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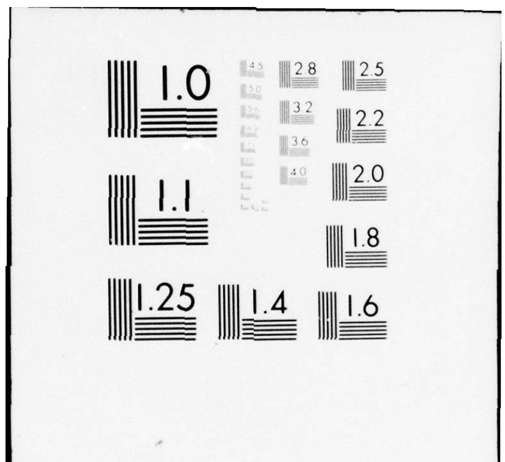

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OPTIMAL LINEAR CONTROL (CHARACTERIZATION OF MULTI-INPUT SYSTEMS)

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Control theory	Multivariable control systems	

20. ABSTRACT (CONTINUE ON REVERSE SIDE IF NECESSARY AND IDENTIFY BY BLOCK NUMBER)

This report presents results on characterizations of optimal multi-input systems. The major results are: (1) equivalence of quadratic state weighting to quadratic weighting of m responses where m is the dimension of the input, (2) characterization of quadratic performance indexes in terms of asymptotic eigenvalues and eigenvectors, (3) characterization of the effects of compensators on high frequency attenuation, and (4) development of a procedure for constructing quadratic performance indexes based on modal design specifications.

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SECTION I

INTRODUCTION AND SUMMARY

This research program's goal is the advancement of optimal linear control design technology to the point where it yields operational control laws that meet conventional design specifications. The results for single input systems are reported in [1]. This report presents results on the characterization of optimal multi-input systems.

The major results established during this phase of the program were:

- Equivalence of quadratic state weighting to quadratic weighting of m responses where m is the dimension of the input,
- Characterization of quadratic performance indices in terms of asymptotic eigenvalues and eigenvectors,
- Characterization of the effect of compensators on high frequency attenuation properties, and
- Development of a procedure for constructing quadratic performance indices based on modal design specifications.

The equivalence result provides a direct generalization of the concept of model response weighting[1] for single input systems and hence serves as an aid in the interpretation of the asymptotic properties of optimal systems. This result also provides a general form for performance indices which have fewer parameters than the state form and hence possess less redundancy.[2]

The asymptotic eigenvalue/eigenvector characterization provides important insight for the weight selection process of linear optimal design. This characterization provides a complete description of the quadratic performance index in terms of nm independent optimization parameters where n is the dimension of the state vector.

High frequency attenuation properties are especially important for operational control systems. Optimal systems without compensators are shown to have first order attenuation characteristics. This may be insufficient for many applications. The characterization of compensator effects on attenuation provides a guide for selection of the proper control system structure to meet design specifications.

The above results are derived in Section II. Section III presents the development of the procedure for constructing quadratic performance indices from design specifications. Application of this procedure to an illustrative example is also described in Section III.

Since transmission zeros of multivariable systems play a key role in the asymptotic characterization, Appendix A summarizes the important concepts of multivariable system zeros.

SECTION II

CHARACTERIZATION OF OPTIMAL MULTI-INPUT SYSTEMS

We consider the linear controllable system

$$\dot{x} = Fx + Gu, \ x(0) = x_o \tag{1}$$

where x is an n-vector and u is an m-vector* with m generally greater than one and less than or equal to n. We are interested in basic characteristics of linear control laws that are optimal with respect to a quadratic performance index of the form

$$J = \int_o^\infty [x' Qx + u' Ru]dt, \ Q \geq 0, \ R > 0 \tag{2}$$

with (Q, F) observable. It is well known that the optimal control is given by

$$u = Kx \tag{3}$$

where the gain matrix K satisfies

$$K = -R^{-1}G' P \tag{4}$$

with P being the positive definite symmetric solution of the Riccati equation

$$PF + F' P - PGR^{-1}G' P + Q = 0 \tag{5}$$

*The components of u are assumed to be independent, i.e. Rank (G) = m.

We will first present the known characterization in terms of the return difference. Then we present a new result on the equivalence of quadratic performance indices of the form (2) to the form

$$\tilde{J} = \int_0^{\infty} (r' r + u' Ru)dt \tag{6}$$

where

$$r = Cx \tag{7}$$

is an m-vector. Optimal systems are then characterized in terms of asymptotic eigenvalues and eigenvectors. A unique relationship between these asymptotic properties and the weighting matrices in the performance index is then established. Finally optimal systems are characterized in terms of their high-frequency attenuation properties.

THE RETURN DIFFERENCE EQUATION

The Riccati equation (5) may be rewritten using (4) as

$$P(sI-F) + (-sI-F')P + K'RK = Q \tag{8}$$

Multiplying by $G' (-sI-F')^{-1}$ on the left and $(sI-F)^{-1}G$ on the right, gives

$$G'(-sI-F')^{-1}PG + G'P(sI-F)^{-1}G + G'(-sI-F')^{-1}K'RK(sI-F)^{-1}G$$
$$= G'(-sI-F')^{-1} Q (sI-F)^{-1}G \tag{9}$$

Since $PG = -K'R$, adding R to both sides, and rearranging yields*

$$[I-K(-sI-F)^{-1}G]' R [I-K(sI-F)^{-1}G]$$

$$= R + G'(-sI-F)^{-1} Q(sI-F)^{-1}G \tag{10}$$

where the return difference matrix is the mxm matrix

$$T(s) = I-K(sI-F)^{-1}G \tag{11}$$

For $s = j\omega$, the last term of the sum on the right hand side of (10) is non-negative for all real ω. Thus, optimal systems may be characterized by the inequality

$$[T(-j\omega)]'RT(j\omega) \geq R \text{ for all real } \omega \tag{12}$$

EQUIVALENCE OF $x'Qx$ TO $x'C'Cx$ IN THE PERFORMANCE INDEX

Assume that we are given the system (1) and performance index (2). Let us first make the change of control variables.

$$v = \sqrt{R}u \tag{13}$$

where $\sqrt{R}$ is the positive definite symmetric square root of R. Then (1) and (2) may be rewritten as

$$\dot{x} = Fx + \hat{G}v, \; x(0) = x_o \tag{14}$$

$$J = \int_o^\infty [x'Qx + v'v]dt \tag{15}$$

where $\hat{G} = G(\sqrt{R})^{-1}$. The optimal control is

---

*I is used in equation (10) and subsequent equations to denote the identity matrices of appropriate dimensions.

$$v = \hat{K}x = (\sqrt{}R)Kx \tag{16}$$

The return difference matrix for the system (14) with control (16) is

$$\hat{T}(s) = I - \hat{K}(sI-F)^{-1}\hat{G} \tag{17}$$

and the return difference equation is

$$[\hat{T}(-s)]' \hat{T}(s) - I = \hat{G}'(-sI-F')^{-1} Q(sI-F)^{-1}\hat{G} \tag{18}$$

For $s = j\omega$, the right hand side of (18) is non-negative for all real ω. According to Molinari [2], Popov [3] has shown that $[\hat{T}(-j\omega)]'\hat{T}(j\omega) - I \geq 0$ implies the existence of an mxm matrix C such that

$$[\hat{T}(-s)]'\hat{T}(s)-I = \hat{G}'(-sI-F')^{-1} C'C(sI-F)\hat{G} \tag{19}$$

For such a matrix C, let

$$v = \tilde{K}x \tag{20}$$

be the optimal control for the system (14) and the performance index

$$\mathcal{J} = \int_0^{\infty} (x'C'Cx + v'v)dt \tag{21}$$

The return difference equation for the system (14) with the control given by (20) is

$$[\tilde{T}(-s)]'\tilde{T}(s)-I = \hat{G}'(-sI-F')^{-1} C'C(sI-F)^{-1}\hat{G} \tag{22}$$

where

$$\tilde{T}(s) = I-\tilde{K}(sI-F)^{-1}\hat{G}. \tag{23}$$

Thus, from (17), (19), (22) and (23), we have

$$[\hat{T}(-s)]'\hat{T}(s) = [I - \hat{K}(-sI-F)^{-1}\hat{G}]'[I - \hat{K}(sI-F)^{-1}\hat{G}]$$

$$= [I - \tilde{K}(-sI-F)^{-1}\hat{G}]'[I - \tilde{K}(sI-F)^{-1}\hat{G}] \qquad (24)$$

The factorization of $[\hat{T}(-s)]\ \hat{T}(s)$ was shown by Youla [4, Theorem 2] to be unique to within a constant unitary matrix, which implies

$$I - \tilde{K}(sI-F)^{-1}\hat{G} = M[I - \hat{K}(sI-F)^{-1}\hat{G}] \qquad (25)$$

where M is constant and $M'M = I$. Equation (25) may be rewritten as

$$I-M = (\tilde{K}-M\hat{K})\,(sI-F)^{-1}\hat{G} \qquad (26)$$

The left hand side of (26) is independent of s, and the limit of the right hand side of (26) as s tends to infinity is

$$\lim_{s\to\infty} (\tilde{K}-M\hat{K})\,\hat{G}/s = 0 \qquad (27)$$

which implies $M = I$. Thus,

$$0 = (\tilde{K} - \hat{K})\,(sI-F)^{-1}G \qquad (28)$$

which is an identity in s. For $|s|$ large, $(sI-F)^{-1}$ may be written as

$$(sI-F)^{-1} = s^{-1} \sum_{j=o}^{\infty} (Fs^{-1})^j \qquad (29)$$

Therefore,

$$(\tilde{K} - \hat{K})F^j\hat{G} = 0,\ j = 0, 1, 2, \ldots \qquad (30)$$

Equation (30) together with the controllability of $(F, \hat{G})$ implies that $\tilde{K} = \hat{K}$. This establishes that the performance indices (15) and (21) are equivalent in the sense that they yield the same optimal controller. Writing these performance indices in terms of the original control vector, u, gives the equivalence of the performance index (2) to (6) for the system (1) with r defined by (7) for some mxn matrix C.

ASYMPTOTIC EIGENVALUE/EIGENVECTOR CHARACTERIZATION

For this discussion we consider the system (1) and the quadratic performance index in the form

$$J = \int_0^\infty (r'r + \rho u'Ru)dt \tag{31}$$

where r is given by (7) and ρ is a positive real scalar parameter. We are interested in the asymptotic properties of the optimal closed-loop system as the parameter ρ tends to zero. A complete characterization will be given for the case when the pair (C, F) is observable and the matrix CG has rank m.

The optimal control is u = Kx where the feedback gain matrix is

$$K = -\rho^{-1} R^{-1}G'P \tag{32}$$

and $P = P' > 0$ is the solution of the Riccati equation

$$PF + F'P - \rho^{-1}PGR^{-1}G'P + C'C = 0 \tag{33}$$

The return difference equation for the optimal system is

$$[T(-s)]'\rho R\, T(s) = \rho R + G'(-sI-F')^{-1} C'C(sI-F)^{-1}G \tag{34}$$

where T(s) is the return difference matrix (11).

The determinant of the return difference matrix is

$$T(s) \quad = \frac{|sI-F-GK|}{|sI-F|} \tag{35}$$

Taking the determinant of (34) and rearranging yields

$$\phi_c(s)\,\phi_c(-s) = \phi_o(s)\,\phi_o(-s)\,|\,I + \frac{1}{\rho}R^{-1}\,H'(-s)\,C'CH(s)\,| \tag{36}$$

where

$$\phi_o(s) = |sI-F|, \quad \phi_c(s) = |sI-F-GK|$$

and $\quad H(s) = (sI-F)^{-1}G$. The polynomials, $\phi_o(s)$ and $\phi_c(s)$, are the open-loop and closed-loop characteristic polynomials respectively. Equation (36) may be analyzed to determine the asymptotic nature of the closed-loop eigenvalues as ρ tends to zero.

The right hand side of (36) is a polynomial in ρ^{-1} which may be written as

$$\psi_o(-s)\,\psi_o(s)\,\sum_{j=o}^{m} a_j(s)\rho^{-j} \tag{36}$$

where $a_o = 1$ and for $j = 1, 2, \ldots, m$,

$$a_j(s) = \Sigma[j^{th} \text{ ordered principal minors of } R^{-1}H'(-s)C'CH(s)] \tag{37}$$

We note that each $a_j(s)$ is a ratio of polynomials in s^2 with the denominator equal to $\psi_o(-s)\,\psi_o(s)$. Thus

$$\psi_o(s)\,\psi_o(-s)\,a_j(s) = n_j(s^2) \tag{38}$$

with the maximum degree of n_j being n-j.

Kwakernaak describes the asymptotic properties of zeros of general polynomials of the form (36) as ρ tends to zero in reference 5. Let us here consider the special case in which the degree of n_m is n-m and the rank of CG is m. In this case, as ρ tends to zero, n-m roots of $\psi_c(s)$ approach the roots of $n_m(s^2)$ which have negative real parts. These are the zeros or the left half plane mirror images of the zeros of the determinant:

$$|C(sI-F)^{-1}G| \tag{39}$$

The zeros of (39) are the transmission zeros associated with the input-output system described by (1) and (7). Appendix A summarizes recent research on system zeros. The remaining m closed-loop eigenvalues approach infinity in multiple Butterworth patterns. To examine the nature of these eigenvalues, we consider $\phi_c(-s)\,\phi_c(s) = 0$ for $|s|$ sufficiently large so that $\phi_o(-s)\,\phi_o(-s) \neq 0$. Then from (36) we are interested in the equation

$$\begin{aligned} 0 &= |I + \rho^{-1}R^{-1}H'(-s)\,C'CH(s)| \\ &= |I + \rho^{-1}R^{-\frac{1}{2}}H'(-s)\,C'CH(s)R^{-\frac{1}{2}}| \end{aligned} \tag{40}$$

For $|s|$ large (40) may be written as:

$$|I - \frac{1}{\rho}s^{-2}(D + 0(|s|^{-2}))| = 0 \tag{41}$$

where $D = R^{-\frac{1}{2}}G'C'CGR^{-\frac{1}{2}}$

and $0(|s|^{-2})$ indicates terms which tend to zero as $|s|^{-2} \to \infty$.

Thus

$$s^2 = \frac{1}{\rho} \lambda(D) + 0(|s|^{-2}) \tag{42}$$

where λ(Matrix) denotes the eigenvalues of the matrix, and in our case $D > 0$ so that $\lambda(D)$ are real and positive. Hence $\lambda[D + 0(|s|^{-2})] = \lambda(D)[1 + 0(|s|^{-2})]$ so that as $\rho \to 0$, m poles approach infinity along the real axis at the rate of $\rho^{-\frac{1}{2}}$.

Kwakernaak [5] points out that if the determinant (39) has less than n-m zeros or if the rank of CG is less than m then the eigenvalues which go to infinity may group into several Butterworth patterns with different growth rates.

It is somewhat disconcerting that when C is of rank m at most n-m asymptotic eigenvalue locations may be specified in the finite plane. It is possible to have more than n-m finite asymptotic eigenvalue locations by choosing C to have rank less than m. Such an input-output system is called degenerate (see Appendix A). If C is chosen to be a qxn matrix where $q < m$, then equation (36) gives

$$\begin{aligned} \phi_c(s)\phi_c(-s) &= \phi_o(s)\phi_o(-s) \;\left| I_m + \frac{R^{-1}}{\rho} H'(-s)C'CH(s) \right| \\ &= \phi_o(s)\phi_o(-s) \;\left| I_q + \frac{1}{\rho} CH(s)\, R^{-1}\, H'(-s)C' \right| \\ &= \phi_o(s)\phi_o(-s) \;\{ I + \cdots + \rho^{-q}\, |CH(s)R^{-1}\, H'(-s)C'| \} \end{aligned} \tag{43}$$

The elements of $CH(s)R^{-\frac{1}{2}}$ are ratios of polynomials, the denominators being of degree n and the numerators of degree less than or equal to n-1. The determinant, $|CH(s)R^{-1}H'(-s)C|$ is a ratio of polynomials, the denominator being of degree 2qn and the numerator being of degree at most 2q(n-1). The degree of $\psi_o(s)\psi_o(-s)$ is 2n. Cancellation of the denominator with $\psi_o(s)\psi_o(-s)$ and common factors in the numerator takes place in (43). Thus the coefficient of ρ^{-q} in (43) is a polynomial of degree at most:

$$2q(n-1) + 2n - 2qn = 2(n-q) \tag{44}$$

The matrix C may be chosen to achieve the maximal degree, 2(n-q). In this case there are n-q finite asymptotic pole locations. It appears that this may be of some advantage as it allows more poles to be specified. If a vector C were used, then q = 1 and it is possible to have n-1 finite asymptotic poles. Unfortunately, in general when $q < m$, all n-q poles may not be specified arbitrarily. Consider the following example

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R = I_2 \qquad C = [c_1\ c_2\ c_3\ c_4]$$

Then

$\psi_c(s)\psi_c(-s) =$

$$s^8 + \rho^{-1} s^8 C \begin{bmatrix} \frac{1}{s}2 & 0 \\ \frac{1}{s} & 0 \\ 0 & \frac{1}{s}2 \\ 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} \frac{1}{s}2 & \frac{-1}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{s}2 & \frac{-1}{s} \end{bmatrix} C'$$

$$= s^8 + \rho^{-1} s^8 C \begin{bmatrix} \frac{1}{s}4 & \frac{-1}{s}3 & 0 & 0 \\ \frac{1}{s}3 & \frac{-1}{s}2 & 0 & 0 \\ 0 & 0 & \frac{1}{s}4 & \frac{-1}{s}3 \\ 0 & 0 & \frac{1}{s}3 & \frac{-1}{s}2 \end{bmatrix} C'$$

$$= s^4 \{s^4 + \rho^{-1} [(c_1^2 + c_3^2) - (c_2^2 + c_4^2)s^2]\}$$

Thus while three poles will be finite as $\rho \to 0$, two will be at zero independent of the choice of C. In this example the pair (C, F) is not observable, but similar examples can be constructed with an observable pair.

No general characterization of the nature of the pole restriction has been found. Apparently the only general statement we can make is that for a qxn matrix C, where q<m, the n-q finite poles cannot be arbitrarily specified.

For an mxn matrix C it is well known that the zeros of (39) may be selected arbitrarily by appropriately choosing C. There are a multitude of methods for selecting C to achieve desired zeros. The difficulty is that the choice of C is not unique since only n-m parameters are needed to specify the n-m zeros. This difficulty with nonuniqueness will be removed by considering the freedom to specify eigenvectors corresponding to these asymptotic eigenvalues. The characterization will be completed with consideration of the m eigenvalues that approach infinity and their associated eigenvectors.

We begin the discussion of eigenvectors with the following result established by Moore [6]:

Lemma 1

The vectors x_i and associated distinct complex numbers s_i are closed-loop eigenvectors and eigenvalues, respectively, of the system (1) with feedback (3) if and only if there exist m-vectors μ_i such that

$$(s_i I - F)^{-1} G \mu_i = x_i \tag{45}$$

$$\mu_i = Kx_i \quad i = 1, 2, \ldots, n \tag{46}$$

This lemma shows that the eigenvectors achievable with linear state variable feedback are confined to m-dimensional subspaces determined by their associated eigenvalues. This eigenvector adjustment plus arbitrary pole placement is precisely the freedom offered by linear state variable feedback.

The m-vectors, μ_i, may be characterized with respect to the return difference as follows:

Lemma 2

The m-vectors μ_i are identical (except possibly for magnitude) to the nonzero vectors ν_i determined by

$$T(s_i)\,\nu_i = 0,\ i = 1, 2, \ldots, n \tag{47}$$

To show this, note that the μ_i's from (45) and (46) satisfy (47), i.e.

$$\begin{aligned} T(s_i)\,\mu_i &= \mu_i - K(s_i I - F)^{-1} G\,\mu_i \\ &= \mu_i - Kx_i \\ &= 0 \end{aligned} \tag{48}$$

Moreover, solutions of (47) are unique except for magnitude as long as the s_i's are distinct. For this case, the s_i's, interpreted as transmission zeros of T(s), have unit geometric and algebraic multiplicity and, hence, the rank deficiency of T(s) at s_i is one. It then follows that with appropriate normalization

$$\nu_i = \mu_i \quad i = 1, 2, \ldots, n \tag{49}$$

For the case in which the rank of CG is m, we may summarize the asymptotic eigenvalue/eigenvector properties in the form of a theorem. With the rank of CG being full and hence the rank of the mxn matrix C being full there is no loss in generality to normalize C to be of the form

$$C = W_o C_o \tag{50}$$

where W_o is a nonsingular mxm matrix and

$$C_o = [C_{11} \;\vdots\; I_m] \tag{51}$$

Such a normalization may be achieved by a state transformation which reorders the state variables. With this notation we may state the following theorem.

Theorem 1

Let $r = Cx$ with C satisfying the conditions

i) the rank of CG is m

ii) C is in the form described by equations (50) and (51)

iii) the zeros of $|C(sI-F)^{-1}G|$ are distinct and have negative real parts.

Then, the optimally controlled system (1) with respect to the performance index (31) has:

n-m asymptotically finite eigenvalues

$$s_i^o, \; i = 1, 2, \ldots, n-m \tag{52}$$

and m asymptotically infinite eigenvalues

$$s_i^\infty \rho^{-\frac{1}{2}}, \; i = 1, 2, \ldots, m \tag{53}$$

where the s_i^o have negative real parts and if for some i, s_i^o is not real there is a s_j^o which is the complex conjugate of s_i^o and the s_i^∞ are real and negative. The s_i^o's are distinct and have associated eigenvectors

$$x_i^o = (s_i^o I-F)^{-1} G \, \nu_i^o \tag{54}$$

where the s_i^o and ν_i^o are defined by

$$C(s_i^o I-F)^{-1} G \, \nu_i^o = 0, \; \nu_i^o \neq 0 \tag{55}$$

The eigenvectors associated with the asymptotically infinite eigenvalues are

$$x_i = G\, \nu_i^\infty \tag{56}$$

where the s_i^∞ and ν_i^∞ are defined by

$$R = (N')^{-1} S^{-2} N^{-1} \tag{57}$$

$$W_o' W_o = [(C_o GN)']^{-1} (C_o GN)^{-1} \tag{58}$$

with

$$N \triangleq [\nu_i^\infty\ \nu_2^\infty \cdots \nu_m^\infty] \tag{59}$$

$$S \triangleq \mathrm{diag}\,(s_1^\infty\ s_2^\infty \cdots s_m^\infty) \tag{60}$$

This theorem combines known results from the literature [5,7] concerning the asymptotic behavior of eigenvalues with new results for the behavior of the corresponding eigenvectors. The following proof emphasizes the new results.

Proof of Theorem:

From Lemma 2, the optimal controller's eigenvalues and eigenvectors are characterized by (47) which may be premultiplied by $\rho T\,(-s_i)R$ yielding

$$\rho T'(-s_i) R\, T(s_i)\, \nu_i = 0 \tag{61}$$

Then from the return difference equation (34), we have

$$[\rho R + H'(-s_i)\, C'CH(s_i)]\, \nu_i = 0 \tag{62}$$

We are interested in the asymptotic behavior of the pairs (s_i, ν_i) as ρ tends to zero. In [7] it is shown that if the rank of CG is m, then there will be 2(n-m) such pairs for which the s_i values remain finite. The s_i values will be symmetric about the imaginary axis. Those in the left half-plane will be the eigenvalues of the stable closed-loop system. Taking the limit of (62) as ρ approaches zero yields

$$H'(-s_i)\, C'CH(s_i)\nu_i = 0 \tag{63}$$

Condition iii) and the symmetry condition on the s_i values imply that $|H'(-s_i)C'| \neq 0$ for s_i with negative real parts. Hence from (63) we have for the stable s_i,

$$C\, H(s_i)\nu_i = 0, \ \nu_i \neq 0, \ i = 1, 2, \ldots, n-m \tag{64}$$

which establishes equation (55), and equation (54) follows from Lemma 1.

It is known[5] that the remaining m closed-loop eigenvalues approach infinity in m first-order Butterworth patterns. Their corresponding eigenvectors can be determined by using a power series expansion in s_i^{-1} of equation (62), i.e.

$$\begin{aligned} 0 &= \{\rho R - (s_i)^{-2}[G'(I-F's_i^{-1} + \ldots)C']C(I+F\, s_i^{-1} + \ldots)G\}\, \nu_i \\ &= \rho\{R - (\rho^{\frac{1}{2}} s_i)^{-2}[(CG)'CG + 0(s_i^{-2})]\}\, \nu_i \\ &= \rho\{R - (\sigma_i)^{-2}\, [(CG)'CG + 0(\rho\sigma_i^{-2})]\}\, \nu_i \end{aligned} \tag{65}$$

where $\sigma_i = s_i\, \rho^{\frac{1}{2}}$. As ρ tends to zero, σ_i approaches s_i^{∞} and the asymptotically infinite pairs satisfy

$$0 = [R - (s_o^{\infty})^{-2}\, (CG)'CG]\nu_i^{\infty}, \ \nu_i^{\infty} \neq 0 \tag{66}$$

From this equation and condition i) we can derive:

$$0 = \{[(CG)']^{-1} R(CG)^{-1} - (s_i^\infty)^{-2} I\} CG\ \nu_i^\infty,\ \nu_i^\infty \neq 0,\ i = 1, 2, \ldots, m \tag{67}$$

This equation implies that the symmetric matrix $[(CG)']^{-1} R(CG)^{-1}$ has eigenvalues $(s_i^\infty)^{-2}$ and orthogonal eigenvectors CG ν_i. If we normalize the ν_i^∞ by the condition

$$|CG\ \nu_i^\infty| = 1,\ i = 1, 2, \ldots, m \tag{68}$$

we have

$$(CGN)'CGN = I \tag{69}$$

and

$$R = (CG)'CGN\ S^{-2}\ (CGN)'CG \tag{70}$$

where the matrices N and S are defined in (59) and (60). From condition ii), equations (69) amd (70) may be rewritten as

$$W_o' W_o = [(C_oGN)']^{-1} (C_oGN)^{-1}$$

and

$$\begin{aligned} R &= (C_oG)' W_o' W_o C_o GNS^{-2}(C_oGN)'W_o' W_o C_o G \\ &= (C_oG)' [(C_oGN)']^{-1} S^{-2}(C_oGN)^{-1} C_oG \\ &= (N')^{-1} S^{-2} N^{-1} \end{aligned}$$

establishing equations (58) and (57). Equation (56) which characterizes the corresponding eigenvectors follows directly from Lemma 1 for $s_i = s_i^\infty \rho^{-\frac{1}{2}}$ tending to infinity in magnitude as ρ tends to zero. This completes the proof of Theorem 1.

This theorem provides important new insights for the weight selection process in linear optimal design.

We note first that the asymptotic properties provide a complete characterization of the performance index (31) in terms of nm free parameters. The finite modes are characterized by (n-m) eigenvalues (s_i^o) plus (m-1) (n-m) parameters for their associated normalized directions (ν_i^o). We will show in Section III that these provide a unique definition of C_o via equation (55).

Similarly, the asymptotically infinite modes are characterized by (m-1) parameters for the normalized eigenvalues (s_i^∞) and (m-1) (m) for their associated normalized directions (ν_i^∞). These provide unique definitions of R_o and W_o via equations (57) and (58). Together with the scalar ρ, these parameters specify the criterion completely. These observations are summarized in Table 1. Note that the scalar ρ is the only remaining "trade-off" parameter in the specification. It indexes the optimal eigenvalue and eigenvector loci implicit in C_o, R_o and W_o, thus trading off control energy against the degree to which the asymptotic properties are achieved.

We note next that the asymptotically finite eigenvalues (s_i^o) correspond to transmission zeros[8] of the square response transfer matrix CH(s) and that the vectors (x_i^o, ν_i^o) are "state- and control-zero directions [9]" associated with these transmission zeros. Moreover, for each fixed finite s_i^o, the eigenvectors achievable with linear optimal design correspond precisely to those achievable with arbitrary linear state feedback (Lemma 1). The existence of zeros and zero directions in CH(s) are algebraic

Table 1. Performance Index Specified in Terms of Asymptotic Properties

Design Parameters	Number of Parameters	Defined Quantities
Finite Modes eigenvalues s_i^o directions ν_i^o	 $(n-m)$ $(m-1)(n-m)$	C_o
Infinite Modes eigenvalues s_i^∞ directions ν_i^∞	 $(m-1)$ $m(m-1)$	R_o, W_o
Control Weight	1	ρ
Total	nm	J

structural properties of the performance index which must be "built into" the criterion if we wish to achieve desirable dominant modes and mode distributions in the final closed-loop system.

As a final observation, we note that Theorem 1 provides for the first time an explicit way to adjust the behavior of asymptotically inifinite modes. These modes have particular significance to designers because they govern the characteristics of control actuation devices. Note that the eigenvectors of asymptotically infinite modes are in the range space of G. Hence, if the state model includes actuator dynamics with G nonzero only for actuator rows, the asymptotically infinite modes are actuator modes. According to equations (53) and (56) - (60), these modes have bandwidth ratios and cross couplings specified entirely by computable matrices R_o and W_o. In the past, designers were forced to struggle iteratively to achieve bandwidth ratios and couplings consistent with hardware constraints.

HIGH-FREQUENCY ATTENUATION CHARACTERISTICS

We will first show that optimal multi-input systems have the property that the return ratio matrix at the input possesses a first order attenuation characteristic for large frequencies. Then we show that including a compensator increases the order of attenuation at the actuator.

For the system (1) with control being optimal with respect to (2), i.e. the control u is determined by (3), (4), and (5), the return ratio matrix at the input u is:

$$A_u(s) = K(sI-F)^{-1}G \tag{71}$$

Expanding the right hand side of (71) in a power series in s^{-1} for $|s|$ large yields

$$A_u(s) = \sum_{j=o}^{\infty} s^{-(1+j)} K F^j G \tag{72}$$

The matrix $KG = -R^{-1} G'PG$ is nonsingular since R and $G'PG$ are positive definite. Thus, we may write (72) as

$$A_u(s) = s^{-1} KG [I + 0(s^{-1})] \tag{73}$$

for $|s|$ large. This demonstrates that $A_u(s)$ possesses the first order attenuation characteristic.

Now let us consider the case in which a compensator is included. For this purpose let us suppose the vector u consists of two subvectors u_1 and u_2 and that the compensator is inserted in the u_2 path. The block diagram of such a system is shown in Figure 1. The differential equation description for the system is

$$\begin{aligned} \dot{x} &= Fx + G_1 u_1 + G_2 u_2 \\ \dot{x}_c &= F_c x_c + G_c v \\ u_2 &= Hx_c \end{aligned} \tag{74}$$

The performance index for this system is

$$J = \int_o^{\infty} \{ [x', x_c'] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + [u_1', v'] \begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ v \end{bmatrix} \} dt \tag{75}$$

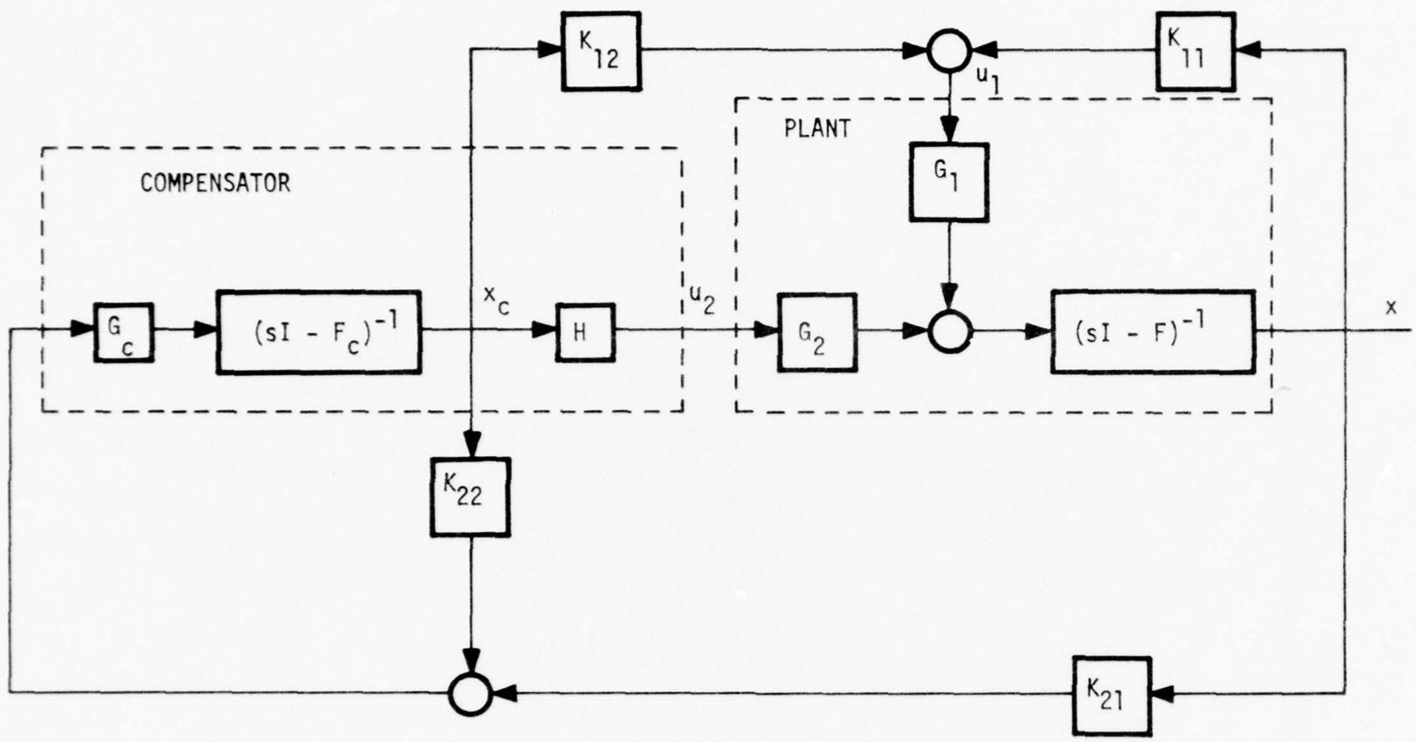

Figure 1. Closed-Loop System with Compensator

The optimal control is

$$\begin{bmatrix} u_1 \\ v \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{76}$$

with

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = -\begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1' & 0 \\ 0 & G_c' \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \tag{77}$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \begin{bmatrix} F & G_2H \\ 0 & F_c \end{bmatrix} + \begin{bmatrix} F & G_2H \\ 0 & F_c \end{bmatrix}' \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & G_c \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix}^{-1} \begin{bmatrix} G_1' & 0 \\ 0 & G_c' \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \tag{78}$$

Note that

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}' & R_{22} \end{bmatrix}^{-1} = \begin{bmatrix} R_1^{-1} & -R_1^{-1}R_{12}R_{22}^{-1} \\ R_{22}^{-1}R_{12}'R_1^{-1} & R_{22}^{-1} + R_{22}^{-1}R_{12}'R_1^{-1}R_{12}R_{22}^{-1} \end{bmatrix} \tag{79}$$

where $R_1 = R_{11} - R_{12}R_{22}^{-1}R_{12}'$. (80)

The matrix A_u may be partitioned consistent with the partitioning of u as

$$A_u = \begin{bmatrix} (A_u)_{11} & (A_u)_{12} \\ (A_u)_{21} & (A_u)_{22} \end{bmatrix}$$

$$= \begin{bmatrix} K_{11} + K_{12}(sI-F_c-G_cK_{22})^{-1}G_2K_2(sI-F)^{-1} \\ H(sI-F_c)^{-1}G_c[I + K_{22}(sI-F_c-G_cK_{22})^{-1}G_c]K_2(sI-F)^{-1} \end{bmatrix} [G_1G_2] \tag{81}$$

Then for large $|s|$,

$$\begin{aligned}(A_u)_{11} &= s^{-1}K_{11}G_1 + 0(s^{-1}) \\ &= -s^{-1}R_1^{-1}[G_1'P_{11}G_1 - R_{12}R_{22}^{-1}G_c'P_{12}'G_1] + 0(s^{-2})\end{aligned} \tag{82}$$

and if $R_{12} = 0$,

$$(A_u)_{11} = -s^{-1}R_{11}^{-1}G_1'P_{11}G_1[I + 0(s^{-2})] \tag{83}$$

Thus if $R_{12} = 0$, $(A_u)_{11}$ has the attenuation characteristic of a system without compensation. If $R_{12} \neq 0$ the order of attenuation of some of the elements might be increased because of cancellations occurring in the first term of equation (82).

The remaining terms in A_u may be written for large $|s|$ as

$$(A_u)_{12} = s^{-1}K_{11}G_2 + 0(s^{-2}) \tag{84}$$

$$(A_u)_{21} = H(sI-F_c)^{-1}G_c[s^{-1}K_{21}G_1 + 0(s^{-2})] \tag{85}$$

$$(A_u)_{22} = H(sI-F_c)^{-1}G_c[s^{-1}K_{22}G_2 + 0(s^{-2})] \tag{86}$$

Equations (85) and (86) show that the compensator's attenuation characteristic is "added" to the first order (at least) characteristics of the u_2 subvector.

SECTION III

SYNTHESIS PROCEDURE BASED ON THE ASYMPTOTIC CHARACTERIZATION

In this section we will describe a procedure for constructing the quadratic weighting matrices of the performance index from desired specifications of closed-loop eigenvalues and eigenvectors. As an illustrative example the results obtained with this procedure for the lateral axis controller design for the F-4 will be described.

WEIGHTING MATRIX CONSTRUCTION

A key feature of the asymptotic interpretations above is the structural requirement that $C(sI-F)^{-1}G$ must possess transmission zeros and zero directions corresponding to the (s_i^o, ν_i^o) pairs. This section provides a construction procedure to "build" such response transfer matrices based on desired closed-loop specifications.

We begin with the assumption that (without loss of generality) the matrix G has been transformed to the form

$$G = \begin{bmatrix} 0 \\ --- \\ G_2 \end{bmatrix}$$

with G_2 an mxm nonsingular matrix.

Further, we assume that for desired closed-loop eigenvalues, s_i^o, the corresponding vectors x_i^* which describe the desired mode distributions in state space† can be specified by the designer, i.e., the desired modes should look like

$$x_i(t) = x_i^* e^{s_i^o t}; \quad i = 1, 2, \ldots, n-m \tag{88}$$

It will generally be true that only a few of the components in x_i^* are actually specified. The rest can be arbitrary. To account for this, we reorder and partition x_i^* as follows:

$$\{x_i^*\}^{R_i} = \begin{bmatrix} \gamma_i^* \\ \hline v \end{bmatrix} \tag{89}$$

where γ_i^* is the specified subvector, v's are the unspecified components, and $\{\ldots\}^{R_i}$ denotes the reordering operation.

In accordance with Lemma 1 and Theorem 1, the desired mode (89) may not belong to the set of eigenvectors achievable by optimal linear design:

$$\begin{bmatrix} \gamma_i^o \\ \hline v \end{bmatrix} = \{(s_i^o I - F)^{-1} G\}^{R_i} \nu_i^o = \begin{bmatrix} L \\ \hline M \end{bmatrix} \nu_i^o \tag{90}$$

The designer is free to select ν_i^o so as to best approximate γ_i^* with γ_i^o. One way to do this is by orthogonal projections,

$$\nu_i^o = (\bar{L}'L)^{-1} \bar{L}'\gamma_i^* \tag{91}$$

---

†A known mode distribution in some output space y = Cx with C non-singular would obviously serve as well.

although other methods may be more suitable in particular applications. We will assume here that $\{\nu_i^o;\ i = 1, 2, \ldots, n-m\}$ have been determined by some procedure and result, via equation (54), in the following eigenvectors

$$X = [x_1^o\ x_2^o \ldots x_{n-m}^o] \tag{92}$$

where care was taken to assure that no linear combination of the x_i^o's lie in the range space of G. We then define the projection matrix

$$P = I_n - X(\bar{X}'X)^{-1}\bar{X}' \tag{93}$$

and note that

$$PG\mu \neq 0 \text{ if } \mu \neq 0 \tag{94}$$

Let us partition the matrix P in the form

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \tag{95}$$

with P_{22} being an mxm matrix. Conditions (87) and (94) and the nonsingularity of G_2 imply that P_{22} is nonsingular. Thus, C_o may be defined as

$$C_o = [0 \,|\, P_{22}^{-1}]P = [P_{22}^{-1}P_{12}' \,|\, I_m] \tag{96}$$

To verify that C_o has the desired structural property, we note that

$$\begin{aligned} C_o(s_i^oI-F)^{-1}G\ \nu_i^o &= [0 \,|\, P_{22}^{-1}]P\,(s_i^oI-F)^{-1}G\ \nu_i^o \\ &= [0 \,|\, P_{22}^{-1}]P\,x_i^o \\ &= 0,\ i = 1, 2, \ldots, n-m \end{aligned} \tag{97}$$

and that for sufficiently large $|s|$, the rank of $C_0(sI-F)^{-1}G$ is m. Thus, there is a rank reduction at s_i^o which means s_i^o is a transmission zero, and hence an asymptotically finite eigenvalue with corresponding eigenvector x_i^o.

ILLUSTRATIVE EXAMPLE

For purposes of illustration we consider the inner loop lateral axis design problem for the F-4 fighter aircraft at a low dynamic pressure flight condition taken from reference [10]. The dynamics are

$$\dot{x} = Fx + Gu$$

with

$$x = \begin{bmatrix} p_s \\ r_s \\ \beta \\ \phi \\ \delta_r \\ \delta_a \end{bmatrix} \qquad \begin{matrix} \text{stability axis roll rate} \\ \text{stability axis yaw rate} \\ \text{angle of sideslip} \\ \text{bank angle} \\ \text{rudder deflection} \\ \text{aileron deflection} \end{matrix}$$

$$u = \begin{bmatrix} \delta_{rc} \\ \delta_{ac} \end{bmatrix} \qquad \begin{matrix} \text{rudder command} \\ \text{aileron command} \end{matrix}$$

Matrices F and G are given in Table 2. Note that they include first order actuator dynamics with 10 and 5 rad/sec bandwidths. These have been made deliberately slower than the hydraulic actuators available on the F-4 (approximately 20 and 10 rad/sec) in order to illustrate the significance of the freedom to choose asymptotically infinite eigenvalues and eigenvectors to achieve desired closed-loop actuator characteristics.

Table 2. F and G Matrices

$$F = \left[\begin{array}{cccc|cc} -.746 & .387 & -12.9 & 0. & .952 & 6.05 \\ .024 & -.174 & 4.31 & 0. & -1.76 & -.416 \\ .006 & -.9994 & -.0578 & .0369 & .0092 & -.0012 \\ 1. & 0. & 0. & 0. & 0. & 0. \\ \hline 0. & 0. & 0. & 0. & -10. & 0. \\ 0. & 0. & 0. & 0. & 0. & -5. \end{array}\right]$$

$$G = \begin{bmatrix} 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 0. & 0. \\ 20. & 0. \\ 0. & 10. \end{bmatrix}$$

From the point of view of fighter handling qualities, all four of the lateral axis closed loop roots have desired values which can be taken from MIL-F8785B, as done, for example, in reference [10]. The results are as follows:

a) Roll subsidence mode -4.0

b) Dutch roll mode $-0.63 \pm j\,2.42$

c) Spiral mode -0.05

Each of these poles can be assigned an asymptotic eigenvector, $x^{(i)}$, which distributes the modal response, $e^{p_i t}$ among the state variables and outputs of the system. However, each eigenvector is constrained to lie in a two dimensional subspace defined by Equation (54). An element in this subspace was selected by finding the best linear projection of an unconstrained desired vector, x*, on the subspace. The results of the eigenvector selection are:

a) Roll subsidence mode $(e^{-4t}x^{(1)})$

$$\begin{array}{lllll|ll} \text{Desired } x^* & = [1. & 0 & 0 & v & v & v] \\ \text{Attainable } x^{(1)} & = [1. & -.007 & 0 & -.25 & .13 & -.56] \end{array}$$

b) Dutch roll mode, real part $(e^{-.63t}(\cos 2.42t)x^{(2)})$

$$\begin{array}{lllll|ll} \text{Desired } x^* & = [0 & v & 1. & 0 & v & v] \\ \text{Attainable } x^{(2)} & = [0 & 15.6 & 1. & 0 & 7.86 & -.103] \end{array}$$

c) Dutch roll mode, imaginary part $(e^{-.63t}(\sin 2.42t)x^{(3)})$

$$\text{Desired } x^* = [0 \quad 1. \quad v \quad 0 \mid v \quad v]$$

$$\text{Attainable } x^{(3)} = [0 \quad 1. \quad 6.16 \quad 0 \mid -9.49 \quad 14.6]$$

d) Spiral mode $(e^{-.05t}x^{(4)})$

$$\text{Desired } x^* = [v \quad v \quad 0 \quad 1. \mid v \quad v]$$

$$\text{Attainable } x^{(4)} = [-.05 \quad .037 \quad 0 \quad 1. \mid -.0014 \quad -.0079]$$

A few comments are in order to explain these choices. Consider, for example, the roll subsidence mode. The desired eigenvector is taken to be $x^* = (1 \quad 0 \quad 0 \quad v \quad v \quad v)$ which means that the mode should show up dominantly on roll rate, but not on yaw rate or sideslip (we want no sideslip buildup during turn entries). These are basically handling quality considerations. The v's in the vector indicate that we do not care how much of the mode shows up on these components. Certainly, since $\phi = \int p_s dt$, some mode content has to be expected on x_4 and similarly, if the surfaces are actually controlling the mode, some mode content should also appear in x_5 and x_6. The linear projection which best achieves these objectives is shown as $x^{(1)}$ above. Note that we can satisfy our desires almost perfectly.

Similar arguments also apply to the dutch roll mode. In this case we want no oscillatory dutch roll content on roll rate and bank angle. This is a key handling quality requirement for all well-behaved lateral control laws [10].

In the case of the spiral we want the mode to show up dominantly on bank angle (corresponding to steady turns) with, again, no substantial sideslip component. The latter is a basic turn coordination requirement.

Once the eigenvectors are specified, it remains only to compute the matrix C_o via Equation (96) and then to select eigenvectors and eigenvalue ratios for the asymptotically infinite roots. The matrix C_o is shown in Table 3.

Table 3. C_o Matrix

$$C_o' = \begin{bmatrix} -.131 & .567 \\ -.612 & .160 \\ 1.64 & -2.39 \\ .0175 & .0303 \\ \hline 1. & 0. \\ 0. & 1. \end{bmatrix}$$

For decoupled actuators the matrix N defined in (59) and matrix S of eq. (60) were chosen to be

$$N = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

to achieve a two-to-one bandwidth ratio. Then from (57) and (58),

$$R = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad \text{and} \qquad W_o' W_o = \frac{1}{100} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

These selections complete the specification of the performance index except for the scalar parameter, ρ. Optimal controllers were computed for several values of this parameter.

Representative results are shown in Table 4 and Figure 2. The eigenvalues and eigenvector characteristics indeed approach their predicted asymptotic values, as can be seen in Table 4. Reasonable actuator bandwidths are obtained with $\rho = 0.0025$.

The controller gains shown in Table 4 indicate that the actuators at this design value are essentially uncoupled (very small cross-feeds from one to the other) and have just enough feedback around themselves to produce the bandwidths available with the aircraft's existing hydraulics.

Transient response characteristics achieved for a range of ρ-values are shown in Figure 2. Although only roll rate and sideslip responses are given for simplicity, dramatic improvement in cross coupling between roll motion and dutch roll is clearly evident. Roll transients approach their intended first order 0.25 second response time, and sideslip responses approach their intended slightly oscillatory second-order characteristics. Cross coupling vanishes as the eigenvectors approach their intended asymptotic directions.

This example illustrates that it is possible to satisfy complex handling quality specifications and actuator bandwidth characteristics with standard quadratic performance indices if the structure of the criterion is based on asymptotic characteristics.

Table 4. Asymptotic Eigenvalue/Vector Characteristics

NODE	CONTROL WEIGHT.	EIGENVALUE	EIGENVECTOR					
			p_s	r_s	β	ϕ	δR	δA
Roll Subsidence	∞	-.776	1.0	-1.501	-2.2340	2.340	0.	0.
	0.0025	-3.810	1.0	-.009	.000	-.262	.119	-.525
	0.000025	-3.998	1.0	-.007	.000	-.250	.131	-.558
Dutch Roll	∞	-.098 +j 2.079	-123.4	45.01	1.0	-16.71	0.	0.
	0.0025	-.727 2.358	-1.439	12.74	1.0	-.491	7.586	.786
	0.000025	-.632 2.419	-.014	15.51	1.0	-.008	7.876	-.096
		-.098 -j 2.079	-40.64	1.0	21.95	60.14	0.	0.
		-.727 2.358	-1.711	1.0	5.096	.761	-6.988	11.33
		-.632 2.419	-.024	1.0	6.141	.008	-9.452	14.51
Spiral	∞	-.0063	.040	-7.500	-1.940	1.0	0.	0.
	0.0025	-.0492	-.049	.037	.000	1.0	-.001	-.007
	0.000025	-.0500	-.050	.037	.000	1.0	-.001	-.008
Rudder Actuator	∞	-10.0	-.000	-.000	-.000	.000	1.0	0.
	0.0025	-22.44	-.012	-.162	-.168	-.514	1.0	-.019
	0.000025	-200.3	-.002	.001	-.063	-.172	1.0	-.003
Aileron Actuator	∞	-5.0	.000	.000	-.000	.000	0.	1.0
	0.0025	-10.43	-.001	-.016	.121	.067	-.029	1.0
	0.000025	-100.0	-.000	-.015	.017	.017	-.004	1.0
Feedback Gains at [illegible] 0.0025								
		δR	.132	.882	-1.576	-.026	-.681	.026
		δA	-.524	-.420	2.827	-.021	.013	-.860

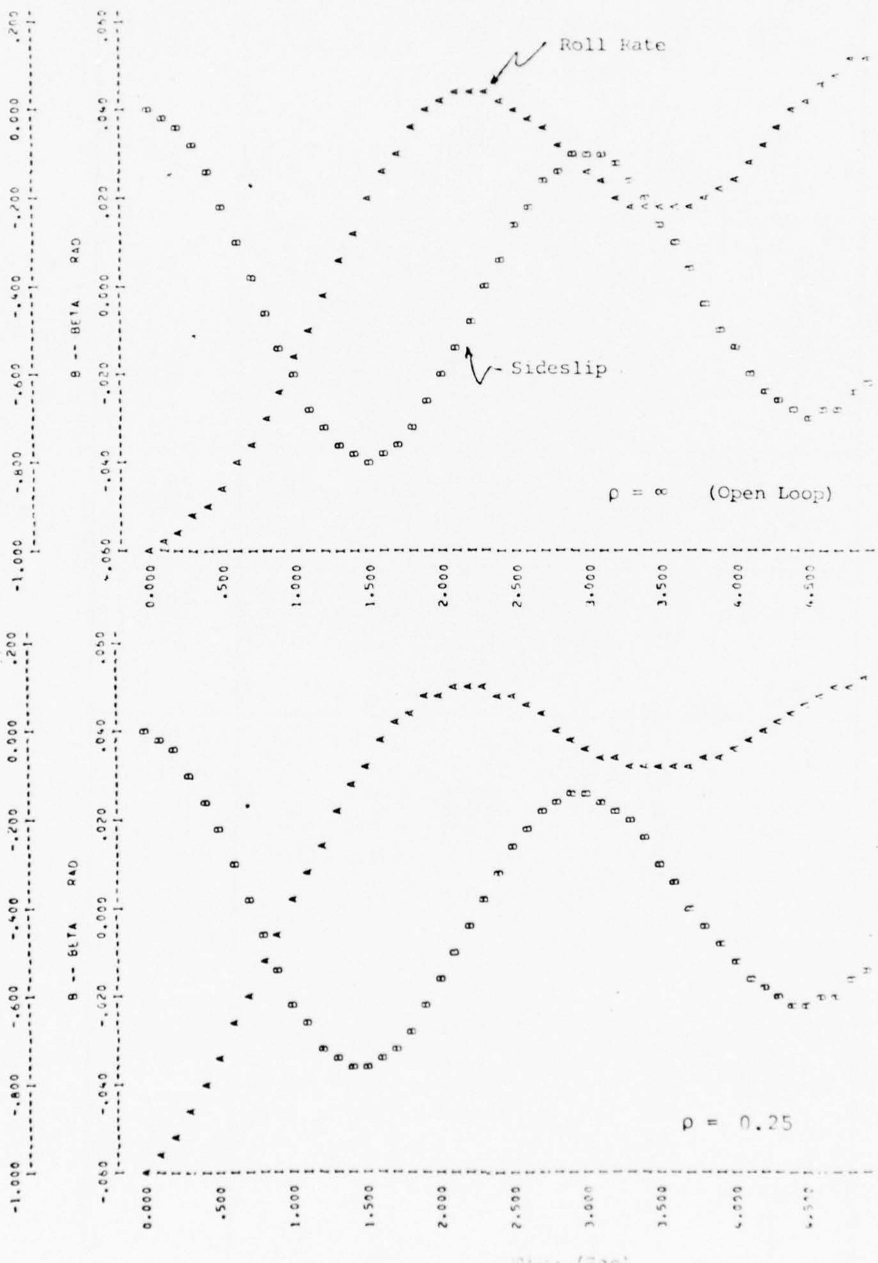

Figure 2. Transient Responses

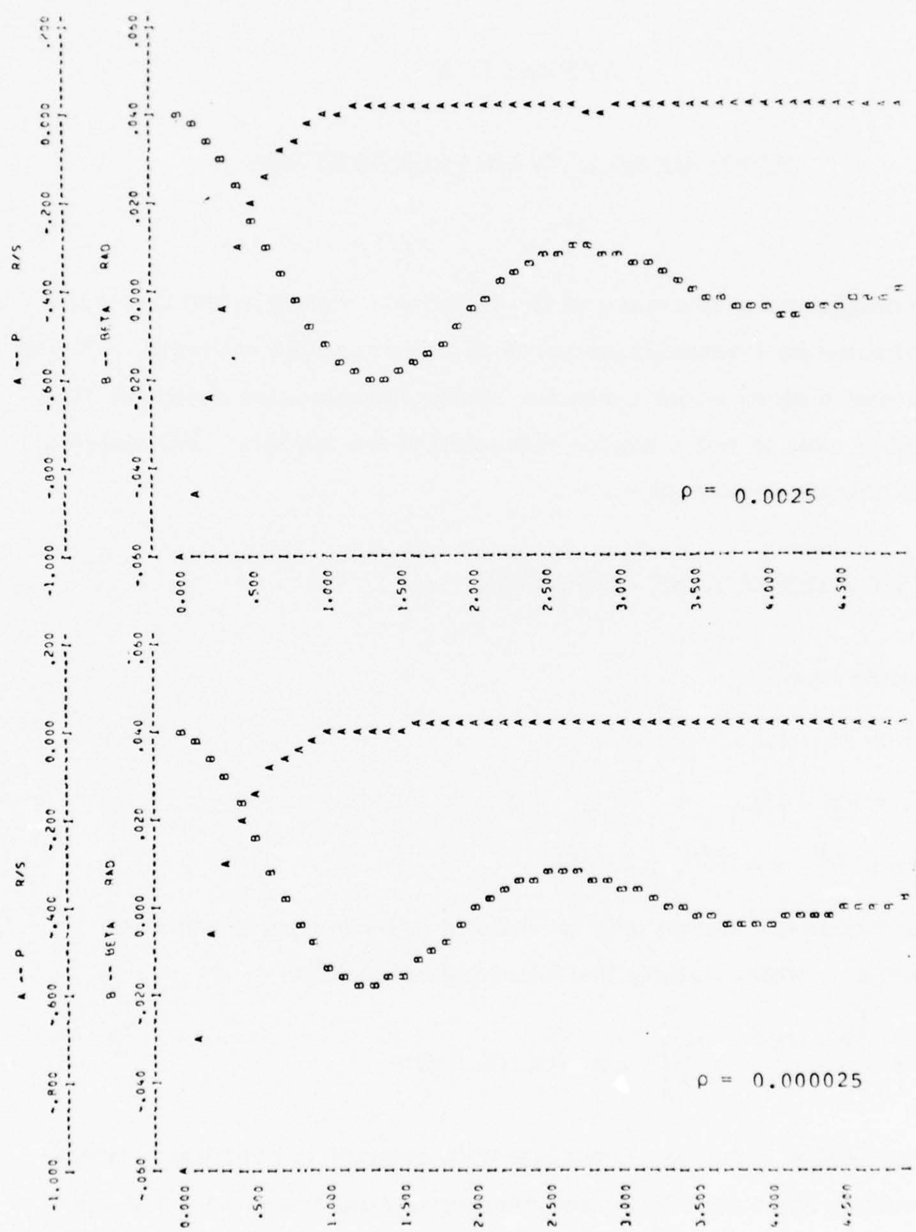

Figure 2. Transient Responses (concluded)

APPENDIX A

ZEROS OF MULTIVARIABLE SYSTEMS

This Appendix gives a summary of the important concepts and an outline of the literature on transmission zeros of multivariable systems. While many intuitive notions about zeros for scalar systems are retained, the multivariable case is not a trivial extension of the scalar. Examples are given to illustrate important ideas.

SUMMARY OF IMPORTANT CONCEPTS

1. Given the system:

$$\dot{x} = Fx + Gu$$

$$y = Cx + Du$$

$$(x \in R^n,\ u \in R^m,\ y \in R^p)$$

the transmission zeros may be defined to be the set of complex numbers λ which satisfy the following inequality:

$$\text{rank} \begin{bmatrix} F & -\lambda I & G \\ & C & D \end{bmatrix} < n + \min(m, p)$$

2. Transmission zeros are invariant with respect to coordinate transformation, state feedback, and nonsingular input and output transformation.

3. Transmission zeros may include the entire complex plane. The remaining points will assume this is not the case.

4. Almost all nonsquare systems ($m \neq p$) have no zeros.

5. For square systems ($m=p$) the definition in [8] is equivalent to defining transmission zeros as zeros of:

$$\det \begin{bmatrix} F - \lambda I & G \\ C & D \end{bmatrix}$$

$$= \det(F - \lambda I) \det[C(\lambda I - F)^{-1} G + D]$$

6. For square systems, the number of zeros is at most $n-m-d$ where d is the rank deficiency of CG.

7. Transmission zeros are <u>not</u> the same as the zeros of the numerators of the various transfer functions relating inputs to outputs. The latter are not invariant with respect to feedback.

8. For high-gain feedback systems where $u = ky$ the system poles migrate to the transmission zeros or infinity as k approaches infinity.

9. A zero at λ corresponds to the system blocking the transmission of certain input signals proportional to $e^{\lambda t}$.

OUTLINE OF LITERATURE ON TRANSMISSION ZEROS

There has been considerable interest recently in transmission zeros of multivariable systems. Rosenbrock [8] apparently was the first to offer a definition. Most of the literature offers little insight and many of the results are incorrect. We will present the most significant results here and try to clarify some of the issues. The interested reader may consult the literature for details, and we will point out the most obvious errors therein.

Much of the difficulty arises because the transmission zeros of a system are, in general, quite different from the zeros of the various scalar transfer functions relating inputs and outputs, as in the single-input single-output controllable-observable case.

Rosenbrock [8] defines zeros of a transfer function matrix in terms of the numerator of the diagonal elements of its McMillan standard form. If $y(s) = G(s)\,u(s)$, (y is length p, u is length m, and G is pxm), then $G(s) = N(s)/d(s)$ where $d(s)$ is the monic least common denominator of the elements of G. Then $N(s)$ can be brought to Smith form by the transformation $L(s)\,N(s)\,R(s) = S(s)$ where L and R are unimodular polynomial matrices. Cancelling any common factors between $S(s)$ and $d(s)$ yields the McMillan form of G which we denote by $M(s)$ as follows:

$$M(S) = [\mathrm{diag}(\epsilon_i(s)/\psi_i(s)) \mid 0_{p,m-p}], \quad m > p \qquad (A.1)$$

or
$$M(s) = \mathrm{diag}(\epsilon_i(s)/\psi_i(s)), \quad m = p \qquad (A.2)$$

or
$$M(S) = \begin{bmatrix} \mathrm{diag}(\epsilon_i(s)/\psi_i(s)) \\ 0_{p-m,m} \end{bmatrix}, \quad m < p \qquad (A.3)$$

The poles of G are the zeros of the denominator polynomials $\psi_i(s)$ in M(s) and the zeros of G are the zeros of the $\epsilon_i(s)$.

The following illustrative example was given in [8].

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)^2} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+3}{(s+2)^2} \end{bmatrix}$$

$$d(s) = (s+1)^2\,(s+2)^2$$

$$N(s) = \begin{bmatrix} (s+2)^2 & (s+1)(s+2) \\ (s+1)(s+2) & (s+1)^2(s+3) \end{bmatrix}$$

$$S(s) = \begin{bmatrix} 1 & 0 \\ 0 & (s+1)^2(s+2)^3 \end{bmatrix}$$

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)^2(s+2)^2} & 0 \\ 0 & s+2 \end{bmatrix}$$

Notice that although G is proper, M is not. Also, G has one zero at -2, and there is not a zero at -3.

Kwakernaak and Sivan [7] define the zeros of the system*

$$\dot{x} = Fx + Gu \tag{A.4}$$

$$y = Cx + Du$$

$$(x \in R^n,\ u \in R^m,\ y \in R^p)$$

for m = p and D = 0 as the zeros of the polynomial det (Is-F)x det$(C(Is-F)^{-1}G)$. If $H(s) = C(sI-F)^{-1}G$ then:

$$\det[H(s)] = \frac{\psi(s)}{d(s)},\ d(s) = \det(sI-F) \tag{A.5}$$

Now:

$$\lim_{|s| \to \infty} \frac{s^m \psi(s)}{d(s)} = \lim_{|s| \to \infty} s^m \det[C(Is-F)^{-1}G]$$

$$= \lim_{|s| \to \infty} \det[Cs(sI-F)^{-1}G]$$

$$= \det(CG) \tag{A.6}$$

This shows that the degree of d(s) is greater than $\psi(s)$ by at least m, hence $\psi(s)$ has degree n-m (when $\det(CG) \neq 0$) or less.

Wolovich [11] attempts to present a somewhat more general state-space approach to multivariable system zeros by essentially defining a zero as a value of s for which the matrix

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*Controllability and observability will be assumed throughout unless otherwise stated.

$$\begin{bmatrix} F-sI & G \\ C & D \end{bmatrix} \tag{A.7}$$

loses rank. The normal rank of a matrix is usually taken to be the maximum rank over s and a zero can be thought of as a value of s for which the rank is less than the normal rank. Wolovich incorrectly states that if the rank of G and the rank of C are both greater than or equal to $r = \min(m, p)$, then the normal rank of (A.7) is n+r. For the case p = m and D = 0, Wolovich's definition is equivalent to that given in [7] because

$$\det \begin{bmatrix} F-sI & G \\ C & 0 \end{bmatrix} = \det(F-sI)\,\det[C(sI-F)^{-1}G] \tag{2.8}$$

However, it is possible, as the following example demonstrates, for observable-controllable systems with C and G each of full rank to have the matrix in (A.7) be of rank less than n+r.

Example: Consider the controllable, observable linear system with n = 3, m = p = 2:

$$\dot{x} = Fx + Gu$$

$$y = Cx$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

For this system, we have

$$\det \begin{bmatrix} F-sI & G \\ C & 0 \end{bmatrix} = 0$$

so the rank of (A.7) is less than $3 + 2 = n + \min(m,p)$

Kouvaritakas and MacFarlane [12, 13] use a geometric approach to describe multivariable zeros. For square systems ($p = m$) they define the system zeros to be the zeros of

$$\det \begin{bmatrix} sI-F & G \\ C & 0 \end{bmatrix} \qquad \text{(A.9)}$$

and almost correctly point out that the number of zeros are $n-m-d$ where d is the rank deficiency of CG. As we shall see, the determinant may be zero for all s rendering the definition somewhat useless in general, but the approach is interesting and can provide worthwhile insights.

Davison and Wang [14] and Desoer and Schulman [15] have provided some apparently correct and definitely useful insights.

Davison and Wang, like Wolovich, define the transmission zeros of (A.4) to be the set of complex numbers λ which satisfy the following inequality

$$\operatorname{rank} \begin{bmatrix} F-\lambda I & G \\ C & D \end{bmatrix} < n + \min(m,p). \qquad \text{(A.10)}$$

They point out that a controllable observable system with G and C full rank may have zeros everywhere in the complex plane. They call these systems degenerate and assume that all their systems are nondegenerate. The above definition is shown to coincide with those of Kwakernaak and Sivan and Rosenbrock.

It is also noted that

$$\operatorname{rank}\begin{bmatrix} F-\lambda I & G \\ C & D \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} T(F+GK-\lambda I)T^{-1} & TGV \\ U(C+DK)T^{-1} & UDV \end{bmatrix} \qquad (A.11)$$

for any nonsingular $T \in R^{nxn}$, nonsingular $V \in R^{mxm}$, nonsingular $U \in R^{pxp}$ and $K \in R^{mxn}$, and hence that the transmission zeros $\{\lambda\}$ are invariant with respect to coordinate transformation, state feedback and nonsingular input and output transformations.

Davison and Wang also prove that the class of square systems having less than n-m zeros must lie on a hypersurface in the parameter space for those systems. Thus, they conclude that for almost all square systems, there are n-m zeros.

Davison and Wang have many other control-related results on zeros, and claim to have as well some efficient algorithms for calculating zeros.

Desoer and Schulman [15] give a very interesting account of zeros using purely algebraic techniques. The rational matrix transfer function $H(\cdot)$ is viewed as a network function of a multiport and $H(s)$ $(n_o x n_i)$ is factored into $D_\ell(s)^{-1} N_\ell(s) = N_r(s) D_r(s)^{-1}$. A zero of $H(\cdot)$ is defined to be a point z where the local rank of $N_\ell(z)$ drops below the normal rank. The intuitive concept that a zero of a multiport corresponds to the system blocking the transmission of signals proportional to e^{zt} is developed rigorously via several theorems.

MacFarlane and Karcanias [16] have written a rather complete survey of poles and zeros of multivariable systems. They treat zeros from the algebraic, geometric and complex-variable points of view and give several apparently new and worthwhile intuitive approaches. Furthermore, there are no glaring errors in the paper. This is definitely good (though lengthy) reading on the subject.

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