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The following report represents the first semiannual review of research conducted under the auspices of the Assoicate Joint Services Electronics Program at the Institute for Electronic Science at Texas Tech University. Specific Topics covered include fault analysis, Computer-aided design, stochastic control and estimation, decentralized control, mathematical system theory, optical noise, and pattern recognition.

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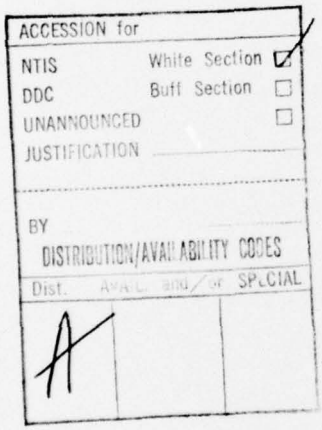
PREFACE

The following report represents the first semiannual review of research conducted under the auspices of the Associate Joint Services Electronics Program at the Institute for Electronic Science at Texas Tech University. Specific topics covered include, fault analysis, computer-aided design, stochastic control and estimation, decentralized control, mathematical system theory, optical noise, and pattern recognition.



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RESEARCH

on

FAULT ANALYSIS

R. SAEKS

DEPARTMENT OF ELECTRICAL ENGINEERING

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ABSTRACT

Research covering several aspects of the fault analysis problem for electronic circuits and systems is reviewed. This includes basic mathematical studies into the concepts of fault diagnosis, fault predictions and self testing. These ideas are then combined to formulate an architecture for an approach to built-in testing.

Introduction

The fault diagnosis problem may sub-divide into three fundamental areas; fault detection, fault diagnosis, and fault prediction. These areas are closely coupled and all three are required for any practical test system. The precise manifistation of any one being dependent on the specific context in which the test system is designed to operate. For instance, in an off-line (periodic) test system the fault diagnosis algorithm must be able to cope with components which have deviated a long way from their nominal values whereas in an on-line (built-in) test system one may assume that a failure is spotted as soon as the components begins to drift away from nominal. As such, a fault diagnosis algorithm for an off-line test system must be designed to cope with significant nonlinearities whereas an algorithm for on-line testing may be formulated in terms of a linearized model.

In the sequel the state of our research in fault diagnosis and fault prediction is reviewed. These ideas are then used to formulate the architecture for a built-in test (BIT) system. Of course, the mathematical theory derived for the fault diagnosis and fault prediction problems is also applicable to off-line test systems, test systems designed for use

at the end of an assembly line, etc. The evnironment in which such test systems operate is, however, quite different then that of the BIT systems and hence the mathematical techniques for fault diagnosis and fault prediction would have to be integrated differently in these other applications.

Fault Diagnosis

For the purposes of doing fault diagnosis we work with a component connection model for the circuit or system under test which takes for form

1. $$b_i = Z_i(s,r)a_i \quad 'i=1,2, \ldots ,n$$

and $$a = L_{11}b + L_{12}u$$

2. $$y = L_{21}b + L_{22}u$$

in the frequency domain[6,12,12]. Here $Z_i(s,r)$ is the transfer function of the ith circuit or system component where $r = \text{col}(r_i)$ is the vector of unknown component parameters and s is the complex frequency variable. The L_{ij} are known connection matrices, $a = \text{col}(a_i)$ and $b = \text{col}(b_i)$ are composite vectors of component inputs and outputs respectively, and u and y are the test input and output signals respectively. In the nonlinear case the component equations are replaced by the state models

3. $$X_i = f_i(X_i,a_i,r)$$
$$; i=1,2, \ldots ,n$$
$$b_i = g_i(X_i,a_i,r)$$

with the connection equations remaining as in 2. Although these component connection models for a circuit or system are nonclassical they are widely used in large-scale system simulation and computer-aided circuit design and are readily amenable to the "computer speed-up techniques" developed for

these applications.[12] As such, they are ideally suited for the fault diagnosis problem.

Combining 1. and 2. yields the fault diagnosis equation.[7]

4. $$S^m = L_{22} + L_{21}(1-Z(s,r)L_{11})^{-1}Z(s,r)L_{12}$$

where $Z(s,r) = \text{diag}(Z_i(s,r))$ and S^m is the measured transfer function relating the input test signal u to the output test signal y. The solution of the fault diagnosis problem therefore amounts to the solution of 4. for the parameters vector, r, given S^m and the connection matrices. Although it is possible to give an analytic description of all possible solutions to this equation[12,13] given any fixed value for the complex frequency variable, s, in a "real world" situation the number of unknowns greatly exceeds the number of equations and, as such, the analytic representation of the solution manifold proves to be of little value. This difficulty is alleaviated via a multi-frequency diagnosis algorithm wherein one writes the set of simultaneous equations

5.
$$\begin{aligned} S(s_1,r) &= L_{22} + L_{21}(1-Z(s_1,r)L_{11})^{-1}Z(s_1,r)L_{12} \\ S(s_2,r) &= L_{22} + L_{21}(1-Z(s_2,r)L_{11})^{-1}Z(s_2,r)L_{12} \\ &\vdots \qquad\qquad\qquad \vdots \\ S(s_k,r) &= L_{22} + L_{21}(1-Z(s_k,r)L_{11})^{-1}Z(s_k,r)L_{12} \end{aligned}$$

where k different complex frequencies are used in equation 4. simultaneously. The interesting and somewhat surprising result is that the additional equations in 5. may be independent thus increasing the number of equations without increasing the number of its unknowns.[7] While the set of simultaneous equations 5, often have a unique solution no analytic solution technique is known and we must restort to time consumming numerical solution procedures

carried out off-line.

Although the multi-frequency fault diagnosis equations of 5. do not admit an analytic solution their numerical solution can be significantly speeded up by careful analysis of the equations. In particular, a little algebra[6,12] will reveal that

$$6. \qquad \frac{dS^m}{dr_s} = L_{21}(1-Z(s_i,r)L_{11})^{-1} \frac{dZ(s_i,r)}{dr_j} [1 + L_{11}(1-Z(si,r)L_{11})^{-1}] L_{12}$$

showing that one can compute the partial derivaties required for the numerical solution to 5. analytically. Moreover, if one observes that the inverse matrix required to compute the partial derivaties in equation 6. is precisely the same inverse matrix required to evaluate the multi-frequency fault diagnosis equations 5. it is seen that the partial derivative information is obtained at virtually no computational cost over and above that required for the evaluation of the equations. In a similar vain one can reduce the computation required to compute the inverses at different complex frequencies by intergrating the differential equation

$$7. \qquad \frac{d(1-Z(s,r)L_{11})^{-1}}{ds} = (1-Z(s,r)L_{11})^{-1} \frac{dZ(s,r)}{ds} L_{11} (1-Z(s,r)L_{11})^{-1}$$

using the inverse computed at one particular frequency as a starting point.[2,14] Although of extremely high dimension this equation is easily integrated without the requirement for any matrix inversions. With the aid of these observations it is therefore possible to carry out an entire interation of a Newton-Raphson algorithm for the solution of the multi-frequency fault diagnosis equations with the aid of only a single matrix inversion.

Although one does not have a "neat" set of equations such as those described above for the solution of the fault diagnosis problem in a nonlinear

circuit or system surprisingly similar computational techniques can be invoked in the nonlinear case. The key to these techniques is the replacement of the multi-frequency information of the linear case by a family of integral performance measures on the test signals, u and y. These play exactly the same role in nonlinear fault diagnosis as played by the frequence information in the linear case, allowing one to formulate multiple independent fault diagnosis equations from the same test signals.

In the nonlinear case the sparse tableau algorithm[3,12] is used to evaluate the fault diagnosis equations at each interation of a Newton-Raphson algorithm. As in the linear case this algorithm allows one to compute the derivative information required for the Newton-Raphson algorithm with essentially no additionally computational cost over and above that required for the evaluation of the equations.[3,4,12] As such, by optimally exploiting the computational efficiencies inherent in the sparse tableau formulation for an electronic circuit or system it is possible to obtain significant computational gains in the solution of the fault diagnosis equations in the nonlinear case as well as the linear case.

Fault Prediction

Although for any particular device one can collect statistical data on which to base a fault prediction algorithm in a practical setting wherein the same fault prediction algorithm is multiplexed through the testing of many different SRA's it is necessary to use an algorithm which is independent of the specific properties of the parameter under test. As such, we employ a curve fitting algorithm.[20] Although less accurate than a statistically based algorithm we have shown by computer experiment[18,19] that such

an algorithm can be employed as a satisfactory fault predictor. Such algorithms are computationally simple thus permitting a single central microprocessor to be multiplexed through the testing of a large number of SRA's.[15]

Over the past several years we have investigated several approaches to the fault prediction problem[5,15,16,18,19,20]. The first is extremely naive but has yielded surprisingly effective results in simulation.[18,19,20] Basically, one collects data at periodic intervals, fits the data with a second order polynomial, and solves the quardradic equation to estimate the time at which the parameter will go out of tolerence. Although such an approach might at first appear to be so naive as to be inapplicable, it has yielded surprisingly good results.[18,19] This is due to the fact that one is not really interested in the accuracy of the failure time estimate but only the accuracy of the binary decision (based on this estimate) as to whether or not replace the SRA. The point is, that this binary decision is only made when failure is expected in the near future, a region of time in which a polynomial extrapolation is reasonably accurate. On the other hand, if failure is estimated to take place in the distant future even though the polynomial estimate may be in significant error this will not lead to an erroneous binary replacement decision. I.e., if failure is estimated to take place in 3 years even if the estimate is off by 90%, the decision to not replace the SRA at this time will still be correct.

A fault prediction algorithm based on the above described second order polynomial extrapolation has been extensively studied by Tung and author and some 10,000 complete operations of the algorithm have been simulated.[18,19]

For the most part these simulations were carried out on artifical data generated by a library of special functions to which a noise term was added. These special functions included some highly complex non-monotonic curves. Additionally, curves based on the empirical drift formula for thin film resistors were studies ($R(t) = At^a$ where a lies between .3 and .5). In both cases random noise with amplitudes of up to 25% of the tolerence interval was added to the data. The result of these simulations, which we believe to represent an environment which is more extreme that the "real world", was that 99.5% of all SRA's were replaced before on-line failure at a cost of about 10% of their lifetime.

At the present time a somewhat more sophisticated fault prediction algorithm is under development.[5] This is still essentially a curve fitting algorithm though one in which a failure model well founded in modern reliability theory[1] is employed. The basic idea for this algorithm is as follows. The drifting SRA parameter, r, is assumed to satisfy a difference equations

8. $$r(k+1) = r(k) = f(k)$$

where the "component time" k represents the number of shocks the SRA has received; switching processes, electrons boiling off a cathode, etc. The relation between component time, i.e. the number of shocks received, and "real time" is assumed to be a Poisson distributed random variable in which the probability of the SRA receiving n shocks in a time interval of length t is

9. $$P_n(t) = (ct)^n e^{-ct} / \lfloor n$$

For the fault prediction algorithm it is assumed that the value of the parameter, r, is known at a fixed set of points in "real time"; $r(t_1)$, $r(t_2)$, ... , $r(t_m)$. Using this data we desire to estimate the unknown failure dynamics, f(k), for the SRA parameter. This is then used in equation 8. to compute the number of shocks required to cause failure; i.e. the smallest value of k for which r(k) is out of tolerence. Finally, this estimate is used to compute the optimal "real time" at which to replace the SRA so as to miniminze the cost functional

10. $$J = c_f P_f + c_w W$$

Here, P_f is the probability of on-line failure, W is the average percentage of SRA lifetime which is wasted by replacing the SRA before its actual failure and c_f and c_w are weight factors.

Note that the implementation of the above described Poission shock based fault prediction algorithm requires that we deal simultaneously with two unknown phenomena; the failure dynamics, f(k), and the random relationship between "real time" and "component time" given by the Poission distribution. Although the required analysis is complex a surprisingly tractable (and optimal in an appropriate sence) fault prediction algorithm can be formulated. Here, one uses the properties of the Poission distribution to estimate the number of shocks which the SRA has received in the time intervals $[t_i, t_{i-1}]$; i=1, 2, ... ,m and combines this data with a generalized inverse algorithm to estimate f(k). Here, f(k) is approximated by a jth order polynomial and one must compute the generalized inverse of an m by j matrix. Fortunately, the algorithm is ideally suited to a

sequential least squares techniques[8] and no matrix inversions need to be carried out on-line. Once f(k) has been estimated to a satisfactory level of accuracy (one keeps on increasing the order of the approximating polynomial until the estimation error is reduced to a prescribed level) it is used with equation 8. to compute the number of shocks, required for the parameter to go out of tolerence. Finally, this value is used in conjunction with the Poission distribution to determine the optimal "real time" at which to replace the SRA. Although apparently complex this latter optimization can be reduced by analytic techniques to the solution of a single nonlinear equation in one variable.[5] As such, the entire fault prediction algorithm may be easily implemented, on-line. Unlike the second order curve fitting algorithm the Poission shock algorithm for fault prediction is still under development and its simulation, hopefully on "real world" data, is just beginning.

Self Testing

An interesting side effect of running a fault analysis system in a predicitve mode is that it opens up the possibility of reliable self testing. The key observation, here, is that to do fault prediction in a digital device one must test its analog parameters such as rise time, power supply voltages, clock speeds, pulse widths, etc., since the digital parameters are either right or wrong and have no gray region within which to extrapolate trends. Now, if one uses a microprocessor to predict its own failure by extrapolating the values of its analog parameters as long as the prediction is made before these parameters go out of tolerence the digital performance of the microprocessor will still be exact and hence one may use the microprocessor as a reliable predictor of its own failure.

The point is that in a predictive mode the microprocessor is still working at the time it predicts its own failure and hence may be used reliable in a self testing mode. Of course, once the analog parameters of the microprocessor have exceeded their tolerence limits it may no longer be trusted as a digital signal processor and hence the device cannot be used to diagnose its own faults after failure.

Although the above described self testing concept is purely conceptual and has yet to be implemented nor even simulated it is indicative of the potential of fault prediction in a BIT system. Indeed, if one can reliably predict failure before it actually takes place such "far out" concepts as self repair move into the realm of feasibility since at the time a replacement decision is made the device under test is still working.

A BIT Architecture

Our basic architecture for a Built-in Test (BIT) system is a two-level hierarchical structure illustrated in Figure 1. Intuitively, the overall system may represent a printed circuit board

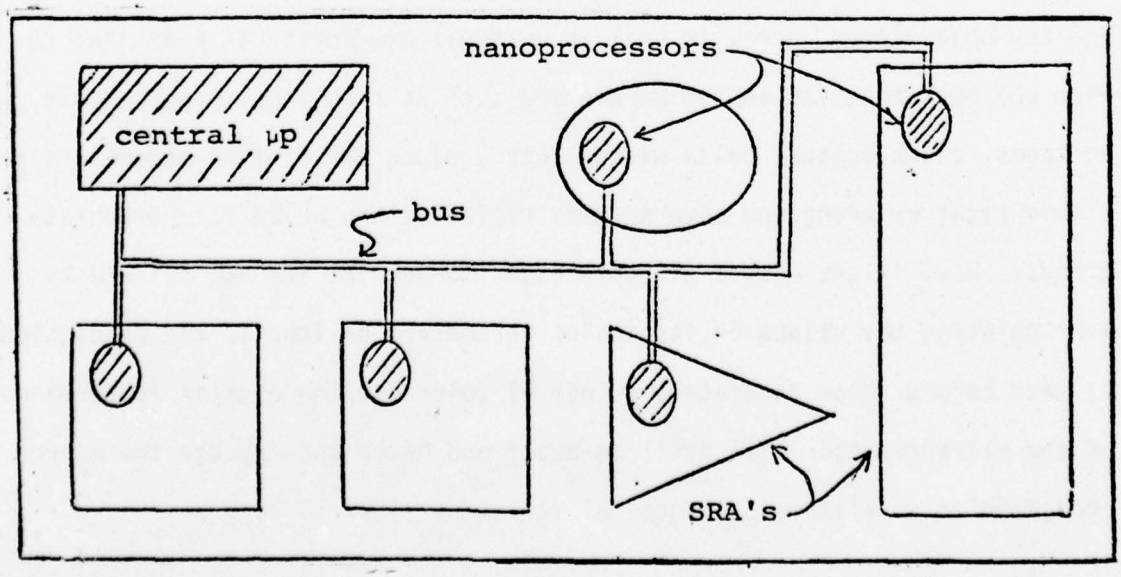

Figure 1: Two-level BIT architecture.

while the subsystems represent various shop replacable assemblies (SRA's) such as integrated circuits, power supplies, SCR's vaccumm tubes, etc. Alternatively, the overall system may represent an entire electronics system with the SRA's being its consituant PC boards. In either case the SRA's may be throw-away units or units intended for off-line repair with build-in test equipment (BITE) designed to detect and/or predict faults in the SRA. For those units intended for off-line repair the BITE may also be used as an interface with an external test stand but will not be capable of isolating the failure within the SRA.

This structure is motivated by the above described research into the relative computational complexity of the three fundamental problems of fault analysis; fault detection, fault diagnosis, and fault prediction.[9] The latter problem requires considerable computational power[18,19] but need only be carried out at widely spaced test intervals, say one test per hour (minute, second, ?). As such, a single central microprocessor can be time multiplexed through the testing of a large number of subsystem parameters thereby achieving the required computational power for the fault prediction algorithm while still holding the amount of dedicated test equipment within reasonable bounds.[10]

While fault diagnosis can be carried out with considerable success the process requires significant computational power (at least a mini by today's standards) and lenghty computer runs[7,11,17]. As such, fault diagnosis within an SRA is done off-line on an external test stand containing the required mini (or maxi) computer. Each SRA, however, will include sufficient BITE,

say a nanoprocessor, to collect and condition test data on the SRA to be periodically communicated to the central microprocessor for purposes of fault prediction and detection.

Fortunately, both of these endeavors may be achieved using a linearized model of the SRA about its nominal values and hence can be implemented with relatively little computational power built into the SRA.[12] In particular, for fault prediction one is interested in tracking various internal parameters of the SRA as they drift from nominal to their tolerence limit. Since the tolerence interval is typically only a few percent this can be achieved with a linearized model. For castastrophic errors a linearized model may be used to detect failures even though it is not sufficiently accurate for fault diagnosis. As such, the BITE within an SRA may be kept within reasonable bounds while still delivering sufficient data to the central microprocessor for its fault prediction and fault detection tasks. If needed fault diagnosis within an SRA will, however, be done off-line with the BITE simply serving as in interface between the SRA and an external test stand.

A final aspect of the BIT architecture is the communication link between the SRA's and the central microprocessor. Here, one desires to keep the wiring between the SRA's and the central microprocessor at a minimum and simultaneously would like to have all data transmitted to the central microprocessor in a uniform format so as to permit interchangability of component parts within the system. Although the details of this communications link have yet to be formalized the existence of an active computing capability in each SRA gives one considerable flexibility. As such, we

believe that is will be possible to work with a single test bus.[16] Here, the central microprocessor requests data from the individual SRA's by transmitting a signal on the bus. This signal is received by the built-in nanoprocessor in the SRA which, in turn, transmits appropriately conditioned test data back to the central microprocessor on the same bus.

The above described BIT architecture would seem to achieve most of the requirements for a built-in testing system.

i). Continous on-line fault prediction and detection, up to an SRA, is achieved.

ii). The system includes an interface for off-line fault diagnosis within an SRA.

iii). Dedicated test equipment represents a small percentage of the total system.

iv). Busing is minimized and test data is transmitted to the central microprocessor in a uniform format thereby facilitating component interchangability.

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RESEARCH

on

COMPUTER-AIDED DESIGN

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Abstract

The use of continuation methods in the computer-aided analysis of electronic circuits is surveyed. Such methods are especially suitable when one desires to compute the solutions to a family of circuit analysis problems as a function of a continuous parameter. Applications of the concept to the location of multiple solutions to nonlinear equations, the computation of input-output characteristics for nonlinear networks, large-change sensitivity analysis, and the computation of multivariable Nyquist plots are reviewed.

Introduction

Over the past decade computer-aided design[5,9,12,27,33] has become a fundamental tool of the electronic circuit designer. Indeed, the design of integrated circuits containing several thousands of elements is now a routine procedure in the semiconductor houses, yet one that could not be successfully carried out without computer aids. The difficulty here does not lie wholely with the complexity of the circuits involved, though that is a major justification for the use of computer aids, but also with the fact that the parasitic effects of the integration process cannot be readily included in a circuit breadboarded from discrete components. As such, the experience (and inertia) of the circuit designer has slowly given way to the onslaught of the computer.

This same period has seen the development of a number of sophisticated codes for the analysis of large electronic circuits such as SCEPTRE, NET, CIRCUS, etc., and an equal number of interactive codes which permit the circuit designer to interface with his computer aids through some sort of

graphic display. For the most part these codes were developed in industry under the pressure of a specific application and thus, except for the use of sparse matrix techniques[#] (required to handle the large systems of equations which are encountered), they use routine numerical methods.

In the past several years this situation has begun to change the advent of new circuit theoretic techniques specifically designed for implementation in circuit analysis codes. Most notable of these have been the sparse tableau algorithms of Hachtel, Brayton,[9,25,26] and the continuation algorithms of Branin,[7,8] and Broyden,[10] Davidenko,[18,19] Chua and Ushida,[17] for the solution of the sets of simultaneous nonlinear equations arising in the analysis of nonlinear circuits. The purpose of the present report is to survey these continuation algorithms and a number of related continuation algorithms applicable to the problems of computeraided analysis of electronic circuits. These latter include an algorithm for the computation of the input-output characteristics and/or the AC analysis of nonlinear resistive networks, an algorithm for the large-change sensitivity analysis of linear circuits, and an algorithm for the computation of multivariable Nyquist plots.

The basic idea of all continuation methods is to convert the solution of a parameterized family of algebraic problems, P(r,), into the solution of a differential equation.

$$\dot{x}(r) = f[x(r), r] \qquad 1.$$

where x(r) is the solution of the rth problem, P(r). Then, if one can find

#This literature has been reported in several contexts. For the general theory of sparse matrices see references, 9, 24, 26, 37, 43, and 49, for application to computer-aided design see refs. 2, 6, 20, 22, 25, 27, 30, 31, 32 and 49; and for applications to power networks see ref. 41, 42, 44 and 45.

the solution, $x(r_o)$, of an initial problem, $P(r_o)$, by classical (or other) methods, the solutions to the other problems can be obtained by integrating equation (1) with $x(r_o)$ as an initial condition.

Although it appears that we have made our problem harder by converting the solution of an algebraic problem into the solution of a differential equation this is not the case, and in fact, when the integration of equation (1) is reduced to a numerical process it often proves to be less cumbersome than direct methods of solution. Indeed, this is illustrated by classical quasi-continuation algorithm* for the solution of a set of simultaneous nonlinear equations. Here we desire to find an n-vector which satisfies

$$f(x) = 0 \qquad 2.$$

where f is a continuously differential function mapping R^n to R^n. Rather than solving (2) directly, however, we integrate the differential equation[4,7,8,12,23]

$$\dot{x}(r) = -(\frac{\partial f}{\partial x})^{-1} \; f[x(r)] \qquad 3.$$

which has a stable equilibrium at the solutions of (2). As such, if we begin with any initial condition, $x(r_o)$, sufficiently close to a solution the trajectory, $x(r)$, will converge to the solution as r goes to infinity. Thus, upon reducing the integration of equation (3) to a numerical algorithm, an algorithm for solving the original set of nonlinear equations is defined in the process. In particular, if Euler integration is used we obtain

*This is not precisely a continuation algorithm in the sense defined above since the solution of the given problem is the limiting value of $x(r)$ and we really have no interest in the intermediary values. The underlying philosophy is, however, the same.

the iterative scheme

$$x(k+1) = x(k) - h \left(\frac{\partial f}{\partial x}\right)^{-1} f[x(k)] \qquad 4.$$

which will be recognized as the classical Newton-Raphson algorithm for the solution of the given equation if the step size h is set equal to one.

Another classical example of the solution of a family of algebraic problems by continuation is the inversion of a family of matrices, M(r), via the solution of

$$\dot{M}^{-1}(r) = -M^{-1}(r)\dot{M}(r)M^{-1}(r) \qquad 5.$$

where $M^{-1}(r_0)$ is computed by classical means. Indeed, the technique can be used to compute the inverse of a single matrix, N, by letting

$$M(r) = rN + (1-r)I \qquad 6.$$

in which case M(0) = I and no initial inverse need be computed. Once again, however, when the integration of the differential equation is reduced to a numerical process a classical series expansion for the matrix inverse results.

The point is, that even though we have taken a roundabout approach to formulating our algorithms, the resultant numerical processes are no more complex than those which might have been formulated directly, Indeed, in the above example our algorithms coincides with a classical algorithm for the solution of the given problem. Moreover, in a true continuation algorithm where one is interested in the entire trajectory, x(r), rather

than only its final value the resultant numerical processes often prove to be more efficient than classical algorithms.

Over the past several years such continuation algorithms have received considerable interest from the numerical analysis community. Applications to polynomial root finding, boundary value problems in ordinary differential equations, parameter identification and eigenvalue problems in differential operators are reviewed in the recent paper of Wasserstrom[48] and will not be discussed here. Other significant applications of the continuation concept include the solution of L_p optimization problems[4] by continuation with the classical L_2 solution taken as an initial condition and the systematic location of multiple solutions to sets of simultaneous nonlinear equations.[7,8,13]

Multiple Solutions of Nonlinear Equations

A systematic search method based on Branin's approach[7,8] has recently been developed by Chao, Liu and Pan[13,14] for obtaining multiple solutions of a nonlinear equation

$$f(x) = 0 \qquad 7.$$

where f is a continuously differentiable function from R^n into itself. The method is capable of finding all of the solutions provided that the solution of any (n-1) equations of (7) is a simple curve - a continuously differentiable curve which does not intersect itself. This technique serves as the key to the development of several of the continuation algorithms to be discussed and is hence reviewed here.

The algorithm involves numerical integration of a set of associated differential equations of the form

$$\dot{f}_i[x(t)] = -f_i[x(t)], \quad f_i[x(0)] = 0, \; i = 1, 2, \ldots, n-1$$

$$\dot{f}_n[x(t)] = \pm f_n[x(t)], \quad f_n[x(0)] = f_{no} \qquad 8.$$

or in the x-space

$$\dot{x} = (\partial f/\partial x)^{-1} \, (-f, \ldots, -f_{n-1} \; \pm f_n)^T, \; x_o \varepsilon \, \ell \qquad 9.$$

along the space curve, ℓ, defined by the intersection of solution manifolds for

$$f_i[x(t)] = 0 \quad i = 1, 2, \ldots, n-1 \qquad 10.$$

The transition in sign of f_n should be made at the solution points and points where the Jacobian determinant changes sign. Equation (9) may be solved by any numerical integration technique. For example, using Euler's method, (9) reduces to an iterative algorithm

$$x_{k+1} = x_k + hJ^{-1}(x_k)[-f_1(x_k), \ldots, -f_{n-1}(x_k), \pm f_n(x_k)]^T$$

$$x_o \varepsilon \ell, \qquad k = 0, 1, 2, \ldots . \qquad 11.$$

It is interesting to note that with only the minus sign and for any initial x_o not necessarily on ℓ, (10) reduces to the well known Netwon"s method. It is observed from (8) that if $x_o \, \varepsilon \ell$, the corresponding trajectory x(t) resulting from (9) remains in ℓ. Since in computational practice, the f_i, for i = 1,2,..., n-1, at each iteration using (9) will not be precisely zero, their signs are kept at the constant negative to insure that the computed trajectory does not stray too far from ℓ. If ℓ is simple, we

can obtain all the solutions by a complete traversal of the space curve ℓ. In applying (9) to locate all the solutions, it is thus essential that there exists such a simple curve. In addition, a starting point lying on or close to this space curve must be used. Theorems that guarantee this existance of a simple curve, ℓ, and the global convergence from any initial guess to a point on ℓ are all detailed in 14 and will not be repeated here. The algorithm is not only capable of finding multiple solutions, it also efficiently computes the solution of any (n-1) equations of (7), the distinct property of which is found useful in many engineering applications.

Input-Output Characteristics

Operating points, driving point plots and transfer characteristic plots are basic concepts of fundamental importance in the analysis of nonlinear resistive networks. The operating point problem is nothing but the determination of the solutions of network equations of the form described in (7). Mathematically speaking, the latter two concepts are essentially the same; they are input-output characteristic plots showing the relationship between a driving source u_i and a certain output variable x_j. Several methods[16,29,34] are available for piecewise-linear analysis of resistive nonlinear networks. Recently, Chua and Ushida[17] developed a switching-parameter algorithm for solving nonlinear equations and the technique is also capable of finding input-output plots.

A technique is proposed[15] for obtaining input-output characteristic plots for nonlinear resistive networks where the characteristic curve of

each nonlinear element is continuously differentiable. The network equation, in general, can be represented by

$$f(x,u) = 0 \tag{12.}$$

where f is a continuously differentiable function, x is an n-vector of network variables, the m-vector u denotes the input sources and the Jacobian matrix $\partial f/\partial x$ is assumed to be nonsingular at the solution points. Since in finding input-output characteristic plots, all input sources except one, say u_i, are considered to be fixed constrants, the network equation (12) thus reduces to

$$f(x,u_i) = 0. \tag{13.}$$

Suppose it is now required to find the input-output characteristic plot x vs. u_i, for all $u_i^- \leq u_i \leq u_i^+$. Instead of solving (13) directly, a set of (n+1) equations consisting of (13) and an auxiliary equation $f_{n+1} = 0$ of the form

$$\begin{aligned} f(x,\ u_i) &= 0 \\ f_{n+1} &\triangleq u_i^+ - u_i = 0 \end{aligned} \tag{14.}$$

is considered. Following the technique described above, a continuous input-output curve can be traced automatically by integrating a set of associated differential equations

$$\begin{aligned} &\frac{df}{dt} = -f, \qquad f[x(0),\ u(0)] = f(x^-,\ u_i^-) = 0 \\ &\frac{df_{n+1}}{dt} = \pm f_{n+1},\ f_{n+1}(0) = u_i^+ - u_i^- \triangleq f_{n+1,0} \end{aligned} \tag{15.}$$

where x^- is a solution of (17) subject ot $u_i = u_i^-$.

The initial conditions on f and f_{n+1} are such that the starting point must lie on the solution manifold of (13) which is the desired TC plot. Since (f, f_{n+1}) is not a function of t explicitly, the chain rule of differentiation is applied so that in the (x, u_i) - space, (15) reduces to

$$\begin{bmatrix} \dot{x} \\ \\ \dot{u}_i \end{bmatrix} = J^{-1} \begin{bmatrix} -f \\ \\ \pm f_{n+1} \end{bmatrix} , \quad \begin{bmatrix} x(0) \\ \\ u_i\ (0) \end{bmatrix} = \begin{bmatrix} x \\ \\ u_i^- \end{bmatrix} \qquad 16.$$

where

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u_i} \\ \\ 0 & 1 \end{bmatrix} \qquad 17.$$

Equation (16) may be solved by any existing numerical integration technique and the sign change should again be made at the Jacobian singularities. From the solution of (15)

$$\begin{aligned} f[x(t), u_i(t)] &= f[x(0), u_i(0)]e^{-t} = 0 \\ f_{n+1}[u_i(t)] &= f_{n+1}[u_i(0)]e^{\pm t} = f_{n+1,0}e^{\pm t} , \end{aligned} \qquad 18.$$

it is seen that for any $[x(0), u_i(0)] \in \ell$, the corresponding trajectory $[x(t), u_i(t)]$ resulting from (16) remains in ℓ. The input-output characteristic curve denoted by ℓ can thus be traced from u_i^- to u_i^+ automatically.

The advantage of this method is that the input-output plot is not required to be either input or output controlled. The only restriction is that the plot has to be a simple curve.

AC-Resistive Network Analysis

Resistive networks containing one or more time-varying sources are known as ac-resistive networks. In this case, the network equations can always be written in the form

$$f[x(t), u(t)] = 0 \qquad 19.$$

where u(t) is an m-vector of known time-varying sources and the n-vector x represents network variables. The operating point of the network, which is the solution of (19), is now a function of t. The presence of the time parameter greatly increases the complexity of the problem. Theroetically, the problem can be treated by simply solving the problem at different instants of time which would probably be a very time-consuming practice. A continuation algorithm is proposed for finding the solution of (19) for all t automatically.

Consider a system of differential equations of the form

$$\begin{aligned} \dot{f} &= -f, \qquad f[x(0), u(0)] = 0 \\ \dot{u} &= w, \qquad u(0) = u_o \end{aligned} \qquad 20.$$

where u is assumed to be continuously differentiable and its time derivative, denoted by w, is a known function of t. The initial condition on f is such that $x(0) = x_o$ must be a solution point of (19) subject to $u(0) = u_o$. In

the x-space (20) is equivalent to

$$\dot{u} = -J^{-1}f - j^{-1}\left(\frac{\partial f}{\partial u}\right) w, \quad x(0) = x_o \tag{21.}$$

where $J \triangleq \partial f/\partial x$.

Equation (21) may now be solved by using any numerical integration technique. If J is never nonsingular at the solution points (as assumed), then $|J|$ will never vanish along the trajectory of (21). Therefore once an initial solution point x_o is found, the solution of (19) for all t can be obtained automatically from (21).

If $f[x, u_0] = 0$ has multiple solutions, then all the initial solution points must be used as initial conditions for (21) in order to obtain a complete family of solution curves.

A more direct approach may be obtained by differentiating (19) with respect to t yiedling

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial u}\dot{u} = 0 \tag{22.}$$

or

$$\dot{x} = -J^{-1}\left(\frac{\partial f}{\partial u}\right) w. \tag{23.}$$

The only difference between the two proposed schemes (21) and (23) is the appearance of the correction term $-J^{-1}f$ in (21). Theoretically along the solution curve of (24), f = 0. In actual computation, however, f may not be identically zero. The use of the minus sign in front of f in (22) tends to prevent divergence and hence to minimize computational error.

Large Change Sensitivity Analysis

The sensitivity of system performance is an important aspect of system design. The system characteristics may vary with environmental changes, such as temperature or uncertainties in the parameters that characterize the system. The analysis of a system's sensitivity to parameter variations is concerned with the determination of changes in network or system behavior which will be produced by changes of one of its parameters. The study of system sensitivity to small parameter variations around a reference position reduces to the well known perturbation problem. For large parameter variations, however, it is an entirely different problem which cannot be treated by perturbation methods.[46] Two approaches for large change sensitivity analysis[21,63] in two different contexts are discussed below.

Sensitivity to large parameter variations, or global sensitivity, for nonlinear resistive networks can be formulated in a manner similar to the one of finding input-output characteristic plots.

Suppose it is now desired to find the operating point of the network as a function of a network parameter α_i can be written in the functional form

$$f(x, \alpha_i) = 0 \qquad 24.$$

where f, as before, is assumed to be continuously differentiable. Since (24) is in the same form as (13), the operating point x as a function of α_i can be obtained by integrating (16) with u_i replaced by α_i.

Our second approach[40] for sensitivity to large parameter variations is based on the component connection model for a large-scale system[35,36,39] whose explicit algebraic connection equations for a large-scale system yield analytic expressions for sensitivity analysis. It is assumed that the system is characterized by a mapping which relates the output y to the input u in the frequency domain by

$$y = Su. \tag{25.}$$

The component connection model of (25) consists of writing a set of linear algebraic equations of the form

$$\begin{bmatrix} a \\ y \end{bmatrix} \quad \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \quad \begin{bmatrix} b \\ u \end{bmatrix} \tag{26.}$$

where the L_{ij} are connection matrices and the component input a is mapped into the component output b by a linear transformation

$$b = Za. \tag{27.}$$

It can be shown by a straight-forward algebraic computation from (25), (26) and (27) that S is given by

$$S = L_{22} + L_{21} \ (I - ZL_{11})^{-1} \ ZL_{12}. \tag{28.}$$

With the assumption that the output is not a function of the input explicitly, i.e., $L_{22} = 0$, (28) reduces to

$$S = L_{21}QL_{12} \tag{29.}$$

where Q is an intermediate matrix defined by

$$Q = (I - ZL_{11})^{-1}Z. \qquad 30.$$

In order to derive an algorithm for computing the transfer function matrix sensitivity to large component variations, we differentiate (30) with respect to a potentially variable parameter r to yield

$$\frac{dQ(r)}{dr} = Q(r)Z^{-1}(r)\frac{dZ(r)}{dr}\ [I + L_{11}Q(r)]. \qquad 31.$$

Since for a fixed system connection, L_{11}, $Z(r)$ and hence $\dot{Z}(r)$ are known analytically, the right-hand-side of (31) can easily be computed for a given r. We first compute $Q(r_o)$ for a nominal value r_o of the parameter r by classical analysis techniques and then integrate equation (36) for all $r \geq r_o$. Once $Q(r)$ is known, then $S(r)$ can be calculated directly from (34). Note that if the variable parameter r is chosen to be the angular frequency ω, then the frequency response $S(j\omega)$ as a function of ω can be computed in a similar manner. Starting with an initial condition $Q(r_o)$, it is seen from (31) that the integration of $Q(r)$ involves only matrix multiplications and a simple inversion of the usually elementary component matrix $Z(r)$. Computationally, the technique thus requires much less time than would be needed to reanalyze the system which would normally require an inversion at each step of a much more complicated matrix $(I-ZL_{11})$ as indicated in (30).

Multivariable Nyquist Plots

Although the classical Nyquist criterion had long been thought to be inapplicable to multivariable systems Barman and Katznelson[1] have recently

formulated a Nyquist-like test for such systems. Here, the classical Nyquist plot is replaced by a plot made up of the eigenvalue loci of the (open loop gain) transfer function matrix $G(j\omega)$. More precisely, for each fixed value of ω, $G(j\omega)$ has n (not necessarily distinct) eigenvalues $\lambda_i(j\omega)$. As such, if one lets ω evlove one can compute n eigenvalues loci for $G(j\omega)$ by plotting the n functions $\lambda_i(\omega)$ as functions of ω. In general the $\lambda_i(\omega)$ are not unique but they can always be formulated as piecewise analytic functions.[1] We form a "multivariable Nyquist plot" from the $\lambda_i(\omega)$ by concatenating the analytic segments obtained by plotting the various $\lambda_i(\omega)$ to form a closed curve in the complex plane. Using this plot it has been shown[1] that the multivariable feedback system with a stable open loop gain $G(j\omega)$ will be stable if and only if its multivariable Nyquist plot does not encircle -1.

In order to apply the multivariable Nyquist criterion, it is thus necessary to compute the eigenvalue loci of $G(\omega)$ as a function of frequency. For a given frequency, the eigenvalues can be calculated by using classical techniques. Since the eigenvalues are functions of frequency, normally one would have to repeat the entire computational procedure for each frequency. In the actual stability analysis, this repetition is, however, impractical. Our first approach[38] to the formulation of a continuation algorithm for computing the eigenvalue loci follows directly from that described by Faddeev and Faddeeva[21] and Van Ness et al[47]. The eigenvalues $\lambda_i(\omega)$ of $G(j\omega)$ and their complex conjugates $\overline{\lambda}_i(\omega)$ satisfy

$$G(j\omega)X_i(\omega) = \lambda_i(\omega)X_i(\omega) \quad i = 1, 2, \ldots, n \qquad 32.$$

and

$$G^*(j\omega)V_i(\omega) = \bar{\lambda}_i(\omega) \quad i = 1, 2, \ldots, n \tag{33.}$$

where $X_i(\omega)$ and $V_i(\omega)$ are the corresponding eigenvectors of $\lambda_i(\omega)$ and $\bar{\lambda}_i(\omega)$ respectively, and $G^*(j\omega)$ is the complex conjugate transpose matrix of $G(j\omega)$. We differentiate (32) with respect to ω to yield

$$\frac{dG}{d\omega}X_i + G\frac{dX_i}{d\omega} = \frac{d\lambda_i}{d\omega}X_i + \lambda_i\frac{dX_i}{d\omega}. \tag{34.}$$

Taking the scalar product on both sides of (34) with V_j gives

$$< \frac{dG}{d\omega}X_i, V_j > + < G\frac{dX_i}{d\omega}, V_j> = \frac{d\lambda_i}{d\omega} < X_i, V_j > + \lambda_i< \frac{dX_i}{d\omega}, V_j>. \tag{35.}$$

Using (33), the identity

$$<G\frac{dX_i}{d\omega}, V_j > = \lambda_j< \frac{dX_i}{d\omega}, V_j> \tag{36.}$$

and letting $j = i$, we obtain

$$\frac{d\lambda_i}{d\omega} = \frac{< \frac{dG}{d\omega}X_i, V_i>}{<X_i, V_i>}, \quad i = 1, 2, \ldots, n. \tag{37.}$$

In deriving a differential equation for X_i, we let

$$\frac{dX_i}{d\omega} = \sum_{j=1}^{n} \alpha_{ij} X_j, \quad i = 1, 2, \ldots, n. \tag{38.}$$

In view of the fact that the eigenvectors are unique, one may assume without loss of generality that $\alpha_{ii} = 0$. The coefficients α_{ij} for $i \neq j$ are obtained by forming the scalar product with V_j on both sides of (37) and using the orthogonality conditions

$$< X_i, V_j > = 0 \quad i \neq j \tag{39.}$$

to yield

$$\alpha_{ij} = \frac{< \frac{dX_i}{d\omega} , V_j >}{< X_j, V_j >} . \qquad 40.$$

Substituting (36) and (39) into (35), leads to

$$< \frac{dX_i}{d\omega}, V_j> = \frac{1}{\lambda_i - \lambda_j} < \frac{dG}{d\omega} X_i, V_j >. \qquad 41.$$

Combining (40) and (41) yields the desired expression for α_{ij}

$$\alpha_{ij} = \frac{< \frac{dG}{d\omega} X_i, V_j >}{(\lambda_i - \lambda_j)<X_j, V_j>} \quad i \neq j. \qquad 42.$$

In a similar manner we obtain

$$\frac{dV_i}{d\omega} = \sum_{j=1}^{n} \beta_{ij} V_j, \quad i = 1, 2, \ldots, n \qquad 43.$$

where

$$\beta_{ii} = 0, \quad \beta_{ij} = \frac{< \frac{dV_i}{d\omega}, X_j >}{<V_j, X_j>} \quad i \neq j. \qquad 44.$$

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RESEARCH

on

STOCHASTIC CONTROL AND ESTIMATION

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Abstract

Once a state model for a stochastic linear plant is obtained, the analytical aspects of designing a feedback controller can be conceptually dichotomized. The first part includes: selection of a performance measure that reflects a-priori design specifications, selection of performance indices, and selection of a controller via optimization of these indices. The second part includes selection of post design performance measures and the attainment of statistical or probabilistic descriptions of these performance measures.

Introduction

We will collectively refer to the first part as design and the second part as performance analysis. For certain classes of stochastic linear control systems the performance analysis problem has been solved; see [7] and [14]. The purpose of this paper is to present the formulation of a complete set of statistics of the integral quadratic design performance measure normally employed in stochastic linear control system design. These statistical formulations are expressed explicitly in terms of dynamical variables related to the plant observations and control action. The ultimate value of these formulations is that they provide the analytical basis for the selection of design indices which can be used to systematically select a controller via optimization.

There are many ways of selecting indices and we do not treat this topic here. In Section II we describe the system, the performance measure and the control objective. Section III contains the development of a complete statistical description of this performance and Section IV contains the formulation of these statistics in terms of the filtered estimate of the plant state. The paper is concluded with comments on how the results might be employed and suggestions for further research.

The System Description and Performance

Let R^n denote the n-fold Cartesian product of the real line, and let I denote the real line interval $[t_o, t_f]$. We wish to control the noisy linear system described on I by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)u(t) + \xi(t), \tag{1}$$

and

$$z(t) = C(t)x(t) + \theta(t), \tag{2}$$

where the state $x(t)\ \varepsilon R^n$, the control action $u(t)\ \varepsilon R^m$, and the observation $z(t)\ \varepsilon R^r$. The initial condition for (1), $x(t_o)$, is assumed to be Gaussian with mean

$$x_o = E\{x(t_o)\} \tag{3}$$

and covariance

$$\Sigma_o = E\{[x(t_o)-x_o][x^T(t_o)-x_o^T]\} \tag{4}$$

where $(^T)$ denotes matrix transposition. The state process noise, $\xi(t)$ and the observation noise are zero-mean Guassian-white with

$$E\{\xi(t)\theta^T(\tau)\} = 0, \tag{5}$$

$$E\{[x(t_o)-x_o]\xi^T(t)\} = 0, \tag{6}$$

$$E\{[x(t_o)-x_o]\theta^T(t)\} = 0, \tag{7}$$

$$E\{\xi(t)\xi^T(\tau)\} = \Xi(t)\delta(t-\tau), \tag{8}$$

and

$$E\{\theta(t)\theta^T(\tau)\} = \Theta(t)\delta(t-\tau), \tag{9}$$

where $\Xi(t)$ and $\Theta(t)$ are positive semi-definite and positive definite, respectively, on I.

We require that the control action, $u(t)$, be a causal function of the observation. That is,

$$u(t) = \psi(t, z(\tau):\tau\varepsilon[t_o,t]) \tag{10}$$

where $\psi(t, \cdot)$ satisfies certain technical assumptions stated in [14]; also see [15]. All matrix functions on I and the mapping $\psi(t,\cdot)$ are assumed to be smooth enough to guarantee mean-square continuity of the state process on I.

For the purposes of design we define a "design-performance-measure", J, by

$$J = x^T(t_f)Sx(t_f) + \int_{t_o}^{t_f} [x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt, \tag{11}$$

where the terminal penalty weighting, S, is symmetric and positive semi-definite as is the weighting $Q(t)$ on I. The weighting $R(t)$ is symmetric and positive definite on I, and both $Q(t)$ and $R(t)$ are continuous on I. These weighting matrices are given values by the designer that reflect a-priori design specifications involving the relative importance of state regulation and control effort. The design objective is to choose $u(t)$ in (10) so system performance is "good" in some sense.

The functional, J, assigns a non-negative real number to each sample run of the control system with small values implying good performance, However, the question of quality of performance is multiply clouded. First, J is random so it is only meaningful to refer to J in a statistical or probabilistic sense. Second, since J is the sum of terms representing measures of state regulation

and control effort, the individual quality of these measures is not apparent in a broad statistical description of J. We will demonstrate these subtleties in another paper. Meanwhile we are content to use J as an a-priori performance measure upon which we will base our selection of control action, u(t). But, before this selection can be made we must describe J statistically or probabilistically. Based upon intuition gained in our previous work [7] we choose a statistical approach.

A Complete Statistical Description of J

Let F_σ be the sigma-algebra induced by the observation $\{z(\tau):\tau\varepsilon[t_o,\sigma]\}$. When $\sigma = t_f$ we shall write F without a subscript. We will now generate conditional statistics of J . Expand the process modeled by (1) in an orthonormal series,

$$x(t) \sim \sum_{i=1}^{\infty} x_i\phi_i(t), \qquad t\varepsilon I, \tag{12}$$

where the x_i's are scalar random variables given by

$$x_i = x^T(t_f)\, S\phi_i(t_f) + \int_{t_o}^{t_f} x^T(t)Q(t)\phi_i(t)dt, \qquad \forall i, \tag{13}$$

and the nonrandom vector valued functions, $\phi_i(t)$, are chosen to satisfy the orthonormality condition

$$\phi_i^T(t_f)S\phi_j(t_f) + \int_{t_o}^{t_f}\phi_i^T(t)Q(t)\phi_j(t)dt = \delta_{ij}, \qquad \forall i, j. \tag{14}$$

In addition, we require that the x_i's be conditionally uncorrelated, that is

$$E\{[x_i-m_i][x_j-m_j]|F\} = \lambda_i\delta_{ij}, \qquad \forall i, j, \tag{15}$$

where m_i is the conditional mean of x_i given by

$$m_i = E\{x^T(t_f)|F\}\, S\phi_i(t_f) + \int_{t_o}^{t_f} E\{x^T(t)|F\}Q(t)\phi_i(t)dt, \qquad \forall i, j. \tag{16}$$

A necessary and sufficient condition for (15) given (13) and (14) is that $\phi_i(t)$ and λ_i satisfy

$$\lambda_i \phi_i(t) = \int_{t_o}^{t_f} \Gamma(t,\tau) Q(\tau) \phi_i(\tau) d\tau + \Gamma(t,t_f) S\phi_i(t_f), \quad t \in I, \quad \forall i, \tag{17}$$

where $\Gamma(t,\tau)$ is the smoothed estimate error covariance kernel of the state process. That is, let the smoothed estimate of $x(t)$ be denoted by

$$\hat{x}(t|t_f) = E\{x(t)|F\}, \quad t \in I, \tag{18}$$

and let $\Gamma(t,\tau)$ be given by

$$\Gamma(t,\tau) = E\{[x(t) - \hat{x}(t|t_f)]\,[x(\tau) - \hat{x}(\tau|t_f)]|F\} \quad t,\tau \in I. \tag{19}$$

As a consequence of the linear-Gaussian assumptions of Section II and the technical assumptions on $\psi(t,\cdot)$,

$$\Gamma(t,\tau) = E\{\Gamma(t,\tau)\} \tag{20}$$

Therefore each λ_i is nonrandom.

Under the assumptions we have made, J is finite with probability one; see Doob [1]. It follows that the series in (12) converges in the square integrable sense. That is,

$$J = \sum_{i=1}^{\infty} x_i^2 + \int_{t_o}^{t_f} u^T(t) R(t) u(t) dt, \tag{21}$$

where convergence is with probability one; see Kolmogorov and Fomin [5].

Since $x(t)$ is conditionally Gaussian each x_i is conditionally Gaussian. But, we have forced the x_i to be conditionally uncorrelated, thus they are conditionally independent as are their squares. The conditional characteristic function of each x_i^2 term in (21) is of the noncentral chi-square type given by,

$$C_{x_i^2|F}(j\omega) = (1-2j\omega\lambda_i)^{-\frac{1}{2}}\exp[j\omega m_i^2(1-2j\omega\lambda_i)^{-1}]. \tag{22}$$

The conditional characteristic function of J follows as,

$$C_{J|F}(j\omega) = \Big[\prod_{i=1}^{\infty}(1-2\omega\lambda_i)^{-\frac{1}{2}}\Big]\cdot\exp\Big[j\omega\int_{t_o}^{t_f}u^T(t)R(t)u(t)dt + \sum_{i=1}^{\infty}j\omega m_i^2(1-2j\omega\lambda_i)^{-1}\Big]. \tag{23}$$

In our previous work [6], [7] we have observed that in the Linear-Quadratic-Gaussian class of systems the second characteristic function generates the most tractable statistical forms. The second conditional characteristic function, $T_{J|F}(j\omega)$, is defined as the natural logarithm of $C_{J|F}(j\omega)$, that is,

$$T_{J|F}(j\omega) = \ln[C_{J|F}(j\omega)]. \tag{24}$$

The MacLaurin series representation of $T_{J|F}(j\omega)$ is

$$T_{J|F}(j\omega) = \sum_{i=1}^{\infty}\kappa_{i|F}\frac{(j\omega)^i}{i!} \tag{25}$$

where the coefficients $\kappa_{i|F}$ are called conditional cumulants. Utilizing (23), it can be easily shown that the first conditional cumulant of J is given by

$$\kappa_{1|F} = \sum_{i=1}^{\infty} \lambda_i + \sum_{i=1}^{\infty} m_i^2 + \int_{t_o}^{t_f} u^T(t)R(t)u(t)dt, \tag{26}$$

while the remaining conditional cumulants are of the form

$$\kappa_{k|F} = (k-1)!2^{k-1} \sum_{i=1}^{\infty} \lambda_i^k + k!2^{k-1} \sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1}, \quad k > 1. \tag{27}$$

Although the conditional cumulants as given by (26) and (27) are complete in the sense that any statistic of J can be derived from them (an interesting exercise) they are not in an attractive form for the control system designer since they are not expressed in terms of system variables. To accomplish this we must attack the series expressions in (26) and (27). The first step is to define "iterated kernels"

$$\Gamma^{(1)}(t,\tau) \triangleq \Gamma(t,\tau) \tag{28}$$

and

$$\Gamma^{(k)}(t,\tau) \triangleq \Gamma(t, t_f)\, S\Gamma^{(k-1)}(t_f,\tau) + \int_{t_o}^{t_f} \Gamma(t,\sigma)\, Q(\sigma)\Gamma^{(k-1)}(\sigma,\tau)\, d\sigma, \quad k > 1. \tag{29}$$

It can be inductively shown using (14) and (17) that

$$\Gamma^{(k)}(t,\tau) = \sum_{i=1}^{\infty} \lambda_i^k \phi_i(t)\phi_i^T(\tau). \tag{30}$$

It follows that the expression, $\sum_{i=1}^{\infty} \lambda_i^k$, can be written as

$$\sum_{i=1}^{\infty} \lambda_i^k = \mathrm{Tr}[S\Gamma^{(k)}(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma^{(k)}(t,t)\, dt], \quad k \geq 1, \tag{31}$$

where Tr [·] denotes the trace of the enclosed matrix. Utilizing (16), (18) and (30) it follows that

$$\sum_{i=1}^{\infty} m_i^2 \lambda_i^{k-1} = \hat{x}^T(t_f|t_f) \; S\Gamma^{(k-1)}(t_f,t_f) \; S\hat{x}(t_f|t_f)$$

$$+ \hat{x}^T(t_f|t_f)S \int_{t_o}^{t_f} \Gamma^{(k-1)}(t_f,t)Q(t)\hat{x}(t|t_f)\,dt$$

$$+ \int_{t_o}^{t_f} \hat{x}(t|t_f)Q(t)\,\Gamma^{(k-1)}(t,t_f)dt\; S\hat{x}(t_f|t_f)$$

$$+ \int_{t_o}^{t_f}\int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt, \quad k>1. \tag{32}$$

For the case, k=1, it is easily seen that

$$\sum_{i=1}^{\infty} m_i^2 = \hat{x}^T(t_f|t_f)S\hat{x}(t_f|t_f)$$

$$+ \int_{t_o}^{t_f} \hat{x}^T(t|t_f)Q(t)\hat{x}(t|t_f)dt. \tag{33}$$

Consider the last term in (32) and note that it contains a symmetric (in argument) integrand. Therefore it may be rewritten as

$$\int_{t_o}^{t_f}\int_{t_o}^{t_f}\hat{x}^T(t|t_f)Q(t)\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt$$

$$= 2\int_{t_o}^{t_f}\hat{x}^T(t|t_f)Q(t)\int_{t_o}^{t}\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau dt \tag{34}$$

Define the new n-vector valued variable

$$\eta_{k-1}(t) \triangleq \int_{t_o}^{t}\Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|t_f)d\tau \tag{35}$$

and note that

$$\eta_{k-1}(t_o) = 0. \tag{36}$$

The conditional cumulants can now be written as

$$\kappa_{1|F} = \hat{x}^T(t_f|t_f)\,S\hat{x}(t_f|t_f) + \int_{t_o}^{t_f}[\hat{x}^T(t|t_f)Q(t)\hat{x}(t|t_f)$$

$$+ u^T(t)R(t)u(t)]dt \tag{37}$$

$$+ \mathrm{Tr}[S\Gamma(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma(t,t)dt],$$

and

$$\kappa_{k|F} = k!2^{k-1}[\hat{x}^T(t_f|t_f)S\Gamma^{(k-1)}(t_f,t_f)S\hat{x}(t_f|t_f)$$

$$+ 2\hat{x}^T(t_f|t_f)S\eta_{k-1}(t_f) + 2\int_{t_o}^{t_f}\hat{x}^T(t|t_f)Q(t)\eta_{k-1}(t)dt]$$

$$+ (k-1)!2^{k-1}\mathrm{Tr}[S\Gamma^{(k)}(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma^{(k)}(t,t)dt], \quad k > 1. \tag{38}$$

Equations (37) and (38) provide us with the conditional cumulants of the design-performance measure, J, expressed in terms of the smoothed estimate of the process and the corresponding error covariance kernel. A complete statistical description of J is now at hand since (as mentioned earlier) any statistic of J can be expressed in terms of the conditional cumulants. For example, denoting the unconditional cumulants of J by κ_k, we have

$$E\{J\} = \kappa_1 = E\{\kappa_{1|F}\} \tag{39}$$

$$\text{Var}\{J\} = \kappa_2 = E\{\kappa_{2|F}\} + \text{Var}\{\kappa_{1|F}\} \tag{40}$$

and in general

$$\kappa_k = E\{\kappa_{k|F}\} + \{\text{statistics of lower order conditional cumulants}\} \tag{41}$$

The relationship between noncentral moments, μ_k, and cumulants is well-known [4] and given by

$$\mu_{k+1} = \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j} \kappa_{j+1} \ . \tag{42}$$

The Filtered Estimate Formulation

Those familiar with the traditional minimum mean LQG problem may be a little suspicious of equation (37) since it is well known that $E\{\kappa_{1|F}\}$ is normally expressed in terms of the filtered estimate and its corresponding error covariance with precisely the same structure as (37) under expectation; see [14]. To demonstrate the equivalence of these two formulations note that in view of (10) the smoothed estimate can be expressed as

$$\hat{x}(t|t_f) = \hat{x}(t|t) + \int_t^{t_f} K(t,\tau)\, C^T(\tau)\theta^{-1}(\tau)\, C(\tau)\, \nu(\tau|\tau)d\tau, \quad (43)$$

see [8], where

$$\hat{x}(t|t) = E\{x(t)|F_t\}, \quad (44)$$

$$K(t,\tau) = E\{[x(t) - \hat{x}(t|t)][x^T(\tau) - \hat{x}^T(\tau|\tau)]\} \quad (45)$$

and the "innovation" [2], $\nu(\cdot|\cdot)$, is given by

$$\nu(t|t) = C(t)[x(t) - \hat{x}(t|t)] + \theta(t). \quad (46)$$

The smoothed error covariance can also be expressed as

$$\Gamma(t,\tau) = K(t,\tau) - \int_{tv\tau}^{t_f} K(t,\sigma)C^T(\sigma)\theta^{-1}(\sigma)\, C(\sigma)\, K(\sigma,\tau)d\sigma \quad (47)$$

where $tv\tau$ means max $[t,\tau]$.

Substitution of (43) and (47) into (37) and application of expectation immediately yields

$$\kappa_1 = E\{\kappa_{1|F}\} = E\{\hat{x}^T(t_f|t_f)S\hat{x}(t_f|t_f) + \int_{t_o}^{t_f}[\hat{x}^T(t|t)Q(t)\hat{x}(t|t)$$

$$+ u^T(t)R(t)u(t)]dt\} + Tr[S\Sigma(t_f) + \int_{t_o}^{t_f} Q(t)\Sigma(t)dt], \qquad (48)$$

where

$$\Sigma(t) \triangleq K(t,t) \qquad (49)$$

and we have utilized the fact that $\nu(\cdot|\cdot)$ is white noise with covariance

$$E\{\nu(t|t)\nu^T(\tau|\tau)\} = \Theta(t)\delta(t-\tau). \qquad (50)$$

In deriving (48) from (37) only one subtlety arises that might be troublesome to the reader. In particular, two terms of the form

$$E\{\int_{t_o}^{t_f} \hat{x}^T(t|t)Q(t) \int_{t}^{t_f} K(t,\tau)C^T(\tau)\Theta^{-1}(\tau)C(\tau)\nu(\tau|\tau)dt\}$$

$$= Tr[\int_{t_o}^{t_f} Q(t) \int_{t}^{t_f} K(t,\tau)C^T(\tau)\Theta^{-1}(\tau)C(\tau)E\{\nu(\tau|\tau)\hat{x}^T(t|t)\}d\tau dt] \qquad (51)$$

arise.

It is well known [14] that under some technical assumptions on the causal mapping $\psi(t, \cdot)$ in (10)

$$\hat{x}(t|t) = \int_{t_o}^{t} G(t,\tau)\nu(\tau|\tau)d\tau + \int_{t_o}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \Phi(t,t_o)x_o \qquad (52)$$

for some $G(t,\cdot)$, where $\Phi(t,\cdot)$ is the transition matrix associated with $A(t)$ in (1). Since the control $u(\cdot)$ is assumed to be a causal function

of the observation $z(\cdot)$ which in turn can be expressed as a causal function of the innovation $\nu(\cdot|\cdot)$, [2], it follows in view of (50) that the inner most integrand in (51) is zero almost everywhere. Consequently the terms in question vanish under expectation.

For higher order cumulants things aren't as nice. Define the following:

$$\hat{\eta}_{k-1}(t) \triangleq \int_{t_o}^{t} \Gamma^{(k-1)}(t,\tau)Q(\tau)\hat{x}(\tau|\tau)d\tau, \tag{53}$$

$$\overset{*}{\rho}(t) \triangleq \int_{t}^{t_f} K(t,\tau)C^T(\tau)\,\Theta^{-1}(\tau)\nu(\tau|\tau)d\tau, \tag{54}$$

and

$$\rho_{k-1}(t) \triangleq \int_{t_o}^{t} \Gamma^{(k-1)}(t,\tau)Q(\tau)\overset{*}{\rho}(\tau)d\tau. \tag{55}$$

Then substitution of (43) into (35) and (38) yields

$$\begin{aligned}
\kappa_{k|F} = k!2^{k-1}[\ & \hat{x}^T(t_f|t_f)\ S\Gamma^{(k-1)}(t_f,t_f)S\hat{x}(t_f|t_f) \\
& + 2\hat{x}^T(t_f|t_f)S\hat{\eta}_{k-1}(t_f) + 2\hat{x}^T(t_f|t_f)S\rho_{k-1}(t_f) \\
& + 2\int_{t_o}^{t_f}\hat{x}^T(t|t)Q(t)\hat{\eta}_{k-1}(t)dt + 2\int_{t_o}^{t_f}\hat{x}^T(t|t)Q(t)\rho_{k-1}(t)dt \\
& + 2\int_{t_o}^{t_f}\overset{*}{\rho}{}^T(t)Q(t)\hat{\eta}_{k-1}(t)dt + 2\int_{t_o}^{t_f}\overset{*}{\rho}{}^T(t)\ Q(t)\rho_{k-1}(t)dt] \\
& + (k-1)!\ 2^{k-1}\ \mathrm{Tr}[S\Gamma^{(k)}(t_f,t_f) + \int_{t_o}^{t_f} Q(t)\Gamma^{(k)}(t,t)dt], k>1.
\end{aligned} \tag{56}$$

The variable $\hat{n}_{k-1}(t)$, defined in (53) can be generated by a physically realizable linear dynamical system with the filtered estimate as input. Consequently it is implicitly affected by the control action u(t). This system is of order 2n and the derivation of its dynamical equations is left as an exercise. On the other hand, the variable, $\overset{*}{\rho}(t)$, defined in (54) cannot be produced by a causal system but by a noncausal linear dynamical system of order n operating on the innovation process. Thus, $\overset{*}{\rho}(t)$ is unaffected by the control action, u(t), as is $\rho_{k-1}(t)$ in (55).

Consequently the conditional cumulants as given by (56) contain four distinct types of terms:

i. terms independent of the dynamical variables and thus uncontrollable,

ii. terms containing only causally produced variables that are controllable,

iii. terms containing only noncausally produced variables that are uncontrollable, and

iv. terms containing both causally produced variables and noncausally produced variables.

The conditional cumulant descriptions that we have derived form an analytical basis for the selection of performance indices which, in turn, can be utilized in the optimal selection of a controller. This dynamical variable formulation is attractive for such optimization but carries with it a "curse of dimensionality" in that system order will increase linearly with the number of higher order pieces of statistical information included in any selection of performance indices. The noncausally produced variables are troublesome. At this point it is not clear how they should be handled in optimization while enforcing (10). Such questions are the subjects of future research.

V. CONCLUSIONS

How might these formulations be utilized? There are many possible answers to this question. For example, let us assume that the design objective is to select a controller that will keep the variance of J small without the mean of J becoming too large. Such an objective suggests that we select a weighted sum of the indices, mean and variance, for optimization. Thus we might choose as our criterion

$$\min_{u} \left[E\{\kappa_{1|F}\} + \alpha(E\{\kappa_{2|F}\} + \mathrm{Var}\{\kappa_{1|F}\}) \right]$$

subject to the obvious dynamical constraints. The nonnegative real parameter, α, here is simply a design-tradeoff parameter between mean and variance. Clearly the expressions included in $\mathrm{Var}\{\kappa_{1|F}\}$ are rather complicated (fourth order) and it might be worthwhile to drop this term with the hope that minimizing the parameterized tradeoff between $E\{\kappa_{1|F}\}$ and $E\{\kappa_{2|F}\}$ will achieve the design objective for some value of α.

Future directions of research are clear. We should attack the questions of index selection and optimization. These questions are fairly difficult, particularly in the continuous time format. Perhaps the discrete time formulation which has not yet been derived will be more transparent.

We feel that the major observation that we have made is that cumulants (in this case conditional cumulants) retain the second order nature of the original performance, J, whereas other statistics do not. The structural simplicity of the conditional cumulants is the key to our success in the

derivation. The simple fact that other investigators have not considered cumulants explains why the formulations have gone undiscovered despite many years of vigorous research activity in the area of Linear Quadratic Gaussian control theory.

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RESEARCH

on

DECENTRALIZED CONTROL

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Abstract

A study directed towards the development of a special purpose microprocessor architecture for application in decentralized control is reviewed. Specific emphasis is placed on assembly language routines for implementing vector-matrix operations on the microprocessor.

Introduction

Many equations of modern control, digital filtering, Kalman filtering and related problems involve matrix-vector multiplications which must be calculated to obtain their solution. These matrix-vector products have the general form shown in Equation 1

$$\underline{A}\ \underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad (1)$$

where $\underline{A}$ is a n x m matrix and $\underline{x}$ is a m x 1 column vector. In computing a solution to problems where the product $\underline{A}\ \underline{x}$ is needed, considerable time is involved for any reasonable size n and m. Hence if a real-time, on-line solution to the problems of modern control and signal processing is to be computed, a method for

calculating $\underline{A}\ \underline{x}$ efficiently must be developed. The method developed is to design an appropriate architecture and firmware which will yield an efficient implementation of matrix-vector product algorithms. The procedure used to derive this appropriate architecture and firmware is outlined below.

First a machine language implementation of the algorithm to compute $\underline{A}\ \underline{x}$ is produced so that comparisons of execution times with subsequently developed microcoded versions can be made. An algorithm is developed to calculate execution times for any n, m, and T where n and m refer to the matrix size and T is the basic machine language instruction execution. Next, various size matrix-vector products are microcoded. A n x 2 matrix times a 2 x 1 vector microprogram is implemented and run. A Hewlett Packard HP 2100, minicomputer was used to determine both machine language and microcode execution times. Algorithms are developed to calculate execution times and actual execution times for various size matrix-vector products are computed for comparison. Finally half-word length (8-bit) versions of a n x 2 matrix times a 2 x 1 vector and a 4 x 4 matrix times a 4 x 1 vector are microcoded and run. Algorithms for execution time calculation are developed and execution times computed for various size products to compare with the aforementioned machine language and full-word length versions. This development of micoroutines reveals the architectural characteristics necessary for an effective implementation of matrix-vector products on microprogram controlled computers.

Assembly Language Routine

The assembly language routine for an n x m matrix times an m x 1 vector is shown flowcharted in Figure 1. The routine was programmed to handle only integers that had been scaled three octal places, thus actually allowing three octal place accuracy. No checking for overflow or underflow was made. The multiplication of integers is certainly justifiable since the main concern is not to just demonstrate feasibility or viability of this approach but to develop routines amenable to real-time, on-line systems applications. In such applications, information into the computer derives from an analog to digital converter which produces integerized digital values of the analog signal. Scaling is normally employed between the computer and the system both on imput and output. Checking for an overflow condition would also normally be carried out, but the action taken upon an overflow detection is system dependent, e.g., an abort might be necessary, further scaling might be sufficient, etc. Therefore, the routine developed is sufficiently general and adequate.

The elements of $\underline{A}$, the a_{IJ}'s were stored sequentially by rows, i.e., first, a_{11}, then a_{12} through a a_{1m}, then a_{21}, etc. The algorithm used for accessing each array element is given in Equation 2.

$$\text{Address}(a_{IJ}) = (I - 1)N + J + \text{Address}(a_{11}) - 1 \qquad (2)$$

The time of execution for this routine can be calculated from the following equation,

$$t_e = 29.5mnT + 10.5nT + T \qquad (3)$$

where n and m refer to the matrix size and T is the basic machine instruction execction time, (1.96 μsec for the HP2100). Equation 3, along with all other equations in this paper relating execution times, was determined by actually summing the times to execute each instruction and noting the dependency on m and n in processing the algorithm. The times for a 2 x 2 matrix times a 2 x 1 vector, a 4 x 4 matrix times a 4 x 1 vector, and an 8 x 8 matrix times an 8 x 1 vector which were calculated for latter comparisons are, respectively,

$$t_{22} = 274.40\ \mu s$$

$$t_{44} = 1009.40\ \mu s$$

$$t_{88} = 3867.08\ \mu s \qquad (4)$$

Microcoded Full-Word Matrix-Vector Multiplication Routines

The flowchart for a microcoded multiplication of an n x 2 matrix times a 2 x 1 vector is shown in Figure 2. To save memory references and unnecessary programming, x_1 and x_2 are first read into WCS and stored in registers F and Q, respectively. All scratch-pad registers are used: S2 and S4 in the multiply subroutine to temporarily hold the multiplicand and multiplier as

are A and B to hold the results; S3 contains the running sum, and; S1 to count n. Each a_{IJ} is retrieved from memory as needed; and the result $a_{I1} x_1 + a_{I2} x_2$ is then stored in memory. The Flag flip-flop is used to determine when this result is completed.

The algorithm for calculation of time of execution was determined to be

$$t_e = 13T + 110nT \qquad (5)$$

where n is the number of rows in the matrix and T is the microinstruction execution time, (196 ns for the HP2100). The time to execute this microprogram for a 2 x 2 matrix times a 2 x 1 vector was calculated from Equation 5 to be 45.688 μs. Comparing this with the time to perform the same operation in assembly language, a savings factor of six (6) or about 230 μs is accomplished.

As was demonstrated, all available registers plus the Flag flip-flop were used to implement this microprogram. Since only microinstructions and no data (other then eight bit constants stored in the least significant eight bits of certain microinstructions) can be stored in WCS, a severe limitation, this is the maximum size matrix-vector product that can be implemented on the HP-2100. However, microprograms of a 4 x 4 matrix times a 4 x 1 vector, an 8 x 8 matrix times an 8 x 1 vector, and an n x m matrix times an m x 1 vector were flowcharted and actually microcoded, but not implemented. These flowcharts are shown in

Figures 3, 4, and 5, respectively. These routines require one storage register for each element of the vector (4, 8, and m), two scratch registers plus the A- and B-registers for the multiply subroutine, one scratch register for the counter for the square matrices and four counters for the n x m matrix, and one register for the running sum; or, 8, 12, and m + 7 scratch registers, respectively. As seen, these are more than are available on the 2100, a limitation that could be overcome if data could be stored and retrieved directly from WCS.

The time of execution algorithms for the first two of these microprograms is given as

$$t_e = 5T + 7nT + 59n^2T + nT \sum_{d=2}^{n-2} (n - d) + 2nT \sum_{d=2}^{n-1} (n - d) \quad , \tag{6}$$

where n refers to the (square) matrix size, T is the micro-instruction execution time, and d is a counter which depends on the number of decisions to be made (n-dependent). The times of execution for these programs are

$$\begin{aligned} &(4 \times 4) \quad t_e = 200.116 \ \mu s \\ &(8 \times 8) \quad t_e = 860.244 \ \mu s \quad . \end{aligned} \tag{7}$$

Comparing these times with those corresponding ones found using the assembly language routine (i.e., 200.116 μs vs. 974.12 μs and 860.244 μs vs. 3867.08 μs), savings ratio of about 4.8 and

4.5 are effected. The algorithm for calculating execution time for the n x m matrix times the m x 1 vector microprogram is

$$t_e = 2T + 5mT + 58nmT + 4nT + nT \sum_{d=1}^{m-2} (m - d) + 2nT \sum_{d=2}^{m-1} (m - d) \quad , \quad (8)$$

Proposed Architecture

From the preceeding discussion concerning implementation of microroutines for matrix-vector products, the limitations of the HP-2100 for efficiently microprogramming such problems are evident. A proposed architecture to alleviate these limitations and allow an efficient implementation of an n x m matrix times an m x 1 vector is shown in Figure 9 and described in the following paragraphs.

Assuming that the bus structure and microinstruction format remain fixed, the most severe limitation of the HP-2100 is a shortage of scratch-pad registers. By providing more scratch-pad registers, larger size matrix-vector products can be implemented, greater computational versatility results, and the execution speed of many programs can be increased considerably. To implement the n x m matrix times m x 1 vector product, m + 7 registers are needed. These m + 7 registers are shown in Figure 6, where m scratch-pad registers are shown as "S-bus" registers

the other seven are shown as "R-bus" registers. Four of the "R-bus" registers are labeled as in the HP-2100, i.e., A, B, Q, F. Providing more registers on the R-bus also results in greater versatility, a reduction in number of microinstructions to implement a given function, and hence, a reduction in execution time. All of these registers are also assumed to be general purpose registers and not latches as the scratch-pad registers in the HP-2100 are likely to develop a "race" condition if loaded while being interrogated. This limitation prevents the microprogrammer from specifying the same scratch-pad register in both the S- and T-bus (in the same microinstruction) which leads to more microinstructions than necessary if these registers are made general purpose.

Figure 7 also shows several additional five bit counters on the S-bus. These counters are not necessary to implement this matrix-vector product since the m + 7 registers include the registers necessary for counting. The counters are shown to indicate that several of the m + 7 registers may be replaced by shorter length (5 bit) registers to be used as counters.

Another limitation of the HP-2100 is that only microinstructions can be stored in and executed from WCS, that is no data can be directly accessed in WCS. The only data available in WCS is stored as eight bit constants in the least significant eight bits of microinstructions containing a "CR" or "CL" micro-order in the S-bus filed. The "CR" and "CL" micro-

orders direct the computer to read the eight bit constants stored in bits 0-7 of the microinstruction onto the least (CR) or most (CL) significant bits of the register specified in the T-bus field. The feature to read and store data directly into WCS locations by microprograms resident in WCS would greatly enhance the capabilities of the machine. However, the need for this feature is mitigated by the addition of sufficient scratch-pad registers but if m is large, the cost of such registers might become prohibitive. So there is a trade-off here, either incorporating as many registers as necessary to implement a problem or incorporating several additional registers and adding the capability to read the store data into WCS. These features would significantly increse speed of execution of many microprograms and greatly enhance the computer's overall capabilities--not only for matrix-vector products but for a wide class of problems. While the research was implemented on a HP-2100 minicomputer, the proposed architecture could be fabricated as a single microprocessor clip using LSI techniques.

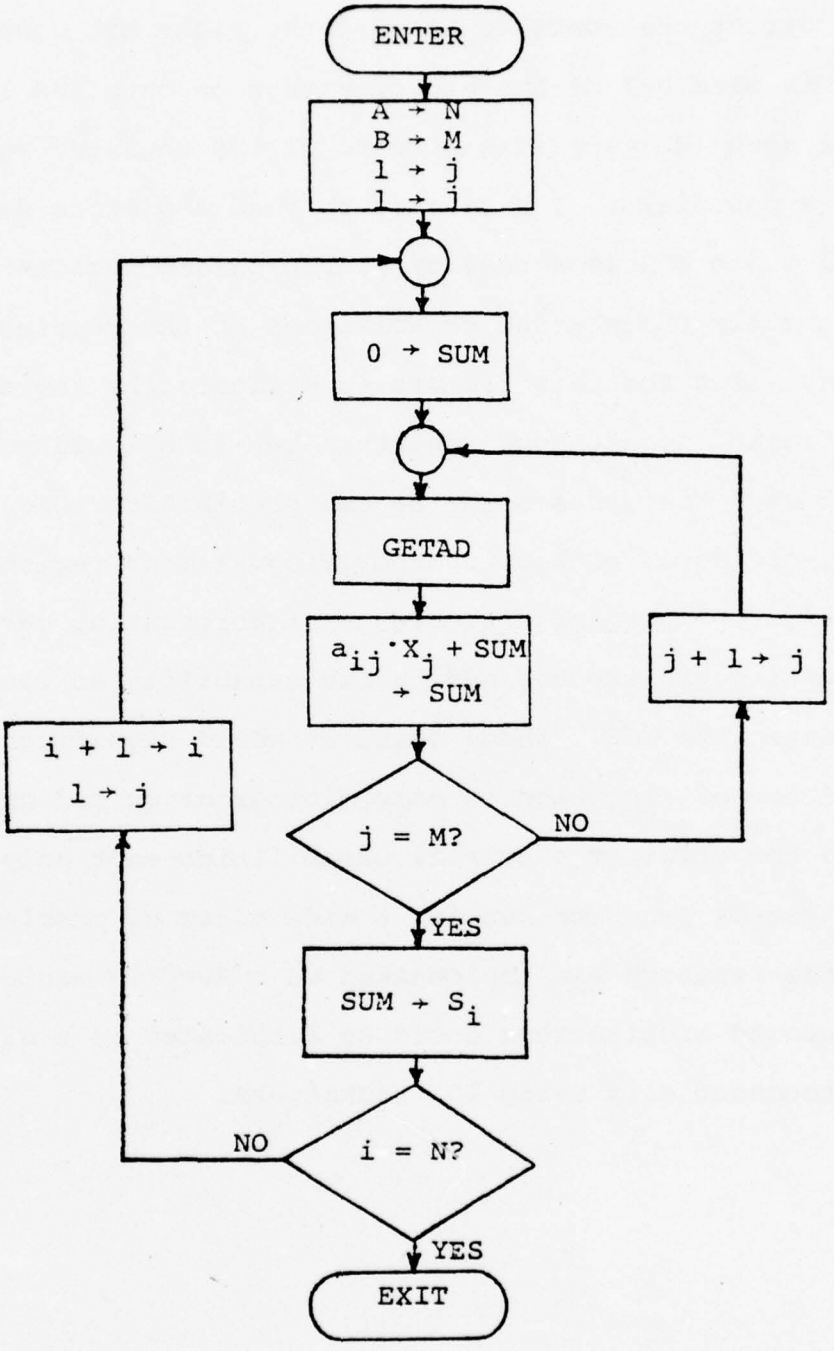

Fig. 11. Flowchart for assembly-language matrix-vector multiplication.

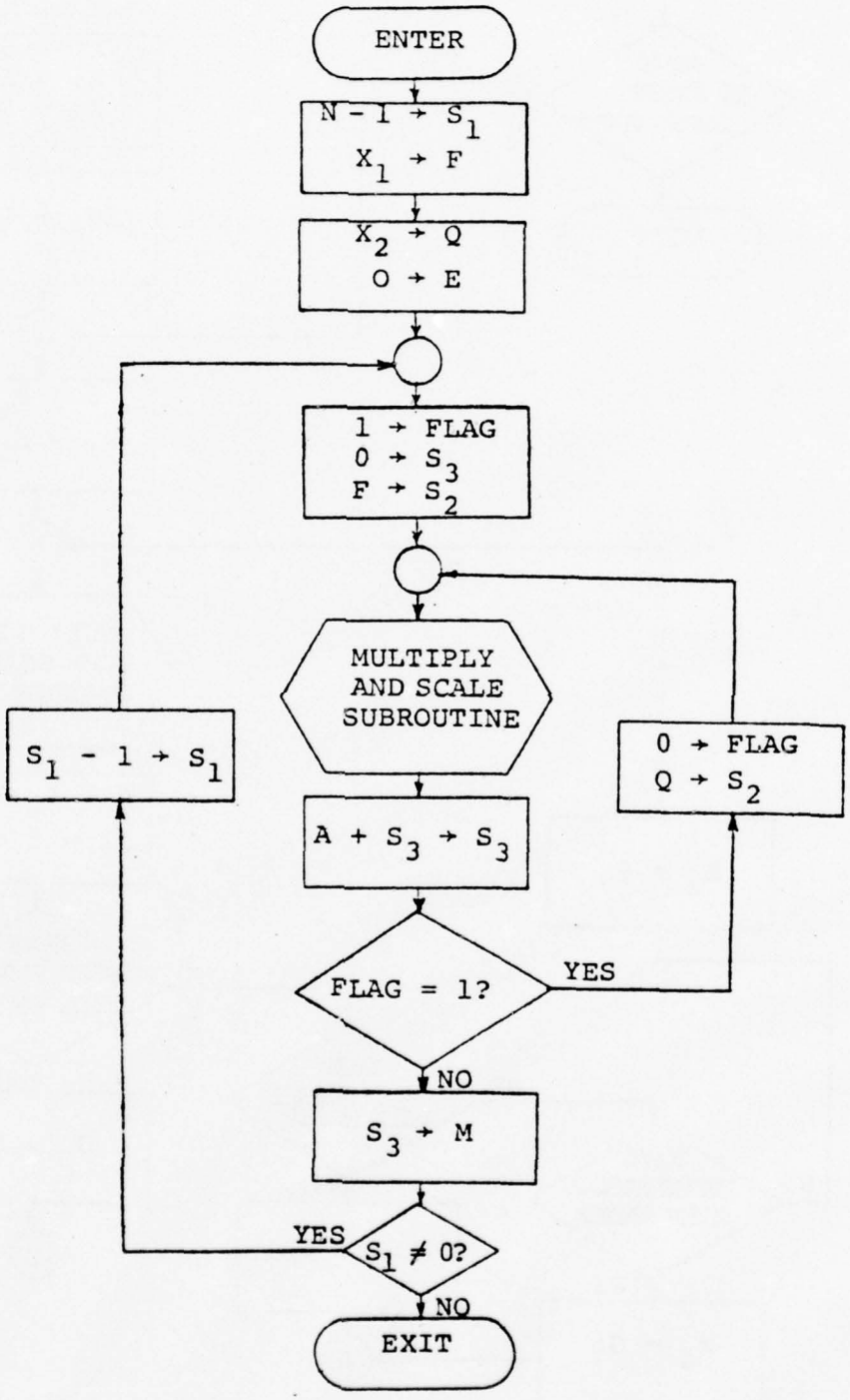

Fig. 2. Flowchart of a n x 2 matrix times a 2 x 1 vector.

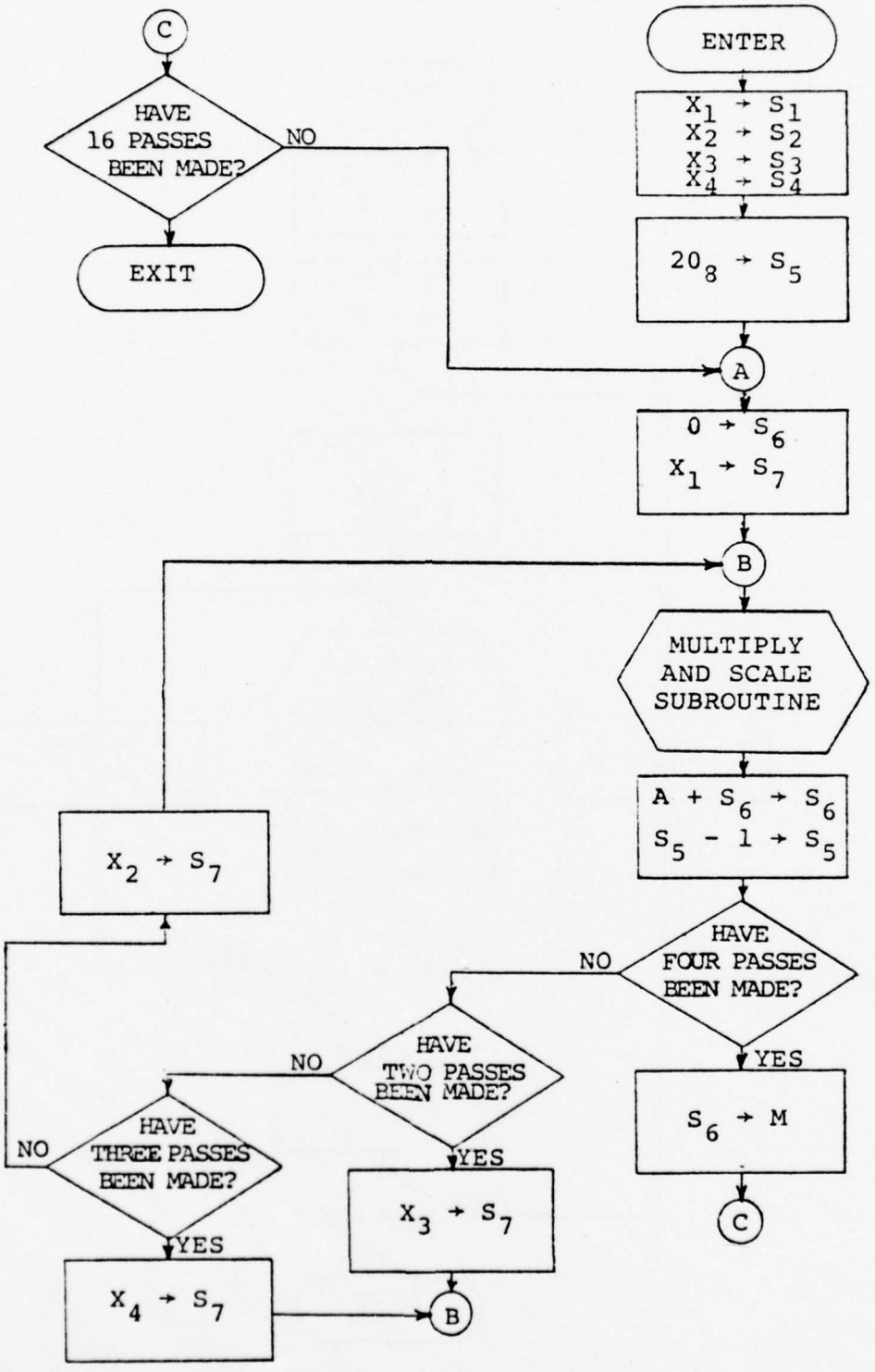

Fig. 3. Microprogram flowchart for a 4 x 4 matrix times a 4 x 1 vector.

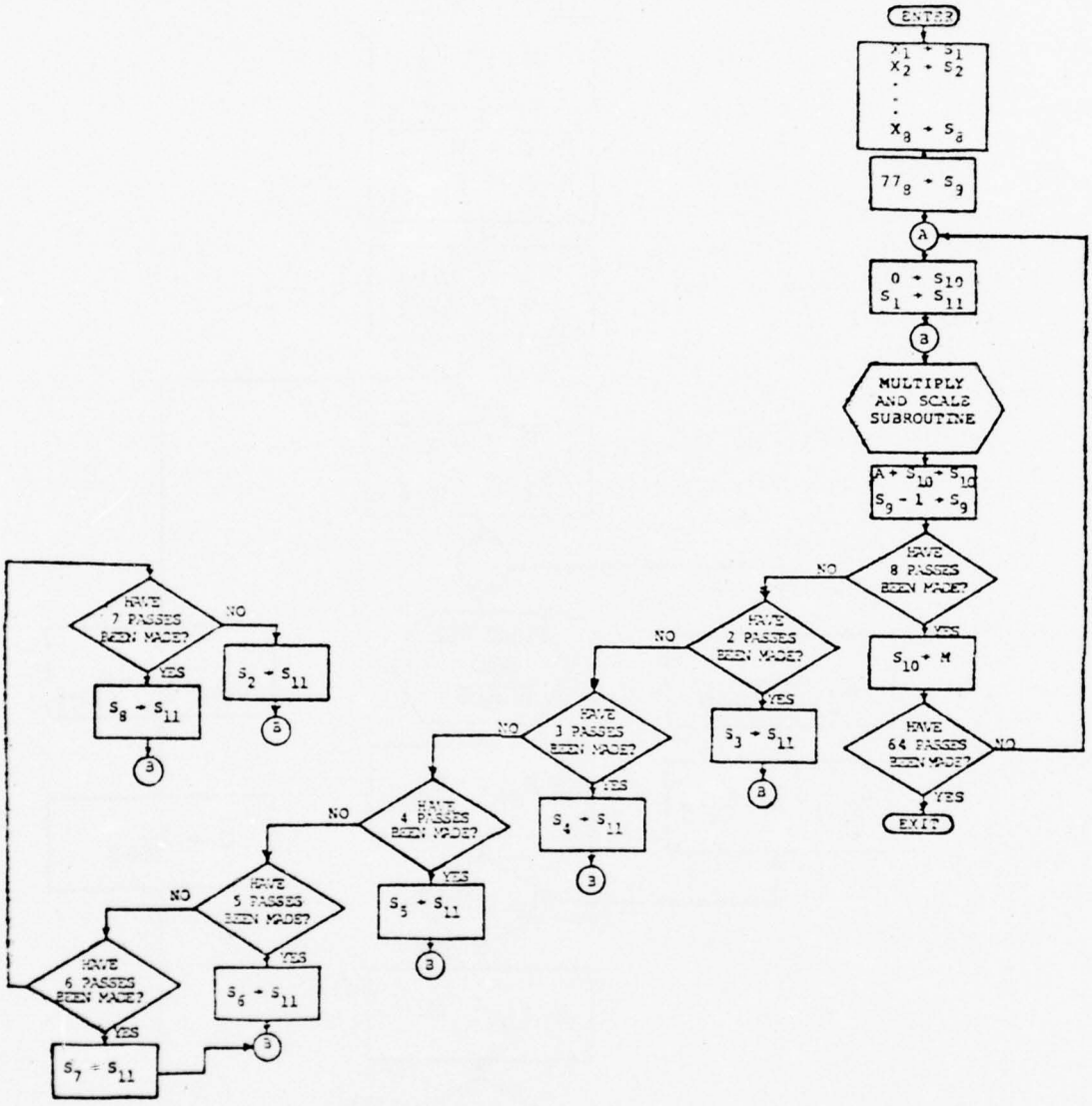

Fig. 4. Microprogram flowchart for an 8 x 8 matrix times an 8 x 1 vector.

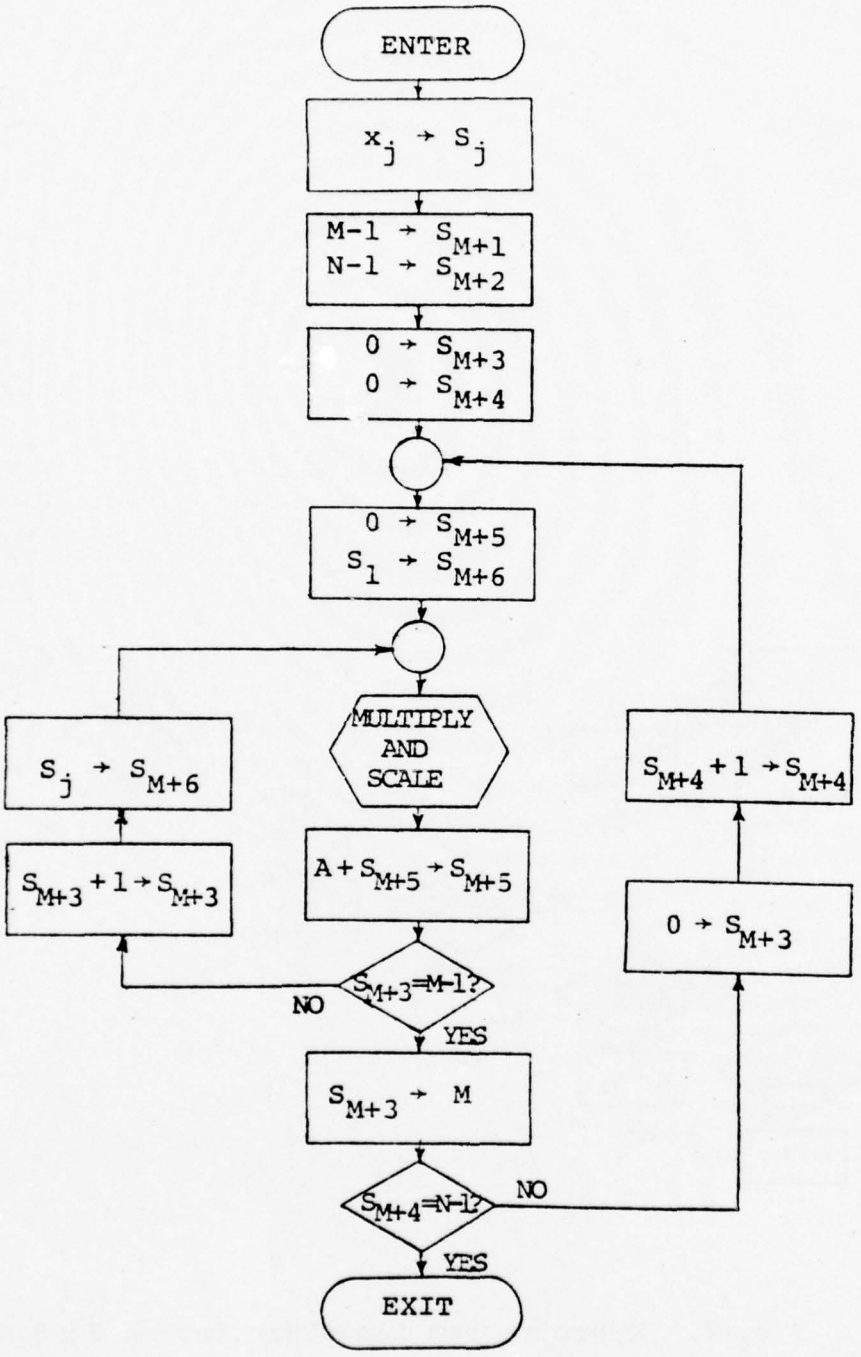

Fig. 5. n x m matrix times m x 1 vector.

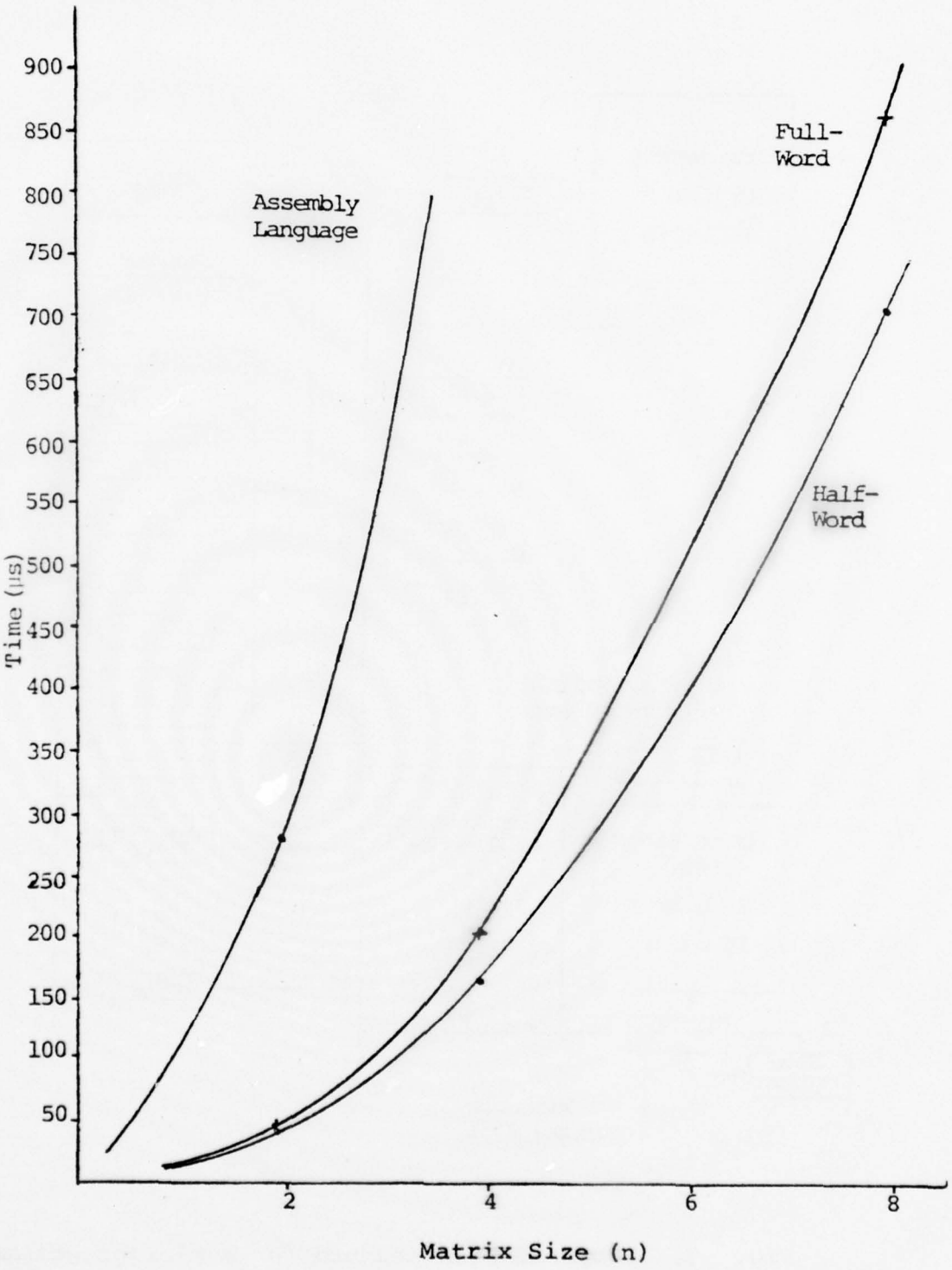

Fig. 6. Time of execution vs. matrix size.

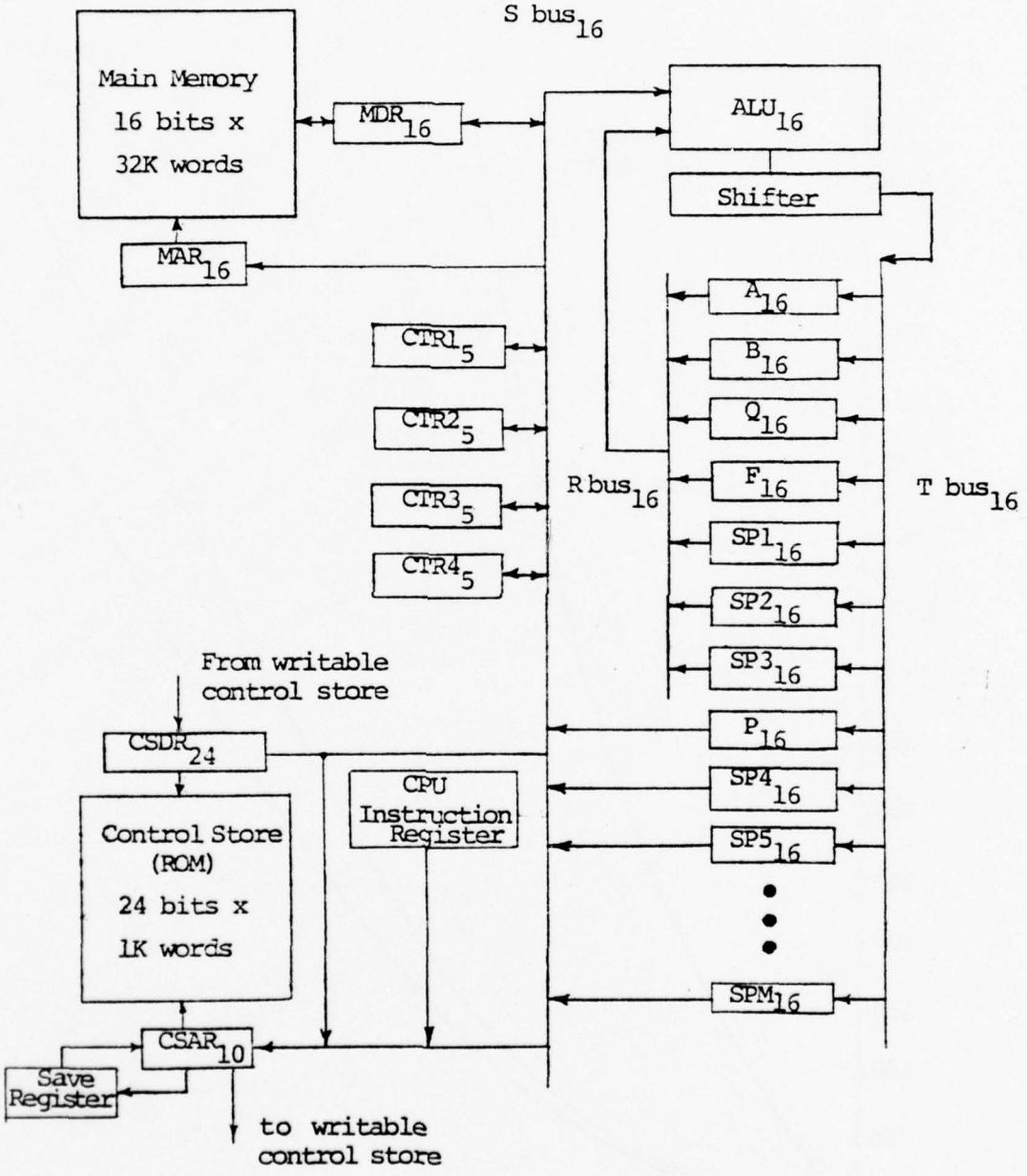

Fig. 7. Proposed architecture for a microprogrammed machine.

RESEARCH

on

MATHEMATICAL SYSTEM THEORY

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Abstract

This research is concerned with the generalization of frequency domain techniques in network and system theory to nonlinear, time-variable, distributed, and digital networks and systems via the techniques of Non-Self-Adjoint spectral theory. Specific projects include a study of the underlying mathematical foundations of the spectral factorization, applications to generalized Wiener filters and optimal feedback controllers, and the formalization of the concept of an Orlicz resoltuion space.

Introduction

The goal of our on-going research project on "Mathematical System Theory" is to extend the frequency domain techniques of classical network and system theory to nonlinear, time-variable, distributed, and digital networks and systems via the methods of non-self-adjoint spectral theory applied to operators defined on an abstract resolution space. As such, the research has been two-fold in nature. One aspect of the work deals with the formulation of the operator theoretic results in resolution space and/or one of classical L_2 spaces while the second aspect of the research is directed toward the reformulation of classical frequency domain results to aid in their generalization and the application of the resultant generalizations in the development of new frequency domain techniques.

This dichotomas approach is illustrated by our recent work on the formulation of a generalized Nyquist stability theory. The

first step was the development of a new proof of the classical theorem which made the homotopic nature of this theorem explicit. This, in turn, led to the formulation of a Nyquist-like theorem for abstract nonlinear operators on a resolution space wherein a system is shown to be stable if its open loop gain is homotopic to the identity operator in an appropriate sense.[19] This condition has been shown to coincide with the classical Nyquist criterion[8] in the linear time-invariant case and is believed to be "tight" in the general case. Indeed, most of the classical sufficient conditions for stability including a generalization of the circle criterion[8] can be derived from the homotopy theorem. Finally, the intuition derived from this general theorem has led to the formulation of a new, and highly surprising, Nyquist-like criterion for multi-dimensional digital filters characterized by functions in several comples variables.[6,7] These results form a totality which, we believe, illustrates the essence of our approach for even though the classical Nyquist criterion could be derived without an explicit call to homotopy theory[8] the homotopic formulation was the direct predecessor to the abstract theorem in resolution space and the multivariable theorem, both of which use homotopic ideas in a highly non-trivial manner.

A second and still incomplete aspect of the research has been the generalization of the spectral factorization theory, classically employed in frequency domain design techniques, to abstract operators defined on resolution space. We have previously demonstrated the existence of a miniphase factorization and ex-

hibited its fundamental relationship to the theory of reproducing kernel spaces.[20] During the past year a number of variations on this theory have been studied and, in particular, it has been shown that much of the theory can be extended to Banach space in such a way that the reproducing kernel space remains a Hilbert space.[24] As such, we have been able to show that the reproducing kernel space for a Banach space valued random variable is a Hilbert space and that the scattering representation of a network with voltage and current vectors defined in Banach space lies in a Hilbert space.[24] Needless to say all of these results have been motivated by classical frequency domain theory and they have led us into the study of the spectral factorization of rational functions in several complex variables. This latter study, which is just beginning, appears to have significant protential both theoretically and practically. We have already shown that scalar rational functions in several comples variables need not have a rational spectral factorization[14] and we are in the process of developing a design criterion for multi-deminsional digital filters which will assure that the resultant rational approximation to the given insertion loss specification admits a factorization (and hence a miniphase realization).[14]

Two other areas of research which have been undertaken during the past year include the extension of the resolution space concept to a relativistic time structure[2] and the formulation of a non-standard model for resolution space. In the former case a theory has been formulated for the case of two-dimensional special relativity[2] whereas in the latter case only partial results have been obtained.

Nyquist Theory: To formulate the frequency domain Nyquist theory in the classical and multivariable cases we require the following notation.

1. $$P^n = \{(z_1,z_2,\ldots,z_n) \in C^n;\ |z_i| \leq 1\}$$

is termed the n-variable polydisc in the space of complex n-tuples and

2. $$T^n = \{(z_1,z_2,\ldots,z_n) \in C^n;\ |z_i| = 1\}$$

is termed its distinguished boundary. Of course, in C^1, P^1 and T^1 reduce to the usual 1-variable disc and its boundary. For a function, f in one complex variable, which is analytic in P^1 its Nyquist plot is the image of f restricted to T^1*. With this notation the classical Nyquist theorem[8] has the following statement.

Theorem: For a function, f, in one complex variable defined as above the following are equivalent.

i. f has no zeros in P^1.

ii. f has no zeros in T^1 and $IND_0 f = 0$.

iii. f has no zeros in T^1 and is homotopically trivial when viewed as a mapping taking its values in $C^1 - 0$.

Here, the index of the complex valued function, f, is defined in the usual manner[13] and the equivalence of ii. and iii. is the classical result attributed to Hopf[13]. Although the proof of the theorem can be obtained as a direct corrollary to the classical argument principal[8] in which the homotopic nature of the result is then implicit in the proof of the argument principal we believe that the homotopic viewpoint is fundamental to the nature of the theorem and thus have formu-

*Actually it suffices for f to be analytic in the interior of P^1 and continuous on T^1.

lated a new explicitally homotopic proof of the theorem.[5] In this proof the necessity follows immediately via a simple continuity argument while the sufficiency proof requires that one carefully modify the domain of f to make it a covering map whence the result follows from classical homotype theory.[13]

The generalization of the Nyquist theorem to functions, f, analytic in P^n takes the following somewhat surprising form.

THEOREM: For a function, f, in several complex variables defined as above the following are equivalent.

i. f has no zeros in P^n.

ii. f has no zeros in T^n and $IND_0 \underline{f} = 0$.

iii. f has no zeros in T^n and $\underline{f}$ is homotopically trivial when viewed as a mapping taking its values in C^1-0.

Here, $\underline{f}$ is the function of one complex variable constructed from f via $\underline{f}(z) = f(z,z, \ldots , z)$.

To our knowledge the theorem is sharper than any known stability test for multivariable digital filters in that the required test is n-dimensional whereas existing stability tests are at least (n+1)-dimensional. Indeed, in applications one often knows a-priori whether or not f has zeros on T^n in which case the remaining part of the test is 1-dimensional.

The proof of the multivariable Nyquist criterion will appear in reference [7] and will not be repeated here. In essence, one first decomposes the multivariable polydisc into a continuum of single variate polydiscs and applies the classical Nyquist criterion to each disc. This then results in a test composed of a continuum of classical Nyquist tests similar to that proposed in reference 6. Fortunately, each of the required plots can be

shown to be homotopic to one of n Nyquist plots and hence this continuum of plots reduces to a finite number of Nyquist plots. Finally the analyticity of f is used to reduce the remaining n plots to the single Nyquist plot for $\underline{f}$.

Our final theorem on Nyquist theory extends the homotopic ideas described above to finite gain nonlinear operators defined on a resolution space.[21] Here, we let T_0 and T_1 be unbiased, finite gain, causal operators on a resolution space and we say that T_0 is homotopic to T_1 if there exists an operator valued function, T, defined on [0,1] such that: T(t) is causal and has a finite gain inverse for each t in [0,1] $T_1(0) = T_0$, and $T(1) = T_1$.

THEOREM: Let T be an unbiased, finite gain, causal operator defined on a resolution sapce which is homotopic to the identity operator. Then T^{-1} is causal.

This theorem is a natural generalization of our earlier result wherein T^{-1} is shown to be causal if 0 is in the unbounded component of the spectrum of T.[22] Indeed, in that case the required homotopy may be constructed via a spectral mapping. The proof of the new theorem[2] follows from a compactness (of [0,1]) argument quite similar to that used in the earlier theorem.[28]

Although the above theorem would seem to be quite abstract most known sufficient conditions for stability[8] follow readily from the theorem. For instance one may derive the circle criterion[19] by constructing a homotopy in two stages. First the nonlinearity is contracted to a scalar with the diameter of the circle being such as to assure the operator is causal and invertable as every state of the contraction. Then, in the second stage of the homotopy the remaining operator is deformed to the idenity using the "non-encirclement" hypothesis.

Factorization: The factorization problem in Hilbert space is essentially the problem of factoring a positive hermitian operator, Q, as

3. $$Q = KK^* \quad (Q = L^*L)$$

where K (L) has an appropriate causality structure. This is an abstractification of several problems commonly encountered in linear system theory such as the factorization of matrix values functions of a complex variable, the solution of Wiener-Hopf equations and the solution of matrix Riccatti equations. As such, a thorough understanding of this problem is essential to the extension of the ideas of linear system theory to new settings.

In reference 20 we proved the existence of a unique (up to a resolution space equivalence) miniphase factorization for an aribtrary positive hermitian operator on a resolution space and, furthermore, showed that the resultant factor, K, could be represented as the injection operator mapping the reproducing kernel resolution space for Q into the space on which Q is defined. The main thrust of our work during the past year has been the extension of this result to operators mapping a Banach resolution space to its dual.[24]

> THEOREM: Let (B,P) be a (reflexive) Banach resolution space and Q be a positive self-adjoint operator mapping B to B^*. Then there exists a unique (up to a resolution space equivalence) miniphase operator, K, Mapping a Hilbert resolution space, (H,E), to (B,P) such that $Q=KK^*$. Furthermore, K may be represented as the injection operator mapping the reproducing kernel resolution space for Q into (B,P).

The proof of the theorem[24] follows lines quite similar to the proof of the Hilbert space theorem of reference 20 as does the definition of the reproducting kernel (Hilbert) resolution space for a positive hermitian operator

mapping B^* to B.[24]

The important and surprising result of the theorem is that even when one is working with operators on a Banach resolution space the factor space is assured to be a Hilbert resolution space. This, in turn implies that when one applies such factorization theorem to systems defined on a Banach resolution space the resultant constructions will take the form of a Hilbert space. Two such applications which we have been investigating are the reproducing kernel space wherein one may represent a Banach resolution space valued random variable as white noise (random variable with identity covariance) in its associated reproducing kernel Hilbert resolution space. Similarly, we have shown that the scattering representation for an electric network defined in Banach space is a mapping between two (possibly different) reproducing Kernel Hilbert resolution spaces.[24]

As indicated earlier the abstract factorization theory subsumes the classical factorization theory for matrix valued functions of a complex variable. It, however, neglects the question of structure. For instance, if Q is represented by a symetric matrix is K represented by a symetric matrix or if Q is represented by a rational matrix is K represented by a rational matrix. Although the answer to the latter question is known to be yes by classical factorization theory[15] we have recently shown that if Q is represented by a rational matrix in two complex variables the answer is no.[14] Indeed, for rational functions the following necessary and sufficient condition may be derived by reformulating well known results on several complex variables.[18]

THEOREM: Let $f(z_1,z_2)$ be a polynomial in two variables.

then f has a factorization in the form

$$f(z_1, z_2) = g(z_1, z_2)h(z_1, z_2)$$

where g is a polynomial having no zeros in P^2 [illegible] h is a polynomial having no zeros in $Q^2 = \{(z_1,z_2) \in C^2;\ |z_1| \geq 1, |z_2| \geq 1\}$ if and only if $\ln|f(e^{i\theta}i, e^{i\theta}2)|$ has all of its non-zero Fourier coeficients in the quadrants; $i \geq 0$, $j \geq 0$ and $i \leq 0$, $j \leq 0$.

Note, that non-polynomial factorizations are possible under weaker hypotheses. In essence, the theorem gives a condition on the insertion loss function to permit it to be factored and as a result a corrollary to the above theorem yields a necessary and sufficient condition for a specified two variable insertion loss to be approximated by a rational miniphase filter.[14]

Relativistic System Theory: Over the past year an investigation into the possibility of formulating a resolution space theory in which a relativistic space-time structure is employed has been investigated.[2] This endeavor has been successful in the case of 2-dimensional (one space and one time dimension) Lorentz space[16] though we have not been able to extend the theory beyond that case. The basic difficulty lies with the need to generate a semiring[23] from those sets in space-time which are the futures and pasts of points.[16] In the case of 2-dimensional Lorentz space the[11] "diamond sets" suffice but no such semiring apparently exists in higher dimensional Lorentz space or general relativistic space-times. In the case of 2-dimensional Lorentz space a resolution space theory closely paralleling the classical theory has been formulated and will be reported in a forthcoming thesis.[9]

Non-Standard Analysis: Some preliminary investigations into the applicability of non-standard analysis[17] to the discretization of resolution space have

been undertaken. Although non-standard analysis is a "logicians game" we believe that the success of such an endeavor may lead to a highly intuitive viewpoint from which to study resolution space. In essence, a resolution space is a "discrete-time" space if the hermitian operator defined by the spectral measure of the resolution space has pure point spectra. When this is the case one can show that most of one's intuition derived from l_2 is valid in the abstract space.

Our main result in this area is a new non-standard proof of a theorem of Berberian.[21]

THEOREM: There exists a "faithful functor" from the catagory of Hilbert spaces and bounded linear maps to itself which takes each normal operator to a normal operator in which the entire spectrum is point spectrum.

By a faithful functor we mean a functor which preserves "most" algebraic and topological Hilbert space properties. Unfortunately, the normal operators of the theorem do not have pure point spectrum and hence cannot be used on our purpose.

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RESEARCH

on

OPTICAL NOISE

J. WALKUP

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Abstract

What we refer here to as "optical noise" actually encompasses a number of noise-related problems in optical information processing. These might include flim grain noise and photoelectronic shot noise in image processing as well as noise-induced limitations in other applications of optical information processing. In these applications we must come to grips with the inherently signal-dependent nature of the noise sources[1-2], in contrast to the signal-independent noise models commonly used in statistical communications problems[3]. It is important to consider the implications of signal-dependent noise models in the detection of signals in the presence of noise, as well as in estimating the parameters of signals. Thus, in effect, we must consider, at the most basic level, the implications of signal-dependent noise models on the issue of optimal and suboptimal detectors and estimators for applications in optical information processing. It is significant to note, moreover, that other (non-optical) noise sources, including magnetic tape recording noise[4] are effectively signal-dependent. Thus it appears that this work should have applications to a broad spectrum of signal-processing problems.

Introduction

To date, the majority of work dealing with signal-dependent noise has been concentrated on rather specialized examples and applications. Using a Poisson point process noise model, Goodman and Belsher[5] have considered the restoration of atmospherically-degraded images using linear minimum mean squared error filters. Walkup and Cheons modified the familiar Wiener filter for various additive, Gaussian signal-dependent

noise models[2], and Naderi did additional work on this problem[6]. On the other hand, Hunt and Trussel have derived a nonlinear maximum a posterori probability (MAP) estimate which can accomodate both signal-dependent and signal-independent noise models[7-8], and have applied this MAP estimator to restoring noise-degraded images. For such applications and in the special case where the images of interest exhibit extremely low contrasts, conventional restoration techniques perform rather poorly. Therefore hueristic (ad hoc) algorithms such as the so-called "noise cheating" algorithm[9], have been developed. Other algorithms, which explicitly include the signal dependence of the noise, as well as incorporating pertinent properties of the human visual system, have also been investigated[6]. Classical likelihood ratio tests have also been derived for some rather general models of signal-dependent noise[10], but the solutions generally require a priori knowledge of the object's exact location, size, intensity, and the additive background present.

In view of the rather specific nature of the topics investigated to date, it is our opinion that investigations of a more general, basic nature should be undertaken. As a result we propose to examine a number of rather general signal-dependent noise models and will address some basic issues. These will include investigations of the different structures of various optimal estimators designed for operation in signal-dependent noise, as compared with the structures of those designed to work with signal-indpendent noise. Of particular interest here are the issues of how and when a priori information concerning the "signal" (e.g. the undegraded image in an image processing application) will enable one to gain additional noise suppression, or achieve a superior detector performance.

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Engineering tradeoffs relating to detectors which do, or do not, take the signal-dependent nature of the noise into account will be considered. Applications to particular problems in areas such as image processing, optical data processing with coherent light, and electronic signal processing will be considered where appropriate.

Illustrative Examples

To illustrate the difference between signal-independent noise problems and those which are fundamentally signal-dependent in nature, we will present some examples of particular optimal estimators for rather simplified noise models. Consider, for example the following two cases:

$$\text{Case (I)} \quad r = s + n,$$

and

$$\text{Case (II)} \quad r = s + ks^{\frac{1}{2}}n,$$

where r represents the received signal, s represents the transmitted signal, and n represents a noise process. Case (I) is a familiar textbook example in statistical communication theory courses[3]. Case (II) comes from image processing, where the observable quantity is photographic density[2]. The constant k might, for example, be a scanning constant which would be related to the ratio of the area of an image scanner's aperture to that of an "average" film grain. In this example we'll assume that s and n are statistically independent, and that n is Gaussian distributed with zero mean and unit variance. Since in Case (II) the "noise" is the term $ks^{\frac{1}{2}}n$, it is clear that the noise is signal-dependent.

The effect of the signal dependence of the noise in Case (II) is readily apparent. The conditional distribution of r given s is Gaussian

with mean s and unit variance in Case (I), whereas Case (II) yields a conditional distribution which is normal with mean s and variance $k^2 s$.

Let's consider the effect of the signal-dependent nature of the noise in Case (II) on the structure of the classical maximum likelihood estimator (MLE). Maximizing the conditional pdf p(r/s) over s, we find in Case (I) that

$$\hat{s} = r \tag{1}$$

where the circumflex denotes our estimate. Maximizing p(r/s) for Case (II), however, we find that

$$\hat{s} = \left[r^2 + \left(\frac{k^2}{2}\right)^2\right]^{\frac{1}{2}} - \frac{k^2}{2}, \tag{2}$$

where a negative root was discarded.

This estimator is strikingly different from that of Eq. (1). Note, however, that as $k \to 0$, Eq. (2) approaches Eq. (1).

Now consider a random signal; for example let s be distributed normally with mean μ and variance σ^2. The appropriate optimal estimator here is the maximum a posteriori probability (MAP) estimator. Maximizing p(s/r) we find in Case (I) that

$$\hat{s} = \frac{\mu}{\sigma^2+1} + \frac{\sigma^2}{\sigma^2+1}\, r. \tag{3}$$

Note that as s becomes fixed (i.e., $\sigma^2 \to 0$), our MAP estimate becomes

$$\hat{s} = \mu, \tag{4}$$

which ignores entirely the received signal r. On the other hand, as p(s) becomes increasingly disperse (i.e., $\sigma^2 \to \infty$), we find that Eq. (3) reduces

to Eq. (1), the signal independent MLE. Maximizing p(s/r) for Case (II), we find our MAP estimate to be a solution of the cubic equation

$$s^3 + \left\{\frac{\sigma^2}{2k^2} - \mu\right\}s^2 + \frac{\sigma^2}{2}s - \frac{\sigma^2 r^2}{2k^2} = 0. \qquad (5).$$

Again note that as s becomes fixed ($\sigma^2 \to 0$), the solution of Eq. (5) is given by Eq. (4) which again ignores the received signal r. Also, as p(s) becomes disperse ($\sigma^2 \to \infty$), the solution of Eq. (5) is given by Eq. (2), the signal dependent MLE.

As another example of the effect of signal dependent noise, consider the familiar Cramer-Rao bounds. If $\hat{s}$ is any unbiased estimate of s, then[11]

$$\text{Var}\,[\hat{s}-s] \geq \{-E\,[\frac{\partial^2 \ln p(r/s)}{\partial s^2}]\}^{-1} \qquad (6)$$

Evaluating Eq. (6) for the two cases above, we find that for Case (I)

$$\text{Var}\,[\hat{s}-s] \geq 1\ , \qquad (7)$$

whereas for Case (II) we have the result

$$\text{Var}\,[\hat{s}-s] \geq \frac{2k^2 s^2}{2s + k^2}\ . \qquad (8)$$

For $s >> \frac{k^2}{2}$, the quantity $\frac{2s}{k^2}$ is large compared to unity, and the right-hand side of Eq. (8) can be approximated by

$$\frac{2k^2 s^2}{2s+k^2} \simeq k^2 s\ . \qquad (9)$$

In image processing applications, k^2 is often much smaller than unity, so the condition $s >> \frac{k^2}{2}$ is not very restrictive[1]; thus, except for the very small signal case, we have

$$\text{Var}\,[\hat{s}-s] \geq k^2 s \quad . \tag{10}$$

Note that the lower bound given by Eq. (10) can be smaller than unity, the lower bound of Eq. (7). In other words, the minimum variance of our unbiased estimate of s might actually be smaller in the signal-dependent noise case than in the signal-independent noise situation. Preliminary results such as these provide ample motivation for further investigations into these problem areas.

Progress on Image "Clutter"/"Contrast" Investigation

One other subproject of interest in the optical noise area is the investigation of quantitative measures of the notions of "contrast" and "clutter" as applied to imagery. During the grant period we have been developing the hardware and software capability (based on usage of an image storage tube interfaced to a minicomputer) of getting imagery into the computer. Our long-range goal is to test various mathematical measures of contrast and clutter on the images, and determine, where possible, those measures which are highly correlated with the subjective notions of image clutter and contrast. Since these notions are related to a definite form of psychovisual "noise" in imagery, it is an appropriate subject for study under the heading of "optical noise". Presumably one application of such measures of contrast and clutter would be in the area of evaluating target location and identification assistance techniques. At present the operating software is 85% complete and the system is awaiting calibration before the entire system is ready for use.

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RESEARCH

on

PATTERN RECOGNITION

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Abstract:

In the following, we present a general model in which many questions arising in pattern recognition may conveniently be phrased. The intention here is to arrive at a realistic model which will allow very powerful mathematical machinery to be utilized. The central thrust, however, of this work is to develop a general theory of invariance sufficient to permit recognition over a broad range of deformations from a standard prototype of the target pattern.

Introduction

The following symbolism will be used consistently:

Ω - the set of patterns;

G - a *group of transformations* acting on the left of Ω;

K - a field of scalars, usually the real or complex numbers;

V - a finite dimensional vector space over K;

R - a map $R: \Omega \to V$, the measurement vector.

In general, if X and Y are sets and G acts on the left of X, then for each $f: X \to Y$ and each $g \in G$ we may define a new function $gf: X \to Y$ by the formula

$$(gf)(x) = f(g^{-1}x) \ , \ x \in X \ .$$

In this fashion G acts on the left of the set $(F(X,Y))$ of functions from X to Y. Note that G need not act on Y for this construction.

With the above facts and notation we may present the basic model. The basic idea is that the natural transformations of the patterns are

assumed to be well modelled by the action of elements of the transformation group G. Thus, if $w \in \Omega$ and $g \in G$, then gw is another pattern which has been obtained as a transform of w. Finally, we perform certain measurements on w, each of which results in a scalar value $x \in K$. The collection of all such measurements is assembled as a measurement vector, $R(w) \in V$.

Now, we may give two interpretations. For the first of these, we regard R as a fixed retina or camera and think of the group of transformations as moving the patterns under the camera. In the second interpretation, through the induced action of G on $F(\Omega,V)$ discussed above, we imagine the group as acting on the camera. Thus, the group G can serve as a parameter set for the control mechanism associated with the measurement apparatus. It is important that we observe that the mathematical formalism is exactly the same irrespective of which interpretation is applicable to a particular problem. In fact, many situations involve some combination of both concepts.

Representation Theory

for $w_1, w_2 \in \Omega$ let us define $w_1 =_R w_2$ if $R(g^{-1}w_1) = R(g^{-1}w_2)$ for all $g \in G$, that is if w_1 and w_2 "look the same" from every possible orientation. This is an equivalence relation on Ω and thus partitions Ω into the classes $\{[w] \mid w \in \Omega\}$, each [w] being one of the equivalence classes. Now, each class [w] determines a unique map $\hat{w} : G \to V$, independent of the representative $w \in [w]$, given by

$$\hat{w}(g) = R(g^{-1}w) \quad , g \in G \quad .$$

Additionally, we have $\hat{w}_1 = \hat{w}_2$ if and only if $w_1 =_R w_2$.

In summary, we have represented Ω as a set of functions, each from G to V. Accordingly, we will refer to a map $R : \Omega \to V$ as a representation of Ω in V.

We have constructed a map $w \mapsto \hat{w}$ from Ω into F(G, V). Let us note that the following property holds:

$$g\,\hat{w} = \widehat{(gw)},\ g \in G,\ w \in \Omega,$$

where $g\,\hat{w}$ is obtained from the action of G on F(G, V). This leads us to introduce the following:

Definition: A functional representation of Ω in a class F of functions from G to V is a map $r : \Omega \to F$, denoted $w \mapsto w^r$, such that

$$g\,w^r = (gw)^r$$

for all $g \in G$ and $w \in \Omega$.

The representation $w \mapsto \hat{w}$ induced by $R : \Omega \to V$ is an example of a functional representation. We can show that this is not coincidental. In fact, we have the following:

Theorem: The functional representations r of Ω as functions from G to V correspond one-to-one with the representations $R: \Omega \to V$ via the correspondence $r \leftrightarrow R$ if and only if $w^r(g) = R(g^{-1}w)$ for all $g \in G$ and $w \in \Omega$.

The connection established by the preceding theorem is a very natural one and has significant applications. The maps R and r of the Theorem will henceforth be assumed related as above.

In effect, this result permits the transfer of structure from the group G (more or less concrete) to the set Ω of patterns (an abstract entity). For example, let us suppose that G is a topological group

and that K is the real or comples numbers. We define a representation $R : \Omega \to V$ to be continuous provided that each w^r, $w \in \Omega$, is continuous in the usual sense. Now let $C_V(\Omega)$ denote the set of representations which are continuous in this sense. It is reasonable to ask if there exists a topology T on the set Ω such that $C_V(\Omega)$ is precisely the class of T- continuous functions from G to V, i.e., that $C_V(\Omega) = C_V(\Omega,T)$. This is indeed the case, and the weakest such T has useful properties with respect to continuity of the action of G on Ω. It is quite likely that our choice of a camera should be restricted to continuous representations. This merits additional consideration.

In general, the representation theorem above, together with the overall smoothing effect normally found in physical measurements justify making mathematically useful assumptions about the set of patterns. For instance, we might begin a particular investigation with "By a pattern we mean a continuous function from a topological group G into a finite dimensional vector space V ...". Likewise, continuity can be replaced by a variety of other properties such as Haar integrable, differentiable, etc.

Formulation of Two Classical Problems

I. Template Matching

The idea here is to compare a pattern against a number (usually finite) of prototypes for goodness of fit. The problem is well worked over in the literature in many situations. However, the work is somewhat limited in the case in which t'.e object is to match the prototypes against a possible transform of the pattern. Many investigations do incorporate a limited amount of correction for such effects as translation

and magnifications. However, rotational effects seem to be generally avoided.

In the context of our model, where a choice of representation R allows us to deal with the corresponding functional representation, the question of matching becomes: given w_1, $w_2 \in \Omega$, find $g \in G$ such that $g\, w_1 = w_2$, i.e., $\hat{w}_1(g^{-1}x) = \hat{w}_2(x)$ for all $x \in G$. Then questions of goodness of fit become questions of approximation theory.

We anticipate results of two types on the template matching problem. First, we should be able to determine the existence or non-existence of a solution for $g \in G$ in the equation $gw_1 = w_2$. Hopefully, we will be able to develop invariant representations or relatively invariant representations for which the solution to the matching problem is straightforward. Finally, we whould be able to show that, in a random environment, the use of Gaussian statistics can be replaced by purely analytic methods involving least squares approximation in an L^2 space.

The major tool here will be the Haar integral and the assumptions necessary for pursuit along the above lines seem to be rather weak. Perhaps a locally compact group is sufficient.

II. Imbedded Patterns

The idea here is to search a pattern for the local presence of a predetermined sub-pattern.

This problem appears to be one which gave much impetus to pattern recognition and artifical intelligence during the early 1960's. However, anticipated success has failed to materialize. Perhaps this is due to the failure to introduce enough powerful mathematical techniques.

In the present context we may formulate a simple version of this problem as follows: Given $w \in \Omega$ and $v_o \in V$, find $g \in G$ (if possible) such that $w^r(g) = V_o$. Thus, the problem reduces to the solution of a functional equation. An analytic structure of some type is in order for this problem, and we propose the use of Lie groups. Upon introduction of coordinates in G we obtain a system of non-linear equations and seek a "best" solution.

An attempt to minimize the objective function

$$\Psi(x) = || w^r(x) - v_o ||^2/2$$

yields the differential equation

$$x'(t) = A^*(x)\,[w^r(x)-v_o]\ ,\ x(0)=x_o,$$

where $x \in K^m$ is the coordinate vector of a group element and $A(x)$ is the derivative of $w^r(x)$. This equation can be solved by ordinary numerical methods, although we need to develop sophisticated methods which utilize the group structure and avoid the differentiation of $w^r(x)$. We fine that $x^* = \lim_{t\to\infty} x(t)$ gives a critical value for the objective function, $\Psi(x)$.

It is hopeful that some combination of techniques such as hill-climbing together with the solution of differential equations will provide a fast and efficient approach to the solution of this problem.

Summary

We hope to further develop the general framework for certain pattern recognition problems involving transformation groups. Additional work is to be done with regard to the transfer to topological and analytical structure from the group to the pattern space. Through the use of invariants, relative invariants, Haar integration, and the solution of

problems of extrema we hope to obtain workable results for use in connection with the template matching problem and the imbedded pattern problem.

The immediate interest will involve compact groups. This is partly because the representation theory for such groups is known in good detail and also because it includes an important special case, namely the rotation groups, which has not received an adequate treatment in the literature.

REVIEW OF RESEARCH

in

ELECTRONICS AND RELATED AREAS

RESEARCH FACULTY

in

ELECTRONICS AND RELATED AREAS

SYSTEMS

Name	Title	Area	Office	Phone
K. S. Chao	Assoc. Prof.-EE	Nonlinear Circuits and Systems	150A-EE	742-3469
R. A. DeCarlo	Lect.-EE	Stability Theory	258-EE	742-3528
D. L. Gustafson	Assoc. Prof.-EE	Microprocessors	150B-EE	742-3530
M.O. Hagler	Prof.-EE	Optical Signal Proc.	103B-EE	742-3470
S. R. Liberty	Assoc. Prof.-EE & Stat.	Stochastic Control and Estimation	201B-EE	742-3441
J. Murray	Research Assoc. in EE	Multi-dimensional Digital Filters	258-EE	742-3528
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F. Williams	Asst. Prof-EE	Interaction of Light with Matter	258B-EE	742-3501

ELECTROMAGNETICS

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R. Cross	Vis. Prof-EE	Plasma	103C-EE	742-3468
M. Hagler	Prof.-EE	Plasma	103B-EE	742-3470
M. Kristiansen	Prof.-EE	Plasma	103A-EE	742-3468
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J. Reichert	Assoc. Prof-EE	Solar Energy	203-EE	742-3502
T. Trost	Assoc. Prof-EE	Magnetic Energy Strg	102-EE	742-3505

ACTIVE GRANTS AND CONTRACTS

in

ELECTRONICS AND RELATED AREAS

Systems

Principal Invest.	Agency	Title	Duration	Total Funding	Present Annual Fund
Saeks	AFOSR	Resolution Space,..	12/73-4/78	$ 80,800	$ 23,500
Liberty	ONR	Kalman Filtering	3/74-3/78	93,300	25,000
Saeks	ONR	Fault Analysis	4/74-12/77	113,000	30,000
Walkup/ Hagler	NSF	Optical Info. Proc.	6/75-1/77	20,600	14,000
Chao/Saeks	NSF	Analysis & Design..	6/75-11/77	30,000	15,000
Walkup/ Hagler	AFOSR	Volume Hologram/ Optical Syst.	6/74-9/77	133,000	67,000
Saeks	ONR	Unified Prg. in Elec..	9/76-2/78	200,000	133,000
Gustafson	SORF	Microprocessors	9/76-8/77	4,500	4,500
			Total Annual Funding in Systems		$312,500

Physical Electronics

Principal Invest.	Agency	Title	Duration	Total Funding	Present Annual Fund
Gundersen	ERDA	Pulsed Molecular..	7/74-9/77	$191,000	$ 75,000
Gundersen	NSF	Infrared Upconversion	11/75-10/77	35,000	18,000
Reichert	NSF	Undergraduate Research	3/71 2/77	90,000	16,000
Portnoy	NASA	Insulated ECG Elect..	7/71-277	85,000	25,000
Reichert	AFOSR	Unstable Opt. Res.	10/72-12/76	200,000	50,000
Portnoy	SORF	Solid State Research	9/76-8/77	5,000	5,000
Williams	SORF	Quantum Electronics	9/76-8/77	5,000	5,000
Thomas	Welch	Low Temp Solid State	6/75-5/78	45,000	15,000
Robinson	ARO	Raman Scattering	9/76-8/77	25,000	25,000
Robinson	Welch	Transient Studies	9/76-8/79	150,000	50,000
Wilde	Welch	Solid State Studies	6/76-5/79	51,000	17,000
			Total Annual Funding in Physical Electronics		$301,000

Electromagnetics

Principal Invest.	Agency	Title	Duration	Total Funding	Present Annual Fund
Kristiansen	AFOSR	Plasma Heat for Therm	3/71-10/77	$410,000	$100,000
Kristiansen	NSR	Toroidal Plasma Heat.	6/76-5/77	24,500	24,500
Trost	SNF	Radio Bursts, Tornado	10/75-5/77	42,500	21,000
Kristiansen	SORF	Toriodal Plasma Heat.	6/76-5/77	24,500	24,500
Kristiansen	NSF	RF Plasma Heating	4/71-5/77	205,600	30,000
			Total Annual Funding in Electromagnetics		$200,000

Power

Principal Invest.	Agency	Title	Duration	Total Funding	Present Annual Fund
Craig	TPL	Power Systems	1/73-12/76	$ 32,000	$ 8,000
Burkes	AFWL	Airborne Power Systs.	2/74-5/77	167,300	57,000
Trost	NSWC	Magnetic Energy Strg.	12/75-12/76	16,645	16,645
Burkes	ERDA	E Beam Laser	3/76-1/77	23,000	23,000
Burkes	AFOSR	Pulsed Power Conf.	6/76-5/77	6,000	6,000
Kristinasen	AFWL	High Power Switch Dev	9/76-9/77	50,000	50,000
Reichert/ Liberty	ERDA	Crosbyton Solar Power	9/76-7/77	880,000	400,000
*Includes sub-contractors funding			Total Annual Funding in Power		$560,645
			Total Annual Funding in Electronics and Related Areas		$1,374,145

RESEARCH LABORATORIES

in

ELECTRONICS AND RELATED AREAS

SYSTEMS

Computer Laboratories:

CDC 1604 facility: hands-on facility for both education and research.....108-EE

Hybrid Computer facility: *minis, micros* and analog facilities.....162-EE

Bio-medical Systems: includes instrumentation and microprocessor application facilities.....208-EE

Circuits and Systems Laboratory: the think tank.....258-EE

Optical Systems Laboratories:

Holographic Optics: primarily used for multiplex holography research.....110-EE

Optical Signal Processing: research in optical and digital image processing.....216-EE

PHYSICAL ELECTRONICS

Laser Laboratory: infrared laser research.....262-EE

Solid State Laboratories:

Solid State Physics: basic semiconductor studies.....34-Sci.

Low Temperature Physics: superconductor studies.....58-Sci.

Integrated Circuit: fabrication facility for SSI and special purpose devices.....209-EE

Spectroscopy Laboratories:

Laser Spectroscopy: interaction of light with matter.....260-EE

Molecular Spectroscopy: semiconductor studies.....35-Chem.

Picosecond Spectroscopy: transient studies.....30-Chem.

X-Ray Spectroscopy: X-ray studies and laser development.....2-Sci.

ELECTROMAGNETICS

Plasma Laboratories:

Laser/Plasma facility: plasma heating via laser plasma interaction.....113-EE

ELECTROMAGNETICS (continued)

Plasma Laboratories (continued):

Tokamak facility: radio frequency heating of toroidal plasmas.....117-EE

Electromagnetics Laboratory: nonlinear wave studies.....111-EE

Antenna Laboratory: radio meteorology and ionspheric studies.....West of the Medical School

Power

High Voltage Laboratory: pulsed power studies...North of Textile Bldg.

Solar Energy Laboratory: another think tank.....205-EE

High Power Switching Laboratory: electron beam initiated spark gap.....Trailer west of EE Bldg.

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in

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