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SINGULARITIES IN THE DISTRIBUTION OF THE INCREMENTS OF A SMOOTH--ETC(U)
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SINGULARITIES IN THE DISTRIBUTION OF THE

INCREMENTS OF A SMOOTH FUNCTION

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§1. By the "distribution of the increments" of a Borel function $F: [0,1] \to \mathbb{R}$, I mean the measure

$$\lambda(B) = \int_0^1 \int_0^1 1_B(F(s)-F(t))\,ds\,dt \quad ,$$

B a Borel set in $\mathbb{R}$. λ is the convolution o[illegible] "occupation measure" $\mu(B) = m\{F^{-1}(B)\}$ with $\mu(-B)$; here m in Lebesgue measure. When $\mu \ll m$, write $\alpha(x)$ for the Radon-Nikodym derivative $\frac{d\mu}{dm}(x)$ (the "local time" of F at x). Of course $\mu \ll m$ implies $\lambda \ll m$ and

(1) $$\Lambda(x) \equiv \frac{d\lambda}{dm}(x) = \int_{-\infty}^{\infty} \alpha(y)\alpha(x+y)\,dy \quad .$$

Although this paper treats only smooth F's (at least C^1), the relevant background consists of two general results from [3]. Throughout, ψ will denote a nonnegative,

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Borel measurable function. Define

$$I(\psi;F) = \int_0^1\int_0^1 \psi(F(s)-F(t))dsdt = \int\psi d\lambda \le \infty .$$

Then (a) if ψ is even, decreasing on $(0,\infty)$, and nonintegrable on $(0,1)$, then $I(\psi;F) = \infty$ for any F; (b) $\mu << m$ with $\alpha\in L^2$ if and only if $I(\psi;F) < \infty$ $\forall\psi\in L^1$.

Now for F differentiable a.e., $\mu << m$ if and only if $D_0 \equiv \{t: F'(t) = 0\}$ has Lebesgue measure 0. Suppose $F\in C^1$ (i.e. has a continuous derivative, with the usual conventions about the endpoints) and $D_0 \neq \emptyset$, $m(D_0) = 0$. Then, as the Theorem states, $\lim_{x\to 0} \Lambda(x) = \infty$. (The additional assumptions made on F below are not needed for this.) Hence $I(\psi;F) = \infty$ for some $\psi\in L^1$ (and so $\alpha\notin L^2$) because $\lambda << m$ implies

(2) $$I(\psi;F) = \int \Lambda(x)\psi(x)dx .$$

So, the question arises: for which ψ's - in particular, which monotone ones - is $I(\psi;F) < \infty$? This depends on the nature of the singular points of Λ.

Assume now $D_0 \neq \emptyset$, F *is* C^2 *and that* $F''(t) \neq 0$ *for all* $t\in D_0$. Then D_0 is finite, say $D_0 = \{a_i\}_{i=1}^N$, $0 \le a_1 < a_2 < \ldots < a_N \le 1$. Let $A_i = F(a_i)$ and let $\{B_i\}_{i=1}^L$ denote the (distinct) elements of $\{A_i - A_j\}$ for which there exist $t_1, t_2 \in D_0$ with $F''(t_1)F''(t_2) > 0$ and $F(t_1)-F(t_2) = A_i - A_j$. $\{B_i\}$ is symmetric about 0 and contains 0. For the version of $\Lambda(x)$ given by (1):

THEOREM. $\Lambda(x)$ *is continuous on* $\mathbb{R}\setminus\{B_i\}_1^L$ *and*

(3a) $$0 < \varliminf_{x\to B_i} \frac{\Lambda(x)}{-\log|x-B_i|} \le \varlimsup_{x\to B_i} \frac{\Lambda(x)}{-\log|x-B_i|} < \infty \qquad 1 \le i \le L.$$

Consequently, for $\psi\in L^1$

(3b) $$I(\psi;F) < \infty \iff \psi\in L^1\{\sum_{i=1}^{L} |\log|x-B_i||dx\} .$$

In particular, if ψ *is even and decreasing on* $(0,\infty)$, *then*

$$I(\psi;F) < \infty \iff \int_0^1 \psi(x)\log 1/x \, dx < \infty .$$

§2. The fact that the singularities of Λ occur among the points $\{\Lambda_i - \Lambda_j\}$ is fairly obvious. Indeed for F as above ([4])

$$\alpha(x) = \sum_{s\in F^{-1}(\{x\})} |F'(s)|^{-1} . \tag{4}$$

(Since $F(D_0)$ has measure zero, it doesn't matter how α is defined there.) Clearly α is well-behaved off $\{A_i\}$, and, in turn, Λ off $\{A_i - A_j\}$. (Actually, (4) is valid for any F such that F' exists a.e., although "$s\in F^{-1}(\{x\})$" must be replaced by "$s\in F^{-1}(\{x\}) \cap \{F'$ exists, finite$\}$" and neither $F(\{|F'| = \infty\})$ nor $F(\{F'$ doesn't exist$\})$ need have measure 0.)

The Co-Area Theorem [2], applied to the Lipschitz function $s,t \to F(s)-F(t)$, *leads to this expression for* Λ:

$$\Lambda(x) = \iint_{U_x} [(F'(s))^2 + (F'(t))^2]^{-1/2} H(dsdt) ; \tag{5}$$

here $U_x = \{(s,t): F(s)-F(t) = x\}$ and H is one-dimensional Hausdorff measure in $\mathbb{R}^2$. This shows clearly where Λ might explode. Nonetheless, I will not refer again to (5), but instead work with the version of Λ given by (1) with α as in (4).

That the singularities of Λ are logarithmic is perhaps not as evident, and emerged in a curious way. To get an idea of when $I(\psi;F)$ is finite, $\psi\in L^1$, choose a convenient random function $X(t,\omega)$, $0 \le t \le 1$, $\omega\in\Omega$, with smooth trajectories and compute the expected value $E\{I(\psi;X(\cdot,\omega))\}$ of the random variable $\omega \to I(\psi;X(\cdot,\omega))$. For instance, let $X(t,\omega)$ be Gaussian, mean 0, $\sigma^2(s,t) = E(X(s)-X(t))^2$. Then

$$E\{I(\psi;X(\cdot,\omega))\} = \int_0^1\int_0^1\int_{-\infty}^{\infty} \psi(x)[2\pi\sigma(s,t)]^{-1}\exp\{-\frac{x^2}{2\sigma^2(s,t)}\}\,dxdsdt\ .$$

For simplicity, and to insure the differentiability of the sample functions, suppose there are constants $0 < C_1 < C_2 < \infty \ni C_1|s-t| < \sigma(s,t) < C_2|s-t| \quad \forall s,t$. (For example, $X(t,\omega)$ is stationary, $r(t) \equiv EX_tX_0 \neq r(0)$, $t \neq 0$, and $-r''(0) < \infty$.) A straightforward computation yields (for ψ even):

$$E\{I(\psi;X(\cdot,\omega))\} < \infty \iff \int_0^{\infty} e^{-y^2}\frac{1}{y}\int_0^y \psi(x)dxdy < \infty\ ;$$

equivalently, $M_\psi(x) \equiv \frac{1}{x}\int_0^x \psi(u)du$ is integrable around the origin, say over $[0,1]$. If ψ is decreasing on $(0,\infty)$, then $M_\psi(x)$ is the usual maximal function:

$$M_\psi(x) = \sup_{0<u<x<v<1} \frac{1}{v-u}\int_u^v \psi(y)dy\ , \qquad 0 < x < 1\ ;$$

hence $M_\psi \in L^1[0,1]$ if and only if $\psi \in L\log L$, i.e.

$$\int_0^1 \psi(x)\log^+\psi(x)dx < \infty\ .$$

Whether or not ψ is monotone, Fubini's theorem shows

$$\int_0^1 M_\psi(x)dx = \int_0^1 \psi(x)\log\frac{1}{x}\,dx\ .$$

Consequently, $I(\psi;X(\cdot,\omega)) < \infty$ a.s. for any $0 \leq \psi \in L^1$ with $\psi(x)\log\frac{1}{x} \in L^1[0,1]$, and likewise for any stochastic process which satisfies several mild conditions concerning the distribution of its derivative $X'(s,\omega)$. This is a "stochastic version" of the real-variable theorem above: only the "fixed" singularity of Λ at 0 is picked up; the others - at $\{B_i\}\backslash 0$ - depend on the specific function and will generally occur at any <u>fixed</u> point x_0 with probability 0.

Rounding out the picture, it follows from a theorem of Bulinskaya [1] that *the hypotheses of the theorem are valid for almost every sample function of a stochastic process* $X(t,\omega)$ *for which:* (i) $X(\cdot,\omega)$ *is* C^2 *a.s.,*

(ii) *for each* $0 \leq t \leq 1$, $X'(t,\omega)$ *has a density* $p_t(x)$ *which is bounded in* t *and* x.

Condition (ii) guarantees that $\{t: X'(t,\omega) = X''(t,\omega) = 0\}$ is empty a.s. Our earlier statement "$I(\psi;X(\cdot,\omega)) < \infty$ a.s. for any $\psi\downarrow$ in LlogL" can then be strengthened to "$I(\psi;X(\cdot,\omega)) < \infty$ for all $\psi\downarrow$ in LlogL, a.s.," i.e. the exceptional ω-set no longer depends on the particular ψ.

§3. Here is the proof of the theorem, which uses little else than ordinary calculus. Recall that Λ is the version of $d\lambda/dm$ given by

$$\Lambda(x) = \int_{-\infty}^{\infty} \sum_{s\in F^{-1}(\{x+y\})} |F'(s)|^{-1} \sum_{s\in F^{-1}(\{y\})} |F'(s)|^{-1} dy\ , \quad -\infty < x < \infty.$$

(i) Λ is continuous off $\{B_i\}_1^L$. I will show that Λ is continuous on $\{A_i-A_j\}_{i,j=1}^N \setminus \{B_i\}_1^L$; the proof of continuity at $x\in \mathbb{R}\setminus\{A_i-A_j\}$ goes about the same, except is easier.

Let $A_0 = \inf_s F(s)$, $A_{N+1} = \sup_s F(s)$, and $\nu(x) = \text{Card}\{s:F(s)=x\} \leq 1 + \text{Card}\{D_0\} < \infty$. First, notice that α is continuous off $\{A_i\}_0^{N+1}$ because F' and F^{-1} (defined piecewise) are continuous, and because $\nu(x+\varepsilon) = \nu(x)$ for all small ε if $x\notin\{A_i\}_0^{N+1}$.

Now fix $A_i-A_j\notin\{B_k\}_1^L$, $1 \leq i,j \leq N$, and let (k_ℓ,r_ℓ), $\ell = 1,\ldots,q$, be those pairs of integers among $\{1,2,\ldots,N\}$ for which $A_{k_\ell}-A_{r_\ell} = A_i-A_j$. Assume $F''(a_i) < 0 < F''(a_j)$; then $F''(a_k) < 0$ (resp. $F''(a_k) > 0$) for each $1 \leq k \leq N$ with $A_k = A_i$ (resp. $A_k = A_j$). It follows that $\alpha(A_j-) = \lim_{\varepsilon\downarrow 0} \alpha(A_j-\varepsilon)$ and $\alpha(A_i+) = \lim_{\varepsilon\downarrow 0} \alpha(A_i+\varepsilon)$ exist, finite. The same argument applies to each A_{k_ℓ}, A_{r_ℓ} and yields:

$$(*) \qquad \alpha(A_{k_\ell}+) < \infty\ , \quad \alpha(A_{r_\ell}-) < \infty, \qquad 1 \leq \ell \leq q.$$

Since $\Lambda(x)$ is an even function and $\{A_i-A_j\}_{i,j=1}^N \setminus \{B_i\}_1^L$ is symmetric about

0, it will be enough to check that Λ is right-continuous at $A_i - A_j$. Set $K(x,y) = \alpha(y)\alpha(y+x+A_i-A_j)$ and let

$$T_\delta = \bigcup_{k=1}^{N} (A_k - \delta, A_k + \delta), \qquad W_\delta = \bigcup_{\ell=0}^{q} (A_{r_\ell} - \delta, A_{r_\ell} + \delta) \quad ;$$

also, let $\eta > 0$ be the distance from $A_i - A_j$ to $\{A_n - A_m\}$, $(n,m) \neq (k_\ell, r_\ell)$. Then

$$\Lambda(x+A_i-A_j) = \int_{W_\delta^c \cap T_\delta} K(x,y)dy + \int_{W_\delta^c \cap T_\delta^c} K(x,y)dy + \sum_{\ell\in\Gamma} \int_{A_{r_\ell}-\delta}^{A_{r_\ell}+\delta} K(x,y)dy$$

$$\equiv P_1(x) + P_2(x) + \sum_{\ell\in\Gamma} P_{3,\ell}(x)$$

where the A_{r_ℓ}, $\ell\in\Gamma\subseteq\{1,\ldots,q\}$, are distinct and δ is small enough that the intervals $(A_{r_\ell}-\delta, A_{r_\ell}+\delta)$, $\ell\in\Gamma$, are disjoint.

If $0 \le x < \delta/2$ and $\delta < \eta/2$, $y\in W_\delta^c \cap T_\delta \Rightarrow y+A_i-A_j \in T_\delta^c \Rightarrow y+x+A_i-A_j \in T_{\delta/2}^c$. In particular, $\sup_{y\in W_\delta^c \cap T_\delta} \alpha(y+x+A_i-A_j) < \infty$ for such x's. Consequently, recalling that $\alpha\in L^1$ and α is continuous a.e., $P_1(x) \to P_1(0) < \infty$ as $x \downarrow 0$ (dominated convergence theorem). Similarly, $P_2(x) < \infty$ $\forall x \ge 0$ and

$$|P_2(x) - P_2(0)| \le \sup_{y\in T_\delta^c} \alpha(y) \int_{\infty}^{\infty} |\alpha(x+y) - \alpha(y)| dy \to 0 \text{ as } x\downarrow 0.$$

Finally,

$$P_{3,\ell}(x) = \int_{A_{r_\ell}-\delta}^{A_{r_\ell}} K(x,y)dy + \int_{A_{k_\ell}}^{A_{k_\ell}+\delta} \alpha(y-A_i+A_j)\alpha(y+x)dy$$

which converges to $P_{3,\ell}(0) < \infty$ as $x\downarrow 0$ by using (*) and arguing as above with P_1 and P_2.

Next, $F' \circ F^{-1}$ satisfies upper and lower Hölder conditions of order 1/2 at each A_i, $1 \le i \le N$. For convenience, assume $0 < a_0 < a_N < 1$; the other cases only need some additional notation. For each $a_i \in D_0$ and $s \in [0,1]$ there are numbers $\xi_s, \bar{\xi}_s$ between s and a_i with $F'(s) = F''(\xi_s)(s-a_i)$ and $F(s) - A_i = \frac{1}{2}F''(\bar{\xi}_s)(s-a_i)^2$. It follows that there are constants $0 < C_1, C_2, C_3, C_4 < \infty$ and a $\tilde{\delta}_0 > 0$ such that for each $1 \le i \le N$ and $\delta \le \tilde{\delta}_0$,

(6a) $$C_2|s-a_i| \le |F'(s)| \le C_1|s-a_i|, \qquad s \in (a_i-\delta, a_i+\delta)$$

(6b) $$C_4|s-a_i|^2 \le |F(s)-A_i| \le C_3|s-a_i|^2, \qquad s \in (a_i-\delta, a_i+\delta) .$$

Let $\hat{F}_i$ denote the inverse of F on $J_i \equiv [a_i, a_{i+1}]$, $1 \le i \le N-1$. From (6b) and the continuity of the $\hat{F}_i$'s, there is a $\delta_0 > 0$ such that, for each $1 \le i \le N-1$,

(7) $$\frac{1}{C_3}|y-A_i|^{1/2} \le |\hat{F}_i(y)-a_i| \le \frac{1}{C_4}|y-A_i|^{1/2}, \qquad y \in (A_i-\delta, A_i+\delta) \cap F(J_i)$$

$$\frac{1}{C_3}|y-A_{i+1}|^{1/2} \le |\hat{F}_i(y)-a_{i+1}| \le \frac{1}{C_4}|y-A_{i+1}|^{1/2}, \quad y \in (A_{i+1}-\delta, A_{i+1}+\delta) \cap F(J_i).$$

Let $D(i,\delta) = (A_i, A_i+\delta)$ if $F''(a_i) > 0$, $= (A_i-\delta, A_i)$ if $F''(a_i) < 0$, $1 \le i \le N$. Combining (6a) and (7), and reducing δ_0 if necessary, there are constants $0 < C_5, C_6 < \infty$ such that for each $1 \le i \le N-1$, $\delta \le \delta_0$,

(8) $$C_5|y-A_i|^{1/2} \le |F'(\hat{F}_i(y))| \le C_6|y-A_i|^{1/2}, \qquad y \in D(i,\delta)$$

and likewise (in case $a_N = 1$) with A_i, $D(i,\delta)$ replaced by A_{i+1}, $D(i+1,\delta)$. We can assume that for each i,j and each small δ, either $D(i,\delta) = D(j,\delta)$ or $D(i,\delta) \cap D(j,\delta) = \emptyset$. Defining $J_0 = [0,a_1]$, $J_N = [a_N,1]$ and the corresponding inverses $\hat{F}_0$, $\hat{F}_N$, it is clear that (8) extends to $F' \circ \hat{F}_0$ and $F' \circ \hat{F}_N$ at the appropriate places. (By the way, both inequalities in (8) depend on $F'' \ne 0$ on D_0.)

(ii) $\underline{\varlimsup_{x\to B_i} \Lambda(x)/-\log|x-B_i| > 0,\ 1 \le i \le L.}$ Suppose $B_i = A_\ell - A_k$, $1 \le \ell, k \le N$, and $F''(a_k) < 0$, $F''(a_\ell) < 0$; the other case, namely $F''(a_\ell)$, $F''(a_k) > 0$ is the same.

$$\Lambda(x+B_i) = \int_{-\infty}^{\infty} \alpha(y+x+A_\ell)\alpha(y+A_k)dy$$
$$\ge \int_{-\varepsilon}^{-|x|} \alpha(y+x+A_\ell)\alpha(y+A_k)dy, \qquad |x| < \varepsilon .$$

Now for ε small, the conditions $|x| < \varepsilon$ and $-\varepsilon < y < -|x|$ together imply that $y+x+A_\ell \in D(\ell,\delta_0)$ and $y+A_k \in D(k,\delta_0)$. Consequently,

$$\Lambda(x+B_i) \ge C_5^2 \int_{-\varepsilon}^{-|x|} |y+x|^{-1/2}|y|^{-1/2}dy$$

$$= C_5^2 \log\left|\frac{2\sqrt{\varepsilon^2-\varepsilon x} + 2\varepsilon - x}{2\sqrt{x^2-|x|x} + 2|x| - x}\right|$$

$$\ge C \log \frac{1}{|x|} ,$$

for all small x, for some $C > 0$.

(iii) $\underline{\Lambda(x) \le \text{const.} \times [1 + \sum_1^L |\log|x-B_i||]\ \forall x.}$ (This is equivalent to the "$\overline{\lim}$" part of (3a).) Evidently,

$$\alpha(y) = \sum_{i=0}^{N+1} 1_{F(J_i)}(y)|F'\circ\hat{F}_i(y)|^{-1} .$$

Off T_δ, α is bounded. Let $y\in T_\delta$, say $A_i-\delta < y < A_i+\delta$, $y\in F[0,1]$. Keeping (8) in mind and that non-identical $D(j,\delta)$'s are disjoint:

$$\alpha(y) = \sum_{j:A_i=A_j} 1_{F(J_j)}(y)|F'\circ\hat{F}_j(y)|^{-1} + \sum_{j:A_i\ne A_j} 1_{F(J_j)}(y)|F'\circ\hat{F}_j(y)|^{-1}$$
$$\le \nu(y)C_5|y-A_i|^{-1/2} + \nu(y)\sup_{s\in H_\delta}|F'(s)|^{-1},\ H_\delta = F^{-1}[\bigcap_{i=1}^{N}(A_i-\delta, A_i+\delta)^c]$$
$$\le \text{const.} \times [1 + \sum_{i=1}^{N} |y-A_i|^{-1/2}] .$$

Let $V = F[0,1]$ and $\mathcal{U} = \bigcup_{i=1}^{N} V - A_i$, which is bounded.

$$\begin{aligned}\Lambda(x) &= \int_V \alpha(y)\alpha(x+y)dy \\ &\le \text{const.}\times[1 + 2\sum_{i=1}^{N}\int_V |y-A_i|^{-1/2}dy + \sum_{i,j=1}^{N}\int_V |y-A_i|^{-1/2}|y+x-A_j|^{-1/2}dy] \\ &\le \text{const.}\times[1 + \sum_{i,j=1}^{N}\int_{\mathcal{U}} |y+A_j-A_i|^{-1/2}|y+x|^{-1/2}dy] \\ &\le \text{const.}\times[1 + \sum_{i,j=1}^{N} |\log|x-(A_i-A_j)||]\end{aligned}$$

since $\int_{\mathcal{U}} |y+\varepsilon|^{-1/2}|y|^{-1/2}dy = O(\log \frac{1}{|\varepsilon|})$ as $\varepsilon \to 0$.

As for (3b), let $H(x) = 1 + \sum_{i=1}^{L} |\log|x-B_i||$. Then $I(\psi;F) < \infty$ $\forall\psi\in L^1(Hdm)$ if and only if

$$\int_{-\infty}^{\infty} \frac{\psi(x)}{H(x)} \Lambda(x)dx < \infty \quad \forall\psi\in L^1(dx) ,$$

if and only if $\text{ess}_x\sup \frac{\Lambda(x)}{H(x)} < \infty$. Since Λ, H are continuous from $\mathbb{R}$ to $\mathbb{R}\cup\{\infty\}$, this is the same as $\sup_x \frac{\Lambda(x)}{H(x)} < \infty$. In other words, the "$\overline{\lim}$" part of (3a) is <u>equivalent</u> to "$I(\psi;F) < \infty$ $\forall\psi\in L^1(Hdm)$." Now if $I(\psi;F) < \infty$ and $\psi\in L^1(dx)$, then it is easy to see, using the "$\underline{\lim}$" part of (3a) that ψH is integrable. The last statement of the theorem follows from (3b) and the aforementioned fact that $I(\psi;F) < \infty$ and $\psi\downarrow$ imply $\psi\in L^1[0,1]$.

§4. Let $F(t) = t^2$. Then $D_0 = \{B_i\} = \{0\}$ and

$$\Lambda(x) = \frac{1}{2}\log\{\frac{1 + \sqrt{1-|x|}}{\sqrt{|x|}}\} , \quad |x| < 1 .$$

For $F(t) = \sin 2\pi t$, $\Lambda(x)$ is an elliptic integral (of the first kind). I would give more examples, especially in "closed form" and with $L > 1$, if I could; the computations (even for F a third degree polynomial) are formidable.

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Let $F(t)$, $0\le t\le 1$, be a real function with two continuous derivatives such that $\{F'=F''=0\}$ is empty. Then $B \to$ meas. $\{(s,t):F(s)-F(t)\in B\}$ in absolutely continuous; its density is continuous on $\mathbb{R}\setminus\{B_i\}$, $\{B_i\} \equiv \{y: y=F(t_1)-F(t_2),\ F'(t_1) = F'(t_2) = 0,\ F''(t_1)F''(t_2) > 0\}$, and has a logarithmic singularity at each B_i.

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