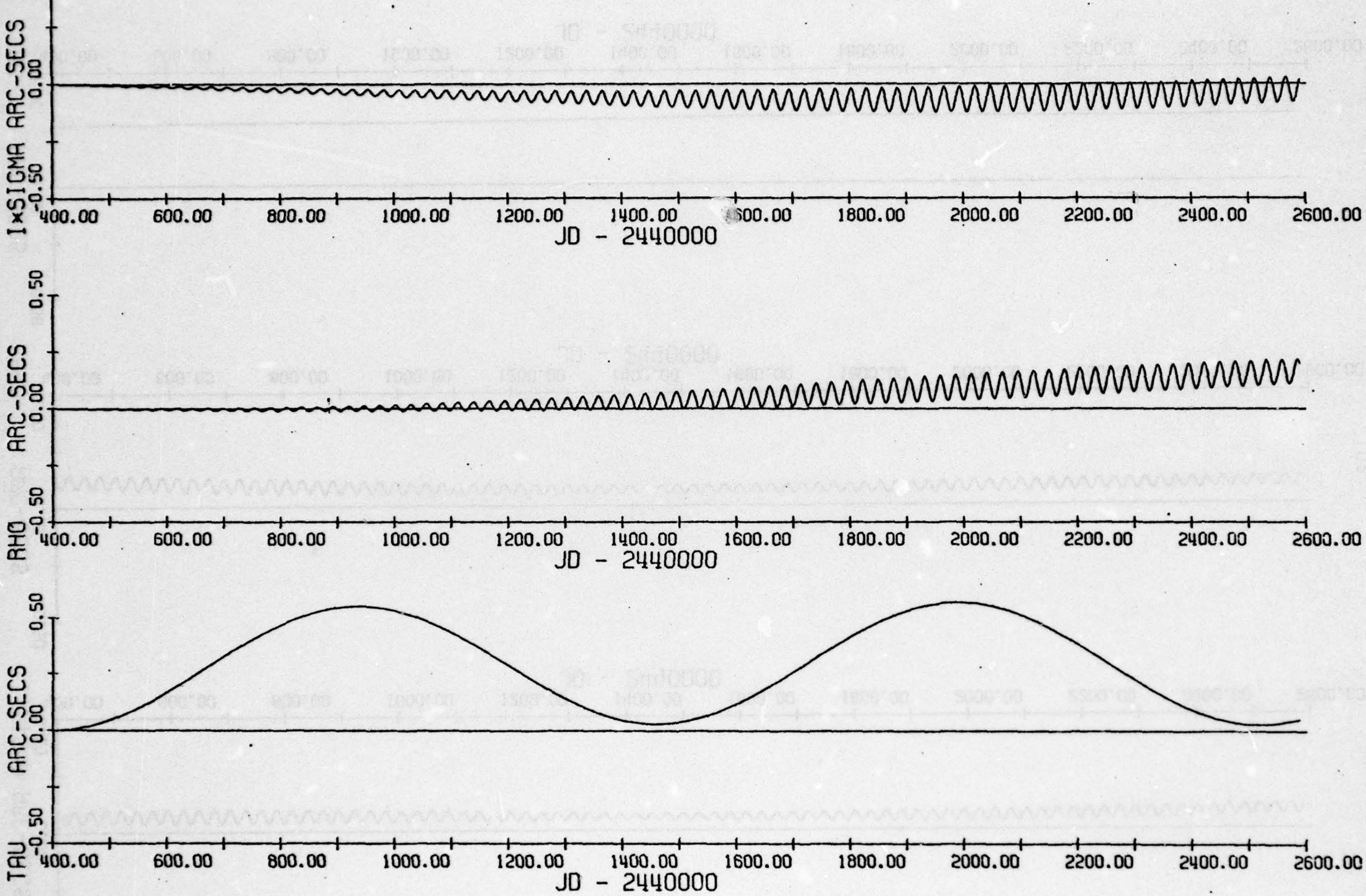

FIGURE 2

EULER - LLB5

INTEGRATION OF ICS FROM FIT TO LLB5 OF 6 ICS AND 3 CASSINI ANGLE BIASES

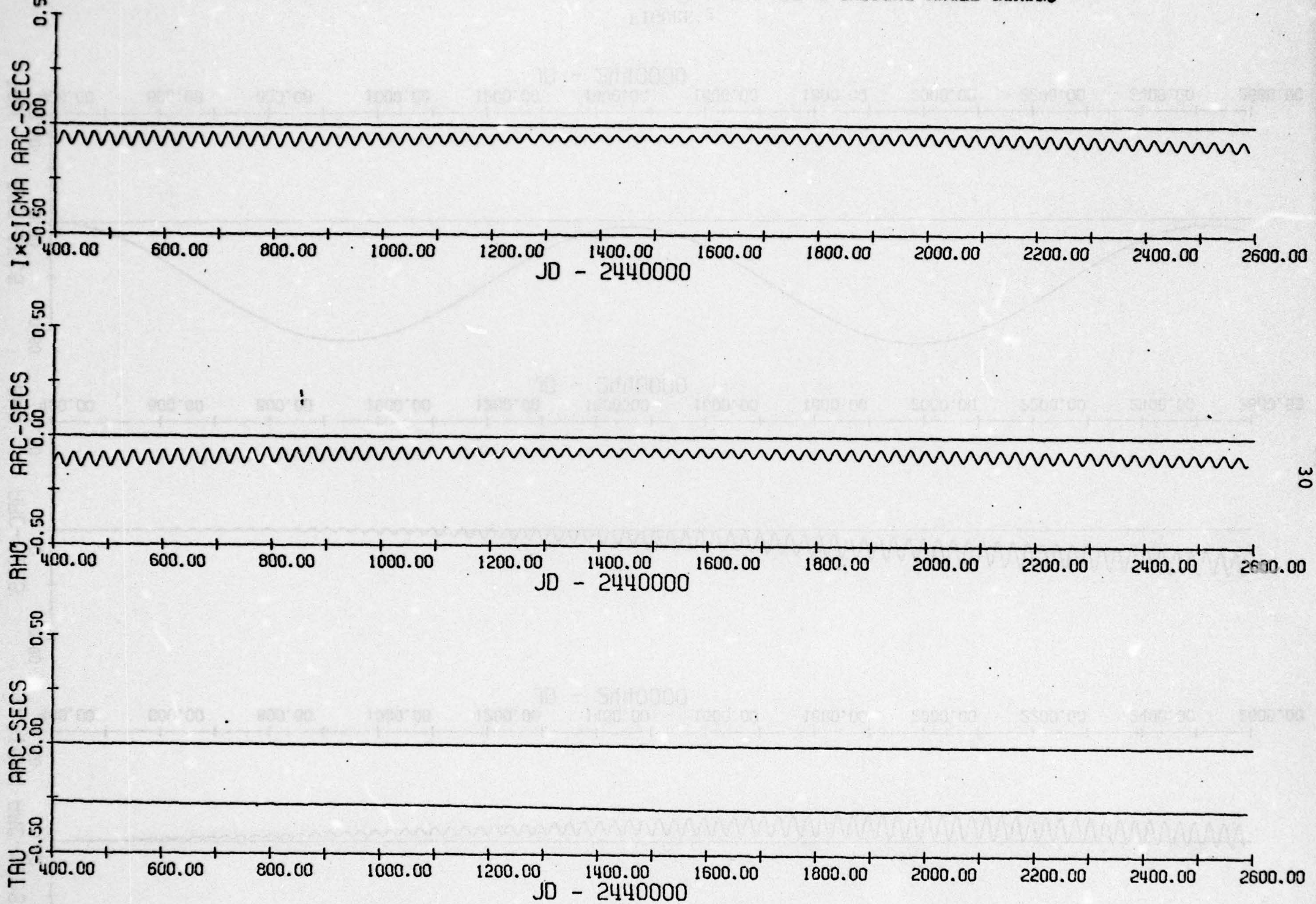

FIGURE 3