AD-A043 634 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 12/2
ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS.(U)
JUN 77 R AL-AYAT, R FARE N00014-76-C-0134
UNCLASSIFIED ORC-77-16 NL

| OF |

AD
A043634

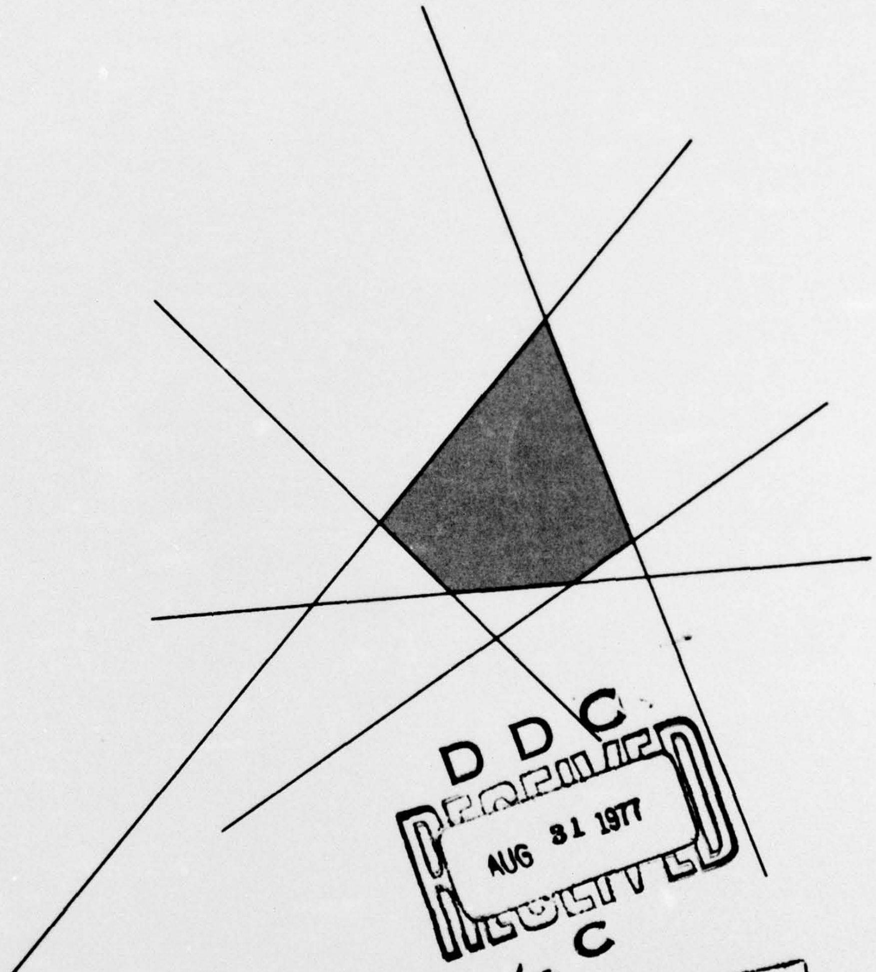
END
DATE
FILMED
9-77
DDC

ORC 77-16
JUNE 1977

ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS

by
ROKAYA AL-AYAT
and
ROLF FÄRE

12
NW



OPERATIONS
RESEARCH
CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS

by

Rokaya Al-Ayat and Rolf Färe
Operations Research Center
University of California, Berkeley

JUNE 1977 ORC 77-16

This research was supported by the Office of Naval Research under Contract N00014-76-C-0134 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC-77-16	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS,		5. TYPE OF REPORT & PERIOD COVERED Research Report,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 Rokaya Al-Ayat [redacted] Rolf Fare		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-76-C-0134
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 047 033
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Department of the Navy Arlington, Virginia 22217		12. REPORT DATE 11 June 1977
		13. NUMBER OF PAGES 11
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 12p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 561		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Isoquant Joint Production Function Strong Disposability		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (SEE ABSTRACT)		

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

270 750 SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LB

ACKNOWLEDGMENT

The authors sincerely thank Professor Ronald W. Shephard for his suggestions and helpful comments.

ABSTRACT

Within a general framework of production correspondences satisfying a set of weak axioms necessary and sufficient conditions for the existence of a joint production function are given. Without enforcing the strong disposability of inputs or outputs it is shown that a joint production function exists if and only if both input and output correspondences are strictly increasing along rays.

ON THE EXISTENCE OF JOINT PRODUCTION FUNCTIONS

by

Rokaya Al-Ayat and Rolf Färe

Joint production functions are frequently used in economics, however, it was not until Shephard in [6] defined such a notion within the general framework of production correspondences that its meaning became clear. The question of existence of these functions, dealt with in this paper, is yet to be settled. On this issue Shephard [8] wrote, "The joint production function is a tricky concept, seemingly simple but not shown to exist except under very restrictive conditions."

For a production technology with strongly disposable inputs and outputs Bol and Moeschlin [2], showed that continuity of both the input and the output correspondences together with *essentiality of all inputs are sufficient* for the existence of a joint production function. Later Bol in [1] showed that such a function would also exist if the essentiality condition is replaced by strict increasancy of the output correspondence in all inputs.

It is to be recalled that an output correspondence $x \to P(x) \in 2^{\mathbb{R}_+^m}$ is a mapping from input vectors $x \in \mathbb{R}_+^n$ into subsets $P(x) \in 2^{\mathbb{R}_+^m}$ of all output vectors obtainable by x . Inversely to $P(x)$ the input correspondence $u \to L(u) := \{x \mid u \in P(x)\}$ is the set of all input vectors x yielding at least an output vector u . In this paper the existence of a joint production function will be considered under the weak axioms as stated in [7]. Specifically neither the strong disposability of inputs or outputs (i.e., $x' \geqq x \in L(u) \Rightarrow x' \in L(u)$, $u' \leqq u \in P(x) \Rightarrow u' \in P(x)$ respectively) nor convexity of $P(x)$ or $L(u)$ are enforced.

Having strong disposability of inputs means that if a subvector of inputs is kept constant while the remaining are increased, output will never decrease implying there can be no congestion in the production system. In addition, strong disposability of outputs excludes their null jointness (see [9]) which is one of the basis for discussions of the external dis-economics. Thus having only weak disposability of inputs (i.e., $P(\lambda \cdot x) \supset P(x)$, $\lambda \geq 1$) and outputs (i.e., $L(\theta \cdot u) \subset L(u)$, $\theta \geq 1$) allow modelling of both congestion and null jointness.

As defined by Shephard [6], the joint production function relates input and output isoquants to each other. Recall that

$$\text{ISOQ } P(x) := \{u \mid u \in P(x) , \theta \cdot u \notin P(x) , \theta > 1\} , P(x) \neq \{0\} ,$$

and

$$\text{ISOQ } L(u) := \{x \mid x \in L(u) , \lambda \cdot x \notin L(u) , \lambda < 1\} , L(u) \not\ni \{0\} , L(u) \neq \phi .$$

Definition:

The function $F : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ such that

(1) for $u^o \geq 0$, $\text{ISOQ } L(u^o) = \{x \mid F(u^o,x) = 0\}$, $L(u^o) \neq \phi$ and

(2) for $x^o \geq 0$, $\text{ISOQ } P(x^o) = \{u \mid F(u,x^o) = 0\}$, $P(x^o) \neq \{0\}$

is a joint production function.

An equivalent statement to the definition, to be used in the sequel, was proved by Bol and Moeschlin [2] namely:

Lemma:

A joint production function $F(u,x)$ exists if and only if for all $x \geq 0$(1) , $P(x) \neq \{0\}$ and $u \geq 0$, $L(u) \neq \phi$, $u \in \text{ISOQ } P(x) \Leftrightarrow x \in \text{ISOQ } L(u)$.

(1) $x \geq 0$ means $x \geqq 0$ but $x \neq 0$.

Theorem:

For all $x \geq 0$, $u \geq 0$ such that $P(x) \neq \{0\}$, $L(u) \neq \phi$ with $x \to P(x)$ $(u \to L(u))$ satisfying the weak axioms, a necessary and sufficient condition for the existence of a joint production function $F(u,x)$ is

$$(*) \quad \text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x) = \text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u) \quad \text{empty}$$

for all positive scalars λ , $\theta \neq 1$.

Proof:

To show the necessity of (*) , assume there is a joint production function $F(u,x)$ and let $u \in \text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x)$. By the lemma, $x \in \text{ISOQ } L(u)$ and $\lambda \cdot x \in \text{ISOQ } L(u)$, $\lambda \neq 1$, which is a contradiction. Thus if a joint production exists, $\text{ISOQ } P(x) \cap \text{ISOQ } P(\lambda \cdot x)$ is empty for all positive scalars λ , $\lambda \neq 1$. A similar argument can be used to show that the existence of $F(u,x)$ implies that for all positive θ , $\theta \neq 1$, $\text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u)$ is empty.

To show the sufficiency, assume that (*) holds, and that for $x \geq 0$, $P(x) \neq \{0\}$, $u \in \text{ISOQ } P(x)$ but $x \notin \text{ISOQ } L(u)$. From the definition of the isoquant, there exists a $\lambda < 1$ such that $\lambda \cdot x \in \text{ISOQ } L(u)$ implying that $u \in P(\lambda \cdot x)$. But from the weak disposability of inputs $P(\lambda \cdot x) \subset P(x)$ which together with (*) implies that $u \notin \text{ISOQ } P(x)$, a contradiction. Similarly it can be shown that having $\text{ISOQ } L(u) \cap \text{ISOQ } L(\theta \cdot u)$ empty would guarantee that $x \in \text{ISOQ } L(u) \Rightarrow u \in \text{ISOQ } P(x)$. Hence the sufficiency of (*) for the existence of a joint production function is proved. See lemma. Q.E.D.

Continuity of the production correspondences has not been enforced. However, following an argument similar to that used by Bol and Moeschlin in [2] one can prove:

Corollary:

If a joint production function exists, then both the input and the output correspondences are continuous along rays i.e., $P(\lambda^{o} \cdot x) = \overline{\bigcup_{0<\lambda<\lambda^{o}} P(\lambda \cdot x)}$ and $L(\theta^{o} \cdot u) = \overline{\bigcup_{\theta>\theta^{o}} L(\theta \cdot u)}$ respectively, with $u , x \neq 0$.

Note that continuity along rays together with strong disposability imply continuity (see [2] for definition).

Next, consider the production technology;

$$P(x_1,x_2) := \{\{(u_1,0)\} \cup \{(0,u_2)\} \mid 0 \leqq u_i \leqq x_i \ , \ i = 1,2\}$$

and inversely

$$L(u_1,u_2) := \{\{(x_1,0)\} \cup \{(0,x_2)\} \mid x_i \geqq u_i \ , \ i = 1,2\} \ .$$

The corresponding isoquants are given by

$$\text{ISOQ } L(u_1,u_2) = \{\{(x_1,0)\} \cup \{(0,x_2)\} \mid x_i = u_i \ , \ i = 1,2\}$$

and

$$\text{ISOQ } P(x_1,x_2) = \{\{(u_1,0)\} \cup \{(0,u_2)\} \mid u_i = x_i \ , \ i = 1,2\} \ .$$

In this example, the production correspondence satisfies the weak axioms, but neither strong disposability of inputs and outputs nor the essentiality condition (i.e., $P(x) \neq \{0\}$ implies $(x_1,x_2) > (0,0)$) used in [2] hold. Yet it is clear that a joint production function exist.

Finally, an example not satisfying the sufficiency conditions applied in [1] and [2] is given. Before introducting it the following proposition to be used, is proved.

Proposition:

If the production function $\phi(x) := \max\{u \mid x \in L(u)\}$, is continuous and strictly increasing along rays in the input space $\mathbb{R}^n_+$, ISOQ $L(u) = \{x \mid \phi(x) = u\}$, $u > 0$.

Proof:

Clearly ISOQ $L(u) \subset \{x \mid \phi(x) \geqq u\}$, $u > 0$; let $x^o \in \{x \mid \phi(x) > u\}$. Since ϕ is continuous along rays, $\{\lambda \mid \phi(\lambda \cdot x^o) > u\}$ is open implying that $x^o \notin$ ISOQ $L(u)$, hence ISOQ $L(u) \subset \{x \mid \phi(x) = u\}$. Next assume $x^o \notin$ ISOQ $L(u)$, $u > 0$, then since ϕ is strictly increasing along rays, if $x^o \in L(u)$, there is a $\lambda < 1$ such that $\phi(\lambda \cdot x^o) = u$ implying that $x^o \notin \{x \mid \phi(x) = u\}$. Q.E.D.

Now, consider the output correspondence $x \to P(x) \subset [0,+\infty)$,

$$P(x) := \left\{u \;\middle|\; 0 \leqq u \leqq A \cdot \left[(1-\delta) \cdot \max\left\{0, (x_1 - \gamma \cdot x_2)^{-\rho}\right\} + \delta \cdot x_2^{-\rho}\right]^{-1/\rho} =: \phi(x)\right\}$$

where the parameters of the WDI - production function $\phi(x)$ are $A > 0$, $\delta \in (0,1)$, $\gamma \in (0,\infty)$ and $\rho \in (-1,0)$ (see [3]). For these values of the parameters, $\phi(x)$ is upper semi-continuous which is equivalent to $P(x)$ being upper hemi-continuous (see [5], p. 22) also $x_2 = 0$ does not imply $P(x) = \{0\}$ and ϕ is not increasing in x_2 . Thus $P(x)$ does not meet the continuity requirement of [1] and [2] nor does it meet the other sufficiency condition of [2] (essentiality of all factors) or [1] (strict increasancy in all factors).

Using the proposition above the isoquants of $P(x)$ and $L(u)$ are easily computed to be,

$$\text{ISOQ } P(x) = \{u \mid u = \phi(x)\} \quad \text{and} \quad \text{ISOQ } L(u) = \{x \mid \phi(x) = u\} \ .$$

Thus, $x \in \text{ISOQ } L(u) \Leftrightarrow u \in \text{ISOQ } P(x)$, showing that under the weak axioms for a production technology, the sufficient conditions found in [1] and [2] need not hold for a joint production function to exist.

REFERENCES

[1] Bol, G., "Produktionskorrespondenzen und Existenz Skalarwertiger Produktionsfunktionen bei der Mehrgüterproduktion," Karlsruhe, (1976).

[2] Bol, G. and O. Moeschlin, "Isoquants of Continuous Production Correspondences," Naval Logistics Research Quarterly, Vol. 22, pp. 391-398, (1975).

[3] Färe, R. and L. Jansson, "On VES and WDI Production Functions," International Economic Review, Vol. 16, pp. 745-750, (1975).

[4] Färe, R. and L. Jansson, "Joint Inputs and the Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 36, pp. 407-416, (1976).

[5] Hildenbrand, W., CORE AND EQUILIBRA OF A LARGE ECONOMY, Princeton University Press, (1974).

[6] Shephard, R. W., THEORY OF COST AND PRODUCTION FUNCTIONS, Princeton University Press, (1970).

[7] Shephard, R. W., "Semi-Homogeneous Production Functions," Lecture Notes in Economics and Mathematical Systems, Volume 99, *PRODUCTION THEORY*, Berlin, Springer-Verlag, (1974).

[8] Shephard, R. W., "On Household Production Theory," ORC 76-24, Operations Research Center, University of California, Berkeley, (1976).

[9] Shephard, R. W. and R. Färe, "The Law of Diminishing Returns," Zeitschrift für Nationalökonomie, Vol. 34, pp. 69-90, (1974).