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UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

DEFECT CORRECTIONS VIA NEIGHBOURING PROBLEMS  
I. GENERAL THEORY

Klaus Böhmer

Technical Summary Report # 1750  
May 1977

ABSTRACT

*This document*  
To solve  $Fy = 0$  numerically ~~we use~~ <sup>using</sup> two different methods, the first of which is sketched already ~~in [3]~~ <sup>and the</sup>. Secondly, ~~we~~ <sup>introduce</sup> a neighbouring problem (N.P.)  $Fu = d$ ,  $\|d\|$  <sup>absolute value of d</sup> small, with known solution. ~~We solve~~ <sup>one</sup> the original problem and the N.P. with the "same discretization method". The known error of the N.P. is used as an estimation for the unknown error of the original problem. These procedures are used iteratively and their relations are discussed. In subsequent papers ~~we will apply~~ <sup>the</sup> our general theory <sup>will be applied</sup> to some special cases and will discuss relations to collocation methods and to PEREYRA's deferred correction methods [8,9].

AMS (MOS) Subject Classifications: 65B99, 65J05, 65L05, 65L10, 65M99, 65R05

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# Defect Corrections via Neighbouring Problems

## I. General Theory.

by

Klaus Böhmer

Summary: To solve a functional equation  $Fy = 0$  numerically we use two different methods: The first one, sketched already in [3], combines Newton's method with a discretization of the linear problem. Secondly, we introduce a neighbouring problem (N.P.)  $Fu = d$ ,  $\|d\|$  small, with known solution. We solve the original problem and the N.P. with the "same discretization method". The known error of the N.P. is used as an estimation for the unknown error of the original problem. Both methods are used iteratively and their relations are discussed. The idea of using N.P. goes back to ZADUNAISKY [15,16,17], again discussed by STETTER [14]. They treated initial value problems of ordinary differential equations where STETTER used our first method, too. The method of N.P. was applied by FRANK, HERTLING, UEBERHUBER [5,6,7] to initial and boundary value problems of ordinary differential equations. In subsequent papers we will apply our general theory to some special cases and will discuss relations to collocation methods and to PEREYRA's deferred corrections [10,11]. We find numerical methods which seem to be as convenient and appropriate than those derived directly (see [5,6,7,14,15,16,17]).

### 1. Introduction

With Stetters [13] notations, slightly modified, a discretization method  $\mathcal{M}$ , applicable to the given problem  $\mathcal{P}$

$$(1.1) \quad F(y) = 0, \quad F : D \rightarrow E^0, \quad D \subseteq E, E^0 \text{ Banach spaces,}$$

is an infinite sequence  $\{E_h, E_h^0, \Delta_h, \Delta_h^0, \phi_h\}_{h \in \mathbb{H}}$ ,  $\mathbb{H} \subset (0, h_0] \subset \mathbb{R}^+$ ,  $\inf \mathbb{H} = 0$ , such that

$$(1.2) \left\{ \begin{array}{l} \Delta_h : E \rightarrow E_h, \Delta_h^{\circ} : E^{\circ} \rightarrow E_h^{\circ}, \Delta_h, \Delta_h^{\circ} \text{ linear bounded,} \\ E_h, E_h^{\circ} \text{ Banach spaces, } \dim E_h = \dim E_h^{\circ} < \infty \\ E \text{ is continuously imbedded into a Banach space } \hat{E} \text{ and} \\ \lim_{h \rightarrow 0} \|\Delta_h y\| = \|y\|_{\hat{E}} \text{ for every fixed } y \in E \subseteq \hat{E} \\ \lim_{h \rightarrow 0} \|\Delta_h z\| = \|z\| \text{ for every fixed } z \in E^{\circ}. \\ \phi_h : C_h \rightarrow (E_h \rightarrow E_h^{\circ}), C_h \subseteq (E \rightarrow E^{\circ}) \text{ and } F \in C_h \text{ for } h \in \mathbb{H}. \end{array} \right.$$

Here  $\|\cdot\|$  means anyone of the norms in  $E, E^{\circ}, E_h, E_h^{\circ}$ .

Applying  $\mathcal{M}$  to problem (1.1) we find the discretization of (1.1) as

$$(1.3) \left\{ \begin{array}{l} \phi_h := \phi_h(F), \phi_h : D_h \rightarrow E_h^{\circ}, D_h \subseteq E_h \text{ for } h \in \mathbb{H}, \\ \phi_h(\eta_h) = 0. \end{array} \right.$$

We assume that (1.1) and (1.3) have unique solutions  $y \in D$  and  $\eta_h \in D_h$  (see [13] pp 12 ff.). Further we use the equivalent notations

$$x_1 = x_2 + O(h^r), r \in \mathbb{R}_+ \text{ iff } \|x_1 - x_2\| = O(h^r) \text{ for } h \rightarrow 0.$$

If there is a sequence of  $\Lambda_h : E \rightarrow E^{\circ}, h \in \mathbb{H}$  such that

$$(1.4) \quad \phi_h(\Delta_h u) = \Delta_h^{\circ} \{F(u) + \Lambda_h(u)\} \text{ for every } u \in E$$

$\Lambda_h$  is called a "local error mapping". In many cases the local error mapping  $\Lambda_h$  admits an asymptotic expansion up to the order  $v_q$ , that is

$$(1.5) \left\{ \begin{array}{l} \Delta_h^{\circ} \Lambda_h(u) = \Delta_h^{\circ} \left\{ \sum_{i=1}^q h^{v_i} f_i(u) + O(h^{v_{q+1}}) \right\}, 0 < v_1 < v_2 < \dots < v_{q+1} \\ \text{for } u \in D_q \subseteq D, f_i : D_q \rightarrow E^{\circ}, f_i \text{ independent of } h. \end{array} \right.$$

If  $y \in D_q$  is the solution of (1.1) and if (1.3) is consistent to (1.1) of order  $v_p$  then the "local discretization error"  $\lambda_h := \Delta_h^o \Lambda_h(y)$  satisfies

$$(1.6) \quad \lambda_h = \Phi_h(\Delta_h y) = \Delta_h^o \left\{ \sum_{i=p}^q h^{v_i} f_i(y) + O(h^{v_{q+1}}) \right\}.$$

Very important for numerical applications is the question, whether (1.6) carries over to the "global discretization error"

$$(1.7) \quad \gamma_h := \eta_h - \Delta_h y, \quad \eta_h, y \text{ solutions of (1.1), (1.3).}$$

Gragg [8] has studied this question first for initial value problems of ordinary differential equations. Stetter [12,13] generalized Gragg's result to functional equations. In these papers we always have  $v_1 = 1$  or  $v_1 = 21$ . Several difficult special cases were treated directly (see for example Benson [1]).

## 2. Asymptotic expansion of the global discretization error

Here we generalize and modify Stetter's [12,13] results a little bit to prepare it for our later applications. We require that the  $v_i$  in (1.5) are such that

$$(2.1) \quad \begin{cases} \mathbb{N}_* := \{v_1, v_2, \dots, v_q, v_{q+1}\}, \quad 0 < v_1 < v_2 < \dots < v_q < v_{q+1} \text{ and} \\ v_i, v_j \in \mathbb{N}_* \text{ implies } v_i + v_j \in \mathbb{N}_* \text{ or } v_i + v_j > v_{q+1}. \end{cases}$$

For this case we give (see Stetter [13], p. 25).

Definition 2.1: We call the asymptotic expansion (1.5) of the local error mapping  $(v_q, v_p)$ -smooth at  $u$ , if the derivatives

$$f_i^{(\sigma)}(u), \sigma=1, 2, \dots, \left[ \frac{v_q - v_1}{v_p} \right] = \max\{n \in \mathbb{Z} \mid n \leq \frac{v_q - v_1}{v_p}\} \text{ exist and if}$$

for  $u, e_k \in D_q = D_{v_q} \subseteq D$ ,  $\left\| \sum_{i=p}^q h^{v_i} e_i \right\|$  small enough,  $k=p(1)q$ , the following relation is valid

$$(2.2) \left\{ \begin{aligned} & \sum_{l=1}^q h^{v_l} \{ f_l(u) + \sum_{\sigma=1}^{\lfloor \frac{v_q - v_1}{v_p} \rfloor} \frac{1}{\sigma!} f_l^{(\sigma)}(u) \left( \sum_{k=p}^q h^{v_k} e_k \right)^\sigma \} \\ & = \Lambda_h(u + \sum_{k=p}^q h^{v_k} e_k) + O(h^{v_{q+1}}) \end{aligned} \right.$$

In many cases the conditions  $u, e_k \in D_q = D_{v_q}$  are too stringent for (2.2). So we give

Definition 2.2: Let  $D_{q,k} \subseteq E_{q,k}$ ,  $k=p(1)q$ , be such that  $D_q \subseteq D_{q,k} \subseteq D$ ,  $D_q \subseteq E_q$ , and that

$$(2.3) \left\{ \begin{aligned} & \Lambda_h(u + \sum_{k=p}^q h^{v_k} e_k) \text{ and} \\ & f_l^{(\sigma)}(u) \left( \sum_{k=p}^{q^*} h^{v_k} e_k \right)^\sigma, \quad l=1(1)q, \sigma=1(1)\lfloor \frac{v_q - v_1}{v_p} \rfloor \end{aligned} \right.$$

with  $q^* = q^*(q, p, l, \sigma)$  such that  $v_{q^*} = v_q - v_1 - (\sigma-1)v_p$

are defined for  $u \in D_q$ ,  $e_k \in D_{q,k}$ . Further let

$$(2.4) \left\{ \begin{aligned} & \Lambda_h^*(u + \sum_{k=p}^q h^{v_k} e_k) := \sum_{l=1}^q h^{v_l} \{ f_l(u) + \\ & + \sum_{\sigma=1}^{\lfloor \frac{v_q - v_1}{v_p} \rfloor} \frac{1}{\sigma!} f_l^{(\sigma)}(u) \left( \sum_{k=p}^{q^*} h^{v_k} e_k \right)^\sigma \}, \end{aligned} \right.$$

satisfy

$$(2.5) \quad \Lambda_h(u + \sum_{k=p}^q h^{v_k} e_k) = \Lambda_h^*(u + \sum_{k=p}^q h^{v_k} e_k) + O(h^{v_{q+1}}).$$



Then the  $D_{q,k}$  are called (asymptotically) admissible sets (for  $\mathcal{M}$  and  $\mathcal{P}$ ).

The summation index  $q^*$  in (2.4) is chosen such that  $e_k$  with  $k > q^*$  only contributes to  $O(h^{v_{q^*+1}})$ . Since further  $v_p \leq v_{q^*}$  iff  $v_1 \leq v_q - \sigma v_p$ , only those  $f_1^{(\sigma)}(u)$  contribute to  $\Lambda_{h^*}$  with  $v_1 \leq v_q - \sigma v_p$ .

Now the following Proposition is simply proved:

Proposition 2.3: Let the operators given in (2.3) be defined and continuous for  $u \in D_q$ ,  $e_k \in D_{q,k}$  with respect to the norms in  $E_q$ ,  $E_{q,k}$ . Then (2.5) is satisfied.

Since we are concerned only with asymptotic expansions,  $h^{v_{q+1}}$ -terms are fairly uninteresting. So in addition to ordinary Frechet-derivatives we introduce the following type:

Definition 2.4:  $\Phi_h$  is called  $r$ -times  $(v_q, v_p)$ -differentiable at  $\eta$ , if there are bounded  $\sigma$ -linear operators  $\Phi_h^{(\sigma)}(\eta)$ :

$(E_h)^\sigma \rightarrow E_h^0$ , such that for  $v_q \geq r v_p$ ,  $e_1 \in D_{q,1}$ ,  $1 = p(1)q$ , and  $\| \sum_{i=p}^q h^{v_i} e_i \|$  small enough

$$(2.6) \quad \left\{ \begin{array}{l} \Phi_h \left( \eta + \Delta_h \sum_{i=p}^q h^{v_i} e_i \right) - \Phi_h(\eta) \\ = \sum_{\sigma=1}^r \frac{1}{\sigma!} \Phi_h^{(\sigma)}(\eta) \left( \Delta_h \sum_{i=p}^q h^{v_i} e_i \right)^\sigma + O(h^{\min\{(r+1)v_p, v_{q+1}\}}). \end{array} \right.$$

The operator  $\Phi_h^{(\sigma)}(\eta)$  is called the  $\sigma$ -th  $(v_q, v_p)$ -derivative of  $\Phi_h$  at  $\eta$ .

Now the following Proposition is a simple consequence of the last two definitions:

Proposition 2.5: Let  $\Lambda_h$  be  $(v_q, v_p)$ -smooth in  $B_\rho(y) := \{u \in D \mid \|u-y\| < \rho\}$  and let, with  $v_q \geq r \cdot v_p$ ,  $F$  have  $r$  Lipschitz-continuous derivatives. Then  $\phi_h = \Delta_h^0 \{F + \Lambda_h\}$  is  $r$ -times  $(v_q, v_p)$ -differentiable in  $\Delta_h B_\rho(y)$  and with  $j$  such that  $v_j < \min\{(r+1)v_p, v_{q+1}\} - \sigma v_p$  we find

$$(2.7) \quad \phi_h^{(\sigma)}(\Delta_h u) = \Delta_h^0 \{F^{(\sigma)}(u) + \sum_{i=1}^j h^{v_i} f_i^{(\sigma)}(u)\}, \quad \sigma = 1(1)r.$$

For  $\sigma \leq r \leq [v_q/v_p]$ , with at most one  $=$ -sign, we always have  $\min\{(r+1)v_p, v_{q+1}\} - \sigma v_p > v_p$ , so  $j \geq p \geq 1$  and the  $\sum$  in (2.7) is well defined. For  $\sigma = r = [v_q/v_p]$  it may be that  $j < 1$ . Then we define  $\sum_{i=1}^j := 0$  in (2.7).

The proof of Proposition 2.5 is obvious with Definition 2.1 and 2.4 and (2.1).

To give the following Theorem we need some formal notations. Let  $F$  be  $[v_q/v_p]$ -times differentiable, let  $y$  be the solution of (1.1) and let  $\Lambda_h$  be  $(v_q, v_p)$ -smooth. Then, with  $F(y) = 0$ ,  $f_1(y) = 0$ ,  $i = 1(1)p-1$ ,  $y, e_i \in D_q$

$$(2.8) \quad \left\{ \begin{aligned} (F + \Lambda_h)(y + \sum_{i=p}^q h^{v_i} e_i) + o(h^{v_{q+1}}) &= \sum_{i=p}^q h^{v_i} f_i(y) \\ + \sum_{\sigma=1}^{[v_q/v_p]} \frac{1}{\sigma!} \{F^{(\sigma)}(y) + \sum_{i=1}^{v_i \leq v_q - \sigma v_p} h^{v_i} f_i^{(\sigma)}(y)\} &\left( \sum_{k=p}^q h^{v_k} e_k \right)^\sigma \end{aligned} \right.$$

Now we introduce operators  $g_j(\dots, \dots)$  by collecting equal powers of  $h$  in

$$(2.9) \quad \left\{ \begin{aligned} \sum_{\sigma=2}^{[v_q/v_p]} \frac{1}{\sigma!} \{F^{(\sigma)}(y) + \sum_{i=1}^{v_i \leq v_q - \sigma v_p} h^{v_i} f_i^{(\sigma)}(y)\} &\left( \sum_{k=p}^q h^{v_k} e_k \right)^\sigma \\ = \sum_{j=p_1}^q h^{v_j} g_j(e_p, \dots, e_{\ell(j)}) &, \quad g_j = 0, \quad j = p(1)p_1-1, \\ \text{with } v_{p_1} = 2v_p, \quad v_{\ell(j)} = v_j - v_p &, \text{ therefore } \ell(j) < j; \end{aligned} \right.$$

in the first line of (2.9)  $e_k$  gives contributions to powers  $\geq v_p + v_k$ . So the biggest  $k$  appearing in  $g_j(e_p, \dots, e_k)$  is  $k = \ell(j)$  with  $v_j = v_p + v_{\ell(j)}$ . For the special case  $v_1 = 1, \ell = 1(1)q$ , we have  $\ell(j) = j - p$ . With the  $g_j$ , defined in (2.9), we finally have

$$(2.10) \left\{ \begin{aligned} (F + \Lambda_h)(y + \sum_{i=p}^q h^{v_i} e_i) + O(h^{v_{q+1}}) &= \sum_{j=p}^q h^{v_j} \{F'(y)e_j + f_j(y) \\ &+ \sum_{\substack{i \geq 1, k \geq p \\ v_i + v_k = v_j}} f'_i(y)e_k + g_j(e_p, \dots, e_{\ell(j)})\}, \quad v_j = v_p + v_{\ell(j)}. \end{aligned} \right.$$

Since we want to have

$$\eta = \Delta_h(y + \sum_{i=p}^q h^{v_i} e_i) + O(h^{v_{q+1}})$$

and since

$$\Phi_h(\Delta_h u) = \Delta_h^0 \{F(u) + \Lambda_h(u)\}$$

the proof of the following Theorem mainly consists in equating to zero all the coefficients of  $h^{v_i}$ ,  $i = p(1)q$ , and can be done exactly in the same way like in Stetter [13]. (Due to  $\mathbb{N}$ :  $v_p([v_q/v_p] + 1) \geq v_{q+1}$ !)

Theorem 2.6: Let the original problem  $\mathcal{P}$  with the exact solution  $y$  and the discretization method  $\mathcal{M}$ , applicable to  $\mathcal{P}$ , satisfy

$$(2.11) \left\{ \begin{aligned} (a) \mathcal{M} \text{ is stable for } \mathcal{P}; & \text{ (see [13]);} \\ (b) \mathcal{M} \text{ is consistent with } \mathcal{P} & \text{ of order } v_p; \\ (c) \text{ an asymptotic expansion (1.5)} & \text{ of the local error mapping} \\ & \Lambda_h \text{ exists and } y \in D_q; \\ (d) \Lambda_h \text{ is } (v_q, v_p)\text{-smooth} & \text{ at } y; \\ (e) F \text{ has } [v_q/v_p] \text{ Lipschitz-continuous} & \text{ bounded derivatives} \\ & \text{ in } B_\rho(y); \\ (f) (F'(y))^{-1} \text{ exists and is bounded.} & \end{aligned} \right.$$



Define the  $e_j$ ,  $j = p(1)q$  by (see (2.9), (2.10))

$$(2.12) \quad F'(y)e_j = -\{f_j(y) + \sum_{\substack{i \geq 1, k \geq p \\ v_i + v_k = v_j}} f_i'(y) e_k + g_j(e_p, \dots, e_{\ell(j)})\},$$

and let  $e_j \in D_{q,j}$ .

Then the global discretization error  $\gamma_h = \eta_h - \Delta_h y$  admits a unique asymptotic expansion

$$(2.13) \quad \eta_h - \Delta_h y = \Delta_h \sum_{i=p}^q h^{v_i} e_i + O(h^{v_{q+1}}).$$

For the rest of the paper we postulate (2.13).

For our later applications we need the structure and the arguments of the  $g_j$  in (2.9) a little bit more completely:

Proposition 2.7: The  $g_j$  in (2.9) are operators acting multilinearly on the  $e_k$ ,  $k=p(1)\ell(j)$ ,  $v_j = v_p + v_{\ell(j)}$ , and linearly on the  $F^{(\sigma)}(y)$ ,  $\sigma=2(1)[v_j/v_p]$ , and the  $f_i^{(\sigma)}(y)$ ,  $\sigma=2(1)[\frac{v_j-v_i}{v_p}]$ .  $f_i^{(\cdot)}(y)$  contributes to  $g_j$  iff  $2 \leq [\frac{v_j-v_i}{v_p}]$  or for  $i=2(1)i(j)$  with  $v_{i(j)} \leq v_j - 2v_p < v_{i(j)+1}$ . So

$$(2.14) \quad \left\{ \begin{array}{l} g_j = g_j(e_p, \dots, e_{\ell(j)}; F''(y), \dots, F^{([\frac{v_j}{v_p}])}(y); f_1''(y), \\ \dots, f_1^{([\frac{v_j-v_1}{v_p}])}(y), f_1''(y), \dots, f_1^{([\frac{v_j-v_1}{v_p}])}(y), \dots, f_{i(j)}''(y)). \end{array} \right.$$

Here  $F^{(\sigma)}(y)$  resp.  $f_i^{(\sigma)}(y)$  has to be applied to those  $e_k$  with

$$(2.15) \quad v_k \leq v_j - (\sigma-1)v_p \quad \text{resp.} \quad v_k \leq v_j - v_i - (\sigma-1)v_p.$$

Proof: We only give the arguments for the  $f_1^{(\sigma)}$ . By (2.9) we have contributions to  $h^j g_j$  by  $f_1^{(\sigma)}(y) \left( \sum_{k=p}^q h^{v_k} e_k \right)^\sigma$  if  $v_1 + \sigma v_p \leq v_j$ , that is for  $\sigma = 2(1) \left[ \frac{v_j - v_1}{v_p} \right]$ . From  $2 \leq \left[ \frac{v_j - v_1}{v_p} \right]$  we get  $\iota(j)$ .  $\square$

### 3. Defect corrections via Newton's method

The results given here are slight modifications and extensions of those given in [3]. We use an operator (see for example [9])

$$(3.1) \quad T_\delta : \begin{cases} E_h \rightarrow E \\ \eta_h \mapsto y_h := T_\delta \eta_h \end{cases}, \quad \delta \in \mathbb{R}_0 := \{t \in \mathbb{R} \mid t \geq 0\}$$

which essentially reproduces the asymptotic expansion of  $\eta_h$  in (2.13):

$$(3.2) \quad \begin{cases} \text{If } \eta_h \text{ satisfies (2.13) then} \\ T_\delta \eta_h = y_h = y + \sum_{l=p}^{\bar{q}} h^{v_l} e_l(y) + O(h^{v_{\bar{q}}^*}), \\ \text{where } v_{\bar{q}} \leq v_{\bar{q}} - \delta < v_{\bar{q}}^* \leq v_{\bar{q}+1}. \end{cases}$$

Usually  $T_\delta$  are interpolation operators and often  $\|\cdot\|_E$  includes derivatives, so one has  $\bar{q} \leq q$ . We will have to compute the defect  $F(y_h) = F(T_\delta \eta_h)$  and we need an asymptotic expansion

$$(3.3) \quad F(y_h) = \sum_{l=p}^{\bar{q}} h^{v_l} g_l(y) + O(h^{v_{\bar{q}}^*})$$

with the  $\bar{q}$  from (3.2) (see (3.7)). Sometimes, for example in spline interpolation,  $y_h = T_\delta \eta_h$  is not smooth enough to admit the full expansions in (3.2) resp. (3.3). In those cases we have to cut down the asymptotic expansion (2.13) to

$$\eta_h - \Delta_h y = \Delta_h \sum_{l=p}^{q_0} h^{v_l} e_l + O(h^{v_{q_0+1}}) \text{ with } q_0 < q$$

such that this expansion admits (3.2), (3.3).

As a consequence of the modified Newton method we find [2]

Theorem 3.1: Let  $y_0 \in D$  be an approximation for the solution  $y$  of (1.1) such that  $F$  is at least  $K := [v_q/v_p]$ -times differentiable with uniformly Lipschitz-continuous derivatives in  $B_\rho(y_0)$  and let  $F^*(y_0) \in \mathcal{L}(E, E^0)$  be invertible with

$$\|(F^*(y_0))^{-1}\| \leq \mu, \quad \mu \cdot \sup_{z \in B_\rho(y_0)} \|F'(z) - F^*(y_0)\| \leq q < 1$$

$$\|F^*(y_0)^{-1} F(y_0)\| \leq \rho \cdot (1-q).$$

Then the modified Newton method

$$(3.4) \quad \left\{ \begin{array}{l} F^*(y_0)(y_\ell - y_{\ell-1}) = -F(y_{\ell-1}), \quad \ell=1,2,3,\dots, \\ \text{with } v_{p_0} := v_p, \quad v_{p_{\ell-1}} + v_{p_0} = v_{p_\ell}, \quad \ell=1,2,3 \end{array} \right.$$

defines a sequence  $y_\ell$  converging to  $y$ ,  $\lim_{\ell \rightarrow \infty} \|y_\ell - y\|_E = 0$ . If further  $F'(y)^{-1}$  exists and is bounded and

$$(3.5) \quad \left\{ \begin{array}{l} y_0 = y + \sum_{l=p_0}^q h^{v_l} e_l(y) + O(h^{v_{q+1}}), \quad e_{l,0}(y) := e_l(y), \\ F^*(y_0) = F'(y) + \sum_{l=p}^{\hat{q}} h^{v_l} C_l(y) + O(h^{v_{\hat{q}+1}}), \quad v_{\hat{q}} + v_p = v_q \\ C_l(y) \in \mathcal{L}(E, E^0), \quad C_l(y) \text{ independent of } h, \end{array} \right.$$

then  $y_\ell$  admits an asymptotic expansion of the form

$$(3.6) \quad \left\{ \begin{array}{l} y_\ell = y + \sum_{i=p_\ell}^q h^{v_i} e_{i,\ell}(y) + O(h^{v_{q+1}}) \text{ for } p_\ell \leq q \\ \text{resp. } y_\ell = y + O(h^{v_{q+1}}) \text{ for } p_\ell > q. \end{array} \right.$$

If  $F^*(y_0) = F'(y_0)$ , the second part of (3.5) follows from the Taylor-formula.

If  $y_0$  and  $F^*(y_0)$  satisfy (3.5) then we have

$$F^*(y_0) = F'(y_0) + \sum_{i=p}^{\hat{q}} h^{v_i} \tilde{\zeta}_i(y) + O(h^{v_{\hat{q}+1}}),$$

$$\tilde{\zeta}_i(y) \in \mathcal{L}(E, E^0).$$

Proof: The convergence of the modified Newton method under the restrictions on  $y_0$  and  $F^*(y_0)$  follows by usual conclusions (see [2]). (3.6) is proved by induction. Let  $y_{\ell-1}$  be of the form (3.6) with  $p_\ell$  and  $e_{i,\ell}$  replaced by  $p_{\ell-1}$  and  $e_{i,\ell-1}$ . Then we have with Taylor's formula and because of the Lipschitz-continuous  $F^{(\kappa)}$  - we suppress the argument  $y$  in the asymptotic expressions -

$$\begin{aligned} F(y_{\ell-1}) &= F\left(y + \sum_{i=p_{\ell-1}}^q h^{v_i} e_{i,\ell-1} + O(h^{v_{q+1}})\right) \\ &= F(y) + (F^*(y_0) + F'(y) - F^*(y_0)) \sum_{i=p_{\ell-1}}^q h^{v_i} e_{i,\ell-1} + O(h^{v_{q+1}}) \\ &\quad + \sum_{j=2}^{\kappa} \frac{F^{(j)}(y)}{j!} \left( \sum_{i=p_{\ell-1}}^q h^{v_i} e_{i,\ell-1} \right)^j + O(h^{(\kappa+1)v_{p_{\ell-1}}}). \end{aligned}$$

Since  $(\kappa+1)v_{p_{\ell-1}} \geq (\kappa+1)v_p = \kappa v_p + v_p \geq v_{q+1}$  and with  $F(y) = 0$  and (3.5) we find

$$(3.7) \quad F(y_{\ell-1}) = F^*(y_0) \sum_{i=p_{\ell-1}}^{p_\ell-1} h^{v_i} e_{i,\ell-1} + \sum_{i=p_\ell}^q h^{v_i} \tilde{e}_{i,\ell} + O(h^{v_{q+1}})$$

with  $e_{i,\ell-1}$  and  $\tilde{e}_{i,\ell}$  independent of  $h$ . Now (3.4), (3.5) and (3.7) imply



$$y_\ell = y + \sum_{i=p_\ell}^q h^{v_i} (e_{i,\ell-1} - F^*(y_0)^{-1} \tilde{e}_{i,\ell}) + O(h^{v_{q+1}}).$$

Again (3.5) implies

$$F^*(y_0)^{-1} = F'(y)^{-1} (\text{id}_{E^0} + \sum_{i=p}^{\hat{q}} h^{v_i} C_i^* + O(h^{v_{\hat{q}+1}}))$$

with  $C_i^*$  independent of  $h$ , and a combination of the two last results proves (3.6). The last statement again follows with the Taylor formula.  $\square$

Since, again, (3.4) is usually not solvable exactly, we have to use a discretization method. The crucial question is, if this discretization method reproduces the asymptotic expansion of the  $y_\ell$ . To guarantee that we need

Definition 3.2: In (1.1) let  $Fu = F_1 u + d$  resp.  $Fu = F_1 u + F_2 u$  with  $F_1, F_2 \in \mathcal{L}(E, E^0)$  and  $d$  independent of  $u$ . Let  $\mathcal{M}$ , applied to  $\mathcal{P}$ , give

$$(3.8) \quad \begin{cases} \phi_h(F_1 u + d) = \phi_h(F_1) \Delta_h u + \Delta_h^0 d \text{ with } \phi_h(F_1), \phi_h(F_2) \in \mathcal{L}(E_h, E_h^0) \text{ resp.} \\ \phi_h(F_1 u + F_2 u) = \phi_h(F_1) \Delta_h u + \phi_h(F_2) \Delta_h u. \end{cases}$$

Then we call  $\mathcal{M}$  a linear discretization method for  $\mathcal{P}$  and  $\phi_h(F_1) \Delta_h u + \Delta_h^0 d = 0$  resp.  $\phi_h(F_1) \Delta_h u + \phi_h(F_2) \Delta_h u = 0$  linear discretizations of  $F_1 u + d = 0$  resp.  $F_1 u + F_2 u = 0$ .

Let

$$(3.9) \quad \begin{cases} y_{\ell-1} := T_{\delta} \eta_{h, \ell-1}, \ell=1, 2, \dots \text{ and} \\ \Phi_h^{**}(\Delta_h y_0)(\eta_{h, \ell} - \eta_{h, \ell-1}) = -\Delta_h^0 F(y_{\ell-1}) \end{cases}$$

be a linear discretization of (3.4). Sometimes we use  $F^{**}(y_0) = F'(y_0)$ , then often  $\Phi_h^{**}$  is the Frechet derivative of  $\Phi_h$  from (1.3), but that is not true in every case. We want to prove a result for the  $\eta_{h, \ell}$  in (3.9) similar to that for the  $y_{\ell}$  in (3.6). Since we have to use in (3.9) the  $T_{\delta} \eta_{h, \ell-1}$  not only the lower index  $p_{\ell}$ , but also the upper index  $q$  has to be changed for every step corresponding to  $T_{\delta}$ , and we define (see (3.2) and (3.4))

$$(3.10) \quad v_{q_{\ell}} \leq v_{q_{\ell-1}} - \delta < v_{q_{\ell}}^{**} \leq v_{q_{\ell}+1}.$$

Now the following Theorem is valid

Theorem 3.3: In addition to the conditions in Theorem 3.1 let

$\Phi_h^{**}(\Delta_h y_0) \Delta_h u = \Delta_h^0 d$  be a stable linear discretization for  $F^{**}(y_0)u = d$ .

Further let in (3.9) the  $\eta_{h, \ell-1}$  and therefore the corresponding defect  $F(y_{\ell-1})$

$$(3.11) \quad \begin{cases} \eta_{h, \ell-1} = \Delta_h \{ y + \sum_{i=p_{\ell-1}}^{q_{\ell-1}} h^{\nu_i} e_{i, \ell-1}(y) \} + O(h^{\nu_{q_{\ell-1}}^{**}}), \ell=1, 2, \dots \\ F(y_{\ell-1}) = F^{**}(y_0) \{ \sum_{i=p_{\ell-1}}^{p_{\ell}-1} h^{\nu_i} e_{i, \ell-1}(y) \} + \sum_{i=p_{\ell}}^{q_{\ell}} h^{\nu_i} e_{i, \ell}(y) + O(h^{\nu_{q_{\ell}}^{**}}) \end{cases}$$

be such that (with  $e_{i, \ell-1}, \tilde{e}_{i, \ell}, k_{i, \ell-1, i}, e_{i, \ell}^+, g_{i, \ell, i}$  independent of  $h$ )

$$(3.12) \left\{ \begin{array}{l} \Phi_h^{**}(\Delta_h y_0) \varepsilon_{1, \ell-1} = \Delta_h^0 F^{**}(y_0) e_{1, \ell-1} \text{ implies} \\ \varepsilon_{1, \ell-1} = \Delta_h \{ e_{1, \ell-1} + \sum_{i=p_0}^{s_{1, \ell-1}} h^{v_i} k_{1, \ell-1, i} \} + O(h^{v_{s_{1, \ell-1}}^{**}}) \\ \text{with } v_1 + v_{s_{1, \ell-1}} = v_{q_{\ell-1}}, v_1 + v_{s_{1, \ell-1}+1} \geq v_{q_{\ell-1}}^{**} \text{ and} \\ \Phi_h^{**}(\Delta_h y_0) \tilde{\varepsilon}_1 = \Delta_h^{o \vee} e_{1, \ell} \text{ implies } \tilde{\varepsilon}_1 = \Delta_h \{ e_{1, \ell}^+ + \sum_{i=p_0}^{s_{1, \ell-1}} h^{v_i} g_{1, \ell, i} \} + \\ + O(h^{v_{s_{1, \ell-1}}^{**}}). \end{array} \right.$$

Then, for  $h$  small enough,  $\eta_{h, \ell}$  from (3.9) satisfies

$$(3.13) \quad \eta_{h, \ell} = \Delta_h \{ y + \sum_{i=p_\ell}^{q_\ell} h^{v_i} e_{1, \ell}(y) \} + O(h^{v_{q_\ell}^{**}}) \text{ for } p_\ell \leq q_\ell$$

resp.

$$(3.14) \quad \eta_{h, \ell} = \Delta_h y + O(h^{v_{q_\ell}^{**}}) \text{ for } p_\ell > q_\ell.$$

Proof: In a totally analogous way to (3.7) we find from (3.2), (3.10), (3.11) and

$$y_{\ell-1} = T_\delta \eta_{h, \ell-1} = y + \sum_{i=p_{\ell-1}}^{q_\ell} h^{v_i} e_{1, \ell-1}(y) + O(h^{v_{q_\ell}^{**}})$$

that

$$(3.15) \quad F(y_{\ell-1}) = F^{**}(y_0) \{ \sum_{i=p_{\ell-1}}^{p_\ell-1} h^{v_i} e_{1, \ell-1}(y) \} + \sum_{i=p_\ell}^{q_\ell} h^{v_i} \tilde{e}_{1, \ell}(y) + O(h^{v_{q_\ell}^{**}}).$$

With the stability, (3.9) and (3.12) we conclude

$$\eta_{h, \ell} - \eta_{h, \ell-1} = -\Delta_h \{ \sum_{i=p_{\ell-1}}^{p_\ell-1} h^{v_i} e_{1, \ell-1}(y) + \sum_{i=p_\ell}^{q_\ell} h^{v_i} \hat{e}_{1, \ell}(y) \} + O(h^{v_{q_\ell}^{**}})$$



or finally with (3.11)

$$\eta_{h,\ell} = \Delta_h \{y + \sum_{i=p_\ell}^{q_\ell} h^{\nu_i} e_{i,\ell}(y)\} + O(h^{\nu_{q_\ell}}) \text{ for } p_\ell \leq q_\ell$$

resp. (3.14) if  $p_\ell > q_\ell$ .  $\square$

To find (3.13) or (3.14) one has to know  $F(y_{\ell-1})$  exactly. In many applications that is not the case. This difficulty is overcome in the following

Theorem 3.4: Define the  $\eta_{h,\ell}^*$  by  $\eta_{h,0}^* = \eta_{h,0}$  and

$$(3.16) \quad \Phi_h^*(\Delta_h y_0)(\eta_{h,\ell}^* - \eta_{h,\ell-1}^*) = -\psi(\eta_{h,\ell-1}^*), \quad \ell=1,2,\dots$$

where

$$(3.17) \quad \psi(\eta_{h,\ell-1}^*) = \Delta_h^0 \{F(T_\delta \eta_{h,\ell-1}^*) + \sum_{i=p_\ell}^{q_{\ell-1}} h^{\nu_i} f_{i,\ell-1}(y)\} + O(h^{\nu_{q_{\ell-1}}}).$$

Further let the conditions of Theorem 3.3 be satisfied with

$\eta_{h,\ell-1}, e_{i,\ell-1}$  a.s.o. in (3.11), (3.12) replaced by  $\eta_{h,\ell-1}^*, e_{i,\ell-1}^*$  a.s.o. Then (3.13) resp (3.14) are valid for  $\eta_{h,\ell}^*$  instead of  $\eta_{h,\ell}$ .

Proof: The proposition follows by comparing (3.15) and (3.17).  $\square$

The question, how exactly  $\eta_{h,0} = \eta_h$  in  $\Phi_h(\eta_h) = 0$  should be known, may be treated exactly in the same way as it is done in [10] by introducing the concept of approximate solutions.

#### 4. Defect corrections via neighbouring problems

If the discretization (1.3) of the original problem (1.1) is good and if  $h$  is small enough then  $y_0 := T_\delta \eta_{h,0}$  (see (3.1), (3.2), (3.9)) will be a good approximation for the solution  $y$  of (1.1) and the

defect  $d_o := F(y_o)$  will be small. We assume that the neighbouring problems (N.P.)

$$(4.1) \quad F(u) = d, \quad \|d\| < c_o \in \mathbb{R}_+ \quad \text{N.P. for (1.1)}$$

and, by discretization of (4.1),

$$(4.2) \quad \Phi_h(\xi_h) = \Delta_h^o d, \quad \|d\| < c_o \in \mathbb{R}_+ \quad \text{N.P. for (1.3)}$$

are uniquely solvable. So  $y_o$  is the unique solution of

$$(4.3) \quad F(u) = F(y_o), \quad \text{if } \|F(y_o)\| < c_o.$$

Though we know the exact solution  $y_o$  of (4.3) we use the corresponding discretization to compute  $\xi_{h,o}$  from

$$(4.4) \quad \Phi_h(\xi_{h,o}) = \Delta_h^o F(y_o).$$

If  $\|F(y_o)\|$  is small enough the known error

$$(4.5) \quad \xi_{h,o} - \Delta_h y_o = \xi_{h,o} - \eta_{h,o} + O(h^{v_{q_o}^*}),$$

see (3.2), (3.9), should be a good estimation for the unknown error  $\eta_{h,o} - \Delta_h y$  of the original problem (1.1) and of its discretization (1.3) and

$$(4.6) \quad \eta_{h,1} := \eta_{h,o} - (\xi_{h,o} - \Delta_h y_o) = \eta_{h,o} - (\xi_{h,o} - \eta_{h,o}) + O(h^{v_{q_o}^*})$$

should be a better approximation than  $\eta_{h,0}$  and the defect  $F(y_1)$  with  $y_1 := T_\delta \eta_{h,1}$ , should be smaller than  $F(y_0)$ . ( $\eta_{h,1}$  in (4.6) is usually different from  $\eta_{h,1}$  in (3.9).) So

$$F(u) = F(y_1) \quad , \quad \Phi_h(\xi_{h,1}) = \Delta_h^0 F(y_1)$$

are closer to (1.1), (1.3) than (4.3), (4.4) and the known error

$$\xi_{h,1} - \Delta_h y_1 = \xi_{h,1} - \eta_{h,1} + O(h^{v_{q_1}^*})$$

should be a better estimation for the error  $\eta_{h,0} - \Delta_h y$  than (4.5) and, corresponding to (4.6), we define

$$\eta_{h,2} := \eta_{h,0} - (\xi_{h,1} - \Delta_h y_1) = \eta_{h,0} - (\xi_{h,1} - \eta_{h,1}) + O(h^{v_{q_1}^*})$$

This process may be used to generate the following iteration method

$$(4.7) \quad \left. \begin{array}{l} \Phi_h(\eta_{h,0}) = 0 \\ y_{\ell-1} := T_\delta \eta_{h,\ell-1} \\ \Phi_h(\xi_{h,\ell-1}) = \Delta_h^0 F(y_{\ell-1}) \\ \eta_{h,\ell} := \eta_{h,0} - (\xi_{h,\ell-1} - \eta_{h,\ell-1}) \end{array} \right\} \ell=1,2,3,\dots$$

Since we hope that the  $y_{\ell-1}$  will approach the exact solution  $y$ , the defect  $F(y_{\ell-1})$  will grow smaller and smaller, so  $\Phi_h(\xi_{h,\ell-1}) = \Delta_h^0 F(y_{\ell-1})$  will differ less and less from  $\Phi_h(\eta_{h,0}) = 0$ . Therefore a very good

way to compute  $\xi_{h,\ell-1}$  from (4.7) is to use a modified Newton method with starting value  $\zeta_0 = \eta_{h,0}$

$$(4.8) \quad \left\{ \begin{array}{l} \phi_h'(\eta_{h,0})(\zeta_i - \zeta_{i-1}) = -\{\phi_h(\zeta_{i-1}) - \Delta_h^0 F(y_{\ell-1})\}, \\ i=1,2,\dots; \zeta_0 = \eta_{h,0}, \phi_h(\zeta_0) = \phi(\eta_{h,0}) = 0, \\ \text{and hopefully } \lim_{i \rightarrow \infty} \zeta_i = \xi_{h,\ell-1}. \end{array} \right.$$

As a consequence of (2.6) and (3.3) we will have  $\zeta_i - \zeta_{i-1} = \sum_{\ell=p_i}^{q_i} h^{v_\ell} g_\ell(y) + O(h^{v_{q_i}^*})$  with appropriate  $g_\ell, p_i, q_i \leq q$ . Similarly as in 3., we may use in (4.8) instead of  $\phi_h'(\eta_{h,0})$  any  $\phi_h^{**}(\eta_{h,0})$  with

$$(4.9) \quad \left\{ \begin{array}{l} \phi_h^{**}(\eta_{h,0}) = \phi_h'(\eta_{h,0}) + \sum_{\ell=p}^{\hat{q}} h^{v_\ell} \phi_\ell(\eta_{h,0}) + O(h^{v_{\hat{q}+1}}), \\ \phi_\ell \in \mathcal{L}(E_h, E_h^0), \quad \ell=p(1)q, v_{\hat{q}} + v_p = v_q. \end{array} \right.$$

If we do so and compute only the first step in (4.8) we find the following simplified version of (4.7)

$$(4.10) \quad \left\{ \begin{array}{l} \phi_h(\eta_{h,0}) = 0 \\ y_{\ell-1} := T_\delta \eta_{h,\ell-1} \\ \phi_h^{**}(\eta_{h,0})(\xi_{h,\ell-1} - \eta_{h,0}) = \Delta_h^0 F(y_{\ell-1}) \\ \eta_{h,\ell} := \eta_{h,0} - (\xi_{h,\ell-1} - \eta_{h,\ell-1}) = \eta_{h,\ell-1} - (\xi_{h,\ell-1} - \eta_{h,0}) \end{array} \right. \quad \ell=1,2,\dots$$



To be able to prove the following Theorems we need

Definition 4.1: Let  $\mathcal{M}$  be  $(v_q, v_p)$ -smooth for  $\mathcal{P}$ . Further let, for  $u \in B_\rho(y)$ ,

$$(4.11) \quad \phi'_h(\Delta_h u)\tau = \Delta_h^{\circ} d$$

be such that the discretization of

$$(4.12) \quad F'(u)v = d$$

by  $\mathcal{M}$

$$(4.13) \quad \phi_h(F'(u))\tau = \Delta_h^{\circ} d$$

satisfies

$$(4.14) \quad \phi_h(F'(u)) = \phi'_h(\Delta_h u) + \sum_{l=p}^q h^{v_l} \phi_l(\Delta_h u) + O(h^{v_{q+1}}),$$

with  $\phi_l \in \mathcal{L}(E_h, E_h^{\circ})$  independent of  $h$  resp.

$$(4.15) \quad \phi_h(F'(u)) = \phi'_h(\Delta_h u) + O(h^{v_{q+1}}).$$

Then  $\mathcal{M}$  is called a differentiable resp. a strongly differentiable discretization method for  $\mathcal{P}$ .

If, in addition to (4.9), we have

$$(4.16) \quad F^{**}(u) = F'(u) + \sum_{l=p}^{\hat{q}} h^{v_l} F_l(u) + O(h^{v_{\hat{q}+1}}), \quad F_l \in \mathcal{L}(E, E^{\circ})$$

then we have an analogous formula to (4.14) with  $\phi_h(F'(u))$  and  $\phi'_h(\Delta_h u)$  replaced by  $\phi_h(F^{**}(u))$  and  $\phi_h^{**}(\Delta_h u)$ . For most of our later applications we have the situation given in

Definition 4.2: Let  $\phi_h^{**}(\Delta_h u)$  in (4.9) and  $F^{**}(u)$  in (4.16) be such that, for  $d$  and  $u$  smooth enough, the solutions  $\tau$  and  $v$  of

$$(4.17) \quad \phi_h^{**}(\Delta_h u)\tau = \Delta_h^{\circ} d \quad \text{and} \quad F^{**}(u)v = d$$

satisfy

$$(4.18) \quad \tau = \Delta_h \left\{ v + \sum_{i=p}^q h^{v_i} f_i(v) + O(h^{v_{q+1}}) \right\}.$$

Then  $(\phi_h^{**}(\Delta_h u), F^{**}(u))$  is called  $\mathcal{M}$ -admissible ( $\mathcal{M}$  a  $(v_q, v_p)$ -smooth discretization method).

It is possible, to give sufficient conditions, depending on  $d, u$  and the  $\phi_i$  in (4.9) and the  $F_i$  in (4.16), for the  $\mathcal{M}$ -admissibility of  $(\phi_h^{**}(u), F^{**}(u))$ . But since these conditions are very complicated and do not save too much work in the special cases to be treated later on, we do not formulate the corresponding theorem. In our later applications we will often use

$$\phi_h^{**}(\Delta_h u) = \phi_h'(\Delta_h u) \text{ or } \phi_h^{**}(\Delta_h u) = \phi_h^{<1>}(\Delta_h u).$$

If  $\mathcal{M}$  is strongly differentiable for  $\mathcal{K}$ , then  $(\phi_h'(\Delta_h u), F'(u))$  is  $\mathcal{M}$ -admissible.

Theorem 4.3: In addition to the conditions of Theorem 2.4 and 3.1 let  $\mathcal{M}$  be a linear discretization for  $\mathcal{K}$ ,  $(\phi_h^{**}(\eta_{h,0}), F^{**}(y_0))$  be  $\mathcal{M}$ -admissible and  $\theta = \{E_h, E_h^0, \phi_h^{**}(\eta_{h,0})\}_{h \in H}$  be stable (see [13] or [10]), that is for this case

$$\|\gamma\| \leq c \|\phi_h^{**}(\eta_{h,0})\gamma\| \text{ for all } \gamma \in E_h \text{ and a fixed } c \in \mathbb{R}_+.$$

Further let (4.2) be the  $\mathcal{M}$ -discretization of (4.1) and let the N.P.s (4.1), (1.1) resp. (4.2), (1.3) be uniquely solvable for  $|d| < c_0 \in \mathbb{R}_+$ . Finally let the implication (3.11), (3.12) be satisfied with  $\phi_h^{**}(\Delta_h y_0)$  replaced by  $\phi_h^{**}(\eta_{h,0})$ .

Then, for  $h$  small enough, the  $\eta_{h,l}$  from (4.10) satisfy

$$(4.19) \quad \eta_{h,l} = \Delta_h \left\{ y + \sum_{i=p_l}^{q_l} h^{v_i} e_{i,l}(y) + O(h^{v_{q_l}^{**}}) \right\} \text{ for } p_l \leq q_l$$

resp.

$$(4.20) \quad \eta_{h,l} = \Delta_h y + O(h^{v_{q_l}^{**}}) \text{ for } p_l > q_l .$$

Proof: For  $h$  small enough,  $T_\delta \eta_{h,0}$ , and by induction,  $T_\delta \eta_{h,l-1} \in B_\rho(y)$ .

Analogously to (3.7) and (3.15) we have

$$(4.21) \quad \left\{ \begin{aligned} F(y_{l-1}) &= h^{v_{p_{l-1}}} F^{**}(y_0) \left\{ \sum_{i=p_{l-1}}^{p_l-1} h^{v_i - v_{p_{l-1}}} e_{i,l-1}(y) \right\} \\ &+ h^{v_{p_{l-1}}} \sum_{i=p_l}^{q_l} h^{v_i - v_{p_{l-1}}} \tilde{e}_{i,l}(y) + O(h^{v_{q_l}^{**}}) . \end{aligned} \right.$$

Since  $F^{**}(y_0)^{-1}$  exists, the problem

$$F^{**}(y_0)v = F(y_{l-1})$$

is uniquely solvable and we find, by (4.21) and arguments similar to 3.,

$$(4.22) \quad \left\{ \begin{aligned} v &= h^{v_{p_{l-1}}} \left\{ \sum_{i=p_{l-1}}^{p_l-1} h^{v_i - v_{p_{l-1}}} e_{i,l-1}(y) + \sum_{i=p_l}^{q_l} h^{v_i - v_{p_{l-1}}} \hat{e}_{i,e}(y) \right\} \\ &+ O(h^{v_{q_l}^{**}}) \\ &= y_{l-1} - y + \sum_{i=p_l}^{q_l} h^{v_i} e_{i,l}^+(y) + O(h^{v_{q_l}^{**}}) . \end{aligned} \right.$$

Now  $(\phi_h^{**}(\eta_{h,0}), F^{**}(y_0))$  is  $\mathcal{M}$ -admissible,  $\mathcal{M}$  is linear and  $\theta = \{E_h, E_h^0, \phi_h^{**}(\eta_{h,0})\}$  stable, so we find with  $\tau := \xi_{h,l-1} - \eta_{h,0}$  from (4.10), (4.17), (4.18), (4.21), (4.22) that

$$(4.23) \quad \xi_{h,l-1} - \eta_{h,0} = \tau = h^{v_{p_{l-1}}} \Delta_h \left\{ \sum_{i=p_{l-1}}^{p_l-1} h^{v_i - v_{p_{l-1}}} e_{i,l-1}(y) + \sum_{i=p_l}^{q_l} h^{v_i - v_{p_{l-1}}} e_{i,l}^+(y) \right\} + O(h^{v_{q_l}^{**}})$$



and so finally

$$(4.24) \quad \left\{ \begin{aligned} \eta_{h,\ell} &= \eta_{h,\ell-1} + (\xi_{h,\ell-1} - \eta_{h,0}) = \\ &= \Delta_h \left\{ y + \sum_{i=p_\ell}^{q_\ell} h^{\nu_i} e_{i,\ell}(y) \right\} + O(h^{\nu_{q_\ell}^*}) \end{aligned} \right.$$

for  $p_\ell \leq q_\ell$  and (4.20) for  $p_\ell > q_\ell$ .  $\square$

Instead of the original iteration method (4.7) we have studied in Theorem 4.3 the method defined by (4.10). What did we lose by this change?

Theorem 4.4: *Under the conditions of Theorem 4.3 one finds, using (4.7) instead of (4.10), again the relations (4.19) and (4.20). In (4.19) only the  $e_{i,\ell}$ , but not the  $p_\ell, q_\ell$  are to be changed. That means, from the viewpoint of asymptotic expansions: The relatively simple method (4.10) works as well as the relatively complicated method (4.7).*

Proof: Since we, usually, cannot solve  $\phi_h(\xi_{h,\ell-1}) = \Delta_h \circ F(y_{\ell-1})$  in (4.7) directly, we do it via (4.8). Let us substitute  $\phi_h^*(\eta_{h,0})$  for  $\phi_h^1(\eta_{h,0})$  in (4.8). In (4.8) we will perform only a few iterations. So the following argument is valid: Comparing (4.8) and (4.10) we find that  $\zeta_0 = \eta_{h,0}$  implies  $\zeta_1 = \xi_{h,\ell-1}$  from (4.10). Now by (4.23)

$$\xi_{h,\ell-1} - \eta_{h,0} = \Delta_h \left\{ \sum_{i=p_{\ell-1}}^{p_\ell-1} h^{\nu_i} e_{i,\ell-1}(y) + \sum_{i=p_\ell}^{q_\ell} h^{\nu_i} e_{i,\ell}^+(y) \right\} + O(h^{\nu_{q_\ell}^*})$$

To compute  $\zeta_2$  from (4.10) we have to determine  $\Phi_h(\zeta_1)$  and we find with  $\xi_{h,\ell-1} = \zeta_1$ ,  $\Phi_h(\eta_{h,0})$  and the  $(v_q, v_p)$ -smoothness of  $\mathcal{M}$

$$\begin{aligned} \Phi_h(\zeta_1) &= \Phi_h^{(1)}(\eta_{h,0})(\xi_{h,\ell-1} - \eta_{h,0}) \\ &+ \sum_{\sigma=2}^{[v_q/v_p]} \frac{1}{\sigma!} \Phi_h^{(\sigma)}(v_q, v_p)(\eta_{h,0})(\xi_{h,\ell-1} - \eta_{h,0})^\sigma + O(h^{v_{q\ell}^*}) . \end{aligned}$$

This and the conditions in Definition 4.1 imply

$$-\{\Phi_h(\zeta_1) - \Delta_h^0 F(y_{\ell-1})\} = \Delta_h \left\{ \sum_{i=p_\ell}^{q_\ell} h^{v_i} g_{i,\ell}(y) \right\} + O(h^{v_{q\ell}^*}) .$$

So again using (4.8) and the  $\mathcal{M}$ -admissibility of  $(\Phi_h^{**}(\eta_{h,0}), F^{**}(y_0))$  we find

$$(4.25) \quad \zeta_2 - \zeta_1 = - \sum_{i=p_\ell}^{q_\ell} h^{v_i} \tilde{g}_{i,\ell}(y) + O(h^{v_{q\ell}^*})$$

where the  $\tilde{g}_{i,\ell}(y) \dagger e_{i,\ell}$ ,  $i=p_\ell(1)q_\ell$ , unless the  $e_{i,\ell-1}$ ,  $i=p_{\ell-1}(1)p_\ell-1$ , are reproduced more exactly than we required and unless additional conditions on the higher derivatives are imposed. If we now define an improved  $\eta_{h,\ell}^{**}$ , analogously to (4.7) and (4.10) by

$$\eta_{h,\ell}^{**} := \eta_{h,\ell-1} - (\zeta_2 - \eta_{h,0}) = \eta_{h,\ell-1} - (\zeta_2 - \zeta_1) - (\zeta_1 - \eta_{h,0}) = \eta_{h,\ell} - (\zeta_2 - \zeta_1)$$

we find with (4.24), (4.25) and  $\tilde{g}_{i,\ell}(y) \dagger e_{i,\ell}(y)$

$$\eta_{h,\ell}^{**} = \Delta_h \left\{ y + \sum_{i=p_\ell}^{q_\ell} h^{v_i} e_{i,\ell}^{**}(y) \right\} + O(h^{v_{q\ell}^*}) .$$

So  $\eta_{h,\ell}^{**}$  has the same asymptotic expansion like  $\eta_{h,\ell}$  and further iterations via (4.8) would give the same result.  $\square$

Again, like in 3., it is not necessary to know  $F(y_{\ell-1})$  in (4.10) exactly, but only approximatively: If we use in (4.10) an approximation  $\psi^*$  to  $F(y_{\ell-1})$  analogous to (3.17), we find

$$(4.26) \left\{ \begin{array}{l} \phi_h(\eta_{h,0}^{**}) = 0, \text{ so } \eta_{h,0}^{**} = \eta_{h,0}, \\ \phi_h^{**}(\eta_{h,0}^{**})(\xi_{h,\ell-1}^{**} - \eta_{h,0}^{**}) = \psi^*(\eta_{h,\ell-1}^{**}) \text{ with} \\ \psi^*(\eta_{h,\ell-1}^{**}) = \Delta_h^0 \{ F(T_\delta \eta_{h,\ell-1}^{**}) + \sum_{i=p_\ell}^{q_\ell} h^{\nu_i} g_{i,\ell-1}(y) \} + O(h^{\nu_{q_\ell}^{**}}) \\ \eta_{h,\ell}^{**} := \eta_{h,\ell-1}^{**} - (\xi_{h,\ell-1}^{**} - \eta_{h,0}^{**}) . \end{array} \right.$$

Theorem 4.5: Define  $\eta_{h,\ell}^{**}$  by (4.26) and let the conditions of Theorem 4.3 be satisfied with  $\eta_{h,\ell-1}$ ,  $e_{i,\ell-1}$  in (3.11) resp. (3.12) replaced by  $\eta_{h,\ell-1}^{**}$ ,  $e_{i,\ell-1}^{**}$ . Then (4.19) resp. (4.20) are valid with  $\eta_{h,\ell}$ ,  $e_{i,\ell}$  replaced by  $\eta_{h,\ell}^{**}$ ,  $e_{i,\ell}^{**}$ .

We now proceed to study the relations between the results in 3. and 4.. (3.9) and (4.10) look very alike and so the following Theorem is not astonishing:

Theorem 4.6: Let  $\mathcal{M}$  be a linear discretization method and let  $F^*(y_0)$  in (3.9) satisfy (3.5) and  $\phi_h(F^*(y_0)) = \phi_h^*(\eta_{h,0}) + O(h^{v_{q+1}})$  with the  $\phi_h^*(\eta_{h,0})$  in (4.10) resp. (4.26) and let  $\psi(\Delta_h u) = \psi^*(\Delta_h u) + O(h^{v_{q+1}})$ . If we write the results of (3.9) resp. (3.16) as  $\eta_{h,\ell}$  resp.  $\eta_{h,\ell}^*$  and the results of (4.10) resp. (4.26) as  $\bar{\eta}_{h,\ell}$  resp.  $\bar{\eta}_{h,\ell}^*$ , then

$$(4.27) \quad \begin{cases} \eta_{h,\ell} = \bar{\eta}_{h,\ell} + O(h^{v_{q\ell}^*}) & , \quad \ell=0,1,\dots \\ \eta_{h,\ell}^* = \bar{\eta}_{h,\ell}^* + O(h^{v_{q\ell}^*}) & , \quad \ell=0,1,\dots \end{cases}$$

Proof: With the denominations in Theorem 4.6 we have by (4.10) resp. (4.26)

$$\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1} = -(\bar{\xi}_{h,\ell-1} - \bar{\eta}_{h,0}) ,$$

$$\xi_{h,\ell-1} = \bar{\xi}_{h,\ell-1}, \eta_{h,0} = \bar{\eta}_{h,0}$$

$$\bar{\eta}_{h,\ell}^* - \bar{\eta}_{h,\ell-1}^* = -(\bar{\xi}_{h,\ell-1}^* - \bar{\eta}_{h,0}^*)$$

and therefore by (4.10) and  $\phi_h(F^*(y_0)) = \phi_h^*(\eta_{h,0}) + O(h^{v_{q+1}})$

$$\begin{aligned} -\phi_h^*(\eta_{h,0})(\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1}) &= -\phi_h(F^*(y_0))(\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1}) + \\ &\quad + O(h^{v_{q+1}}) \end{aligned}$$

$$= \Delta_h F(y_{\ell-1}) .$$

So, if (4.27) is already proved for  $\ell-1$ , we have

$$\phi_h(F^*(y_0))(\bar{\eta}_{h,\ell} - \eta_{h,\ell-1}) = -\Delta_h F(y_{\ell-1}) + O(h^{v_{q+1}}) ,$$



and so

$$\bar{\eta}_{h,\ell} = \eta_{h,\ell} + O(h^{v_{q+1}}) .$$

The results for  $\eta_{h,\ell}^{\ddot{}}$  are obtained in the same way.  $\square$

The idea of using N.P. goes back to ZADUNAISKY [13,14,15], again discussed by STETTER [12]. They treated initial value problems of ordinary differential equations where STETTER used our first method, too. The method of N.P. was applied by FRANK, HERTLING, UEBERHUBER [4,5,6] to initial and boundary value problems of ordinary differential equations. In the next papers we will give corresponding results for non-smooth starting values  $y_0$  and present some examples of our general theory.

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