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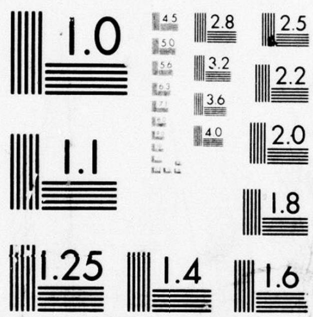

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GERSTNER WAVES AT THE INTERFACE BETWEEN STRATIFIED FLUIDS

by

Erik Mollo-Christensen‡

ABSTRACT

Paper extends the theory of Gerstner waves by the addition of a fluid layer in rigid body motion above the surface of the conventional Gerstner wave field. The propagation speed is then decreased by a factor of $(\Delta\rho/\rho)^{1/2}$. The same result is also valid for the system turned upside down with a reversal of the density difference. Combining the flows, one can also construct a three-fluid system with a middle layer in uniform motion in the plane of the Gerstner waves, while the flow normal to this plane can be arbitrary and must be steady.

INTRODUCTION.

Dubreil-Jacotin (1932) found that Gerstner's (1802) rotational free surface waves can exist independently of the stratification of the fluid below the free surface. This is with the reservation that the stratification must be stable. Yih (1966) showed that Gerstner waves can occur as boundary waves near an inclined solid boundary, the particle motions being in planes parallel to the boundary.

The conditons under which Gerstner waves can exist on the boundary between fluids have not been covered in the literature. We shall, by mathematically trivial extensions of the analysis of Gerstner waves given by Rankine (1863) and Lamb (1932) obtain some examples of Gerstner waves on interfaces. The examples are not of much current interest, because we do not have a theoretical description of how Gerstner waves may arise naturally as, for example, a finite amplitude limiting stage of shear flow instability. This note is written in the hope that others may be encouraged to search for processes that may generate Gerstner waves.

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(See 1473)

DISPLACEMENT FIELD.

Use Lagrangian variables $\mathbf{r} = \mathbf{r}(\boldsymbol{\gamma},t)$ as functions of labelling variables $\boldsymbol{\gamma} = (\xi,\eta,\zeta)$ and time t. Let $\mathbf{r} = (x,y,z)$ and define initial particle position $\mathbf{r}_o = \mathbf{r}(\boldsymbol{\gamma},0)$. Gravity acts in the negative z-direction.

$$x = \xi + (e^{k\zeta}/k) \sin k(\xi - Ut) - Ut \tag{1}$$

$$y = \eta + V(\xi,\zeta)t \tag{2}$$

$$z = \zeta - (e^{k\zeta}/k) \cos k(\xi - Ut) \tag{3}$$

The flow is incompressible, since the Jacobian

$$\begin{aligned} &\partial(x,y,z)/\partial(x_o,y_o,z_o) = \\ &[\partial(x,y,z)/\partial(\xi,\eta,\zeta)]/[\partial(\xi,\eta,\zeta)/\partial(x_o,y_o,z_o)] = 1. \end{aligned} \tag{4}$$

Let the fluid surface be at $\zeta = s$, so that the fluid extends over the domain $-\infty < \xi < +\infty$, $-\infty < \eta < +\infty$, $-\infty < \zeta < s$.

We next add a fluid of constant density ρ_o above $z_s = z(\eta,s,t)$; This fluid extends to $z = h$, where $h > z_s$. $z = h$ is the upper surface of the upper fluid, where we specify that the pressure is constant. The waves stand still in (x,y,z). For example, the interface wave crests are where $\cos k(\xi - Ut) = -1$, which is where $k(\xi - Ut) = (2n+1)\pi$; here, the phase is independent of time. We therefore specify that the upper fluid shall have no motion in the (x,z)-plane.

THE PRESSURE FIELD.

The pressure in the upper fluid is purely hydrostatic, and the pressure in the interface at a point (ξ_A,η_A,s) is p_A:

$$p_A = p(\xi_A,\eta_A,s) = p_o + \rho_o\, g\,[h - z(\xi_A,\eta_A,s)] \tag{5}$$

The pressure difference between point A and another point, B, on the interface is calculated from the momentum equation for the lower fluid and is found to be

$$\begin{aligned} p_A - p_B &= p(\xi_A,\eta_A,s) - p_B(\xi_B,\eta_B,s) \\ &= \int_A^B \rho_L(\ddot{x}x_\xi + \ddot{y}y_\xi + \ddot{z}z_\xi + g\,z_\xi)\,d\xi \\ &= \rho_L e^{ks}(U^2 - g/k)\,[\cos k(\xi - Ut]_{\xi_A}^{\xi_B} \\ &= g\,\rho_o(z_B - z_A) = g\,\rho_o\,(e^{ks}/k)\,[\cos k(\xi - Ut)]_{\xi_A}^{\xi_B}. \end{aligned} \tag{6}$$

Eq. (6) is satisfied provided:

$$U^2 = g(\rho_L - \rho_o)/(\rho_L k) \tag{7}$$

This gives the average horizontal speed of the fluid in the (x,z)-plane with respect to the waves.

Eq. (7) is also valid and yields real values for U if one reverses the signs of both g and $(\rho_L - \rho_o)$. This describes the flow of an infinitely deep fluid over a lower homogeneous fluid with its lower horizontal boundary at a constant pressure.

TWO STRATIFIED DEEP FLUID WITH AN INTERMEDIATE LAYER.

Superimposing a horizontally moving stratified fluid upon a layer of fluid of constant density which again rests on another stratified fluid with another mean horizontal motion, one can define the following displacement field:

Upper fluid:

$$\zeta_1 < \zeta < \infty$$

$$\begin{aligned} x &= \xi - U_1 t - (e^{-k\zeta}/k) \sin k(\xi - U_1) \\ y &= \eta + V_1(\xi,\zeta)t \\ z &= \zeta + (e^{-k\zeta}/k) \cos k(\xi - U_1 t) \end{aligned} \tag{8}$$

Middle fluid:

$$\zeta_2 - e^{k\zeta_2} \cos[k(\xi + U_2 t)]/k < \zeta < \zeta_1 + e^{-k\zeta_1} \cos[k(\zeta - U_1 t)]/k$$

$$\begin{aligned} x &= \xi \\ y &= V_M(\xi,\zeta)t \\ z &= \zeta \end{aligned} \tag{9}$$

Lower fluid:

$$-\infty < \zeta < \zeta_2$$

$$\begin{aligned} x &= \xi + U_2 t + e^{k\zeta} \sin[k(\xi + U_2 t)]/k \\ y &= \eta + V_2(\xi,\zeta)t \\ z &= \zeta - e^{k\zeta} \cos[k(\xi + U_2 t)]/k \end{aligned} \tag{10}$$

In order for the center fluid to have no x-velocity, and for all pressure conditions to be satisfied one finds:

$$U_1{}^2 = g(\rho_M - \rho_u)/(\rho_M k) \tag{11}$$

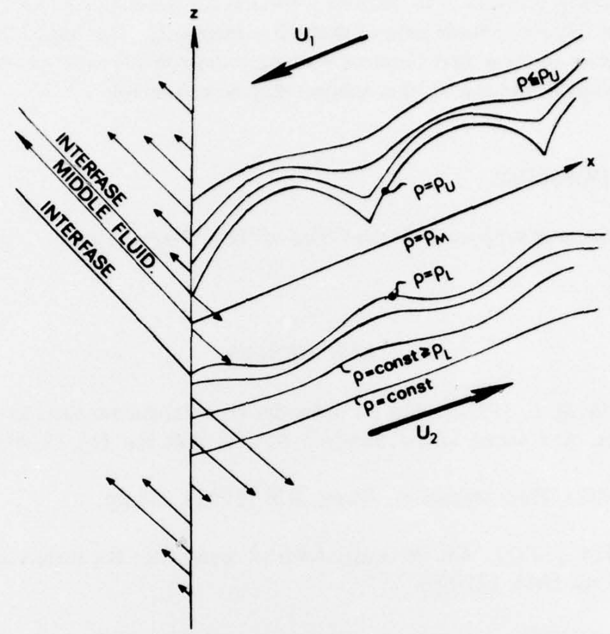

Fig. 1. The flow field in a coordinate system where the wave pattern is stationary.

$$U_2{}^2 = g(\rho_L - \rho_M)/(\rho_L k). \tag{12}$$

Fig. 1 shows a sketch of the flow field for the above two-dimensional case. The velocities are shown in a coordinate system where the wave pattern is stationary. The arbitrary steady velocity in the y,z-plane is indicated.

DISCUSSION.

The conditions under which one may observe Gerstner waves seem so special that there is little hope of encountering them in nature. The hope of observing infinitesimal amplitude sinusoidal waves with a stratified shearing fluid seems so much more justified, until one remembers the difficulty of observing infinitesimal disturbances.

But there also are theories for finite amplitude periodic waves after Stokes (1847), as reviewed by Whitham (1975).

Although there is no stability theory for Gerstner waves, we will argue that since the wave phase speed is independent of amplitude, the near-linear analysis of Whitham (1975) would suggest that modulations of Gerstner waves would be neutrally stable. That means that a modulation would persist but not grow or decay. This is worthy of further exploration.

A remaining problem is to discover a process for generation of Gestner waves by, for example, a finite amplitude limit of shear flow instability. It is hoped that the present results may serve to show that Gerstner waves can exist in a broader range of flows than previously accepted, and that further analysis may be worthwhile.

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