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THE THEORETICAL BASIS OF THE CODE 50 NUCLEAR EXCHANGE MODEL

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Research Contribution 173

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ABSTRACT

The Code 50 Nuclear Exchange Model is a war game model produced by the LAMBDA Corporation. This Research Contribution derives and explains the basic mathematical models used in the computer programs of that model, including models of missile and bomber penetration as well as damage calculation, weapon allocation, and kill probability models. Model implementation and integration into the Code 50 program are also demonstrated.

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SECTION I

INTRODUCTION

GENERAL

This Research Contribution presents the derivations and explanations of the basic mathematical models used in the computer programs of the Code 50 nuclear exchange model and includes modifications by CNA to permit exercise of the Code 50 model and submodels on computers of limited core-storage capacity. The models included in the Code 50 nuclear exchange model are concerned with:

- Missile penetration
- Bomber penetration
- Damage calculation
- Weapon-target allocation
- Single-shot kill probability.

The final section of this Research Contribution shows how the various models are implemented and integrated into the Code 50 program. Additional documentation on the exact details of computer operations of the Code 50 program is published in Research Contribution 133, <u>The Calculation Procedure for the CNA Version of the Code 50 Nuclear Exchange Model</u>. AD 370 017

NUCLEAR EXCHANGE MODELS

A nuclear exchange model simulates the result of strategic warfare in a 2-player game in which nuclear weapons only are employed. There are basically two types of such models:

The first includes highly detailed simulations, used to establish targeting strategy for various contingencies. Such simulations consider many variables, requiring, in some cases, weeks of data preparation and many hours of computer time.

The second type includes the many aggregated gross effectiveness models in use today. In this type of model, many individual targets are grouped in a single target class if their individual characteristics are similar. Such details as range constraints, attack azimuth, attack timing, etc., are not considered. The Code 50 exchange model is of the latter type.

In the play of a nuclear exchange war game, it is first necessary to decide which side strikes first, second, third, etc., and how many strikes each delivers. It is also necessary to specify which types of target are attacked in each strike. For example, targets might be the strategic force, population centers, or mixes of military targets and population centers. The terms "counter-force," "counter-value," and "mixed" are used, respectively, to define these types of target.

In a counter-force attack, the attacker attempts to minimize future damage to himself by striking at the other side's strategic nuclear force. This is also known as a "damage-limiting" attack.

In a counter-value strike, the attacker attempts to maximize the damage to some measure of value on the other side. Generally, this measure of value is expressed in terms of either population or industrial floor space. Mixed targets present some of the characteristics of both counter-value and counter-force targets.

Since mixed targets present complications, particularly with respect to target value assignments, the Code 50 nuclear exchange model considers pure counter-force and counter-value strikes only.

CODE 50 BACKGROUND

Code 50 is an early version of a nuclear war game developed by the Lambda Corporation for the Office of the Assistant Secretary of Defense (Systems Analysis). Throughout this model, expected values that tend to be conservative from the point of view of the offense ("offense-conservative") are used. For example, the probability of penetrating defenses tends to be offense-conservative. Blast damage to the intended target is the only damage considered, with the result that both collateral damage and damage resulting from effects other than blast are not included. Hence, total damage inflicted also tends to be offense-conservative.

Code 50 allows a maximum of three strikes. The choice of the first attacker is determined arbitrarily; on the second strike, however, the roles of the attacker and defender are reversed. The roles are reversed again on the third strike. The type of strike is also arbitrary on the first two strikes (i.e., counter-value or counter-force), but the third strike is always a counter-value strike.

The forces and cities of the two sides are agglomerated, each side being restricted to a total of 48 target types. These include all the weapon types and all the city types for the side. The program does not require that the proportion of weapon types to city types conform to any pre-established ratio.

CODE 50 EXPERIENCE AT CNA

Under Task 1 of the Strategic Force Study, the Center for Naval Analyses was charged with evaluating Code 50. This included program adaptation to exercise the Code 50 model on the CNA computer, as well as documentation and evaluation of the program. Because the original Code 50 program required more core storage than the CNA computer could provide, modifications were made to reduce the core requirements. The resulting program, designated "CNA 50", carried out generally the same operations and yielded approximately the same results as earlier implementations of the original Code 50 program.

SECTION II

CODE 50 MATHEMATICAL MODELS

GENERAL

This section explains and derives the appropriate mathematical expressions for each of the Code 50 models, to promote a fuller understanding of the underlying implications of the models themselves and to provide greater insight into the meaning of the overall results obtained when Code 50 models are used.

In both the missile and bomber penetration models, the derivation is in two parts. The first is concerned with penetration of area defenses; the second is concerned with the problem of penetrating terminal defenses. In the case of bombers, the area defenses consist of waves of fighters sent out to intercept and shoot down the bombers. In the case of missiles, the area defenses consist of a nationwide area ABM defense. Terminal defenses against bombers consist of surface-to-air missiles (SAMs); terminal defense against missiles is assumed to be provided by a terminal ABM system around the target. An additional complication is provided in the case of bombers: those which penetrate terminal defenses are allowed, in certain cases, to attack the SAM sites directly, rather than merely suffer attrition in exhausting the supply of SAMs.

The single-shot kill probability model covers the mechanics of kill, taking into account target hardness and weapon yield. The model considers the effects of probabilistic delivery errors and the geographic distribution of an individual target's value, under the assumption that both delivery error and target value can be approximated by circular normal distributions.

The damage assessment models recognize two types of targets:

The first is a point target, that is, a target with no appreciable area. Weapons and other military targets are considered point targets within the meaning of the program. Damage to point targets is assessed, using an exponential law (Bernoulli trial methodology), since each weapon is assumed capable of destroying the target if placed correctly.

The second type is an area target. All cities are treated as area targets. For area targets, the square root law is employed, since it is assumed that a single weapon may not be able to destroy the entire target, even if placed optimally.

The weapon-target allocation model in Code 50 has a two-fold purpose: to inflict the maximum amount of damage on the whole target base and to inflict the required amount of damage on preferred targets. Preferred targets are defined as non-military targets that have terminal defenses. The generalized LaGrange multiplier technique is used, along with some heuristic non-linear programming logarithms to improve the efficiency of the program in finding a new optimum solution that satisfies the maximization problem and the preferred target damage constraint.

MISSILE PENETRATION MODELS

Offensive missile penetration in Code 50 consists of a two-part problem. First, there is the problem of missiles penetrating an area defense system that is assumed capable of protecting (up to the level of available interceptors) all probable targets. The second problem is the penetration of terminal defenses at the individual targets by missiles that succeed in penetrating the area defenses. In this subsection, the area and terminal penetration models used in Code 50 are described, beginning with the basic area penetration model and proceeding to the terminal penetration and entry price models. Subsequently, sample calculations are made to demonstrate the operation of the models.

The subscript conventions in this subsection are as follows:

IW	Weapon type subscript (missile types only)
KS	Strike subscript

To simplify notation, no target type subscripts are used in this development. However, because the equations do not vary with target type, no confusion should result.

Other symbols used in the equations are listed below:

AIN	Number of interceptors used on a given strike
AINX	Total number of area interceptors available
AMP_{IW}	Price in missiles successfully launched which just exhausts area interceptors
AOB_{IW}	Number of area objects per type IW missile
$ARR_{IW,KS}$	Number of missiles of type IW successfully launched and arriving at defender's area defenses on strike KS
ARR^*_{IW}	Number of missiles arriving at area defenses, just exhausting area defenses but resulting in no payoff
$ARRMX_{IW,1}$	Maximum number of type IW missiles that could be successfully launched by attacker and arrive at defender's area defenses on first strike
$FSENT_{IW,KS}$	Fraction of missiles of type IW sent on strike KS
$NLEFT_{IW}$	Number of offensive missiles of type IW from original inventory that have been withheld from prior strikes
NMINT	Number of terminal interceptors
$NNWP_{IW}$	Number of offensive missiles of type IW in attacker's inventory
NP_{IW}	Price in terms of independent targetable weapons successfully launched
NWP_{IW}	Total number of independent targetable weapons available
$OARR_{IW}$	Number of area objects successfully launched by the attacker and arriving at defender's area defenses on strike KS
OBJ_{IW}	Maximum number of area objects that could be launched by the attacker and arrive at the defender's area defenses on the first strike

OBJD	Number of area objects that can be destroyed by area interceptors
OPEN	Number of area objects penetrating the area defense
PEN_{IW}	Equivalent number of type IW missiles penetrating area defense
PENPROB	Probability that missile area objects will penetrate the area defense
PKMIS	Probability of kill of an individual interceptor against an individual area object
RELAI	Interceptor reliability
$RN_{IW,KS}$	Probability that missile type IW does not fail during flight on strike KS
$RR_{IW,KS}$	Probability that missile type IW does not fail during launch on strike KS
$SURV_{IW}$	Probability of $NLEFT_{IW}$ missiles surviving prior strikes by enemy
TOB_{IW}	Terminal objects per missile
TOS_{IW}	Number of terminal objects surviving
$TPRWPN_{IW}$	Number of independently targetable weapons per missile for weapon type IW

Assumptions Concerning Penetration of Area Defenses

Basic assumptions regarding area defenses are as follows:

- Area defenses are capable of defending all targets and may not be bypassed.
- Area defenses are so balanced that there is no more advantage in attacking the area defense weapons than in attacking targets directly.
- The area defenses do not preferentially defend but randomly destroy the incoming offensive weapons.
- The defense is unable to discriminate between warheads and decoys.
- Offensive weapon spacing is assumed such that a single interceptor may destroy only a single area object.
- On the first strike, the defender's strategy is to use only a fraction of his area interceptors; that fraction is equal to the ratio of area objects actually arriving to the number of area objects that could arrive.
- The offense knows exactly how many of its weapons have survived prior strikes.
- Losses because of unreliability occur before the missile area objects reach the area defenses and do not draw down the area interceptors.

The Area Defense Penetration Model

The significant parameter in the area defense penetration model is PENPROB, the probability that an individual independent weapon in the remaining inventory will penetrate the area defense.

Initially, there are $NNWP_{IW}$ offensive missiles of type IW in the attacker's inventory. At present, $NLEFT_{IW}$ offensive missiles of type IW from the original inventory have been withheld from prior strikes. The probability that these $NLEFT_{IW}$ missiles survived prior strikes by the other side is $SURV_{IW}$. The conditions on the three strikes are:

$$\left.\begin{array}{l} NLEFT_{IW} = NNWP_{IW} \\ SURV_{IW} = 1 \end{array}\right\} KS = 1$$
$$\left.\begin{array}{l} NLEFT_{IW} = NNWP_{IW} \\ SURV_{IW} \le 1 \end{array}\right\} KS = 2 \qquad (1)$$
$$\left.\begin{array}{l} NLEFT_{IW} = NNWP_{IW}\,(1- FSENT_{IW,1}) \\ SURV_{IW} \le 1 \end{array}\right\} KS = 3$$

The reprogrammable reliability (the launch reliability or the probability of failing such that remaining weapons can be retargeted, taking the failure into account) for missile type IW on strike KS is $RR_{IW,KS}$. The non-reprogrammable reliability (in-flight reliability or the probability of not failing after it is too late to retarget, taking the failure into account) for missile type IW on strike KS is $RN_{IW,KS}$. The fraction of the remaining missiles of type IW sent on strike KS is $FSENT_{IW,KS}$.

The maximum number of missiles of type IW that could be successfully launched by the attacker and arrive at the defender's area defenses on the first strike is:

$$ARRMX_{IW} = NNWP_{IW} \cdot RR_{IW,1} \cdot RN_{IW,1} \qquad (2)$$

Now, for missile type IW there are AOB_{IW} area objects per missile. Therefore, the maximum number of area objects that could be launched by the attacker and arrive at the defender's area defenses on the first strike is:

$$OBJ = \sum_{IW} ARRMX_{IW} \cdot AOB_{IW} \qquad (3)$$

The number of missiles of type IW successfully launched and arriving at the defender's area defenses on strike KS is:

$$ARR_{IW} = NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} \cdot RR_{IW,KS} \cdot RN_{IW,KS} \qquad (4)$$

The number of area objects successfully launched by the attacker and arriving at the defender's area defenses on strike KS is then:

$$OARR = \sum_{IW} ARR_{IW} \cdot AOB_{IW} \quad (5)$$

On the first strike, the area interceptors are "shared out," so that the fraction of area interceptors expended is identical to the ratio of offensive area objects arriving in strike one to the maximum number of offensive area objects that could arrive. Thus, for any remaining interceptors, AIN will be expended on strike one according to the relationship:

$$\frac{AIN}{AINX} = \frac{OARR}{OBJ} : KS = 1 \quad (6)$$

The number of area interceptors expended on a given strike is, then:

$$AIN = AINX \cdot OARR/OBJ : KS = 1 \quad (7)$$

$$AIN = AINX : KS > 1$$

The number of area interceptors remaining is, then:

$$AINX \rightarrow AINX - AIN \quad (8)$$

The destruction of area objects by area interceptors is determined on the basis of zero leakage until the number of area objects exceeds the expected kill potential of the area interceptors. Thereafter, the number of area objects destroyed is random and proportional to the number of interceptors (AIN), interceptor reliability (RELAI), and the probability of kill of an individual interceptor against an individual area object (PKMIS). The maximum number of area objects that can be destroyed is therefore:

$$OBJD = AIN \cdot RELAI \cdot PKMIS \quad (9)$$

The number of area objects penetrating the area defense is, then:

$$OPEN = MAX \langle 0 : OARR - OBJD \rangle \quad (10)$$

The probability that missile area objects penetrate the area defense is, then:

$$PENPROB = \frac{OPEN}{OARR} \quad (11)$$

The equivalent number of missiles of a given type penetrating the area defense, then, is:

$$PEN_{IW} = ARR_{IW} \cdot PENPROB \quad (12)$$

Terminal Defense Penetration and Total Price to Attack a Target

These are the assumptions regarding terminal defenses:

A. The number of terminal intercepts is equal to the total number of terminal interceptors. (This implies either that each terminal interceptor is perfectly reliable or that a reduced number of interceptors is assigned to compensate for reliability losses.)

B. Terminal interceptors have a probability of kill of unity against all terminal objects arriving at the target.

C. The terminal defense may not be suppressed, and no terminal objects penetrate to the target until the terminal interceptor supply is exhausted.

D. The defense is unable to discriminate between warheads and decoys.

E. Offensive weapon spacing is assumed such that a single interceptor destroys exactly one terminal object.

F. All offensive terminal objects have perfect reliability, all unreliability losses having occurred before penetration of area defenses.

As presented earlier, the number of missiles that penetrate the area defenses is:

$$PEN_{IW} = ARR_{IW} \cdot PENPROB \tag{13}$$

Based on the assumption of TOB_{IW} terminal objects per missile, NMINT terminal interceptors with unity kill probability and perfect reliability can destroy NMINT offensive terminal objects.

The number of terminal objects surviving is, then:

$$TOS_{IW} = MAX \langle 0 : \left(ARR_{IW} \cdot PENPROB \cdot TOB_{IW}\right) - NMINT \rangle \tag{14}$$

Now, if TOS_{IW} is exactly zero, the number of missiles arriving at the area defenses that just exhausts the area interceptors but results in no payoff is ARR_{IW}^*.

$$ARR_{IW}^* = NMINT/(PENPROB \cdot TOB_{IW}) \tag{15}$$

Another way of expressing the price is in terms of missiles successfully launched. If equation (15) is divided by $RN_{IW,KS}$, the price will be in these terms:

$$AMP_{IW} = ARR_{IW}^*/RN_{IW,KS} \tag{16}$$

Finally, if this result is multiplied by the number of independently targetable weapons per missile, and the result is converted to an integer, the price will be, in terms of independently targetable weapons successfully launched:

$$NP_{IW} = I_U [AMP_{IW} \cdot TPRWPN_{IW}] \tag{17}$$

The total number of independently targetable weapons available, NWP_{IW}, is just the integer number of missiles that will be successfully launched on this strike (the number left multiplied by the fraction sent and the launch reliability), multiplied by the number of independently targetable weapons per missile, with the results truncated to an integer value.

$$NWP_{IW} = I_D\left[I_D[NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} \cdot RR_{IW,KS}] \cdot TPRWPN_{IW}\right] \tag{18}$$

Now, if $NP_{IW} \geq NWP_{IW}$, the price of entry exceeds or equals the available resources and no payoff from the target can be obtained by weapon type IW alone.
If $NP_{IW} < NWP_{IW}$ the entry price allows a payoff from weapon type IW .

It should be noted that the preceding calculations of entry price presumes that only a single weapon type will be used on a terminally defended target, since there are no provisions for allocating the total price among several weapon types.

Sample Problem of Missile Penetration

A series of cases will now be considered and the necessary calculations carried out in a tabular manner, with one case explained in detail. The parameter values for the cases are as follows:

$KS > 1$

$NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} = 10, 20, 30, 40, 50$

$RR_{IW,KS} = 0.9$

$RN_{IW,KS} = 0.8$

$AOB_{IW} = 3$

$TOB_{IW} = 3$

$TPRWPN_{IW} = 3$

$AIN = 30$

$RELAI = 0.8$

$PKMIS = 1.0$

$NMINT = 30$

The case of $NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} = 50$ will now be carried out in detail.

The number of missiles arriving at the area defenses is:

$$ARR_{IW} = \left(NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS}\right) \cdot RR_{IW,KS} \cdot RN_{IW,KS}$$
$$= (50) \cdot (.9) \cdot (.8) = 36$$

The 14 missiles lost consist of 5 launch failures and 9 inflight failures. Because only a single missile type is being analyzed, the number of area objects arriving is:

$$OARR = ARR_{IW} \cdot AOB_{IW}$$
$$= (36) \cdot (3) = 108$$

The number of effective area interceptors is:

$$OBJD = AIN \cdot RELAI \cdot PKMIS$$
$$= (30) \cdot (.8) \cdot (1.0) = 24$$

Therefore, the number of area objects penetrating the area defenses is:

$$OPEN = MAX \langle 0 : OARR - OBJD \rangle$$
$$= MAX \langle 0 : 108 - 24 \rangle = 84$$

The penetration probability may then be found:

$$PENPROB = \frac{OPEN}{OARR}$$
$$= \frac{84}{108} = .7778$$

The number of missiles penetrating is, then:

$$PEN_{IW} = ARR_{IW} \cdot PENPROB$$
$$= (36) \cdot (.7778) = 28$$

The number of terminal objects surviving terminal defenses is:

$$TOS_{IW} = MAX \langle 0 : \left(ARR_{IW} \cdot PENPROB \cdot TOB_{IW}\right) - NMINT \rangle$$
$$= MAX \langle 0 : \left((36) \cdot (.7778) \cdot (3)\right) - 30 \rangle$$
$$= 54$$

The price in missiles arriving at area defenses that was necessary to gain a payoff was:

$$ARR_{IW}^{*} = NMINT/(PENPROB \cdot TOB_{IW})$$
$$= 30/(.7778 \cdot 3) = 12.86$$

This translates into the price of missiles successfully launched:

$$AMP_{IW} = ARR_{IW}^{*}/RN_{IW,KS}$$
$$= 12.86/.8 = 16.08$$

In terms of an integer number of independently targetable weapons, the price is:

$$NP_{IW} = I_U \left[AMP_{IW} \cdot TPRWPN_{IW} \right]$$
$$= I_U \left[16.08 \cdot 3 \right]$$
$$= 49$$

Now, the number of allocatable weapons was:

$$NWP_{IW} = I_D \left[I_D \left[NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} \cdot RR_{IW,KS} \right] \cdot TPRWPN_{IW} \right]$$
$$= I_D \left[I_D \left[(50) \cdot (.9) \right] \cdot (3) \right]$$
$$= 135$$

Since $NP_{IW} \leq NWP_{IW}$, a payoff may be secured from the target.

The number of weapons providing the payoff is:

$$N_{IW} = MAX \langle 0 : (NWP_{IW} - NP_{IW}) \cdot PENPROB \cdot RN_{IW,KS} \rangle$$
$$= MAX \langle 0 : (135 - 49) \cdot (.7778) \cdot (.8) \rangle$$
$$= 54$$

Notice that this result is the same as the result obtained for the number of terminal objects surviving terminal defenses since the terminal objects per missile and the independently targetable weapons per missile are both set to the same value (3). The results for the other cases are shown in table 1.

TABLE I

SUMMARY OF RESULTS FOR SAMPLE CASES

Number of missiles launched	Number of missiles successfully launched	Number of allocatable weapons, NWP_{IW}	Number of missiles that reach area defense, ARR_{IW}	Probability of a missile or weapon penetrating area defense, PENPROB	Number of missiles that reach terminal defense, PEN_{IW}	Number of missiles that arrive at target	Number of targetable weapons on target
10	9	27	7.2	0	0	0	0
20	18	54	14.4	.4444	6.4	0	0
30	27	81	21.6	.6296	13.6	3.6	11
40	36	108	28.8	.7222	20.8	10.8	32
50	45	135	36.0	.7778	28.0	18.0	54

BOMBER PENETRATION MODELS

As with missile penetration, bomber penetration in Code 50 consists of a two-part problem. First, there is the problem of the bombers that penetrate the interceptor defense system.* The second problem is penetration of the SAM defenses.** The remainder of this subsection discusses the penetration models, beginning with the model of interceptor defenses, and then proceeds to the SAM defense penetration model. Some sample calculations, showing how the models function in Code 50, are then presented.

The subscript conventions used in this subsection are as follows:

IW	Weapon type subscript (bomber types only)
IB	Bomber class subscript
IF	Fighter type subscript
KS	Strike subscript

*This corresponds to the penetration of area defenses in the missile penetration problem although, mathematically, the missile penetration and bomber penetration are handled much differently.

**This corresponds to penetration of the terminal defenses in the missile penetration problem.

To simplify notation, no target type subscripts are used in this development. However, because the equations derived do not vary with target type, no confusion should result.

Other symbols used in the equations are listed below:

AMP_{IW}	Price in bombers successfully launched
$AREL_{IW}$	ASM reliability
ARR_{IW}	Number of bombers and decoys successfully launched and arriving at defender's fighter defense system on strike KS
ARR_{IW}^*	Price in bombers that arrive at defenses
$ASMDEC_{IW}$	Number of ASM decoys per ASM warhead
B	Total number of bombers and decoys arriving at fighter defense system
$BDES_{IW}$	Number of bombers destroyed by all NSAMX missiles
$BDPEN_{IW}$	Number of bombers and decoys of weapon type IW expected to penetrate fighter defenses
$BMRDEC_{IW}$	Number of bomber decoys per bomber of type IW
F	Fraction of bombers that attack through given corridor; also fraction of fighters that attack bombers
$FKBD_{IW}$	Fraction of type IW bombers killed before weapon delivery
$FSENT_{IW,KS}$	Fraction of remaining type IW bombers sent on strike KS
IWP_{IW}	Number of warheads required to produce at least 95 percent kill probability
NI_{IF}	Number of fighters in defense system of type IW
$NLEFT_{IW}$	Number of bombers that have not been sent in prior strikes
$NNWP_{IW}$	Number of type IW bombers in attacker's inventory
NP_{IW}	Total price in bombers, or in subsonic cruise armed decoys (SCADs)
$NPSN_{IW}$	Number of independently targetable weapons successfully launched
$NPSS_{IW}$	Number of independently targetable weapons for suppressing SAM sites
$NSAMD_{IT}$	Number of suppressible SAM sites at target IT

NSAMDM	Number of suppressible SAMs per site
NSAMX	Number of non-suppressible SAMs
NWP_{IW}	Number of independently targetable weapons available
PEN_{IW}	Expected number of bombers penetrating defenses
$PENPROB_{IW}$	Overall penetration probability for type IW weapon
$PK_{IF, IB_{IW}}$	Probability that a type IF attacking fighter kills a type IB_{IW} bomber
$PKILL_{IB_{IW}}$	Probability that a bomber or decoy is killed
$PSEFF_{IW}$	Probability of bomber survival until weapon launch
$PSURV_{IB_{IW}, IF}$	Probability that none of the $F \cdot NI_{IF}$ fighters kills a given bomber of same type
R_{IF}	Ratio of type IF interceptors to total number of bombers in the corridor
$RN_{IW, KS}$	Probability that bomber type IW does not fail during flight on strike KS
$RR_{IW, KS}$	Probability that bomber type IW does not fail during launch on strike KS
SSK	Single-shot kill probability for ASM bomber, taking into account SAM site hardness, SRAM yield, CEP, and height of burst
SSKK	Overall kill probability of ASM bomber against suppressible SAM site
$SURV_{IW}$	Probability that $NLEFT_{IW}$ bombers survive prior enemy strikes
$TPRWPN_{IW}$	Number of independently targetable weapons per missile for weapon type IW
$WHDS_{IW}$	Number of ASM warheads per bomber for weapon type IW

Interceptor Penetration Model

Before a discussion of the penetration equations themselves, we shall list the basic assumptions regarding the interceptor defenses:

A. The interceptors are assumed capable of defending all targets and may not be bypassed.

B. There is no way of destroying the interceptors on the ground before the bombers attempt penetration.

C. The interceptors attack the bomber force randomly in waves, each wave consisting of a single type of interceptor.

D. The interceptor/bomber ratio is constant for a single interceptor type in all attack corridors.

E. There is no coordination between interceptor waves; i.e., the whole force of interceptors of each type attacks the whole bomber force on a random basis.

F. The defense is unable to discriminate between bombers and bomber decoys and between ASMs and ASM decoys.

G. The offense knows exactly how many of its weapons have survived prior strikes.

H. Losses because of unreliability occur before the bombers reach the interceptors.

These assumptions understood, we now present the interceptor defense penetration model. The significant parameter is $PENPROB_{IW}$, the probability that an individual independently targetable weapon in the remaining inventory penetrates the interceptor defenses.

Initially, there are $NNWP_{IW}$ bombers of type IW in the attacker's inventory. At present, there are $NLEFT_{IW}$ bombers that have not been sent in prior strikes. The probability that these $NLEFT_{IW}$ bombers have survived prior strikes by the other side is $SURV_{IW}$. The conditions existing on the three strikes are:

$$\left.\begin{aligned} NLEFT_{IW} &= NNWP_{IW} \\ SURV_{IW} &= 1 \end{aligned}\right\} \quad KS = 1$$
$$\left.\begin{aligned} NLEFT_{IW} &= NNWP_{IW} \\ SURV_{IW} &< 1 \end{aligned}\right\} \quad KS = 2 \qquad (19)$$
$$\left.\begin{aligned} NLEFT_{IW} &= NNWP_{IW}(1 - FSENT_{IW,1}) \\ SURV_{IW} &\le 1 \end{aligned}\right\} \quad KS = 3$$

The reprogrammable reliability (the launch reliability or the probability of failing such that remaining weapons can be retargeted, taking the failure into account) for bomber type IW on strike KS is $RR_{IW,KS}$. The non-reprogrammable reliability (in-flight reliability or the probability of not failing after it is too late to retarget, taking the failure into account for bomber type IW on strike KS is $RN_{IW,KS}$. The fraction of the remaining bombers of type IW that are sent on strike KS is $FSENT_{IW,KS}$. The number of bomber decoys per bomber for bomber type IW is $BMRDEC_{IW}$.

The number of bombers and decoys of type IW successfully launched and arriving at the defender's fighter defense system on strike KS is:

$$ARR_{IW} = NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} \cdot RR_{IW,KS} \cdot RN_{IW,KS} \cdot (1 + BMRDEC_{IW}) \tag{20}$$

The total number of bombers and decoys arriving at the fighter defense system is, then:

$$B = \sum_{IW} ARR_{IW} \tag{21}$$

Now, the defense includes NI_{IF} fighters of type IF . It is assumed that a fraction F of the bombers is to attack through a given corridor and that the same fraction F of the fighters is pre-positioned to counter the $F \cdot B$ bombers in the corridor. If the probability that an attacking fighter of type IF will kill a bomber of type IB_{IW} is $PK_{IF, IB_{IW}}$, the probability that a specific one of the $F \cdot NI_{IF}$ fighters encounters and kills a specific bomber out of the $F \cdot B$ in the corridor is $PK_{IF, IB_{IW}}/(F \cdot B)$.

The probability that none of the $F \cdot NI_{IF}$ fighters kills a given bomber of this type is, then:

$$PSURV_{IB_{IW}IF} = \left[1 - \left(PK_{IF, IB_{IW}}/(F \cdot B)\right)\right]^{F \cdot NI_{IF}} \tag{22}$$

Let the ratio of interceptors of type IP to the total number of bombers in the corridor be R_{IF} .

$$R_{IF} = F \cdot NI_{IF}/(F \cdot B) \tag{23}$$

The probability of survival is, then:

$$PSURV_{IB_{IW}IF} = \left\{ \left[1 - \left(PK_{IF, IB_{IW}}/(F \cdot B)\right)\right]^{F \cdot B} \right\}^{R_{IF}} \tag{24}$$

As an approximation of (24) as $(F \cdot B)$ becomes large:

$$\begin{aligned} PSURV_{IB_{IW}IF} &= [\exp(-PK_{IF, IB_{IW}})]^{R_{IF}} \\ PSURV_{IB_{IW}IF} &= \exp[-PK_{IF, IB_{IW}} \cdot NI_{IF}/B] \end{aligned} \tag{25}$$

When all interceptor types are considered, the probability that a given bomber or decoy penetrates the defenses is of the form:

$$PSURV_{IB_{IW}} = \prod_{IF} \exp[-A_{IF, IB_{IW}} \cdot (NI_{IF}/B)^{C_{IF, IB_{IW}}}] \tag{26}$$

Although equation (26) represents the survival probability of a bomber or decoy, the important issue is whether a bomber successfully delivered its weapon and, if so, the reliability of those weapons after release. These parameters are taken into account in calculations of the overall penetration probability ($PENPROB_{IW}$) of the weapon type. From equation (26), the probability that a bomber or decoy of type IB_{IW} is killed is:

$$PKILL_{IB_{IW}} = 1 - PSURV_{IB_{IW}} \tag{27}$$

The fraction of bombers of weapon type IW killed before weapon delivery is $FKBD_{IW}$. The effective probability of survival to weapon launch is, then:

$$PSEFF_{IW} = PSURV_{IB_{IW}} + \left(PKILL_{IB_{IW}} \cdot (1 - FKBD_{IW})\right) \tag{28}$$

Finally, for ASMs the probability that the weapon is successfully delivered must take the ASM reliability ($AREL_{IW}$) into account as a multiplication factor. The overall penetration probability for weapon type IW is, then:

$$\begin{aligned} PENPROB_{IW} &= PSEFF_{IW} \cdot AREL_{IW} \\ &= AREL_{IW} \left[1 - \left(FKBD_{IW}(1 - PSURV_{IB_{IW}})\right)\right] \end{aligned} \tag{29}$$

The number of bombers and decoys of weapon type IW expected to penetrate is:

$$BDPEN_{IW} = ARR_{IW}\ PENPROB_{IW} \tag{30}$$

However, since only the fraction $1/(BMRDEC_{IW} + 1)$ are actually bombers, the expected number of bombers penetrating is:

$$PEN_{IW} = ARR_{IW} \cdot PENPROB_{IW}/(1 + BMRDEC_{IW}) \tag{31}$$

Terminal Defense Penetration and Total Price to Attack a Target

The assumptions regarding surface-to-air-missile (SAM) defenses are as follows:

A. Gravity bombers do not have to pay a definite price to attack a SAM-defended target; their kill probability against that target takes into account, as a multiplicative factor, the probability that the target will shoot down the bomber with SAMs.

B. ASM bombers with short-range attack missiles (SRAMs) must destroy with a probability of .95 the requisite number of suppressible SAM sites and exhaust the requisite number of missiles from non-suppressible SAM sites before attacking the target.

C. Subsonic cruise armed decoys (SCADs) must exhaust the requisite number of suppressible and non-suppressible SAM sites before attacking the target.

D. Each SAM may kill only one attacking SRAM or SCAD.

E. Each SRAM may attack only a single suppressible SAM site.

Based on the above assumptions, gravity bombers have no entry price to pay at a SAM-defended target. Therefore:

$$NP_{IW} = 0 \ : \ \text{gravity bombers} \tag{32}$$

For the ASM bomber case, the price for suppressing suppressible sites is calculated first. A single-shot kill probability (SSK) is calculated for the SAM site hardness, the SRAM yield, CEP, and height of burst. The probability of kill must then be degraded by the probability that the SRAMs arrive. The resulting overall kill probability is, then:

$$SSKK = SSK \cdot PENPROB_{IW} \cdot RN_{IW,KS} \tag{33}$$

For this SSKK value, the number of warheads required to produce at least a 95 percent kill probability (IWP_{IW}) is obtained from the relationship:

$$.95 \leq 1 - (1 - SSKK)^{IWP_{IW}} \tag{34}$$

Therefore:

$$IWP_{IW} = I_U [\log(.05)/\log(1 - SSKK)] \tag{35}$$

Now, since there are $NSAMD_{IT}$ suppressible SAM sites at target IT, the total number of warheads required to suppress the sites is $IWP_{IW} \cdot NSAMD$.

This result must now be translated into independently targetable weapons required per target. We may do this by dividing through by the number of warheads per bomber of weapon type IW and multiplying by the number of independently targetable weapons of weapon type IW per bomber, and taking this to the next larger integer value, to obtain the price of independently targetable weapons for suppressing SAM sites:

$$NPSS_{IW} = I_U [IWP_{IW} \cdot NSAMD \cdot TPRWPN_{IW}/WHDS_{IW}] \tag{36}$$

Next, the price for non-suppressible SAM sites must be calculated. This is very similar to the missile price calculation, since, in both cases, the specified number of interceptors must be exhausted.

The number of ASM warheads per bomber for weapon type IW is $WHDS_{IW}$. The number of ASM decoys per ASM warhead is $ASMDEC_{IW}$. The total number of warheads and decoys per ASM bomber of weapon type IW is, then, $WHDS_{IW}(1 + ASMDEC_{IW})$. Then, since the NSAMX non-suppressible SAMs are capable of destroying a single decoy or warhead each, all NSAMX missiles are capable of destroying $BDES_{IW}$ bombers of weapon type IW:

$$BDES_{IW} = NSAMX/[WHDS_{IW}(1 + ASMDEC_{IW})] \tag{37}$$

Taking into account the penetration probability, the price in bombers of type IW arriving at the interceptor defense is:

$$ARR^*_{IW} = NSAMX/[PENPROB_{IW} \cdot WHDS_{IW}(1 + ASMDEC_{IW})] \tag{38}$$

In terms of bombers successfully launched, thus taking inflight reliability into consideration, the price is:

$$AMP_{IW} = ARR^*_{IW}/RN_{IW,KS} \tag{39}$$

Then, as in the case of missiles, if this result is multiplied by the number of independently targetable weapons per bomber, the result is expressed in terms of independently targetable weapons successfully launched:

$$NPSN_{IW} = I_U\Big[NSAMX \cdot TPRWPN_{IW}/\Big(RN_{IW,KS} \cdot PENPROB_{IW} \cdot WHDS_{IW}\,(1 + ASMDEC_{IW})\Big]\Big) \tag{40}$$

The total price for ASM bombers, then, is just the sum of the prices against suppressible and non-suppressible SAM sites:

$$NP_{IW} = NPSS_{IW} + NPSN_{IW} \tag{41}$$

For the SCAD case, since all SAMs must be exhausted and there are NSAMDM suppressible SAMs per site, the total number of SAMs to be exhausted is (NSAMD · NSAMDM + NSAMX). Additionally, for the SCAD there are no secondary decoys, and $ASMDEC_{IW} = 0$. The overall price for SCADs, thus, is:

$$NP_{IW} = I_U\Big[(NSAMD \cdot NSAMDM + NSAMX) \cdot TPRWPN_{IW}/(RN_{IW,KS} \cdot PENPROB_{IW} \cdot WHDS_{IW})\Big] \tag{42}$$

The number of independently targetable weapons available, NWP_{IW}, is just the integer number of bombers that will be successfully launched on this strike (the number left, multiplied by the fraction sent and the launch reliability), multiplied by the number of independently targetable weapons per bomber, with the results truncated to an integer value:

$$NWP_{IW} = I_D\Big[I_D\big[NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} \cdot RR_{IW,KS}\big] \cdot TPRWPN_{IW}\Big] \tag{43}$$

Now, if $NP_{IW} \geq NWP_{IW}$, the price of entry exceeds or equals the available resources, and no payoff from the target can be obtained by weapon type IW alone. If $NP_{IW} < NWP_{IW}$, the entry price allows a payoff from weapon type IW.

Sampie-Bomber Penetration Problem

A series of cases will now be considered and the necessary calculations carried out in a tabular manner, with one case explained in detail. The results will be presented both in a table and graphically. The parameter values for the cases are as follows:

$KS > 1$

$NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} = 10, 20, 30, 40, 50$

$RR_{IW,KS} = .99$

$RN_{IW,KS} = .95$

$BMRDEC_{IW} = 1$

$IB_{IW} = 1$

$WHDS_{IW} = 5$

$TPRWPN_{IW} = 5$

$ASMDEC_{IW} = 1$

$NI_1 = 50$

$A_{1,1} = .5$

$C_{1,1} = 1$

$FKBD_{IW} = .9$

$AREL_{IW} = .9$

$SSK = .5$

$NSAMX = 5$

$NSAMD = 2$

$NSAMDM = 5$

The case of $NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS} = 50$ will now be carried out in detail.

The expected number of bombers and decoys arriving at the interceptor defenses is:

$$ARR_{IW} = (NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS}) \cdot RR_{IW,KS} \cdot RN_{IW,KS}$$

$$\cdot (1 + BMRDEC_{IW})$$

$$= (50) \cdot (.99) \cdot (.95) \cdot (2) = 94.05$$

Because only one weapon type is considered, the total number of bombers and decoys arriving at the interceptor defenses is:

$$B = 94.05$$

The probability that a decoy or bomber survives the interceptor attack is:

$$PSURV_1 = \exp[-.5 \cdot (50/94.05)^1]$$

$$= .7666$$

The penetration probability, then, is:

$$PENPROB_{IW} = (.90) \cdot \left[1 - \left((.9) \cdot (1 - .7666)\right)\right]$$

$$= .7109$$

The expected number of bombers penetrating the interceptor defense, then, is:

$$PEN_{IW} = \left((94.05) \cdot (.7109)\right)/(2)$$

$$= 33.43$$

Now, the SAM defense must be handled. First, against the suppressible sites, the overall kill probability is:

$$SSKK = (.5) \cdot (.7109) \cdot (.95)$$

$$= .3377$$

The number of ASMs required per suppressible site is:

$$IWP_{IW} = I_U\,[\log(.05)/\log(1 - .3377)]$$

$$= I_U\,[7.27] = 8$$

The price to overcome suppressible SAM sites, then, is:

$$NPSS_{IW} = I_U\,[(8) \cdot (2)(5)/5]$$

$$= I_U\,[16] = 16$$

The price to exhaust the non-suppressible SAMs is:

$$NPSN_{IW} = I_U \left\{(5) \cdot (5)/[(.95) \cdot (.7109) \cdot (5) \cdot (2)\ ]\right\}$$
$$= I_U\ [3.70] = 4$$

The total entry price to the target, then, is:

$$NP_{IW} = 16 + 4 = 20$$

Now, the number of allocatable weapons was:

$$NWP_{IW} = I_D \left[I_D\ [(NLEFT_{IW} \cdot SURV_{IW} \cdot FSENT_{IW,KS}) \cdot RR_{IW,KS}] \cdot TPRWPN_{IW} \cdot TPRWPN_{IW} \right]$$
$$= I_D \left[I_D [(50) \cdot (.99)] \cdot (5) \right]$$
$$= 245$$

Since $NP_{IW} \le NWP_{IW}$, a payoff may be secured from the target. The results for the other cases are shown in table 2.

TABLE 2

SUMMARY OF RESULTS FOR SAMPLE CASES IN BOMBERS UNLESS OTHERWISE NOTED

NWP_{IW} Independently targetable weapons	ARR_{IW}	$PSURV_{IB_{IW}}$	$PENPROB_{IW}$	PEN_{IW}	SSKK	IWP Independently targetable weapons	$NPSS_{IW}$ Independently targetable weapons	$NPSN_{IW}$ Independently targetable weapons	NP_{IW} Independently targetable weapons
45	18.81	.2647	.3044	2.86	.1446	20	40	9	49*
95	37.62	.5145	.5088	9.53	.2407	11	22	6	28
145	56.44	.6421	.6101	17.22	.2898	9	18	5	23
195	75.24	.7173	.6710	25.24	.3187	8	16	4	20
245	94.05	.7666	.7109	33.43	.3377	8	16	4	20

*No payoff from this case, since $NP_{IW} > NWP_{IW}$

SINGLE-SHOT KILL PROBABILITY MODEL

The single-shot kill probability model has several distinct parts. The first involves determination of a 50 percent single-shot kill radius for weapon type IT when used against target type IW. The second part involves the introduction of aiming errors and, for area targets, distribution of the target, to determine a probability of kill for weapon type IW against target type IT, on the premise that the weapon type is successfully delivered. The third part involves strategy considerations and the probability that weapon type IW can be delivered against target type IT.

Symbols used in the equations are listed below:

A	Parameter for generalized kill function
CD	Drag coefficient
CEP	Circular error probable - radius of the circle centered at the mean and containing 50 percent of the impact points
F	Adjustment factor; ratio of FM for any yield to FMO for the datum yield
FM	Initial value of the force applied to the target
FM'	Peak force per unit area
FM"	Force per unit area of blast wave
FMO	Peak damage pressure for a given level of damage
F(t)	Force resulting from dynamic pressure and as a function of time
GZ	Ground zero
H	Lever arm
HOB	Height of burst
IMP	Impulse of the system
K	Target factor used in reference (c)
KN	Similar to the K factor in reference (c), except for a factor of 10
M	Effective mass of the system
MU	Ductility ratio
$P_{0.5}$	Peak overpressure associated with a .5 probability of obtaining a desired degree of damage
PAT	Atmospheric pressure ahead of blast wave
$PKILL_{IW}$	Probability of target being killed
PK(R)	Probability of damage as a function of distance R
$PSURV_{IW,IT}$	Probability of surviving attack by the target terminal defense
PT	Point at which effective mass is assumed to be concentrated

P(t)	Overpressure
$PTGT_{IW,IT}$	Probability that there is a surviving target at the site
$Q_{0.5}$	Peak dynamic pressure associated with a .5 probability of damage
Q(t)	Dynamic pressure
R	Distance (see figure 7)
R'	Distance to the target element from (0, 0)
R"	Distance to the impact point from (0, 0)
RE	Structural resistance
R(x)	Resistance of target to displacement surface area exposed to force
S	Surface area exposed to blast
$SSKP1_{IW,IT}$	Single-shot kill probability for a single warhead
$STRAT_{IW,IT}$	Strategy factor
$SURVX_{IT}$	Probability that target type IT has not already been destroyed
T	Natural period of elastic vibration
TC	Period of a compound pendulum
TD	Effective time duration
TDO	Effective time duration for a reference yield YO
TS	Artificial "period" for system
V	Velocity of the system
VF	Maximum and final velocity of system
VN	Vulnerability number
VO	Initial velocity of the system
W	Weight of the system
WB	Work done by blast wave in moving the structure
WR	Weapon radius
WS	Work done by the system in resisting deflection
X	Target displacement or deflection
XE	Yield deflection
XM	Maximum deflection
Y	Weapon yield
YO	Datum yield

α	Angular acceleration
η	Coefficient of friction between the sliding surfaces
Ω	Kinetic energy of the system
Ω'	Work done to overturn the system
ω	Angular velocity
σ	Measure of gradual fall-off in probability of damage with distance
σ', σ''	Standard deviations of the distributions
τ	Torque

Determination of the 50 Percent Kill Radius*

Blast Phenomena - The blast from a nuclear explosion, which constitutes the foremost cause of damage to structural targets, is the result of two related blast phenomena: overpressure, P, and dynamic pressure, Q. Overpressure may damage targets by several different modes: by crushing, by causing severe horizontal deflections, by overturning, etc. By contrast, dynamic pressure acts in one direction and can be thought of as wind pressure. It results from the velocity and density of the strong transient winds that accompany the blast wave. Both P and Q are usually expressed as pressure, in units of pounds per square inch (PSI).

When the blast wave arrives at a target, both P and Q rise abruptly to their maximum values and then decay as functions of time, in the manner shown by the solid lines in figure 1.

It is more convenient to work with pressures which vary linearly with time instead of the shapes shown by the solid lines in figure 1. The linear method of representing the pressure-time curves uses triangles with the same maximum pressure value as in the ideal case, but lasting for a new duration, termed effective duration, TD . This representation is indicated by the dotted lines in figure 1. The effective duration is usually so determined as to provide the same impulse in the triangular representation as contained in the ideal case; that is, the areas of triangles P-TD-0 and Q-TD-0 are equal to the areas under the respective ideal curves. Pressures represented in this triangular manner are referred to as "initially peaked triangular pressure pulses."

The two blast phenomena, overpressure and dynamic pressure, cause damage to targets by several different types of loadings. Thus, most targets may be classified into two types, overpressure-sensitive (or P-type) targets and dynamic-pressure-sensitive (or Q-sensitive) targets.

Externally Applied Forces - Although many targets are influenced by more than one form of blast loading, it is convenient to consider targets to be solely either overpressure or dynamic-pressure-sensitive. Also, the complex decaying pressures that result in complex forces can be approximated with sufficient accuracy by the triangular representation discussed previously if the target response time is long compared with the duration of

*Except as noted, this derivation is based on reference (a).

the loading. Since the loaded area of many targets remains constant during the significant damage phase, these pressures are directly proportional to forces on the target. Therefore, these forces may be represented triangularly, as were the pressures.

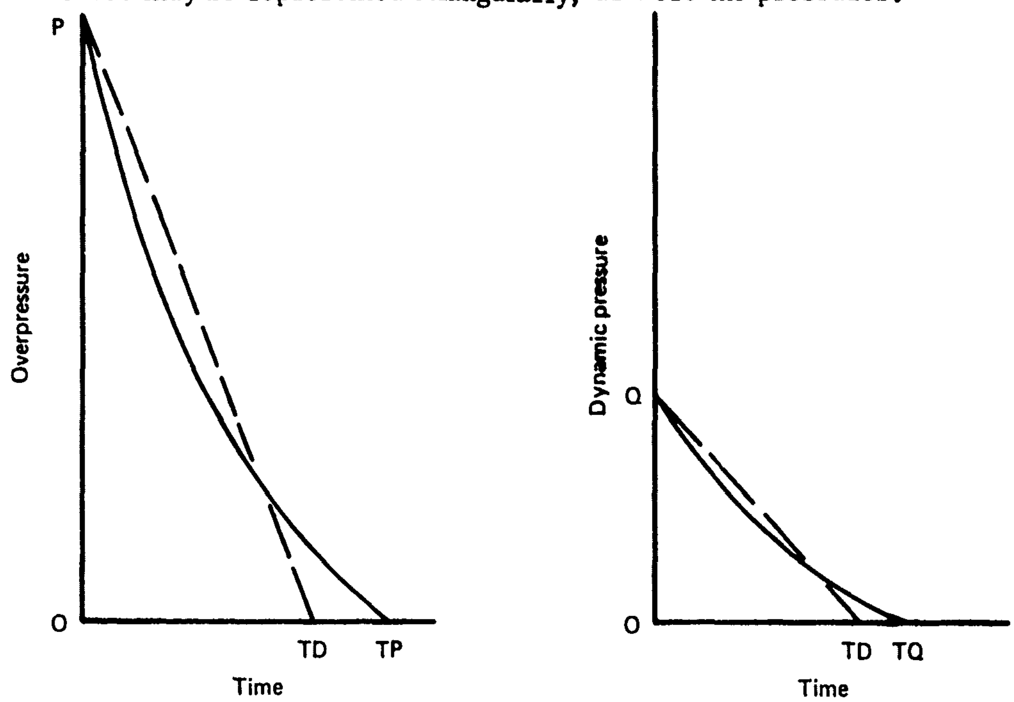

FIG. 1: OVERPRESSURE AND DYNAMIC PRESSURE AS FUNCTIONS OF TIME

Figure 2 is a simplified force-time diagram, where F(t) is the force (pressure times unit area), varying linearly with time, and FM is the initial value of the force applied to the target. This peaked force could result from either of the two pressures discussed previously.

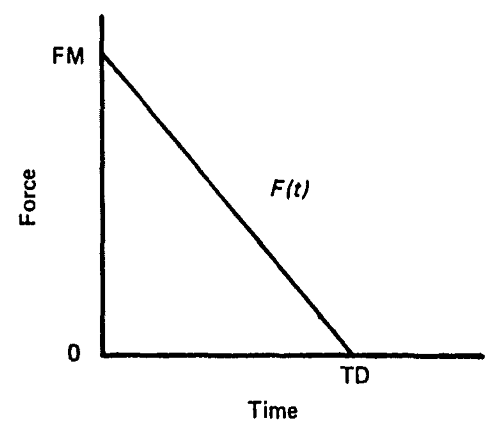

FIG. 2: SIMPLIFIED FORCE-TIME RELATIONSHIP

As mentioned above, the forces on the target during the significant damage phase are equal to the product of the particular pressure times the area existing at that time. For example, the existence of frangible sheathing may be an important determinant of the area loaded. In the special case of a loading resulting from dynamic pressure, the shape of the target is important. In general, the force resulting from the dynamic pressure is given by·

$$F(t) = S \cdot Q(t) \cdot CD \tag{44}$$

where F(t) is force at time t, S is the presented area, Q(t) is the dynamic pressure at time t, and CD is the drag coefficient. This coefficient, a property primarily of target shape, ranges in value from 0.35 for cylinders to 2 for unshielded structural members (I-beams, etc.). It also depends on Reynolds Number for most objects.

Resistance of Target - Complex targets resist displacement in the same manner as a beam resists bending, or a desk resists sliding, or a box resists overturning. The relationship between the resistance R(X) that many targets offer to displacement, and target displacement or deflection, X may be expressed by a bi-linear resistance curve as shown in figure 3. In figure 3, the point (RE, XE) is the yield point. Resistance functions are shown for: (a) a strain-hardening situation in which, for some reason, the resistance of the structure increases beyond the yielding point of (RE, XE), (b) a conventionalized elasto-plastic relationship with constant RE after the yield deflection XE is reached, and (c) an unstable situation in which the resistance decreases after the yield point. Although resistance curves take many forms, as shown in figure 3, they may be approximated by an "equivalent" elasto-plastic resistance-deflection relation. In such equivalent cases, shown in figure 4, fictitious values of RE and XE are used so that the area under the equivalent curve equals the area under the original elasto-plastic curve; thus the work required to deflect the structure is the same. To assist in relating external force and resistance, the structural resistance, R(X), should, for convenience, be defined in the same units as F(t), i.e., pounds or psi.

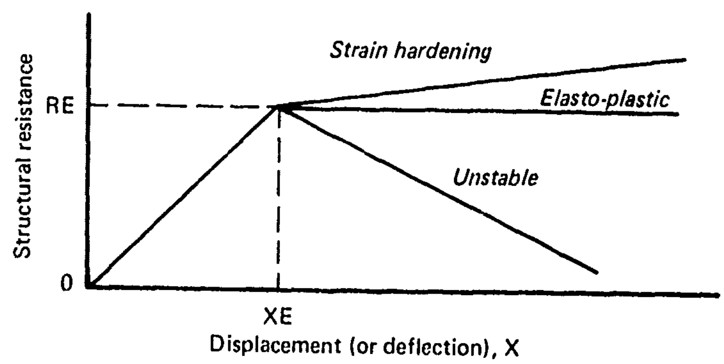

FIG. 3: RELATIONSHIP BETWEEN STRUCTURAL RESISTANCE AND DISPLACEMENT

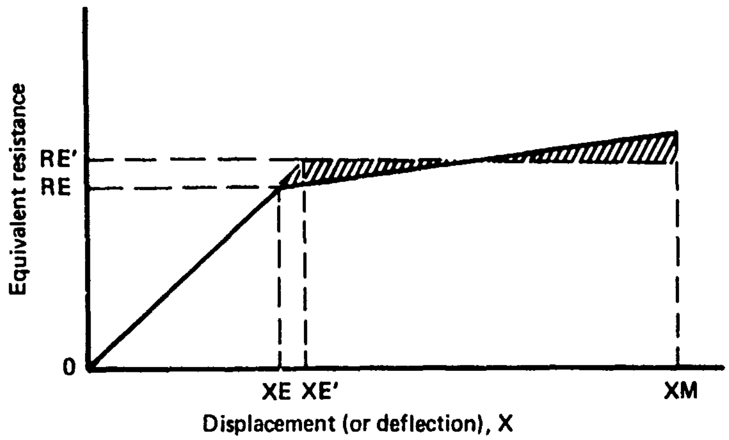

FIG. 4: RELATIONSHIP BETWEEN EQUIVALENT RESISTANCE AND DISPLACEMENT

Figure 4 introduces a new term, XM, called the "maximum deflection," which is the deflection at which a target is damaged to the extent desired. For example, in overturning a target, XM will be the point at which the target is balanced with its center of gravity over the point of rotation. In another case, XM might represent the distance a bridge slides before falling off its supports; or XM might be the deflection of the upper end of a building column at which it will fall by its own weight or the building will be unusable to a specified degree.

In order to describe XM conveniently, the term ductility ratio, MU, is used. It is defined as follows:

$$MU = XM/XE \tag{45}$$

The ductility ratio is a measure of the inelastic action required to damage a target.

It is convenient also to describe structures by their natural periods of elastic vibration, T. For a single-degree-of-freedom system, the period is represented by:

$$T = 2\pi(M \cdot XE/RE)^{1/2} \tag{46}$$

where M is the mass of the structure, and RE/XE is the spring constant of the structure. When an equivalent elasto-plastic resistance deflection relationship is considered as shown in figure 4, the resulting period is then termed the "effective period."

Equation of Motion of a Single-Degree-of-Freedom System - The mass and resistance of a structure may be represented by the single degree-of-freedom system shown in figure 5.

Newton's equation of motion for this single-degree-of-freedom system may be expressed as:

$$F(t) - R(x) = Md^2X/dt^2 \tag{47}$$

where F(t) is the externally applied force as a function of time, R(x) is the resistance of the system as a function of displacement, M is the effective mass of the system and d^2X/dt^2 is the second derivative of displacement with respect to time (acceleration).

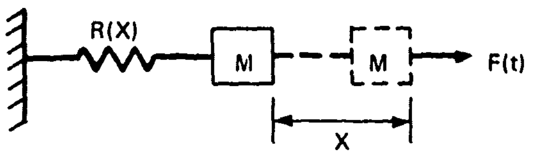

FIG. 5: A SINGLE-DEGREE-OF-FREEDOM SYSTEM

The displacement of this model can correspond to the displacement of some point on a structure, for example, point PT on the roof of the one-story structure shown in figure 6. The effective mass is assumed to be concentrated at this point PT. The resistance of the structure is related to the sum of the resistances of the individual components (the columns) of the structure. The external force, acting on the concentrated mass, may be an initially peaked triangular force of the type described in figure 6.

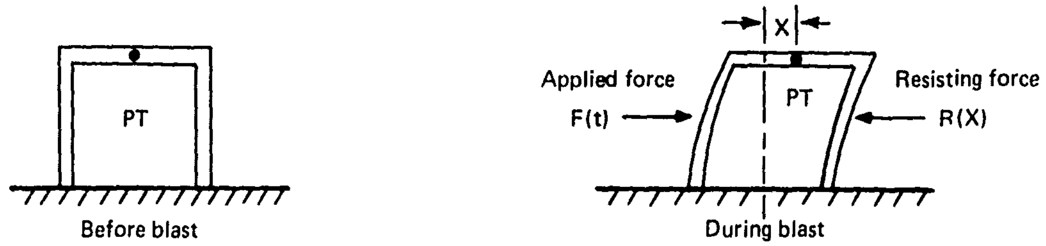

FIG. 6: STRUCTURE BEFORE AND DURING THE BLAST

The solution to equation (47) for structures represented by an elasto-plastic resistance (figure 4) was solved by reference (b); the derivation is given below.

Derivation of the Damage Pressure Level Equation - The damage pressure duration for most targets has a value somewhere between zero (impulse load) and infinity (steady-state load). The approach is to add together the determined peak damage pressure equation for each of these two duration limits. This results in a single equation for a damage pressure level with a duration somewhere between zero and infinity. The mathematical technique of adding the two equations results in a solution that closely agrees with the exact solution.

The basic concept of the derivation is the Law of Conservation of Energy, that is, the work done by the external forces from the blast wave must equal the energy absorbed by the structure in the process of deformation. This means that the work done by the external forces is equal to: (1) the work done in deflecting the target, or (2) the work done on the system resisting the deflection, or (3) the kinetic energy of the system for impulse loading.

Using these facts, the peak damage pressure level equation for impulse loading and then for the steady-state loading will be developed.

For the impulse loads, let the kinetic energy of the system, Ω, equal the work done by resisting deflection, WS. The kinetic energy of a system is:

$$\Omega = M \cdot V^2/2 \tag{48}$$

where Ω is the kinetic energy, M is the mass per unit area perpendicular to the blast wave, and V is the velocity of the system. To solve the equation, the velocity, V, is first found. To do this, the impulse, IMP, is first considered. The impulse is equal to the area under the pressure-time curve (figure 2).

$$IMP = FM' \cdot TD/2 \tag{49}$$

In equation (49), FM' is the peak force per unit area, and TD is the effective time duration. Impulse is also equal to the change in momentum of a system for impulse loading.

$$IMP = M \, \Delta V = M(VF - VO) \tag{50}$$

Also, where M is the mass per unit area, VO is the initial velocity of the system (VO=0 when at rest) and VF is the maximum and final velocity.

Thus:

$$FM' \cdot TD/2 = M(VF - VO) \tag{51}$$

or: $$\Delta V = FM' \cdot TD/2M \tag{52}$$

Substituting this in equation (48), then:

$$\Omega = (M/2) \cdot (FM' \cdot TD/2M)^2 \tag{53}$$

Referring to figure 4: The work done in resisting the external force to the maximum deflection, XM is equal to the area under the elasto-plastic resistance curve:

$$WS = RE \cdot XM - (RE \cdot XE)/2 \tag{54}$$

According to equation (47):

$$RE \cdot XM - RE \cdot XE/2 = (M/2) \cdot (FM' \cdot TD/2M)^2 \tag{55}$$

Dividing both sides by RE^2 and rearranging:

$$FM'/RE = (2/TD) \cdot \left\{(M \cdot XE/RE) \cdot [(2XM/XE) - 1]\right\}^{1/2} \tag{56}$$

Equation (56) contains the terms: M, XE, and RE. Conveniently, these same terms are included in the fundamental period of vibration, an expression that may be used to describe a structure.

Substituting the ductility ratio MU from equation (45) and the period T from equation (46) into equation (56) yields:

$$FM'/RE = (T/(\pi \cdot TD) \cdot (2MU - 1)^{1/2} \tag{57}$$

This completes the first portion of the damage pressure level equation for impulse loading.

A steady-state pressure (or force per unit area) means that the pressure does not decay with time and that the pressure is of the same intensity at zero deflection as at maximum deflection, SM . Thus, the work done by the blast wave in moving the structure a distance XM by a force FM" is equal to:

$$WB = FM'' \cdot XM \tag{59}$$

where WB is the work done by the blast wave in moving the structure and FM" is the force per unit area. As stated before:

$$WB = W$$

Thus: $$FM'' \cdot XM = RE \cdot XM - RE \cdot XE/2 \tag{60}$$

or: $$FM''/RE = 1 - XE/(2XM) \tag{61}$$

Substituting the ductility ratio MU = XM/XE in equation (61) yields:

$$FM''/RE = 1 - 1/(2MU) \tag{62}$$

This completes the second portion of peak damage pressure level equation for a loading of infinite duration.

As stated before, the sum of the two limiting expressions is a good approximate solution over the whole range of durations. Thus:

$$\begin{aligned} FM/RE &= FM'/RE + FM''/RE \\ &= (T/\pi TD) \cdot (2\ MU - 1)^{1/2} - 1/(2MU) + 1 \end{aligned} \tag{63}$$

In usage, equation (63) has been found to be in error by less than 20 percent in most cases and less than 32 percent in all cases when compared with numerical solutions of equation (47).

In order to use this expression, some datum yield YO is selected for which the peak force per unit area, FMO, and the duration, TDO, are known. Then equation (63) can be rewritten for the datum yield as follows:

$$FMO/RE = (T/\pi TDO) \cdot (2\ MU - 1)^{1/2} - 1/(2MU) + 1 \tag{64}$$

The ratio of FM (for any yield) to FMO (for the datum yield) is termed "Adjustment factor." This ratio may be either less than or greater than unity. Thus:

$$R = FM/FMO \tag{65}$$

The following expression for R can be derived by substitution of equations (63) and (64) into equation (65):

$$\begin{aligned} R &= \frac{FM/RE}{FMO/RE} \\ &= \frac{(T/\pi TD) \cdot (2\ MU - 1)^{1/2} - 1/(2\ MU) + 1}{(T/\pi TDO) \cdot (2\ MU - 1)^{1/2} - 1/(2MU) + 1} \end{aligned} \tag{66}$$

By rearranging and simplifying, the influence of weapon yield in terms of duration is given by the following expression:

$$R = 1 - \left\{ \frac{1}{1 + \dfrac{1 - 1/(2 \cdot MU)}{(T/\pi TDO) \cdot (2\ MU - 1)^{1/2}}} \right\} \cdot (1 - TDO/TD) \tag{67}$$

Since the expression within the brackets in equation (67) is a function of the target type's physical characteristics and the effective duration for the datum yield, equation (67) may be rewritten as:

$$R = 1 - KN \cdot (1 - TDO/TD) \tag{68}$$

The term, KN, is similar to the K factor in AFM 200-8, except for a factor of 10.

Thus:

$$K = 10 \; KN \tag{69}$$

Then, equation (68) can also be written as:

$$R = 1 - .1K \cdot (1 - TDO/TD) \tag{70}$$

and:

$$K = \frac{10}{1 + \dfrac{1 - 1/(2 \cdot MU)}{(T/(\pi \cdot TDO)) \cdot (2\;MU - 1)^{1/2}}}$$

All of the factors included in K are constant after the datum yield is selected. Since much of the original target damage data was obtained from the Japanese explosions and from nominal yield experiments, a datum yield of 20 kilotons was selected for AFM 200-8.

Hence, KN or K is a constant for a given damage level to a given target. This constant K then can be used to describe the way in which target damage pressure (FM) changes as a function of weapon yield with respect to the damage pressure level at 20 kilotons.

The K factor above is for a target with an elasto-plastic or an equivalent elasto-plastic resistance. Different K factor relations are required for sliding and overturning targets because they have other than elasto-plastic resistance. These relations are derived in the next two subsections.

Derivation of K for Overturning Targets - The overturning K factor equation is different from the one just described primarily because the overturning target does not follow the conventional elasto-plastic resistance pattern. This subsection will develop the overturning K factor equation which will be used in the same way as the previously developed K factor.

Here again, the approach used in reference (b) is to determine the peak damage pressure equation for an impulse load and then for a steady-state load. These two equations are added, to provide a single equation for a peak damage pressure for a duration between zero and infinity. This single equation will be used to determine an adjustment factor relationship after which the overturning target K factor may be isolated.

Conservation of energy shows that the work done by an external force from the blast wave is equal to: (1) the work done in overturning the target and (2) the kinetic energy of the system for impulse loading.

To obtain the damage pressure level equation for impulse loading, the kinetic energy equation will first be developed. Then the kinetic energy will be equated to the work done in overturning the target.

Consider the overturning target described in figure 7.

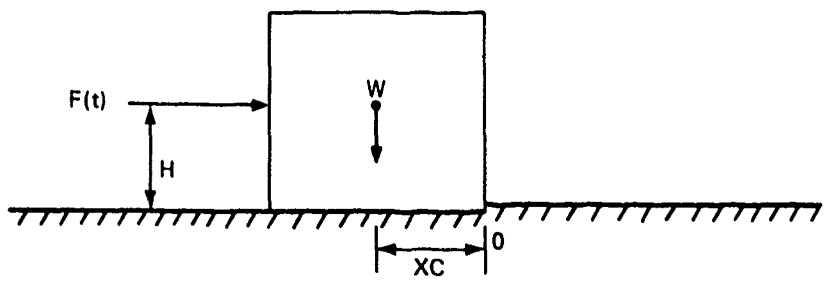

FIG. 7: FORCES ON AN OVERTURNING TARGET

The following assumptions are made: (1) the target overturns about point 0; (2) the target does not slide, and (3) the duration of the shock is short compared with the time required for the target to overturn.

From energy considerations, the following kinetic energy equation is given for an impulse load for an initially peaked triangular pulse that is very short (impulsive) in duration:

$$KE = \Omega = I\omega^2/2 \tag{71}$$

where I is the polar moment of inertia and ω is the angular velocity of the target. For convenience:

$$\Omega = (I\,\omega)^2/2I \tag{72}$$

Another relationship from physics is:

$$\tau = \alpha I \tag{73}$$

where α is the angular acceleration of the target, I the polar moment of inertia of the rotating system, and τ the torque (or the moment equal to the force times the lever arm). In the overturning problem, the force, or force/unit area, i.e. pressure, is a function of time as shown in figure 1 or 2. Thus, the torque in equation (73) may be expressed in terms of a moment:

$$I \cdot \alpha = F(t) \cdot H \tag{74}$$

Since the integration of angular acceleration results in angular velocity, then by integrating both sides, equation (74) becomes:

$$I \cdot \omega = \int F(t) \cdot H \cdot dt \tag{75}$$

The pressure or force at any time, t, between the limits of t = 0 and t = TD, is represented in figure 8 and by equation (77).

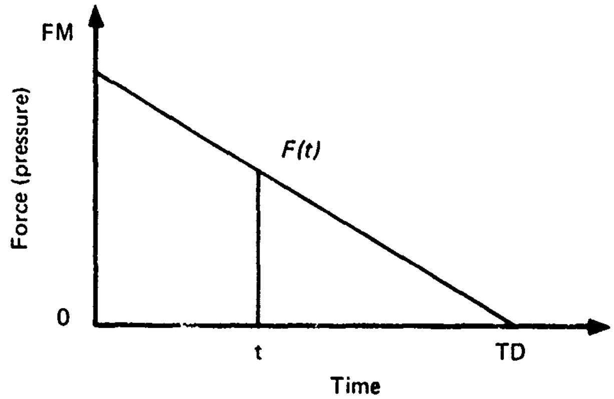

FIG. 8: RELATIONSHIP BETWEEN FORCE (PRESSURE) AND TIME

$$FM/F(t) = TD/(TD - t) \tag{76}$$

or

$$F(t) = FM \cdot (TD - t)/TD \tag{77}$$

Substituting the force (or pressure) at any time, t, between the limits of t = 0 and t = TD into equation (75) yields the result:

$$\begin{aligned} I \cdot \omega &= \int_0^{TD} FM \cdot H \cdot [(TD - t) - /TD] \cdot dt \\ &= FM \cdot H \cdot t - FM \cdot H \cdot t^2/(2 \cdot TD)\Big|_0^{TD} \end{aligned} \tag{78}$$

Consider the static moments about point 0 in figure 7. The resultant static force acts at a distance H above the surface. This force must be large enough to overcome the static resistance in order to overturn the target.

Thus:

$$RE \cdot H = W \cdot XC \tag{79}$$

where RE is considered the static force required for overturning, (or the internal resistance), H is the point of application, and W is weight of the target. Substituting equation (79) into (78), and modifying FM to FM' to indicate a very short pulse, the equations become:

$$I \cdot \omega = (FM'/RE) \cdot RE \cdot H \cdot TD/2$$
$$= (FM'/RE) \cdot (W \cdot XC/H) \cdot (H \cdot TD/2) \tag{80}$$

Substituting equation (80) into (72), the solution is:

$$\Omega = \left(1/(2 \cdot I)\right) \cdot (FM'/RE)^2 \cdot (W \cdot XC \cdot TD/2)^2 \tag{81}$$

With the kinetic energy solved, the work done in overturning must be determined. The work done on the system to overturn it is equal to lifting its weight from the position of rest to a position such that the center of gravity is directly above the center of rotation as shown in figure 9.

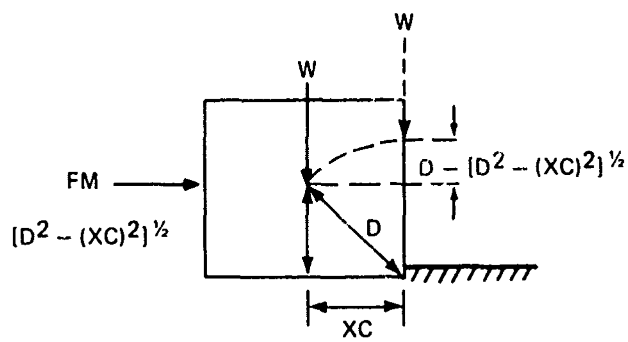

FIG. 9: POSITIONAL RELATIONSHIPS FOR AN OVERTURNING TARGET

$$\Omega' = W\left\{D - [D^2 - (XC)^2]^{1/2}\right\}$$
$$= W \cdot D \cdot \left\{1 - [1 - (XC/D)^2]^{1/2}\right\} \tag{82}$$

Expanding the square root in binomial series and dropping the higher order terms:

$$\Omega' = W \cdot D \cdot \left\{1 - 1 + (XC/D)^2/2 - \ldots\right\}$$
$$\doteq W \cdot D \cdot (XC/D)^2/2 \tag{83}$$
$$\doteq W \cdot (XC)^2/(2 \cdot D)$$

As stated previously, Ω from equation (81) and Ω' from equation (83) may be equated:

$$(1/2I) \cdot (FM'/RE)^2 \cdot (W \cdot XC \cdot TD/2)^2 = W \cdot (XC)^2/(2 \cdot D) \tag{84}$$

Therefore:

$$FM'/RE = 2 \cdot (1/(D \cdot W))^{1/2}/TD \tag{85}$$

When the duration of the force pulse is very long, the peak force must at least equal the initial resistance (RE) or the system will not overturn. Thus FM" = RE:

$$FM''/RE = 1 \tag{86}$$

where FM" indicates a force of great duration.

The damage pressure for an initially peaked triangular force pulse with a duration between that of an impulse (TD = 0) and infinity (TD = 0) can be approximated by the sum of equations (85) and (86).

$$\begin{aligned} FM/RE &= (FM'/RE) + (FM''/RE) \\ &= 1 + 2 \cdot (1/(D \cdot W))^{1/2}/TD \end{aligned} \tag{87}$$

To arrange the solution in terms of a K factor, consider the period of an overturning object. This period corresponds to that of a compound pendulum supported at the point about which it pivots when it overturns.

A compound pendulum (or physical pendulum) is any body that vibrates in a manner of a pendulum but whose mass is distributed throughout the body, and not a concentrated mass at the end of a cord of negligible weight, as is the case with the simple pendulum. Thus, the period of a compound pendulum is:

$$TC = 2\pi \cdot \left(I/(M \cdot D \cdot g)\right)^{1/2} \tag{88}$$

since:

$$M = W/g \quad . \tag{89}$$

Then:

$$TC/2\pi = (I/(W \cdot D))^{1/2} \tag{90}$$

Conveniently, the terms necessary to describe an overturning target in the period expression are also in the force calculations for overturning. Therefore by substituting equation (90) into (87), the result is:

$$FM/RE = 1 + TC/(\pi \cdot TD) \tag{91}$$

The adjustment factor (which leads to the K factor) for overturning targets may be solved by substituting equation (91) into the previously described adjustment factor expression, equation (65).

$$\begin{aligned} F &= FM/FMO \\ &= (FM/RE)/(FMO/RE) \\ &= \frac{1 + TC/(\pi \cdot TD)}{1 + TC/(\pi \cdot TDO)} \end{aligned} \tag{92}$$

By rearranging and simplifying:

$$F = 1 - \left[\frac{1}{1 + (\pi \cdot TDO/TC)} \right] \cdot (1 - TDO/TD) \tag{93}$$

Since the expression within the brackets in equation (93) is a function of the target type's physical characteristics and the effective duration for the datum yield, equation (93) can be rewritten in the form of equation (68).

$$F = 1 - KN \cdot (1 - TDO/TD) \tag{94}$$

By again making K = 10 • KN, equation (94) becomes:

$$F = 1 - .1K \cdot (1 - TDO/TD) \tag{95}$$

$$K = 10/[1 + (\pi \cdot TDO/TC)]$$

Derivative of K for Sliding Targets - Sliding targets have a different type K factor expression from the one described previously, because the sliding system has a rigid-plastic resistance (figure 10). To develop the sliding target K factor, the approach used will be the same as in for overturning targets, which is to determine the peak damage pressure equation for an impulse load and then for a steady-state load. These two equations are added together to provide a single equation for a peak damage pressure for a duration between zero and infinity. This single equation will be used to determine an adjustment factor relationship after which the sliding target K factor may be isolated.

When considering sliding as the mode of damage, the target will not begin to slide (deflection remains zero) until the applied external force becomes greater than the sliding resistance which is presumed to remain constant. An example of a sliding target would be a bridge, which may slide a maximum distance of XM on its abutment.

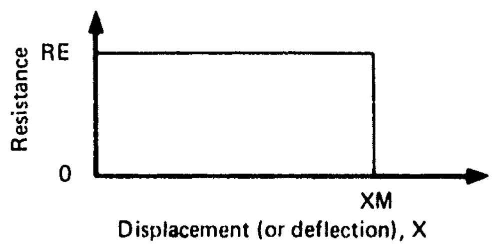

FIG. 10: RELATIONSHIP BETWEEN RESISTANCE AND DISPLACEMENT FOR A SLIDING TARGET

The energy absorbed by the system is equal the area under the resistance curve, or:

$$\Omega = RE \cdot XM \tag{96}$$

If the initially peaked triangular force pulse is of very short duration (impulsive) then the energy absorbed, Ω, in deformation must be equal to the impulsively applied kinetic energy, Ω', or:

$$\Omega = \Omega' \tag{97}$$

Since $I = M \cdot V$ for impulsive loadings:

$$\begin{aligned} \Omega' &= M \cdot V^2/2 \\ &= I^2/(2 \cdot M) \end{aligned} \tag{98}$$

Also, impulse is equal to the area under the initially peaked triangular pressure-time curve, or considering the pressure as a force per unit area, then:

$$I = FM' \cdot TD/2 \tag{99}$$

Substituting equation (99) into equations (97) and (98) yields:

$$\Omega = \Omega' = (FM' \cdot TD/2)^2/2M \tag{100}$$

or

$$RE \cdot XM \cdot 2M = (FM' \cdot TD/2)^2$$

and therefore:

$$FM'/RE = 2 \cdot (2 \cdot M \cdot XM/RE)^{1/2}/TD \tag{101}$$

where FM' indicates a very short duration loading.

When the duration of the loading is very long ($TD \rightarrow \infty$), then the peak pressure or peak force (FM) must be equal to or greater than the sliding resistance; otherwise, the structure will not move.

Thus:

$$FM''/RE = 1 \tag{102}$$

According to reference (b), the damage pressure level for an initially peaked triangular force pulse with a duration between that of an impulse and infinity can be approximated by the sum of equations (101) and (102), or:

$$\begin{aligned} FM/RE &= (FM'/RE) + (FM''/RE) \\ &= 2 \cdot (2 \cdot M \cdot XM/RE)^{1/2}/TD + 1 \end{aligned} \tag{103}$$

The incorporation of this expression in terms of the previously developed adjustment factor will provide a K factor for sliding targets. The adjustment factor expression, equation (65) and equation (103) may be written as follows:

$$\begin{aligned} F &= FM/FMO \\ &= (FM/RE)/(FMO/RE) \\ &= \frac{2 \cdot (2 \cdot M \cdot XM/RE)^{1/2}/TD + 1}{2 \cdot (2 \cdot M \cdot XM/RE)^{1/2}/TDO + 1} \end{aligned} \qquad (104)$$

As shown in figure 10, the sliding resistance is constant, or XE can be taken to be zero. Therefore from equations for the ductility ratio (equation (45)) and natural period of vibration (equation (46)) $T = 0$ and $MU = \infty$. However, to incorporate the sliding target into the K factor system, the following conveniently artificial "period" may be used:

$$TS = 2 \cdot \pi \cdot [2 \cdot (W/g) \cdot (XM/RE)]^{1/2} \qquad (105)$$

or

$$TS/(2\pi) = [2 \cdot (W/g) \cdot (XM/RE)]^{1/2} \qquad (106)$$

Substituting equation (106) into equation (104):

$$F = \frac{2 \cdot (TS/2\pi)/TD + 1}{2 \cdot (TS/2\pi)/TDO + 1} \qquad (107)$$

Rearranging and simplifying yields:

$$F = 1 - \left[\frac{1}{1 + \pi \cdot TDO/TS}\right] \cdot (1 - TDO/TD) \qquad (108)$$

As in the previous cases, equation (108) can be separated into two parts, one of which is only a function of the target type's physical characteristics and the effective duration for the datum yield and can be rewritten as:

$$F = 1 - KN \cdot (1 - TDO/TD) \qquad (109)$$

For $K = 10KN$ equation (109) becomes:

$$F = 1 - .1K \cdot (1 - TDO/TD) \qquad (110)$$

$$K = 10/[1 + (\pi \cdot TDO/TS)]$$

Considering equation (105) once again, W is the weight of the target, XM is the displacement of the target and RE is the sliding resistance offered by the target. Assuming the only sliding resistance is that offered by friction, then $RE = \eta W$ where η is the coefficient of friction between the sliding surfaces. Using this assumption, the TS term in equation (110) becomes:

$$TS = 2 \cdot \pi \cdot [(2 \cdot W/g) \cdot (XM/(\eta \cdot W))]^{1/2}$$
$$= 2 \cdot \pi \cdot (2 \cdot XM/(g \cdot \eta))^{1/2} \quad (111)$$

Effective Time Durations and VN Number Adjustments - As is well known, a target's susceptibility to nuclear weapon blast effects can be described in terms of a vulnerability number, consisting of three parts: a number, a target type (i.e., P or Q) and a K factor. The latter terms have been described previously but a cursory review of the vulnerability number (VN) scheme is in order.

The numerical value scale of the VN is an arbitrary classification scheme describing a target's vulnerability. (The basic VN is referenced to 20 kilotons while the "adjusted VN" is adjusted to some other particular yield.) Actually, the numerical values are linear functions of the logarithm of either the peak overpressure or the peak dynamic pressure that is required to achieve a given probability of a defined level of damage to a randomly orientated target. In the special case of a .5 probability of damage, the numerical value of the "P" type VN is defined as:

$$\text{"P"VN} = \frac{\log(P_{0.5}) - \log(1.1216)}{\log(1.2)} \quad (112)$$

where $P_{0.5}$ is the peak overpressure associated with a .5 probability of obtaining a desired degree of damage such as: "severe," "light," "depot repair," "30-day recovery time," etc. In a similar way, the "Q" type VN is defined as:

$$\text{"Q"VN} = \frac{\log(Q_{0.5}) - \log(0.02893)}{\log(1.44)} \quad (113)$$

where $Q_{0.5}$ is the peak dynamic pressure associated with a .5 probability of damage. These two relations are illustrated in figure 11.

As a point of interest, it may be noted that for a given numerical value of either type VN (e.g., VN-12P and VN-12Q) the corresponding overpressure ($P_{0.5}$ = 10 psi) and dynamic pressure ($Q_{0.5}$ = 2.3 psi) are related by an approximation given by equation (114)

$$Q = 5 \cdot P^2/(2 \cdot (7\ PAT + P))$$
$$= .023 \cdot P^2 \quad (114)$$

In equation (114) PAT is the atmospheric pressure ahead of the blast wave. This means that under nearly ideal surface burst conditions and with no K factor considerations, in the example cited, VN-12P and VN-12Q targets will be damaged with a probability of .5 about the same distance from ground zero (no weapon delivery errors considered).

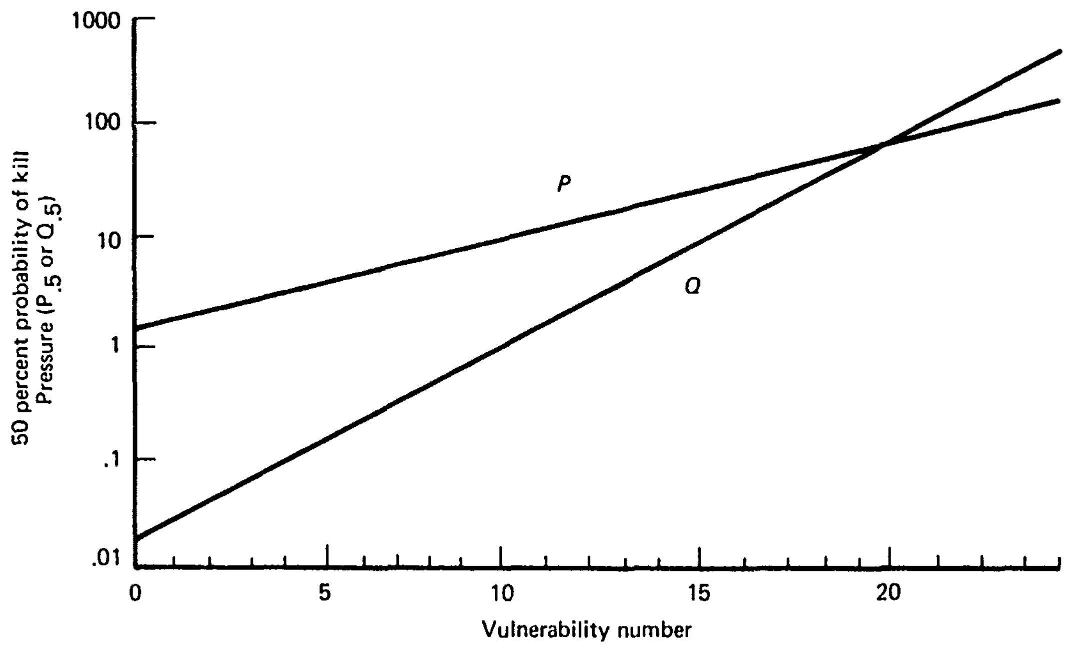

FIG. 11: RELATIONSHIP BETWEEN 50 PERCENT PROBABILITY OF PRESSURE DAMAGE AND VULNERABILITY NUMBER

Occasionally, the .5 probability of damage-distance is used erroneously to indicate a "go-no go" or "cookie-cutter" probability of damage-distance situation. Actually, a target's destruction is a "go or no-go" proposition only when considering a given damage criterion, such as 75 percent building floor space unusable or a vehicle is overturned. This means that a target is either damaged or not damaged to the desired extent. However, achievement of this criterion must be considered on a probabilistic basis because of variabilities in: weapon output and blast phenomena, intrinsic target structural characteristics, mutual target shielding, terrain, target orientation, etc. When a specific target is considered, these factors lead to a variation in probability of damage versus distance from ground zero (GZ), as shown in figure 12.

It may be noted that the distance, σ, between a probability of .69 and .31 is a measure of this gradual fall-off in probability of damage with distance. Although a .5 probability of damage distance can be used as a "probable radius," the VN system employs another distance, which is the weapon radius or WR .

$$WR = \left[\int_0^{\infty} 2\, PK(R) R dR\right]^{1/2} \tag{115}$$

where PK(R) is probability of damage as a function of the distance R . The WR may be thought of as defining the radius of a circle centered on the GZ . Assuming an infinite array of identical targets, the WR circle will enclose just as many undamaged targets as will be damaged outside the circle. Special "cookie cutter" cases of $\sigma = 0$ result in WR equal to the "probable radius." For real conditions, such as existed at Hiroshima and Nagasaki, σ/WR may very well be 0.20 or 0.30 ("Sigma-20" or "Sigma-30"). For Sigma-20, the solution of equation (115) leads to a WR equal to a distance R from the GZ corresponding to a probability of damage, PK(R), of .473 . Thus, for many real-life situations, WR is slightly greater than the .5 probability radius.

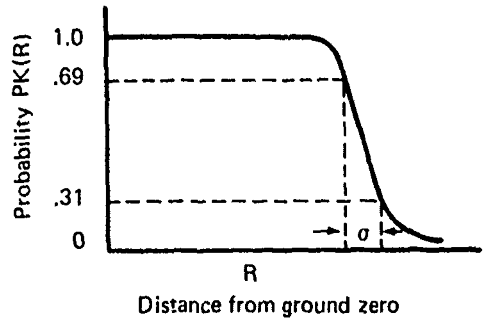

FIG. 12: EFFECT OF DISTANCE FROM GROUND ZERO ON PROBABILITY OF KILL

Obviously, the described probability of damage-distance relationship may be expressed in a probability of damage-pressure relationship shown in figure 13 for either P or Q, by converting distance into pressure by empirical overpressure or dynamic pressure-distance relations.

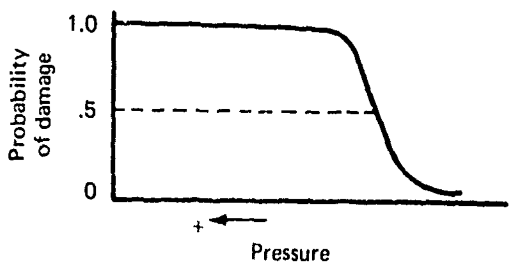

FIG. 13: EFFECT OF PRESSURE ON PROBABILITY OF DAMAGE

A statistical analysis of the probability of damage-pressure data obtained from Hiroshima and Nagasaki was performed for the VN system. These results were combined with the previously described VN-pressure definitions which resulted in probability of damage-VN-pressure relationships.

As a matter of interest, it may be noted that in reference (c), these two probability-pressure-VN curves were "tied together," that is, the dynamic pressure and overpressure were related at all points by the approximate form of the Rankine-Hugoniot equation ($Q = .023P^2$). Originally, only the overpressure relation was based upon empirical evidence and the dynamic pressure relation merely calculated from it. A recent restudy of Hiroshima-Nagasaki data combined with DASA field test data resulted in confirmation of the probability-overpressure curves but provided new probability-dynamic pressure curves. These curves are interrelated through the approximate Rankine-Hugoniot relation at the $P_{.5}$ and $Q_{.5}$ points only.

Therefore, a given VN represents a line on a probability-pressure graph or a curve on a probability-distance graph, rather than a specific point, as may seem to have been implied in equations (112) and (113). Thus, while reference is often made to the $P_{.5}$ or $Q_{.5}$ corresponding to a VN, pressures associated with other probabilities of damage may be read from probability of damage-VN-pressure curves.

Of more ultimate value is the use of these curves to calculate WRs for a wide range of weapon height of burst-VN combinations. This is done by merely retracing the steps described above. Specifically, the relationships are determined at one height of burst (HOB) and are assumed to remain the same regardless of HOB. Then, P and Q versus distance data for one kilotron, as determined by field measurements for various HOBs are used to get probability-versus-distance curves analogous to those in figure 12. The integral of the WR equation (114) is then found. In practice, this has been accomplished by electronic computers and the results are presented in graphs which are shown in the forms in figure 14.

These one-kiloton curves can be used for any nuclear weapons yield by the proper scaling. WR scales with the cube root of the weapon yield, so that:

$$WR = (WR_{1KT})(Y^{1/3}) \qquad (116)$$

K factor equations using the approximate effective time duration for both dynamic pressure and overpressure of $TD = 0.1Y^{1/3}$ were formerly used. The use of this simplified relation for TD was entirely satisfactory for targets vulnerable to pressures of the order of 10 psi overpressure and for the pressure data available at that time. The availability of more recent pressure data plus the increased interest in hard targets as well as fractional KT nuclear weapons necessitated the improvement of the approximate effective duration equation. In this subsection the necessary modification of the effective duration equations for both overpressure and dynamic pressure will be explained.

Since the effective duration of the overpressure and the dynamic pressure differ, separate adjustment factor relationships will be introduced for each. Using the modified effective durations, revised K factor equations will be provided.

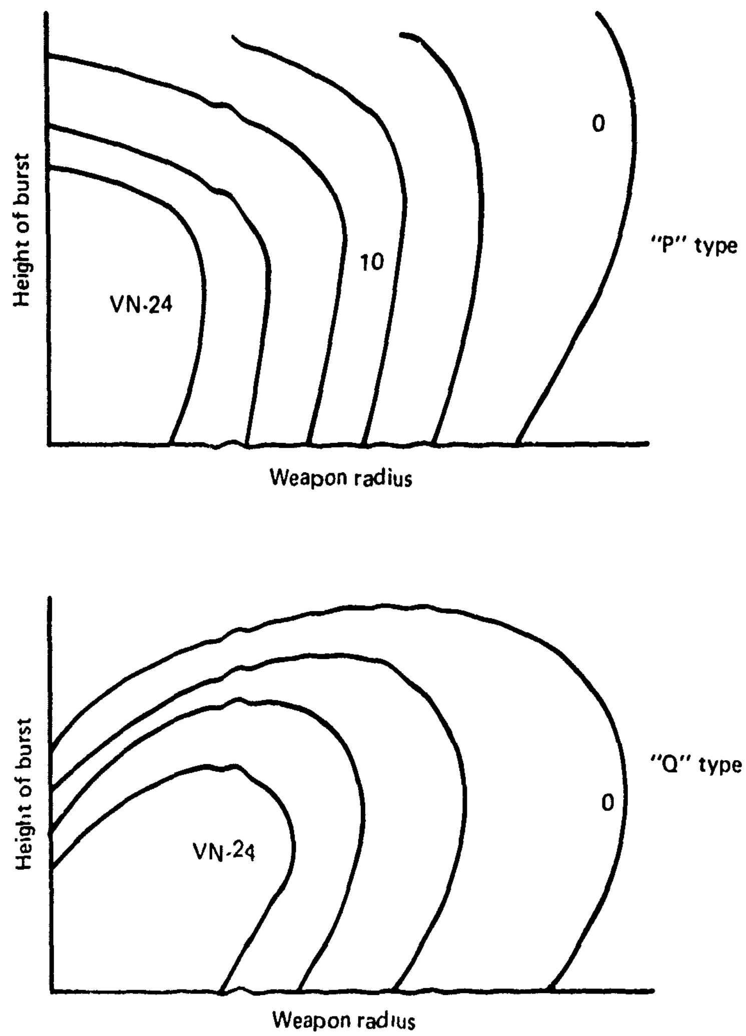

FIG. 14: EFFECT OF HEIGHT OF BURST ON WEAPON RADIUS FOR A ONE-KILOTON YIELD

Overpressure Effective Duration and Adjustment Factor - The meaning of effective duration, TD, was explained in the subsection on determining the 50 percent kill radius and its relationship to the positive phase duration shown in figure 1.

Empirical data (reference (d)) indicate the following equation, for any height of burst to be a better approximation for a burst of yield of Y kilotons.

$$TD = .45 \cdot P^{-1/2} Y^{1/3} \tag{117}$$

As a result of equation (117), the ratio of effective duration for yield Y to the effective duration TDO for a reference yield YO is given by:

$$TD/TDO = (PO/P)^{1/2} \cdot (Y/YO)^{1/3} \tag{118}$$

The adjustment factor for damage by elasto-plastic deformation, overturning and sliding was shown to be:

$$F = 1 - .1K(1 - TDO/TD) \tag{119}$$

By substituting equation (118) into equation (119):

$$F = 1 - .1K\,[1 - (P/PO)^{1/2} \cdot (YO/Y)^{1/3}] \tag{120}$$

Now, by the definition of the adjustment factor R:

$$\begin{aligned} R &= FM/FMO \\ &= S \cdot P/(S \cdot PO) \end{aligned} \tag{121}$$

Therefore, equation (120) can also be written:

$$F = 1 - .1K\,[1 - F^{1/2} \cdot (YO/Y)^{1/3}] \tag{122}$$

By letting $X = F^{1/2}$, equation (122) may be written as:

$$X^2 - .1K \cdot (YO/Y)^{1/3} \cdot X + .1K - 1 = 0 \tag{123}$$

Because the values of K are $0 \le K \le 9$, equation (123) can be shown to have two roots, one positive and one negative. Of these two, only the positive root fits the physical conditions. Therefore:

$$X = \frac{.1K \cdot (YO/Y)^{1/3} + \{[.1K \cdot (YO/Y)^{1/3}]^2 + 4\,(1 - .1K)\}^{1/2}}{2} \tag{124}$$

and as a result:

$$F = 1/4\left\{.1K \cdot (YO/Y)^{1/3} + \{[.1K \cdot (YO/Y)^{1/3}]^2 + 4(1 - .1K)\}^{1/2}\right\}^2 \tag{125}$$

The Dynamic Pressure Effective Duration and Adjustment Factor - Empirical data for determination of the effective dynamic pressure duration are not available. However, theoretical free air blast data (reference (e)) as modified by other theoretical relationships (reference (d)) have provided the following expressions for the dynamic pressure effective duration.

$$TD = .105Q^{-1/3}W^{1/3} \tag{126}$$

As a result of equation (126), the ratio of effective duration for yield Y to the effective duration TDO for a reference yield YO is given by:

$$TD/TDO = (QO/Q)^{1/3}(Y/YO)^{1/3} \tag{127}$$

The equation for the dynamic pressure adjustment factor becomes:

$$\begin{aligned} F &= 1 - .1K(1 - TDO/TD) \\ &= 1 - .1K[1 - (Q/QO)^{1/3} \cdot (YO/Y)^{1/3}] \end{aligned} \tag{128}$$

Now, because of the definition of F:

$$\begin{aligned} F &= FM/FMO \\ &= S \cdot Q/(S \cdot QO) \end{aligned} \tag{129}$$

Therefore, by using the ratio of Q/QO from equation (128) in equation (129):

$$F = 1 - .1K[1 - F^{1/3} \cdot (Y/YO)^{1/3}] \tag{130}$$

By letting $Z = F^{1/3}$, equation (130) may be written as:

$$Z^3 - .1K \cdot (YO/Y)^{1/3} \cdot Z + .1K - 1 = 0 \tag{131}$$

First note that equation (131) has only a single change of sign since:

$$\begin{aligned} &- .1K(YO/Y)^{1/3} < 0 \\ &.1K - 1 < 0 \end{aligned} \tag{132}$$

Then, applying Descartes' rule of signs, there is only a single positive root of equation (131). Since, by definition $F > 0$ then $Z > 0$ and hence the positive root is the only one desired from equation (131).

VN Number Adjustment and Application - Solutions for the overpressure adjustment factor from equation (125) or for the dynamic adjustment factor from equation (131) may now be used to find $(P_{.5})_Y$ or $(Q_{.5})_Y$ respectively for a weapon of yield Y. First, an adjusted VN number is defined for the P and Q cases respectively as:

$$ADJ"P"VN = \frac{\log (P_{.5})_Y - \log (1.1216)}{\log (1.2)} \qquad (133)$$

$$ADJ"Q"VN = \frac{\log (Q_{.5})_Y - \log (0.02893)}{\log (1.44)}$$

Rearranging equations (133) yields:

$$(P_{.5})_Y = (1.1216) \cdot (1.2)^{ADJ"P"VN}$$

$$(Q_{.5})_Y = (0.02893) \cdot (1.44)^{ADJ"Q"VN} \qquad (134)$$

Now, by the definition of F :

$$F = \frac{(P_{.5})_Y}{(P_{.5})_{20KT}} = \frac{(1.1216) \cdot (1.2)^{ADJ"P"VN}}{(1.1216) \cdot (1.2)^{"P"VN}} \qquad (135)$$

$$F = \frac{(Q_{.5})_Y}{(Q_{.5})_{20KT}} = \frac{(0.02893) \cdot (1.44)^{ADJ"Q"VN}}{(0.02893) \cdot (1.44)^{"Q"VN}}$$

Solving for the adjusted VN number:

$$ADJ"P"VN = "P"VN + \log (F)/\log (1.2)$$

$$ADJ"Q"VN = "Q"VN + \log (F)/\log (1.44) \qquad (136)$$

Now, tables relating weapon radii, WR , to VN number for various heights of burst based on a one-kiloton yield are readily available.* The radii obtained from these tables, however, must be scaled using the cube root scaling laws for yield. Thus, the weapon radius for yield related to a one-kiloton weapon radius is:

$$WR_Y/WR_1 = Y^{1/3} \qquad (137)$$

Combining equations (136) and (137) as advocated by equation (116):

$$WR_Y(\text{"P" CASE}) = Y^{1/3} \cdot WR_1 \text{ (BASED ON "P" CASE ADJ VN)}$$

$$WR_Y(\text{"Q" CASE}) = Y^{1/3} \cdot WR_1 \text{ (BASED ON "Q" CASE ADJ VN)} \qquad (138)$$

The Application of R and K to Find Weapon Radius - In the previous paragraphs, individual components of the VN system are discussed. Now these components will be

*See reference (a).

integrated into a useful system. A target's vulnerability may be described by a VN number consisting of 3 parts: a number, a target type symbol and a K factor. The number expresses the peak damage pressure (FMO) for a given level of damage for a 20-kiloton weapon, and for a given probability, say .5; the target type symbol indicates the most dominant damage pressure type; the K factor in effect adjusts the number (or FM) for weapon yields other than 20-kilotons. Also, the adjustment factor as described previously, is the ratio of FM to FMO which can be converted to an Adjustment Value. The Adjustment Value, as explained previously, is applied to the basic datum yield VN. Thus, the purpose of the whole K factor scheme is to derive this Adjusted VN, which is the tool used to determine the usable and meaningful value of Weapon Radius (WR) for a given yield.

The Adjusted VN may be calculated by use of the formulas just given. Having the Adjusted VN, one obtains the conveniently selected WR_{1KT} at the proper HOB. The WR_{1KT} is then scaled to the proper WR by means of the cube root scaling law with the desired yield.

$$WR = WR_{1KT} \cdot Y^{1/3} \tag{139}$$

<u>Aiming Errors, Target Distributions, Kill Functions and the Determination of a Conditional Single Shot Kill Probability</u>

The preceding discussion has shown how the 50 percent kill radius is determined from semi-theoretical relationships, on the basis of target hardness (expressed as a VN number or in psi) and weapon yield. In this subsection, aiming errors, the distribution of the target, and a set of kill functions will be explored, and functions for a conditional single-shot kill probability will be derived. It is a conditional probability, because weapon delivery to the target is assumed. Removal of this condition to find an overall single-shot kill probability will be covered later.

<u>Aiming Error, Target Distribution, and Impact Point-Target Element Distances</u> - First, consider the situation in figure 15 in which the targets are continuously distributed and this distribution is circular normal with center (0,0). Now, assume that the center of the target distribution is also the aim point for weapon delivery and that the weapon delivery error is also circular normal and centered about (0,0). For a differential element of the target area located at (X'_1, X'_2), the probability of its being killed is directly related to its distance from the differential element of area at the impact point (X''_1, X''_2). This distance has components (XA_1, XA_2) related to (X_1, X_2) and (X'_1, X'_2) as follows:

$$\begin{aligned} XA_1 &= X''_1 - X'_1 \\ XA_2 &= X''_2 - X'_2 \end{aligned} \tag{140}$$

Now, the situation portrayed in figure 15 and equation (140) must be generalized if one is to find the probability density distributions of XA_1 and XA_2 from the probability density distributions of X'_1, X'_2, X''_1 and X''_2 . By the definition of a circular normal distribution centered at (0, 0):

$$
\begin{aligned}
P(X'_1) &= 1/(\sigma'\sqrt{2\pi})\exp[-(X'_1/\sigma')^2/2] \\
P(X'_2) &= 1/(\sigma'\sqrt{2\pi})\exp[-(X'_1/\sigma')^2/2] \\
P(X''_1) &= 1/(\sigma''\sqrt{2\pi})\exp[-(X''_1/\sigma'')^2/2] \\
P(X''_2) &= 1/(\sigma''\sqrt{2\pi})\exp[-(X''_2/\sigma'')^2/2]
\end{aligned}
\qquad (141)
$$

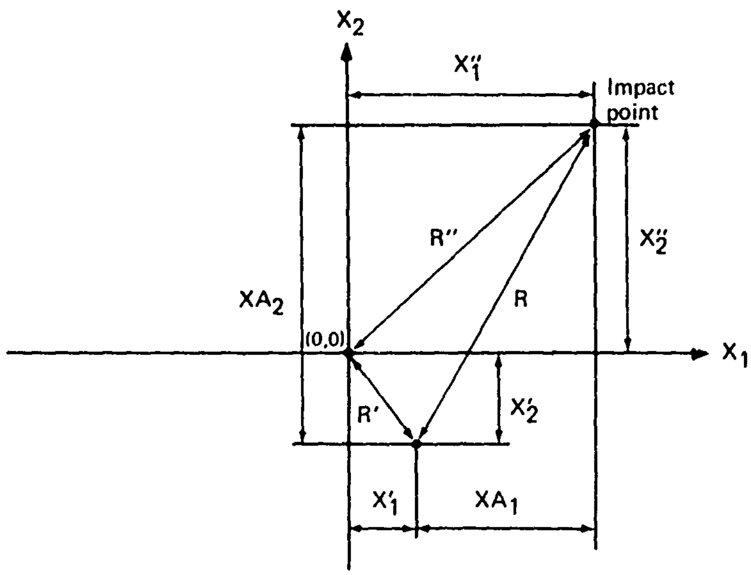

FIG. 15: AREA TARGET SPATIAL RELATIONSHIPS

In equation (141), σ' and σ'' are the standard deviations of the distributions. Generally, the delivery error is specified in terms of circular error probable (CEP). This is just the radius of the circle centered at the mean (in this case (0, 0)) which contains 50 percent of the impact points. For finding σ'' from CEP it is first necessary to find the distance to the impact point, R'' .

$$R'' = [(X''_1)^2 + (X''_2)^2]^{1/2} \qquad (142)$$

It can be shown that the probability density distribution for R'' developed from the density distribution of X''_1 and X''_2 and equation (3) is:

$$P(R'') = (R''/\sigma''^2) \cdot \exp[-(R''/\sigma'')^2/2] \tag{143}$$

Then, by the definition of CEP:

$$\begin{aligned} .5 &= \int_0^{CEP} P(R'')dR \\ &= 1 - \exp[-(R''/\sigma'')^2/2]\Big|_0^{CEP} \end{aligned} \tag{144}$$

Therefore, the standard deviation of delivery error is:

$$\sigma'' = CEP/1.1774 \tag{145}$$

Now, a similar development for the standard deviation for the target distribution will be carried out.

For distributed targets, the appropriate parameter for determining σ' is R_{95}, the radius of the circle centered at the mean (0,0) which contains 95 percent of the target area. Paralleling the previous development:

$$R' = [(X'_1)^2 + (X'_2)^2]^{1/2} \tag{146}$$

Then:

$$P(R') = (R'/\sigma')^2 \exp[-(R'/\sigma')^2/2] \tag{147}$$

Then, by the definition of R_{95}:

$$\begin{aligned} .95 &= \int_0^{R_{95}} P(R')dR \\ &= 1 - \exp[-(R'/\sigma')^2/2]\Big|_0^{R_{95}} \end{aligned} \tag{148}$$

Therefore, the standard deviation of the target distribution is:

$$\sigma' = R_{95}/2.4477 \tag{149}$$

Now that the standard deviations for the equation (141) distribution have been developed, it is possible to proceed to find distributions for XA_1 and XA_2. It can be shown that:

$$P(XA_1) = 1/\left\{2\pi[(\sigma')^2 + (\sigma'')^2]\right\}^{1/2} \exp\left\{-(XA_1)^2/\left\{2[(\sigma')^2 + (\sigma'')^2]\right\}\right\}$$
$$P(XA_2) = 1/\left\{2\pi[(\sigma')^2 + (\sigma'')^2]\right\}^{1/2} \exp\left\{-(XA_2)^2/\left\{2[(\sigma')^2 + (\sigma'')^2]\right\}\right\} \quad (150)$$

Now, since the probability of killing an element of the target is directly related to the distance between the impact point and the target element's location, it would be useful to convert equation (150) to a single equation for the distribution of impact point - target element range R. First:

$$R = [(XA_1)^2 + (XA_2)^2]^{1/2} \quad (151)$$

As in the previous cases of delivery error and target distribution, the probability density distribution for R may be written as :

$$P(R) = \left\{R/[(\sigma')^2 + (\sigma'')^2]\right\}\exp\left\{-R^2/2[(\sigma')^2 + (\sigma'')^2]\right\} \quad (152)$$

Substituting for σ' and σ'' in equation (152), then provides the desired distribution of impact point - target element distances:

$$P(R) = \left\{R/[(CEP/1.1774)^2 + (R_{95}/2.4477)^2]\right\}$$
$$\cdot \exp\left\{-R^2/\left\{2[(CEP/1.1774)^2 + (R_{95}/2.4477)^2]\right\}\right\} \quad (153)$$

Generalized Kill Functions and Weapon Radius

A set of generalized probability of kill functions have been postulated in reference (f) for a target at a distance R from ground zero. These are of the form:

$$PK(R) = \exp(-K \cdot R^2/A^2) \sum_{J=0}^{K-1} (K \cdot R^2/A^2)^J/J! \quad (154)$$

In particular, values of K = 3 and K = 6 in equation (154) have been found to approximate the dynamic pressure and overpressure kill curves of reference (g).

Since the only parameter in equation (154) not defined is A, this will now be derived.

Weapon radius is defined in reference (g) as:

$$WR = \left\{\int_0^\infty 2PK(R)RdR\right\}^{1/2} \quad (155)$$

In equation (155) PK is the probability of damage as a function of distance from ground zero, R . Now, by squaring equation (155) and substituting equation (154) for PK(R):

$$(WR)^2 = \int_0^{\infty} \left\{ 2R \exp(-K \cdot R^2/A^2) \cdot \sum_{J=0}^{K-1} (K \cdot R^2/A^2)^J/J! \right\} \cdot dR \tag{156}$$

Equation (156) can also be written as:

$$(WR)^2 = \sum_{J=0}^{K-1} \int_0^{\infty} 2[(K \cdot R^2/A^2)^J/J!] \exp(-K \cdot R^2/A^2) \cdot R \cdot dR \tag{157}$$

Consider the J^{th} term of equation (157). By making the substitution $K \cdot R^2/A^2 = X$, the J^{th} term may be written as:

$$J^{th} \text{ TERM} = \int_0^{\infty} (A^2/KJ!) \cdot X^J \cdot \exp(-X)dX \tag{158}$$

Carrying out the integration yields:

$$J^{th} \text{ TERM} = A^2/K \tag{159}$$

Integrating and summing all K terms of equation (157) provides the following result:

$$\begin{aligned} (WR)^2 &= \sum_{J=0}^{K-1} A^2/K \\ &= A^2 \cdot K/K \\ &= A^2 \end{aligned} \tag{160}$$

Therefore:

$$A = WR \tag{161}$$

The probability of killing a target at a distance R from ground zero is, then:

$$PK(R) = \exp[-KR^2/(WR)^2] \sum_{J=0}^{K-1} [K \cdot R^2/(WR)^2]^J/J! \tag{162}$$

K = 3 dynamic pressure kill

K = 6 overpressure kill

The Probability of Kill for a Successful Weapon Delivery

The preceding two subsections have dealt with two aspects of the kill problem: The probability that a differential element of target is a distance R from ground zero and the probability of an element of target at a distance R from ground zero being killed. It remains in this subsection to tie these two together and obtain the probability of kill for a single weapon successfully delivered. For this purpose conditional probability and marginal probability relationships will be used.

Consider two random variables Φ and ψ defined in the same sample space with fixed values $\Phi = \Phi_1, \Phi_2 \ldots \Phi_M$ and $\psi = \psi_1, \psi_2 \ldots \psi_N$. The marginal probability of $\Phi = \Phi_J$ is defined as the probability that $\Phi = \Phi_J$ and $\psi = \psi_K$ summed over all K. Symbolically:

$$P(\Phi = \Phi_J) = \sum_{\text{All } K} P(\Phi = \Phi_J, \ \psi = \psi_K) \tag{163}$$

The conditional probability of $\Phi = \Phi_J$ is defined as the probability that $\Phi = \Phi_J$ given that $\psi = \psi_K$. Symbolically:

$$P(\Phi = \Phi_J \mid \psi = \psi_K) = P(\Phi = \Phi_J, \ \psi = \psi_K)/P(\psi = \psi_K) \tag{164}$$

Combining both equations (163) and (164):

$$P(\Phi = \Phi_J) = \sum_{\text{All } K} P(\psi = \psi_K) \cdot P(\Phi = \Phi_J \mid \psi = \psi_K) \tag{165}$$

In the particular case under consideration, the probability of the target's being killed is equal to the summation of conditional probabilities of kill given a ground zero distance of R times the probability of ground zero being at a distance R. Applying this to equations (153) and (162) and replacing the summation with an integration:

$$\begin{aligned} \text{PKILL}_{IW} = \int_0^\infty & \left\{ R/[(\text{CEP}/1.1774)^2 + (R_{95}/2.4477)^2] \right\} \cdot \\ & \exp\left\{ -R^2/\left\{2 \cdot [(\text{CEP}/1.1774)^2 + (R_{95}/2.4477)^2]\right\}\right\} \\ & \left\{ \exp[-KR^2/(WR)^2] \cdot \sum_{J=0}^{K-1} [K \cdot R^2/(WR)^2]^J/J! \right\} dR \end{aligned} \tag{166}$$

Now, for simplicity,

let

$$\begin{aligned} a &= K/(WR)^2 \\ b &= 1/\left\{2[(\text{CEP}/1.1774)^2 + (R_{95}/2.4477)^2]\right\} \end{aligned} \tag{167}$$

Equation (166) then becomes:

$$PKILL_{IW} = \int_0^\infty \left\{ (2bR \cdot \exp[-R^2(a+b)] \cdot \sum_{J=0}^{K-1} [(aR^2)^J/J!] \right\} dR \tag{168}$$

The J^{th} term of equation (168) is:

$$J^{th}\ TERM = \int_0^\infty (a^J b/J!) R^{2J} \exp[-R^2(a+b)]\, 2RdR \quad (169$$

Integrating (169) yields:

$$J^{th}\ TERM = a^J b/(a+b)^{J+1} \tag{170}$$

The complete series therefore, is:

$$PKILL_{IW} = [b/(a+b)] \sum_{J=0}^{K-1} [a/(a+b)]^J \tag{171}$$

By taking the sum of the geometric series in equation (171):

$$PKILL_{IW} = \frac{[b/(a+b)] \cdot \left\{ 1 - [a/(a+b)]^K \right\}}{1 - a/(a+b)}$$
$$= 1 - [a/(a+b)]^K \tag{172}$$

By multiplying the numerator and denominator of the term within the brackets by K/ab:

$$PKILL_{IW} = 1 - \left[\frac{K/2b}{K/2b + K/2a} \right]^K \tag{173}$$

Now, substituting equation (167) into equation (173) provides:

$$PKILL_{IW} = 1 - \left[\frac{K \cdot [(CEP/1.1774)^2 + (R_{95}/2.4477)^2]}{K \cdot [(CEP/1.1774)^2 + (R_{95}/2.4477)^2] + (WR)^2/2} \right]^K \tag{174}$$

Of course, when a point target is being considered, $R_{95} = 0$ in equation (174).

Strategy Considerations and Weapon Delivery Probabilities

Thus far, the probability of kill development has taken into account only those factors of significance for a weapon type actually delivered to the target. Now, the probability of being able to deliver a weapon to the target must be considered. The overall single-shot kill probability of weapon type IW against target type IT depends on inflight reliability (or the probability that the weapon will arrive at the area defenses), the probability of penetrating the area defenses, and the probability that there will be a target available to hit.* Therefore, the single-shot kill probability may be written as:

$$SSKP_{IW,IT} = \text{(INFLIGHT RELIABILITY)} \cdot \text{(PENETRATION PROBABILITY)} \cdot \text{(PROBABILITY OF SURVIVING ATTACK BY RANDOM TARGET TERMINAL DEFENSE)} \cdot \text{(PROBABILITY THERE IS AN AVAILABLE TARGET)} \cdot \text{(PROBABILITY OF KILL WHEN WEAPON IS DELIVERED)} \text{(ALLOWABLE STRATEGY FACTOR)}$$

$$= RN_{IW} \cdot PENPROB_{IW} \cdot PSURV_{IW,IT} \cdot PTGT_{IW,IT} \cdot PKILL_{IW,IT} \cdot STRAT_{IW,IT} \quad (175)$$

The only elements of equation (175) not previously considered are the probability that there is an available target, the probability of surviving attack by target terminal defense when the terminal defense is random, and the allowable strategy factor.

The probability of surviving attack by the target terminal defense is specified as:

$$PSURV_{IW,IT} = (1 - PK_{IT,IW}) \quad (176)$$

The probability that there is a target left at the site is the fraction of weapons withheld from a prior strike plus the fraction of weapons sent but not successfully launched.

The probability that the target type has not been destroyed previously is $SURVX_{IT}$. Therefore, the probability that there is a surviving target at the site is:

$$\begin{aligned} PTGT_{IT,IW} &= [(1 - FSENT_{IT}) + FSENT_{IT} \cdot (1 - RR_{IT})]\, SURVX_{IT} \\ &= (1 - FSENT_{IT} \cdot RR_{IT}) \cdot SURVX_{IT} \end{aligned} \quad (177)$$

Now, the strategy factor, $STRAT_{IW,IT}$, enters. If the weapon type is allowed to attack the target type $STRAT_{IW,IT} = 1$. Therefore, the single-shot kill probability for a single warhead is:

$$\begin{aligned} SSKP1_{IW,IT} &= RN_{IW} \cdot PENPROB_{IW} \cdot (1 - PK_{IT,IW}) \\ &\cdot (1 - FSENT_{IT} \cdot RR_{IT}) \cdot STRAT_{IW,IT} \cdot PKILL_{IW,IT} \end{aligned} \quad (178)$$

*For example, a bomber arriving at an ICBM site may find that all ICBMs have already been fired.

Finally, the number of warheads per independently targetable weapon is used to find the single-shot kill probability per independently targetable weapon. Since the number of warheads ($WHDS_{IW}$) per independently targetable weapon ($TPRWPN_{IW}$) may be considered as an equivalent number of Bernoulli trials, then:

$$SSKP_{IW,IT} = 1 - (1 - SSKP1_{IW,IT})^{WHDS_{IW}/TPRWPN_{IW}} \tag{179}$$

This completes the derivation of the single-shot kill probability relationship.

DAMAGE CALCULATION MODELS

The damage calculation models are important, since in many cases more than a single weapon will be used against a target and the single-shot kill probability model previously developed will, therefore, not be enough for total damage calculations.

There are two different models for calculating damage. The first is for calculating damage to point targets, the second for calculating damage to area targets with a circular normal distribution of target value, e.g., an approximation of fatalities in an attacked city.

Symbols used in the equations are listed below:

A	Target area
K	Expected lethal area of one weapon
N	Number of weapons, or the number of Bernoulli trials
$NP_{IW,IT}$	Price in independent targetable weapons for each weapon type
P	Probability of the weapon's being delivered to the target area
$PKILL_I(N)$	Overall probability that the Ith increment of target within area A is killed for N weapons
R	Distance from the target center
RK	Lethal radius of one weapon
$SSKP_{IW,IT}$	Single-shot kill probability of weapon type IW against target type IT
$SURV_{IT}$	Overall survival of the target type
$SURV_I(N)$	Overall probability of survival for Ith increment of target within area A for attack by N weapons
$TPKILL_{IW,IT}$	Total probability of a target's being killed
TVKILL	Total value of target killed
V_{IT}	Value given to target
$VKILL_{IW,IT}$	Expected value killed per target

$VKILLED_I$	Expected value of target killed by N weapons
$W_{IW,IT}$	Number of weapons of type IW used against target type IT
$W^*_{IW,IT}$	Effective number of weapons
ω	Weapon density

Point-Target-Damage Model - No Terminal Defense - Single Weapon Type

In attempting to destroy a point target, a single successful weapon is assumed to be all that is required to destroy the target completely. The probability of success with a single weapon of type IW against target type IT is $SSKP_{IW,IT}$. When $W_{IW,IT}$ weapons of type IW are used against a single target of type IT, the total probability of kill is:

$$TPKILL_{IW,IT} = 1 - (1 - SSKP_{IW,IT})^{W_{IW,IT}} \tag{180}$$

Considered over many targets of type IT the expected value killed per target when each target has the value V_{IT} is:

$$VKILL_{IW,IT} = V_{IT}[1 - (1 - SSKP_{IW,IT})^{W_{IW,IT}}] \tag{181}$$

This is the final expression for damage to point targets.

Area-Target-Damage Model for Circular Normal Distribution of Target Value - No Terminal Defense - Single Weapon Type

For area targets such as cities, where target damage is measured in fatalities and the lethal radius of a single weapon is small compared with the size of the target, the point-target-damage model is not applicable, since no single weapon may be capable of destroying the entire target and if a single weapon is not capable of destroying the entire target, the weapons should be distributed in a way to maximize expected destruction. The problem then becomes (see reference (e)): maximize the target value destroyed at a target of type IT, subject to the constraint that only $W_{IW,IT}$ weapons of type IW may be used. The solution, when obtained, will be in terms of two limit curves for expected damage. The lower-value limit curve for damage will then be used for damage calculations involving area targets, since it yields a more conservative result.

First, assume that the lethal area, $\pi(RK)^2$, of a single weapon is small with respect to the total target area A. For a single weapon, the probability of the weapon being delivered to the target area A is P. The probability that an increment of target at an arbitrary point within A is killed by a weapon delivered to target area A is $\pi(RK)^2/A$. The overall probability that the Ith increment of target within A is killed is:

$$PKILL_I(1) = P \cdot \pi RK^2/A \qquad (P \cdot \pi \cdot RK^2/A < 1)$$
$$PKILL_I(1) = 1 \qquad (P \cdot \pi \cdot RK^2/A \geq 1) \tag{182}$$

The corresponding probability of survival for the Ith increment is:

$$SURV_I(1) = 1 - PKILL_I$$
$$= 1 - P \cdot \pi \cdot RK^2/A \qquad (P \cdot \pi \cdot RK^2/A < 1)$$
$$SURV_I(1) = 0 \qquad (P \cdot \pi \cdot RK^2/A \geq 1) \tag{183}$$

For N weapons delivered, corresponding to N Bernoulli trials, the probabilities of survival and destruction for target increment I are:

$$SURV_I(N)=(1 - P \cdot \pi \cdot RK^2/A)^N$$
$$(P \cdot \pi \cdot RK^2/A < 1)$$
$$PKILL_I(N) = 1 - (1 - P \cdot \pi \cdot RK^2/A)^N$$
$$SURV_I(N) = 0 \tag{184}$$
$$(P \cdot \pi \cdot RK^2/A \geq 1)$$
$$PKILL_I(N) = 1$$

Now, let K be the expected lethal area of one weapon and ω be the weapon density. Then:

$$K = P \cdot \pi RK^2$$
$$\omega = N/A \tag{185}$$

Substituting equation (185) into equation (184) yields:

$$SURV_I(N) = (1 - K\omega/N)^N$$
$$(K\omega/N < 1)$$
$$PKILL_I(N) = 1 - (1 - K\omega/N)^N$$
$$SURV_I(N) = 0$$
$$(K\omega/N \geq 1) \tag{186}$$
$$PKILL_I(N) = 1$$

Now, assume that there are M increments of the target, the Ith increment containing an area ΔA_I and having a value density of V_I per unit area. The expected value killed by N weapons is:

$$\text{VKILLED}_I = V_I \cdot [1 - (1 - K\omega/N)^N]\,\Delta A_I \qquad (K\omega/N < 1)$$

$$\text{VKILLED}_I = V_I\,\Delta A \qquad (K\omega/N \geq 1) \tag{187}$$

For all M increments the total value killed is:

$$\text{TVKILL} = \sum_{I=1}^{M} V_I\,[1 - (1 - K\omega/N)^N]\,\Delta A_I \qquad (K\omega/N < 1)$$

$$\text{TVKILL} = \sum_{I=1}^{M} V_I \Delta A_I \qquad (K\omega/N \geq 1) \tag{188}$$

It is now desired to maximize the total value killed by adjusting the value of ω in some manner corresponding to the value of the target increments. Thus ω_I replaces ω in equation (188). Now, introduce the constraint of total weapons expended, W , by the relationship:

$$W = \sum_{I=1}^{M} \omega_I\,\Delta A_I \tag{189}$$

While it is noted that W and N are actually the same, it is useful in this parametric approach to keep them separate for the theoretical development.

The problem is now to maximize:

$$\text{TVKILL} = \sum_{I=1}^{M} V_I[1 - (1 - K\omega_I/N)^N]\,\Delta A_I \qquad (\text{For } K\omega_I/N < 1)$$

$$\text{TVKILL} = \sum_{I=1}^{M} V_I\,\Delta A_I \qquad (\text{For } K\omega_I/N \geq 1) \tag{190}$$

Subject to:

$$W = \sum_{I=1}^{M} \omega_I\,\Delta A_I$$

Since the first of equations (190) is a concave function in N for each increment, the total value killed will be maximized when the marginal kills in each increment are equal. To establish the marginal kill for the Ith increment, both equations (190) are differentiated with respect to ω_I. This yields:

$$d(TVKILL)/d\omega_I = V_I\, K(1 - K\omega_I/N)^{N-1}\,\Delta A_I \qquad (K\omega_I/N < 1)$$

$$d(TVKILL)/d\omega_I = 0 \qquad (K\,\omega_I/N \geq 1) \tag{191}$$

$$dW/d\omega_I = \Delta A_I$$

Dividing the first two equations of (191) by the third provides the marginal kill relationship:

$$d(TVKILL)/dW = V_I \cdot K(1 - K\omega_I/N)^{N-1} \qquad (K\omega_I/N < 1)$$

$$d(TVKILL)/dW = 0 \qquad (K\omega_I/N \geq 1) \tag{192}$$

Letting $d(TVKILL)/dW = \lambda$, equation (192) may be solved for ω_I:

$$\omega_I = \frac{N}{K}\left\{1 - [\lambda/(V_I \cdot K)]^{\frac{1}{N-1}}\right\} \qquad [\lambda/(V_I \cdot K) < 1]$$

$$\omega_I = 0 \qquad [\lambda/V_I \quad K) \geq 1] \tag{193}$$

Substituting this result into equations (190):

$$TVKILL = \sum_{I=1}^{M}\left\{1 - [\lambda/(V_I \cdot K)]^{\frac{N}{N-1}}\right\} V_I \Delta A_I \qquad [\text{For } \lambda/(V_I \cdot K) < 1]$$

$$TVKILL = \sum_{I=1}^{M} V_I \Delta A_I \qquad [\text{For } \lambda/(V_I \cdot K) \geq 1] \tag{194}$$

$$W = \sum_{I=1}^{M} \frac{N}{K}\left\{1 - [\lambda/(V_I \cdot K)]^{\frac{1}{N-1}}\right\} \Delta A_I \qquad [\text{For } \lambda/(V_I \cdot K) \geq 1]$$

Now, for area targets, the value is assumed to have a circular normal distribution, in which R is the distance from target center and σ_V is the standard deviation:

$$V = [1/(2\pi\sigma^2_V)]\exp[-R^2/(2\sigma^2_V)] \tag{195}$$

Rewriting equation (195) and expressing R^2 in terms of V yields:

$$R^2 = -2\sigma^2_V \ \log_e(2\pi\sigma^2_V V) \qquad V \leq 1/(2\pi\sigma^2_V) \tag{196}$$

Multiplying equation (196) by π would yield the enclosed area as a function of the enclosed value. By differentiating the results dA may be obtained in terms of V :

$$d(\pi R^2) = dA = -2\pi\sigma^2_V\, dV/V \tag{197}$$

Now, allowing the area increment to decrease to differential size and allowing $M \to \infty$, equation (194) may be written as:

$$\begin{aligned} TVKILL &= \int_A \left\{1 - [\lambda/(V \cdot K)]^{\frac{N}{N-1}}\right\} V dA \\ W &= \int_A \frac{N}{K}\left\{1 - [\lambda/V \cdot K)]^{\frac{1}{N-1}}\right\} dA \end{aligned} \tag{198}$$

Substituting equation (197) into equation (198) and using the limits dictated by equation (194) and (196) provides:

$$\begin{aligned} TVKILL &= 2\pi\sigma^2_V \int_{\lambda/K}^{1/(2\pi\sigma^2_V)} \left\{1 - [\lambda/(V \cdot K)]^{\frac{N}{N-1}}\right\} dV \\ W &= 2\pi\sigma^2_V \int_{\lambda/K}^{1/(2\pi\sigma^2_V)} \frac{N}{K}\left\{1 - [\lambda/V \cdot K)]^{\frac{1}{N-1}}\right\} \frac{dV}{V} \end{aligned} \tag{199}$$

Integrating equation (199) yields:

$$\begin{aligned} TVKILL &= 1 - 2\pi\sigma^2_V \lambda/K + (N-1)\left[(2\pi\sigma^2_V\lambda/K)^{\frac{N}{N-1}} - 2\pi\sigma^2_V\lambda/K\right] \\ W &= (2\pi\sigma^2_V N/K)\left\{-\log_e(2\pi\sigma^2_V\lambda/K) + \left[(2\pi\sigma^2_V\lambda/K)^{\frac{1}{N-1}} - 1\right](N-1)\right\} \end{aligned} \tag{200}$$

Now, since N and W have been kept separate, it is possible to define two limit curves for the extreme conditions of N , namely $N \to 1$ and $N \to \infty$. These two conditions imply the situation of a single weapon or an infinite number of weapons assigned to the target. To find the equation applicable to these two limiting cases, however, requires some effort since by inspection it can be verified that both of equation (200) have indeterminate forms for N=1 and $N = \infty$.

First, in equation (200) the substitution is made:

$$\beta = (2\pi\sigma^2_V \lambda/K)^{\frac{1}{N-1}} \tag{201}$$

Equation (200) may now be rewritten as:

$$\begin{aligned} TVKILL &= 1 - \beta^{N-1}[1 + (N-1)(1-\beta)] \\ W &= \frac{2\pi\sigma^2_V N(N-1)}{K} [\beta - 1 - \log_e(\beta)] \end{aligned} \tag{202}$$

The second of equation (202) may also be rearranged to yield:

$$\frac{KW}{2\pi\sigma^2_V N(N-1)} = \beta - 1 - \log_e(\beta) = X \tag{203}$$

By noting the original limiting conditions in equation (199) and elsewhere, it may be declared that the range for β is $(0 \le \beta \le 1)$. For the limiting values of $\beta=0$ and $\beta=1$, it is useful to tabulate the corresponding values of N and X from equations (201), (202) and (203):

β	N	X
0	1	∞
1	∞	0

Thus, the conditions of $\beta=0$ and $\beta=1$ correspond to the condition of $N \to 1$ and $N \to \infty$, respectively, the conditions of interest as stated earlier.

The $N \to 1$ case will now be listed in detail.

Rewriting equation (203), an expression for $N-1$ may be obtained:

$$\begin{aligned} N-1 &= C/(NX) \\ C &= KW/(2\pi\sigma^2_V) \end{aligned} \tag{204}$$

Using equation (204) in the first of equation (202) results in:

$$\begin{aligned} TVKILL(1) &= 1 - \beta^{\frac{C}{NX}}\left\{1 + [C/(NX)][1-\beta]\right\} \\ &= 1 - \beta^{\frac{C}{NX}}\left\{1 + C/(NX) - \beta C/(NX)\right\} \end{aligned} \tag{205}$$

Now, as established previously, as $N \to 1$, $\beta \to 0$ and $X \to \infty$. Therefore the last two terms go to zero as $N \to 1$ and:

$$\mathrm{TVKILL}(1) \underset{N \to 1}{\to} 1 - \beta^{\frac{C}{X}} \tag{206}$$

Equation (206) may now be rewritten and the natural logarithms of both sides taken:

$$\log_e[1 - \mathrm{TVKILL}(1)] = (C/X)\log_e(\beta) \tag{207}$$

Using the equivalent expression for X from equation (203) in equation (207) results in:

$$\log_e[1 - \mathrm{TVKILL}(1)] = C\log_e(\beta)/[\beta - 1 - \log_e(\beta)] \tag{208}$$

Now, in the limit as $N \to 1$, $(\beta \to 0)$:

$$\log_e[1 - \mathrm{TVKILL}(1)] \underset{\beta \to 0}{\to} C\left\{1/[-1 + (\beta - 1)/\log_e(\beta)]\right\} \to -C \tag{209}$$

Then:

$$1 - \mathrm{TVKILL}(1) = \exp(-C) = \exp(-KW/(2\pi\sigma^2_V)) \tag{210}$$

Therefore, the N=1 bound of the circular normal target destruction case is:

$$\mathrm{TVKILL}(1) = 1 - \exp(-KW/(2\pi\sigma^2_V)) \tag{211}$$

The $N \to \infty$ case will now be treated in detail.

First of all, when $N \to \infty$, equation (203) may be approximated by:

$$\frac{KW}{2\pi\sigma^2_V N(N-1)} \approx \frac{KW}{2\pi\sigma^2_V N^2} = X \tag{212}$$

Now:

$$N = (C/X)^{1/2}$$

$$C = VW/(2\pi\sigma^2_V) \tag{213}$$

By substituting equation (213) in the first of equation (202):

$$\begin{aligned} \text{TVKILL}(\infty) &= 1 - \beta^{\left(\frac{C}{X}\right)^{1/2}-1}\left\{1 + [(C/X)^{1/2} - 1](1-\beta)\right\} \\ &= 1 - \beta^{\left(\frac{C}{X}\right)^{1/2}}[1 + (C/X)^{1/2}(1 - 1/\beta)] \end{aligned} \tag{214}$$

The last term in equation (203) is indeterminate as $N \to \infty$, since $X \to 0$ and $\beta \to 1$. The limit for this last term may be found, however, by substitution of the equivalent expression for X from equation (203) and then, for ease of operation, evaluation of the limit of the last term squared as $N \to \infty (\beta \to 1)$.

$$\begin{aligned} \text{LAST TERM} &= (C/X)^{1/2}(1 - 1/\beta) \\ &= \frac{C^{1/2}(1 - 1/\beta)}{[\beta - 1 - \log_e(\beta)]^{1/2}} \end{aligned} \tag{215}$$

Then:

$$(\text{LAST TERM})^2 = \frac{C(1 - 1/\beta)^2}{\beta - 1 - \log_e(\beta)} \tag{216}$$

Differentiating numerator and denominator with respect to β (l'Hospital's Rule) to find the limit of equation (216) as $\beta \to 1$ yields:

$$\begin{aligned} (\text{LAST TERM})^2_{\beta \to 1} &\to \frac{2C(1 - 1/\beta)\cdot(1/\beta)^2}{1 - 1/\beta} \\ &\to 2C \end{aligned} \tag{217}$$

Therefore:

$$\text{LAST TERM} \to (2C)^{1/2} \tag{218}$$

In equation (207) the positive root was selected since C, $(1-\beta)$ and X are all ≥ 0. Now equation (203) has become:

$$\text{TVKILL}(\infty) = 1 - \beta^{\left(\frac{C}{X}\right)^{1/2}}[1 + (2C)^{1/2}] \tag{219}$$

It is now necessary to find the limit of the indeterminate form $\beta^{\left(\frac{C}{X}\right)^{1/2}}$ as $N \to \infty$ $(B \to 1,\ X \to 0)$. This can be accomplished most efficiently by taking the natural logarithm of the expression in question, squaring it, substituting for X from equation (203), evaluating the squared expression in the limit, and then extracting the appropriate square root. Thus

$$\left\{\log_e\left[\beta^{\left(\frac{C}{X}\right)^{1/2}}\right]\right\}^2 = (C/X)[\log_e(\beta)]^2 = \frac{C[\log_e(\beta)]^2}{\beta - 1 - \log_e(\beta)} \tag{220}$$

Now differentiating numerator and denominator twice (double application of l'Hospital's Rule) to find the limit of equation (220) as $\beta \to 1$:

$$\left\{\log_e\left[\beta^{\left(\frac{C}{X}\right)^{1/2}}\right]\right\}^2_{\beta \to 1} \to \frac{2C\log_e(\beta)}{1 - 1/\beta} \to \frac{2C\,(1/\beta)}{(1/\beta)} \to 2C \tag{221}$$

Now, since $0 \leq \beta \leq 1$, the negative square root from equation (221) is desired:

$$\log_e \beta^{\left(\frac{C}{X}\right)^{1/2}}_{\beta \to 1} \to -(2C)^{1/2} \tag{222}$$

Therefore:

$$\beta^{\left(\frac{C}{X}\right)^{1/2}}_{\beta \to 1} \to \exp[-(2C)^{1/2}] \tag{223}$$

Using the result in equation (219) results in:

$$\begin{aligned} TVKILL(\infty) &= 1 - [1 + (2C)^{1/2}]\exp[-(2C)^{1/2}] \\ &= 1 - \left\{1 + [KW/(\pi\sigma^2{}_V)]^{1/2}\right\}\exp\left\{-[KW/(\pi\sigma^2{}_V)]^{1/2}\right\} \end{aligned} \tag{224}$$

Comparative kill results from equation (211) for $N \to 1$ and from equation (224) for $N \to \infty$ are shown in figure 16. Since the kill values obtained by using TVKILL(∞) are always less than the values obtained by using TVKILL(1), TVKILL(∞) will provide a conservative estimate of kill and has been used as the basic model for evaluating kill for area targets.

Although equation (224) has incorporated weapon delivery probability, delivery error has not been included. It therefore remains to integrate this into the damage model. To do this, the value of W is set equal to one in equation (224) and the result set equal to the single-shot kill probability with K' used in place of $[K/(2\pi\sigma^2_V)]^{1/2}$. Thus:

$$SSKP_{IW,IT} = (1 + K'_{IW,IT}) \exp(-K'_{IW,IT}) \tag{225}$$

The resulting value of K' is then found by an appropriate method (such as Newton's method). The final damage model for area targets is therefore:

$$VKILL_{IW,IT} = V_{IT}[1+K'_{IW,IT}(W_{IW,IT})^{1/2}] \exp[-K'_{IW,IT}(W_{IW,IT})^{1/2}] \tag{226}$$

Using this approach the damage from equation (226) and from equation (181) as a function of the number of weapons employed would be as shown in figure 17. It can also be shown that for some $K''_{IW,IT}$ (in place of $K/(2\pi\sigma^2_V)$) determined from $SSKP_{IW,IT}$, equation (181) and equation (211) yield identical results for the same single shot kill probability. Thus, the optimistic area target damage model and point target damage model are identical for identical single-shot kill probabilities.

Point-Target-Damage Model - Terminal Defense - Multiple Weapon Types

In this extension of the point-target-damage model developed previously, it can be considered that each weapon type must pay a price at the target of $NP_{IW,IT}$ independently targetable weapons. For $W_{IW,IT}$ weapons of type IW allocated to target type IT, the effective number of weapons is:

$$W^*_{IW,IT} = W_{IW,IT} - NP_{IW,IT} \tag{227}$$

The overall survival of the target type is, then, the probability that it survives each weapon type's raid or:

$$SURV_{IT} = \prod_{\substack{\text{ALL} \\ \text{WEAPON} \\ \text{TYPES} \\ \text{ASSIGNED}}} SURV_{IT,IW} \tag{228}$$

Since:

$$PKILL_{IT} = 1 - SURV_{IT} \tag{229}$$

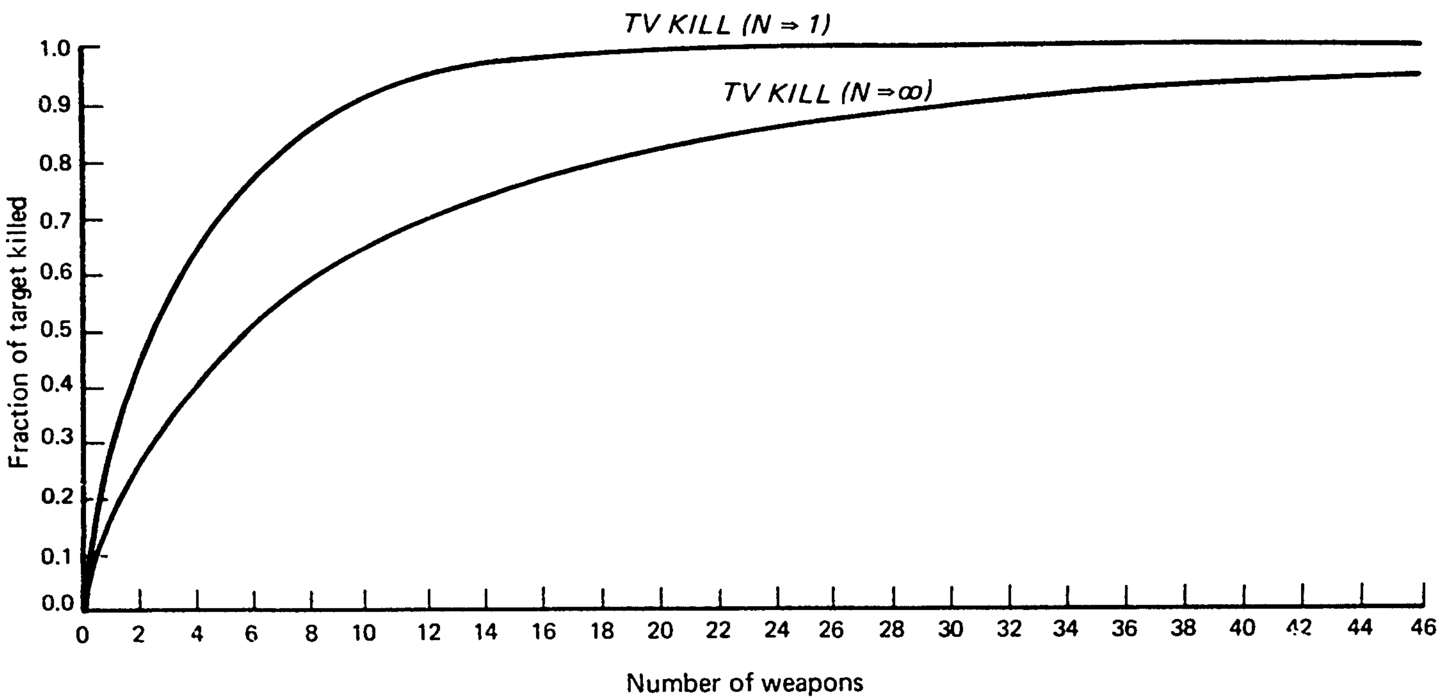

FIG. 16: EXAMPLE OF THE PROBABILITY OF KILL AS A FUNCTION OF THE NUMBER OF WEAPONS DELIVERED WITH ZERO DELIVERY ERROR

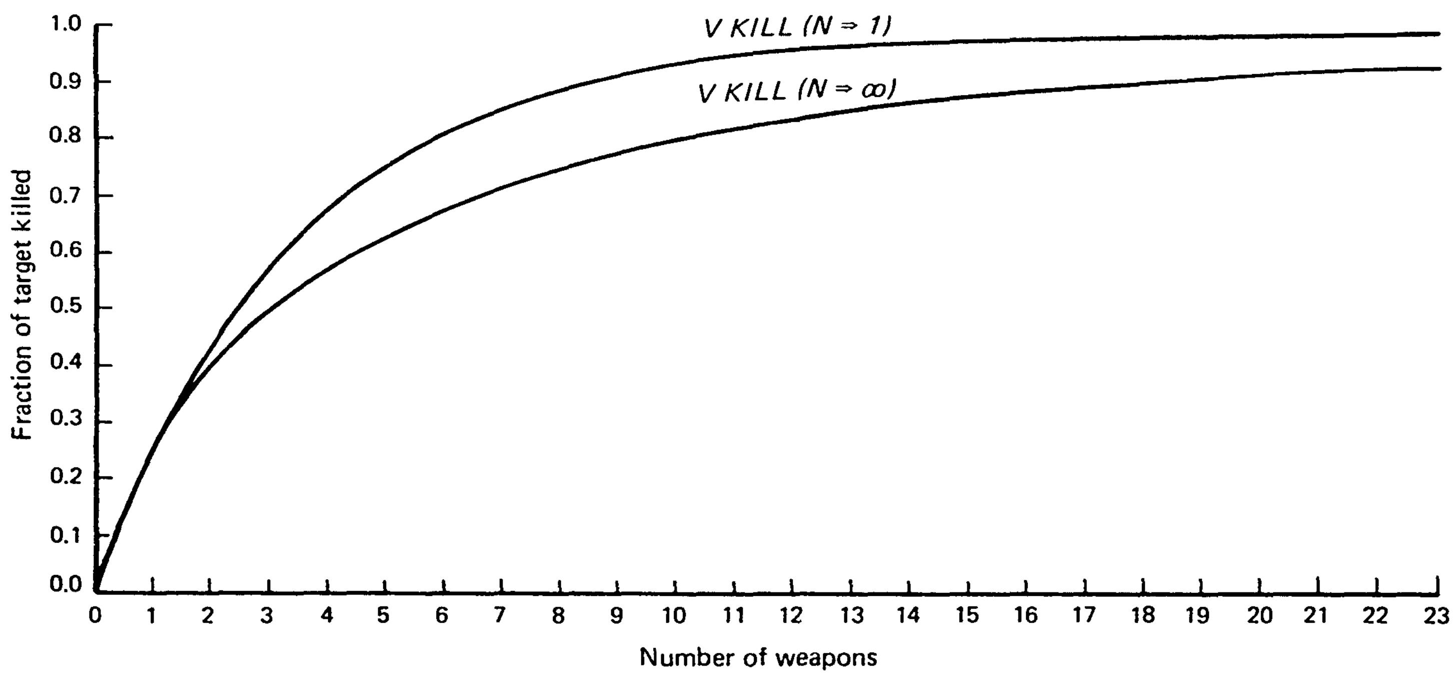

FIG. 17: EXAMPLE OF THE PROBABILITY OF KILL AS A FUNCTION OF THE NUMBER OF WEAPONS DELIVERED WITH DELIVERY ERROR INCLUDED

Then equation (181) may be extended by using equations (227), (228) and (229):

$$VKILL_{IT} = V_{IT}\left[1 - \prod_{\substack{\text{ALL}\\ \text{WEAPON}\\ \text{TYPES}\\ \text{ASSIGNED}}} (1-SSKP_{IW,IT})^{W_{IW,IT} - NP_{IW,IT}}\right] \qquad (230)$$

Area-Target-Damage Model - Terminal Defense - Multiple Weapon Types

In extending the area-target-damage model, the process is not so direct as in the point-target case. While it is true that equation (227) applies in arriving at the effective number of weapons, equation (228) does not. Presumably, the total number of weapons should be applied in some optimum density pattern over the target. How the parameters can be combined can best be seen by consideration of two weapon types with identical kill probabilities against target type IT . Then $W_{1\ IT}$ weapons of the first type and $W_{2\ IT}$ weapons of a second type, in the absence of a terminal defense should yield the same results as if $W_{1\ IT} + W_{2\ IT}$ weapons of either type had been used. This leads by heuristic argument to:

$$VKILL_{IT} = V_{IT}\left\{1 + \Big[\sum_{\substack{\text{ALL}\\ \text{WEAPON}\\ \text{TYPES}\\ \text{ASSIGNED}}} (K'_{IW,IT})^2\, W_{IW,IT}\Big]^{1/2}\right\} \cdot \left\{\exp\left\{-\Big[\sum_{\substack{\text{ALL}\\ \text{WEAPON}\\ \text{TYPES}\\ \text{ASSIGNED}}} (K'_{IW,IT})^2\, W_{IW,IT}\Big]^{1/2}\right\}\right\} \qquad (231)$$

When the terminal defense is taken into account by use of the effective number of weapons $W^*_{IW,IT}$ from equation (227), equation (231) becomes:

$$VKILL_{IT} = V_{IT}\left\{1 + \Big[\sum_{\substack{\text{ALL}\\ \text{WEAPON}\\ \text{TYPES}\\ \text{ASSIGNED}}} (K'_{IW,IT})^2 (W_{IW,IT} - NP_{IW,IT})\Big]^{1/2}\right\} \cdot \left\{\exp\left\{-\Big[\sum_{\substack{\text{ALL}\\ \text{WEAPON}\\ \text{TYPES}\\ \text{ASSIGNED}}} (K'_{IW,IT})^2 (W_{IW,IT} - NP_{IW,IT})\Big]^{1/2}\right\}\right\} \qquad (232)$$

This completes the derivation of the damage calculation models.

WEAPON-TARGET ALLOCATION MODEL

The efficient allocation of resources has always been one of the core problems of economic theory. To solve such problems, a great body of theoretical literature in the fields of linear and non-linear programming has been generated in recent years. One set of methods within this body of literature makes use of the so-called LaGrange Multiplier Theory as a convenient means of solution for resource allocation problems.

The basic resource allocation problem to be solved in Code 50 by the weapon-target allocation model is the allocation of weapons (resources) to targets so as to maximize the destruction of enemy target value subject to the constraints of weapon inventory, integer number of weapons assigned to each target, and lower limits for the destruction of defended targets (if achievable). The solution is provided by a procedure making use of LaGrange Multiplier Theory, steepest ascent methods of mathematical programming and certain heuristic mathematical programming algorithms which have proved sufficient in obtaining a solution to this problem.

Symbols used in the equations of this section are listed below:

CRIT	Maximum value for $CRIT_{IT}$
$CRIT_{IT}$	Penalty parameter for target type IT for using the second best weapon type
CRITFAC	Assured destruction modifier for CRIT based on whether assured destruction criteria have been satisfied
D_{IT}	Damage to target type IT due to weapon types $W_{1,IT},\ldots W_{N,IT}$ assigned to target type IT
DPREF	Total defended target value destroyed
DSUM	Total damage
$DTOT_1$	Initial allocation of damage
$DTOT_2$	Second allocation of damage
FRACINV	Class of weapons
FRACP	Fraction of defended target value specified for destruction; also called the preferred target destruction criterion
FRACPA	Fraction of defended target value destroyed
FRL	Fraction of defended target value destroyed for undefended target multiplier value of α_L
FRU	Fraction of defended target value destroyed for undefended target multiplier value of α_U
$H_{IW,IT}$	Number of weapons which maximizes the LaGrangian profit for single target type

H^*_{IT}	Upper bound of the LaGrangian profit for target type IT
$HMAXC_{IT}$	Best LaGrangian profit per target for target type IT
$HMAXN_{IT}$	Second best LaGrangian profit for target type IT
HSUM	Overall upper bound to the LaGrangian profit
ITC	Most critical class of targets
IWM	Best type of missile
$K_{IW,IT}$	Expected lethal area of weapon type IW used on target type IT
MAXWPS	Number of weapons applied which are of best type for use against at least a single target in the critical class
$NI_{IW,IT}$	Integer number of weapons for each weapon type that will maximize the expected value destroyed per weapon
$NOTAS_{IW}$	Number of unassigned weapons
NP_{IW}	Price in terms of independent targetable weapons successfully launched
NWP_{IW}	Total number of independent targetable weapons available
$NWPD_{IW,IT}$	Total desired inventory for weapon type IW
$PERR_1$	Percentage error between initial allocation of damage, $DTOT_1$, and upper bound of damage, UBOUND
$PERR_2$	Percentage error between second allocation of damage, $DTOT_2$, and upper bound of damage, UBOUND
$PERR_3$	Percentage error between second allocation of damage, $DTOT_2$, and upper bound of damage, UBOUND, based on revised shadow values
$SSKP_{IW,IT}$	Single shot kill probability for target type IT
STOGO	Total shadow value of the unassigned weapons
SVAL	Shadow value of the weapon inventory
SVDL	Shadow value for the aggregate desired inventory
UBOUND	Upper bound for target damage considered over the target set
V_{IT}	Value of target type IT
VPREF	Total defended target value
$W^*_{IW,IT}$	Optimum number of weapons
$W^*_{IWM,ITC}$	Optimum number of weapons to be used against a single target of the critical type

XMAXL	Absolute value of the maximum adjustment of λ_{IW}
α_L	Highest prior value of α_N to produce an acceptable level of assured destruction
α_N	Undefended target value multiplier
α_U	Lowest value of α_N to produce an unacceptable level of assured destruction
$\Delta\lambda'$	Correction factor
$\Delta NWPD_{IW,IT}$	Inventory increment for weapon type IW against target type IT
λ_{IWM}	Shadow value for the best weapon

LaGrange Multipliers from the Classical View

The mathematical formulation of the allocation problem may be stated as:

Maximize:

$$y = y(x_1, x_2, \ldots x_N)$$

subject to the constraints:

$$\begin{aligned} &\Phi_1(x_1, x_2, \ldots x_N) = 0 \\ &\Phi_2(x_1, x_2, \ldots x_N) = 0 \\ &\vdots \\ &\Phi_M(x_1, x_2, \ldots x_N) = 0 \end{aligned} \tag{233}$$

It has been shown in references (h) and (i) that when $x_1, x_2, \ldots x_N$ are continuous variables, $y(x_1, x_2, \ldots x_N)$ is a concave function and $M \leq N$ (there are no non-binding constraints) a global solution to the above problem can be found by finding a solution to the unconstrained problem:

$$Y = y(x_1, x_2, \ldots x_N) - \sum_{K=1}^{M} \lambda_K \Phi_K(x_1, x_2, \ldots x_N) \tag{234}$$

where the λ_K's are non-negative multipliers. The solution to equation (234) is then generated by differentiating Y with respect to each x_I and setting the derivatives to zero. Thus solving the M constraint equations and the N equations

$$\partial Y/\partial x_I = \partial y(x_1, x_2, \ldots x_N)/\partial x_I - \sum_{K=1}^{M} \lambda_K(\partial \Phi_K/\partial x_I) = 0 \tag{235}$$

$$I = 1, 2, \ldots N$$

results in the desired solution, a set of x^o_I's and λ_K's which yield a $y(x^o_1, x^o_2, \ldots x^o_N)$ that is a maximum.

As Samuelson noted in reference (j), the values of the λ_K's in equation (234) are equal to marginal costs or a unit price for each of the M facilities. However, since these are not necessarily real prices in problems of this type, the λ_K's are considered "shadow prices". The interpretation of the summation on the right side of equation (234) is then the "shadow value" expended. The composite view of equation (234), then, is that the difference between the "payoff" and the "shadow value" expended may be termed a "profit". It is this "profit" that is to be maximized.

The Generalized LaGrange Multiplier Theory

For the allocation of weapons in Code 50, the classical problem may be stated as:

Maximize the total damage DSUM:

$$\mathrm{DSUM} = \sum_{\substack{\text{ALL}\\ \text{TARGETS}\\ \text{IN}\\ \text{TARGET}\\ \text{SET}}} D_{IT}(W_{1,IT} \ldots W_{N,IT}) \tag{236}$$

Subject to the N constraints:

$$\sum_{\substack{\text{ALL}\\ \text{TARGETS}\\ \text{IN}\\ \text{TARGET}\\ \text{SET}}} W_{IW,IT} \le NWP_{IW}, \quad IW = 1, 2, \ldots N \tag{237}$$

$$W_{IW,IT} = \text{integers}$$

and subject to the strategy constraints of no mixing of weapon types on terminally defended targets and a minimum level of damage (if achievable) of x percent against that subset of targets with terminal defenses.

The classical theory would then indicate that an augmented equation of the form of equation (234) would be formed. Then the augmented equation would be differentiated with respect to each $W_{IW,IT}$ and the result solved for the weapon allocation. This procedure, however, is not tractable in this case because (1) the numbers of weapons allocated to individual targets must be integers and (2) the form of the destruction equations from the previous subsection is such that solutions are not directly available by these means. To get around this difficulty, Everett, in reference (k), showed that the theory of LaGrange Multipliers is not limited in usefulness to differentiable functions. Essentially Everett's method is based on two theorems. These two theorems are related to the use of the LaGrange Multipliers and to the computation of an upper bound with which to test a particular allocation. These theorems are:

Theorem 1: Let S be the domain of an objective function $F(x)$. If λ_K, $K = 1, \ldots, N$ are non-negative real numbers and if $x^* \epsilon S$ maximizes the function:

$$H = F(x) - \sum_{K=1}^{K=N} \lambda_K C_K(x) \tag{238}$$

Where the C_K's denote the constraints, over all $x \epsilon S$, then x^* maximizes $F(x)$ over all those $x \epsilon S$ such that $C_K(x) \leq C_K(x^*)$ for all K, regardless of the nature of the set S .

As demonstrated earlier, a LaGrange Multiplier problem is normally solved by differentiating the LaGrangian function, setting the results equal to zero, and solving for both the λ_K's and the resource allocation that maximizes the objective function, given a set of constraints. This theorem says that if the LaGrangian is maximized, given a set of λ_K's (i.e., given the shadow prices of the resources), the resulting allocation, x^* , implies a set of constraints, $C_K(x^*)$. Thus, automatically there is an optimum allocation, x^* , for a constrained problem, namely the problem with constraints $C_K(x^*)$.

Thus, there is a great advantage in starting with the multipliers and deducing the constraints, rather than deducing the multipliers given the constraints, in that this approach allows one to place no restriction on the domain of the objective function. Therefore, having an objective function that is neither continuous nor differentiable no longer presents an insurmountable problem.

The disadvantage of this modified method is that unless the correct set of λ_K's is chosen all the constraints will not be satisfied. However, one can prove that if all but one constraint are held constant, the remaining constraint is a monotone decreasing function of its associated multiplier. Thus, after a wrong solution has been reached, the direction to adjust the λ_K's in order to obtain the desired constraints is known. Algorithms have been developed which make this adjustment relatively efficient.

In many cases, one cannot find a set of multipliers which imply the desired constraints. The above theorem insures only that if one finds a solution, it is a correct solution; it does not guarantee that a solution can be found. The existence of optimum solutions that can be found by adjusting the multipliers depends upon the objective function being approximately concave in the region of the solution. Such concavity is not always present; however, the following theorem allows bounds to be placed on the solution to problems where a direct adjustment of the multipliers will not work.

Theorem 2: If x' comes within ϵ of maximizing the LaGrangian, i.e., if for all $x \epsilon S$, where S is the set of all possible strategies

$$F(x') - \sum \lambda_K C_K(x') > F(x) - \sum \lambda_K C_K(x) - \epsilon , \tag{239}$$

then $F(x')$, the payoff at x' , is within ϵ of the maximum payoff for those constraints.

The proof of this theorem gives some insight into its usefulness. Rearranging the above inequality:

$$F(x) - F(x') + \sum \lambda_K [C_K(x') - C_K(x)] < \epsilon \tag{240}$$

Keeping the λ_K's fixed, the allocation x' implies constraints $C_K(x')$ and, obviously, x' is a feasible solution to the problem:

Maximize: $F(x)$

subject to:

$$C_K(x) \le C_K(x') \tag{241}$$

For any other feasible solution to this problem, $C_K(x) \le C_K(x')$ by definition, implying $C_K(x') - C_K(x) \ge 0$. Thus, substituting into inequality equation (239) above, $F(x) - F(x') < \epsilon$ for any x that is a feasible solution to the problem defined by equation (241) above. In practice, the value of ϵ is restricted by imposing the constraint that the number of weapons available for allocating to any one target is restricted to the number of weapons in the stockpile. No payoff can be more than ϵ greater than the payoff given by the allocation x' .

Thus, this theorem allows an upper bound to the maximum payoff to be computed for any constraints. A set of λ_K's is picked and both the maximum LaGrangian and the implied constraints are computed. A new allocation, x' , is then found that implies the desired constraints, and, keeping the λ_K's fixed, the LaGrangian at x' is evaluated ϵ is then the difference between the maximum LaGrangian and the value of the LaGrangian at $x = x'$. This is the maximum additional payoff above $F(x')$ which could ever be obt ned while meeting the desired constraints. Geometrically, Theorem 2 says that the hyperplane with slopes defined by the λ_K's , tangent at x_0 to the envelope of the maximum payoff surface in the space of maximum payoff versus resources expended, is an upper bound to $F(x)$ at any point if x_0 maximizes $F(x) - \sum \lambda_K C_K(x)$.

The Mathematical Programming Procedure in Code 50*

The mathematical programming procedure in Code 50 for weapon allocation involves three fairly separate steps. These are: (1) the initial allocation, (2) testing of the initial allocation, and (3) computation of a second and, if necessary, a third allocation if the prior allocation is determined to be unsatisfactory. These steps will be discussed in turn.

*The remainder of this section parallels the discussion in reference (1).

The Initial Allocation - The first of these allocations is a heuristic process making use of a steepest ascent algorithm. The various operations involved are listed and discussed in the order in which they are carried out.

The first operation that is carried out is to find the integer number of weapons $NI_{IW,IT}$ for each weapon type that will maximize the expected value destroyed per weapon. Of course, for targets without terminal defenses a single weapon produces this result since the destruction functions (see figure 16) are concave functions and each additional weapon after the first adds progressively less value destroyed. For defended targets the number of weapons producing this result is some number greater than the price paid to the terminal interceptors. When the price to be paid to terminal interceptors exceeds the number of weapons in inventory, the target cannot be attacked by that weapon type. These relationships are summarized in equation (242).

$$\begin{aligned} &NI_{IW,IT} = 1 \;:\; NP_{IW,IT} = 0 \\ &NI_{IW,IT} \ni \; D_{IT}(NI_{IW,IT})/NI_{IW,IT} \text{ is MAXIMIZED} \\ &NI_{IW,IT} = \infty : NP_{IW,IT} > NWP_{IW} \end{aligned} \tag{242}$$

Now the best payoff target class for the first weapon type is selected. And a maximum of 1/5 of the weapons of this type are assigned to the target class provided this is enough to attack at least one target of the class under the maximum efficiency attack relationships established in the first two of equation (242). If 1/5 of the weapons cannot produce a maximum efficiency attack on a single target of the class, then enough weapons are allocated to the class for attacking one target. Destruction for the target class attacked is then computed. When all targets of the class are not attacked, the class is split into subclasses — the targets of the class not attacked with the weapon type forming the new subclass. This procedure continues, all target subclasses being considered for remaining weapon increments (1/5 inventory or number for maximum efficiency attack on one target of highest payoff target subclass remaining as outlined above) of the first weapon type until all weapons of this type have been allocated. The procedure above is then continued (except terminally defended subclasses already attacked with another weapon type are not considered) for the second weapon type, then the third until all types have been allocated.

Testing of the Initial Allocation - After all weapons of all types have been assigned the maximum additional payoff available from an additional weapon of each type is computed, with the stipulation that no weapon mixing against terminally defended targets is allowed. This maximum incremental payoff can be considered the value of a single weapon in its best alternative use and in accordance with economic theory, this is its value, or as used here the "shadow value" of the weapon type. In mathematical notation using a difference equation approach (since integer number of weapons are considered) and noting that the "shadow value" is the current value of the LaGrange Multiplier, λ_{IW}.

$$\lambda_{IW} = \text{MAX}\langle \Delta \text{DSUM} \rangle = \underset{\substack{\text{OVER ALL} \\ \text{PERMISSIBLE} \\ \text{TARGET CLASSES}}}{\text{MAX}} \langle D_{IT}(W_{IW,IT}+1) - D_{IT}(W_{IW,IT}) \rangle \tag{243}$$

This result also approximates the result which might have been obtained by formally differentiating the augmented destruction equation, of the form of equation (234) with respect to $W_{IW,IT}$. Noting that the augmented function, exclusive of strategy constraints is of the form:

$$Y = \sum_{IT} D_{IT}(W_{1,IT}, W_{2,IT} \ldots W_{N,IT}) - \sum_{IW} \lambda_{IW} \sum_{IT} W_{IW,IT} - NWP_{IW} \qquad (244)$$

Differentiating equation (244) with respect to each $W_{IW,IT}$ and setting the derivative equal to zero results in:

$$d[D_{IT}(W_{1,IT}, W_{2,IT}, \ldots W_{N,IT})]/dW_{IW,IT} - \lambda_{IW} = 0 \qquad (245)$$

If the derivative in equation (245) is approximated by a difference in which $dW_{IW,IT} = 1$, then it is seen that the two results evaluated for this allocation are approximately equal.

The formal evaluation process now continues by evaluating how much LaGrangian profit (damage minus weapon shadow value) would result from employing the best weapon type on each target (provided that the total inventory would not be exceeded for any one target of the class). This provides an "upper bound" to the LaGrangian profit against which the initial allocation may be evaluated. If the upper bound and the initial allocation fall within a specified amount of each other, the allocation process can be terminated at this point and the initial allocation adopted as the final allocation. The upper bound determination is based upon pure weapon allocations, i.e., a single weapon type per target of each type. The computation of the upper bound is based upon the steps outlined below.

For each weapon type, the number of weapons that maximize the LaGrangian profit $H_{IT,IW}$ is found for a single target of that type:

$$H_{IT,IW} = D_{IT}(W_{IW,IT}) - \lambda_{IW} W_{IW,IT} \qquad (246)$$

For point targets and area targets $D_{IT}(W_{IW,IT})$ is:

Point targets:

$$D_{IT}(W_{IW,IT}) = V_{IT}\left\{ 1 - (1 - SSKP_{IW,IT})^{W_{IW,IT} - NP_{IW,IT}} \right\}$$

Area targets: (247)

$$D_{IT}(W_{IW,IT}) = V_{IT}\left\{1 - \left\{1 + [(W_{IW,IT} - NP_{IW,IT})(K_{IW,IT})^2]^{1/2}\right\} \cdot \exp\left\{ - [(W_{IW,IT} - NP_{IW,IT})(K_{IW,IT})^2]^{1/2}\right\}\right\}$$

Differentiating equation (246) for the two types of targets and setting the results to zero yields:

Point targets:

$$\frac{dH_{IT}}{dW_{IW,IT}} = V_{IT}\left\{[\log_e(1 - SSKP_{IW,IT})](1 - SSKP_{IW,IT})^{W_{IW,IT} - NP_{IW,IT}}\right\} - \lambda_{IW} = 0$$

Area targets:

$$\frac{dH_{IT}}{dW_{IW,IT}} = V_{IT}\left\{[(K_{IW,IT})^2]\right\} \cdot \exp -\left\{[(K_{IW,IT})^2 W_{IW,IT} - NP_{IW,IT})]^{1/2}\right\} - \lambda_{IW} = 0 \qquad (248)$$

Using the form of equation (248) appropriate to the target type and the value of λ_{IW} previously found in equation (243), the optimum number of weapons $W_{IW,IT}{}^*$ may be found as:

Point targets:

$$W_{IW,IT}{}^* = \frac{NP_{IW,IT} + \log_e\left\{-\lambda_{IW}/[V_{IT}\log_e(1 - SSKP_{IW,IT})]\right\}}{\log_e(1 - SSKP_{IW,IT})}$$

Area targets:

$$W_{IW,IT}{}^* = NP_{IW,IT} + \left\{\frac{\log_e[V_{IT}(K_{IW,IT})^2/(2\lambda_{IW})]}{K_{IW,IT}}\right\}^2 \qquad (249)$$

The value of $W_{IW,IT}{}^*$ can then be used to find $H_{IT,IW}{}^*$ from equations (246) and (247).

$$H_{IT,IW}{}^* = D_{IT}(W_{IW,IT}{}^*) - \lambda_{IW}W_{IW,IT}{}^* \qquad (250)$$

The upper bound of the LaGrangian profit $H_{IT}{}^*$ for target type IT is the maximum value of $H_{IT,IW}{}^*$ produced by any single weapon type. Therefore:

$$H_{IT}^{*} = \underset{\substack{\text{OVER}\\\text{ALL}\\\text{WEAPON}\\\text{CLASSES}}}{\text{MAX}} \langle H_{IT,IW}^{*} \rangle \tag{251}$$

Extending this to the complete target set, the overall upper bound to the LaGrangian profit, HSUM, is just the sum of all such H_{IT}^{*}'s .

$$\text{HSUM} = \sum_{IT} H_{IT}^{*} \tag{252}$$

Now HSUM is the difference between the upper bound of target damage UBOUND considered over the target set and the shadow value SVAL of the weapon inventory.

$$\begin{aligned} \text{HSUM} &= \text{UBOUND} - \text{SVAL} \\ \text{SVAL} &= \sum_{IW} \lambda_{IW} \text{NWP}_{IW} \end{aligned} \tag{253}$$

Therefore

$$\text{UBOUND} = \text{HSUM} + \text{SVAL} \tag{254}$$

The expected total damage to the target set provided by the initial allocation, DTOT_1, may now be compared with the upper bound of damage UBOUND, using Theorem 2 and the difference ϵ computed.

$$\epsilon_1 = \text{UBOUND} - \text{DTOT}_1 \tag{255}$$

This may also be expressed as a percentage error PERR_I .

$$\text{PERR}_1 = 100\ (\epsilon_1 / \text{DTOT}_1) \tag{256}$$

If PERR is sufficiently small, then the initial allocation by steepest ascent is satisfactory with respect to total damage and only the level of damage to the defended target subset must be investigated. If PERR_1 is not sufficiently small, then a new allocation must be made using the procedure outlined below.

<u>The Second Allocation</u> - The second allocation makes use of the same λ_{IW}'s as the initial allocation. The general procedure is to find the target type for which there is the greatest penalty incurred when the second-best -- rather than the best -- weapon type is employed against it. The algorithm to determine this penalty includes the fraction of the remaining inventory (initially this is the entire inventory) of the best weapon (in the LaGrangian profit sense) required to produce this profit against all targets of the class, FRACINV , the number of weapons of the best type IWM needed for a

payoff at all, $NP_{IWM,IT}+1$, the best LaGrangian profit per target for target type IT, $HMAXC_{IT}$ and the second best LaGrangian profit for target type IT, $HMAXN_{IT}$. Making use of all the above, the penalty parameter for target type IT, $CRIT_{IT}$, is defined as:

$$CRIT_{IT} = (NP_{IWM,IT} + 1) \cdot FRACINV(1.1\ HMAXC_{IT} - HMAXN_{IT}) \tag{257}$$

The maximum value of $CRIT_{IT}$ over the target set is then selected and the target type which produced it is considered the most critical type with respect to the allocation

$$CRIT = \underset{\substack{\text{OVER}\\ \text{ALL}\\ \text{TARGET}\\ \text{SUBCLASSES}}}{MAX} \langle CRIT_{IT} \rangle \tag{258}$$

Provided there are enough weapons of the best type for use against at least a single target in the most critical class ITC, these are applied in accordance with an algorithm relating (1) the total shadow value of the $NOTAS_{IW}$ unassigned weapons, STOGO, (2) the total shadow value of all weapons, SVAL, (3) the shadow value for the best weapon λ_{IWM}, and (4) the optimum number of weapons to be used against a single target of the critical type, $W_{IWM,ITC}{}^*$. The number applied according to the algorithm is:

$$MAXWPS = MAX \langle [1 + (STOGO - .3\ SVAL)/\lambda_{IWM}] : W_{IWM,ITC}{}^* \rangle \tag{259}$$

Where

$$STOGO = \sum_{IW} NOTAS_{IW}\, \lambda_{IW}$$

$$SVAL = \sum_{IW} NWP_{IW}\, \lambda_{IW} \tag{260}$$

Now, MAXWPS of the type IWM are applied to target type ITC and the destruction calculated. $NOTAS_{IWN}$ and STOGO are updated to reflect the expenditures. In addition, $HMAXC_{IT}$ and $HMAXN_{IT}$ are modified as necessary, to reflect the fact that there may now not be enough weapons if type IWM was the best or second-best weapon for target type IT.

The entire procedure is now repeated, beginning at equation (257) until STOGO ≥ .3 SVAL. When enough weapons have been allocated to critical target types this way to make STOGO < .3 SVAL, the remainder of the weapon inventory is allocated by the steepest ascent method used for the initial allocations.

Testing of the Second Allocation - The second allocation is testeu using the total expected value destroyed on the second allocation, $DTOT_2$, and UBOUND from the first allocation (λ_{IW}'s and unchanged):

$$\epsilon_2 = UBOUND - DTOT_2$$

$$PERR_2 = 100\ (\epsilon_2/DTOT_2) \tag{261}$$

If $PERR_2$ is sufficiently small, then the second allocation is satisfactory with respect to total damage and only the level of damage to the defended target subset must be investigated. If $PERR_2$ is not sufficiently small, then the shadow values (λ_{IW}'s) are adjusted using the procedure outlined below.

Weapon Shadow Value Adjustment - If the second allocation outlined is not satisfactorily close to optimum, the optimum criterion is adjusted by adjusting the weapon shadow prices (λ_{IW}'s). This is carried out by comparing the desired inventory of each weapon type (as derived by a heuristic algorithm) with the actual inventory of the weapon type. If more weapons of type IW are desired than currently are in inventory, the price is too low and λ_{IW} should be increased. Conversely, if fewer of weapon type IW are desired than there are in inventory, λ_{IW} should be decreased. In addition to changing the individual λ_{IW}'s, the total shadow value of the desired inventory should match the total shadow value of the actual inventory. If the total desired inventory shadow value is greater than the actual inventory shadow value, it indicates that the imputed weapon values (λ_{IW}'s) are too high and conversely. When the desired and actual inventories match or the λ_{IW}'s are sufficiently close and the desired inventory shadow value is sufficiently close to the actual inventory, the λ_{IW}'s are used to re-evaluate the second allocation.

Now that an overview of the procedure has been given, the actual steps and algorithms will be presented.

The first step is to determine the desired inventory for each weapon based on the λ_{IW}'s derived in evaluating the second allocation.

The inventory increment for weapon type IW against target type IT is:

$$\Delta NWPD_{IW,IT} = \frac{W_{IW,IT}{}^{*}\ MAX\langle 0: H_{IT,IW} - .95H_{IT}{}^{*}\rangle}{\sum\limits_{IW} MAX\langle 0: H_{IT,IW} - .95H_{IT}{}^{*}\rangle} \tag{262}$$

The total desired inventory $NWPD_{IW}$ for weapon type IW is then:

$$NWPD_{IW} = \sum_{IT} \Delta NWPD_{IW,IT} \tag{263}$$

Now if the desired inventory for each weapon is compared with the actual inventory, and if the desired inventory is less than the actual inventory for weapon type IW , λ_{IW} is reduced by 5 percent.

$$\begin{aligned} &\lambda_{IW} \rightarrow .95\lambda_{IW} \\ &\qquad NWPD_{IW} < NWP_{IW} \end{aligned} \tag{264}$$

A new desired inventory is now computed using the new value of λ_{IW} . Following this, the values of increments to the λ_{IW}'s , ($\Delta\lambda_{IW}$'s), are generated using a converging iterative process. Initial values for the $\Delta\lambda_{IW}$'s are:

$$\begin{aligned} \Delta\lambda_{IW} &= .05 \qquad & NWPD_{IW} > NWP_{IW} \\ \Delta\lambda_{IW} &= -.05 \qquad & NWPD_{IW} \le NWP_{IW} \end{aligned} \tag{265}$$

Further adjustment is then accomplished as follows:

$$\begin{array}{ll} \Delta\lambda_{IW} \rightarrow -\Delta\lambda_{IW}/4 & NWPD_{IW} < NWP_{IW} \text{ and } \Delta\lambda_{IW} > 0 \\ & \qquad\text{or} \\ & NWPD_{IW} = NWP_{IW} \\ & \qquad\text{or} \\ & NWPD_{IW} > NWP_{IW} \text{ and } \Delta\lambda_{IW} < 0 \\ \Delta\lambda_{IW} \rightarrow 1.3\Delta\lambda_{IW} & NWPD_{IW} < NWP_{IW} \text{ and } \Delta\lambda_{IW} \le 0 \\ & \qquad\text{or} \\ & NWPD_{IW} > NWP_{IW} \text{ and } \Delta\lambda_{IW} \ge 0 \end{array} \tag{266}$$

Further refinements are now made to the $\Delta\lambda_{IW}$'s based in part on the absolute value of the maximum adjustment made so far:

$$XMAXL = \underset{\substack{\text{OVER} \\ \text{ALL} \\ \text{WEAPON} \\ \text{TYPES}}}{MAX} \langle |\Delta\lambda_{IW}| \rangle \tag{267}$$

When XMAXL $\leq .001$, (which it will not be the first time) sufficient adjustment has been made. If not, the $\Delta\lambda_{IW}$'s are checked and adjusted if necessary to keep them within preset bounds in two steps as follows:

Step I Adjustment

$$\begin{aligned} &\Delta\lambda_{IW} \rightarrow -.8, && \Delta\lambda_{IW} < -.8 \\ &\Delta\lambda_{IW} \rightarrow XMAXL/20, && 0 \leq \Delta\lambda_{IW} < XMAXL/20 \end{aligned} \tag{268}$$

Step II Adjustment

$$\begin{aligned} &\Delta\lambda_{IW} \rightarrow -XMAXL/20, && -XMAXL/20 < \Delta\lambda_{IW} < 0 \\ &\Delta\lambda_{IW} \rightarrow .5 && \Delta\lambda_{IW} > .5 \end{aligned}$$

The values of the λ_{IW}'s are then recalculated taking into account the above $\Delta\lambda_{IW}$'s and a simultaneous correction factor $\Delta\lambda'$. Initially $\Delta\lambda' = -.03$.

$$\lambda_{IW} \rightarrow \lambda_{IW}(1 + \Delta\lambda_{IW})(1 + \Delta\lambda') \tag{269}$$

The shadow values for both the aggregate desired inventory and the aggregate actual inventory are now computed for the purpose of adjusting $\Delta\lambda'$.

$$\begin{aligned} SVAL &= \sum\lambda_{IW}NWP_{IW} \\ SVDL &= \sum\lambda_{IW}NWPD_{IW} \end{aligned} \tag{270}$$

The adjustments to $\Delta\lambda'$ are now made similar to the way adjustments were made to the $\Delta\lambda_{IW}$'s in equation (266):

$$\begin{aligned} &\Delta\lambda' \rightarrow -\Delta\lambda'/4 && SVDL < SVAL \text{ and } \Delta\lambda' \geq 0 \\ & && SVDL > SVAL \text{ and } \Delta\lambda' \leq 0 \\ &\Delta\lambda' \rightarrow 1.3\,\Delta\lambda' && SVDL < SVAL \text{ and } \Delta\lambda' < 0 \\ & && SVDL > SVAL \text{ and } \Delta\lambda' > 0 \end{aligned} \tag{271}$$

As in the case of the $\Delta\lambda_{IW}$'s the value of $\Delta\lambda'$ is limited as follows:

Step I Adjustment

$$\begin{aligned} &\Delta\lambda' \rightarrow -.8 && \Delta\lambda' < -.8 \\ &\Delta\lambda' \rightarrow XMAXL/20, && 0 \leq \Delta\lambda' < XMAXL/20 \end{aligned} \tag{272}$$

Step II Adjustment

$$\Delta\lambda' \rightarrow -\text{XMAXL}/20, \qquad -\text{XMAXL}/20 < \Delta\lambda' < 0$$
$$\Delta\lambda' \rightarrow .5, \qquad \Delta\lambda' > .5$$

The above sequence is now repeated beginning at equation (262) and continued until either a present number of iteration (generally 100) has been completed of XMAXL < .001 as determined in equation (267).

Retesting the Second Allocation Based on Revised Shadow Values - The second allocation is now tested again using the new shadow values (λ_{IW}'s) just derived. The procedure is identical to the procedure followed in evaluating the initial allocation in equations (249)-(256). The results based on a new value of UBOUND are:

$$\begin{aligned} \epsilon_3 &= \text{UBOUND} - \text{DTOT}_2 \\ \text{PERR}_3 &= 100\,(\epsilon_3/\text{DTOT}_2) \end{aligned} \qquad (273)$$

If PERR_3 is small enough, the second allocation is satisfactory with respect to total damage and only the level of damage to the defended target subset must be investigated. If PERR_3 is not small enough, then a third allocation must be made, by means of the procedure outlined below.

The Third Allocation - The third allocation is generated using exactly the same procedure which was used in generating the second allocation but utilizing the adjusted λ_{IW}'s discussed earlier.

Testing the Third Allocation - The third allocation is tested in the same way as the second allocation, except that the restriction on PERR_4 for a satisfactory solution is changed from 1 percent to 3 percent (or 6 percent if it is not desired to take the time necessary for 3 percent convergence). If PERR_4 meets the criterion with respect to total damage, then only the level of damage to the defended target subset must be investigated. If PERR_4 does not meet the relaxed criterion, the above procedure must be repeated, as outlined below.

Additional Adjustment to Shadow Values Re-evaluation, Additional Allocations and Tests of These Allocations - When the third allocation has not produced satisfactory results, the λ_{IW}'s are again modified, the third allocation is then retested and, if necessary, a fourth allocation is generated and tested. Similarly, if this is not satisfactory, the process of λ_{IW} adjustment, re-evaluation, new allocations, and allocation testing is repeated until a satisfactory allocation is found.

Satisfying the Assured Destruction Constant - When a satisfactory allocation has been found -- whether it be on the first, second, third, etc. allocation -- an additional constraint must also be satisfied. That constraint is the need to destroy at least some stipulated fraction (FRACP) of the total value of the subset of defended targets. The fraction actually destroyed is termed FRACPA. This is also known as the preferred-target-destruction criterion. Thus, starting with the total defended target value, VPREF, and the total defended target value destroyed, DPREF:

$$\begin{aligned} VPREF &= \sum_{\substack{\text{SUBSET OF}\\ \text{DEFENDED}\\ \text{TARGETS}}} V_{IT} \\ DPREF &= \sum_{\substack{\text{SUBSET OF}\\ \text{DEFENDED}\\ \text{TARGETS}}} D_{IT} \end{aligned} \tag{274}$$

Then:

$$FRACPA = DPREF/VPREF \tag{275}$$

Now, if FRACPA < FRACP , all non-defended target types are being valued too highly and a new allocation is made. In this allocation all undefended target values are multiplied by $\alpha_X = .003$ to reduce their values on this next allocation to .3 percent of their original values.

$$\underset{\substack{\text{TARGET}\\ \text{TYPE IT}\\ \text{UNDEFENDED}}}{V_{IT}' \rightarrow .003\, V_{IT}} \tag{276}$$

The allocation procedure is the same as that previously outlined, beginning with a steepest ascent initial allocation and proceeding to second, third, etc., allocations as necessary. The only difference occurs in the penalty algorithm, equation (257), in which a multiplicative factor is added. The value of this factor depends on whether the allocation just previous destroyed enough preferred target value.

$$\begin{aligned} &CRITFAC = .1 \qquad FRACPA > FRACP \\ &CRITFAC = 10 \qquad FRACPA \le FRACP \\ &CRIT_{IT} = CRITFAC(NP_{IWM,IT} + 1) \cdot FRACINV(1.1\ HMAXC_{IT} - HMAXN_{IT}) \end{aligned} \tag{277}$$

After the allocation procedure is again completed the defended target value destroyed is recomputed using equation (274) and the fraction of total defended value destroyed is recomputed using equation (275). If FRACPA and FRACP are now within 0.02 of each other, or if FRACPA < FRACP (assured destruction is inachievable), the allocation process is complete. If not, the undefended target value multiplier α_N for the Nth assured destruction iteration is varied using a linear interpolation method. The essentials of the process are explained below using figure 18.

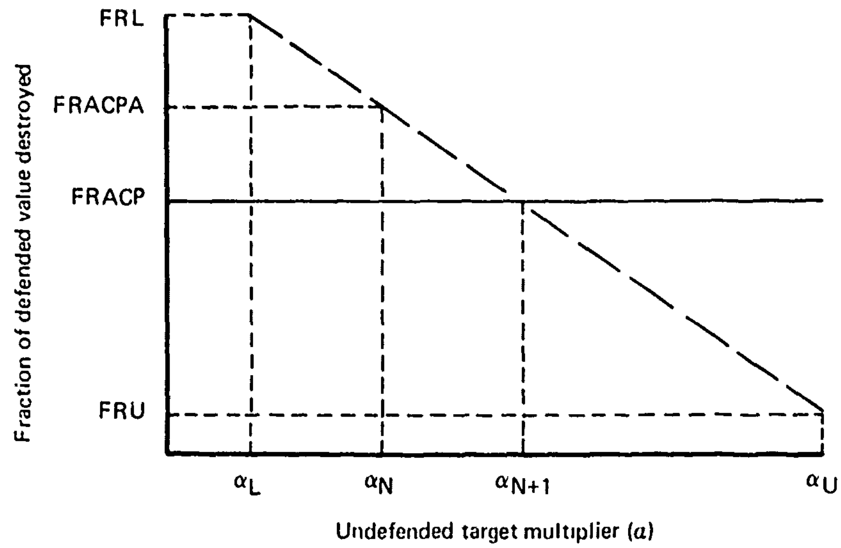

FIG. 18: RELATIONSHIP BETWEEN FRACTION OF DEFENDED VALUE DESTROYED AND UNDEFENDED TARGET MULTIPLIER

In figure 18, α_L is defined as the highest value of α_N that has thus far produced an acceptable level of assured destruction. α_U is the lowest value of α_N that has thus far produced an unacceptable level of assured destruction. Initially, since the first acceptable allocation assumed no preference for defended-target values, $\alpha_U = 1$. Similarly since the first iteration using $\alpha_1 = 0.003$ was successful in producing an acceptable level of assured destruction, then initially $\alpha_L = 0.003$. FRL and FRU, then, are defined as the fractions of defended target value destroyed (assumed destruction level), when the undefended target multiplier takes on values of α_L and α_U, respectively.

Following the first and higher iterations, α_U or α_L and FRU or FRL are updated reflecting the value of FRACPA obtained on the Nth iteration.

$$\left.\begin{array}{l}\alpha_U \rightarrow \alpha_N \\ \text{FRU} \rightarrow \text{FRACPA}\end{array}\quad N \geq 1\right\} \left\{\begin{array}{l}\text{FRACPA} < \text{FRACP} \\ \left|\text{FRACPA} - \text{FRACP}\right| > .02\end{array}\right. \tag{278}$$

$$\left.\begin{array}{l}\alpha_L \rightarrow \alpha_N \\ \text{FRL} \rightarrow \text{FRACPA}\end{array}\quad N \geq 1\right\} \left\{\begin{array}{l}\text{FRACPA} > \text{FRACP} \\ \left|\text{FRACPA} - \text{FRACP}\right| > .02\end{array}\right.$$

Based on figure 18 and the above, the linear interpolation value of α_{N+1} for iteration N+1 is then:

$$\alpha_{N+1} = \alpha_L + (\alpha_U - \alpha_L)(FRL - FRACP)/(FRL - FRU) \tag{279}$$

The value of undefended targets for assured destruction iteration N+1 are then updated:

$$\underset{\substack{\text{TARGET}\\\text{TYPE IT}\\\text{UNDEFENDED}}}{V_{IT}'} \rightarrow \alpha_{N+1} V_{IT} \tag{280}$$

A new allocation is then processed based on the new values of undefended targets. When this allocation is completed, it is tested to see if the defended target fraction FRACPA is within 0.02 of the desired fraction, FRACP . If so, the allocation is complete. If not, but the limit on the number of assured destruction iterations has been reached, then the assured destruction iteration allocation in which FRACP was exceeded by the smallest amount is taken to be the final allocation. If FRACPA is not sufficiently close to FRACP (differing by more than 0.02) and the limit for assured destruction iterations has not yet been reached, then another iteration is made beginning with the updating of the various α's in equation (278).

SECTION III

SUMMARY OF SUBROUTINE OPERATIONS

GENERAL OVERVIEW

As a point of reference, it is first useful to review the typical Code 50 scenario. While many other possibilities exist, this is the most extensively used. First, it is assumed that the Soviet Union attacks the United States in a counter-force-only strike hitting U.S. land-based missile sites and non-alert bombers and ballistic missile submarines. The Soviets are assumed to hold back all bombers and a small number of missiles for the third strike. The Soviet attack is optimized to maximize the U.S. weapon value destroyed. After the destruction is taken into account, all surviving U.S. forces attack the Soviet cities in an attack that is optimized to maximize fatalities, subject to the assured destruction constraint that X percent of the value in terminally defended cities must be destroyed. After the second strike, the Soviets make a third strike against U.S. cities with their remaining weapons, to kill a maximum number of people. On this basis, the discussion can now proceed into how Code 50 operates to work through this scenario.

The Code 50 program is divided into two parts. The first consists of the war-game model itself and the data input systems. This segment of Code 50 reads in data, initializes for the ensuing computations, calculates single-shot kill probabilities, analyzes weapon-interceptor engagements for both missiles and bombers, and in general, prepares the weapon and target data for the second part of the program which will determine the weapon-to-target allocation. The various components of this part of Code 50 are Code 50, the main executive program; GENX and DATAX, the input routines; STRIKE, a secondary executive and general calculating routine; MBPEN, a single-island random area penetration subroutine for both missiles and bombers; SSKCALC, which calculates single-shot kill probabilities; and MINNUM, which computes the minimum number of weapons of each type which must be invested at each type of target in order to obtain a payoff.

The game model segment of Code 50 is interfaced with the weapon-to-target allocation through subroutine TRANS. Control then passes to subroutine ASSIGN which is the executive subroutine for the allocator section of Code 50. When the allocation is complete, the damage is computed, control is returned to Code 50, the attacker becomes the defender and vice-versa, and subroutine STRIKE is called again. The procedure is carried out three times or less, although the initializing and data-reading subroutines are only called on the first strike.

THE WAR GAME SEGMENT

The basic purpose of the war game segment of Code 50 is to feed to the inside (or allocator) segment lists of available weapons to allocate to the various targets along with all requisite information concerning both weapons and targets. The subroutine calling sequence for the war game is as shown in figure 19, the events represented by this sequence are as follows. The executive routine Code 50 sets the attacker and defender indices and calls the data input subroutines, GENX and DATAX. These read in information and also compute some parameters such as the number of weapon and target types for each side. Control then returns to Code 50 where an initializer transfers data from the fixed arrays to the working arrays. Then a secondary executive routine, STRIKE, is called. It sets the number of weapon classes and target classes for attacker and defender for later use. It then obtains the target values. There is an option to either read in target values or calculate them. The calculation, of course, applies only to weapons considered as targets since the value of a city has been taken as its population. The other task of STRIKE is to set flags in arrays to indicate missiles, targets with extended areas to which the square-root law is applied to calculate survival, targets which are terminally defended, and to update the weapons left.

The subroutine that calculates the area penetration probabilities, MBPEN is now called. The bomber and missile area penetration models covered in section II are included in MBPEN. MBPEN is called with an argument so it can be used for either side at any time, not merely the attacker. This capability is required if one assumes that the value of a weapon as a target is in part a function of its delivered yield. Thus, given the index for one side, the first step is to find the other index. MBPEN then proceeds to find the number of bombers of each type arriving at the area defenses as well as the total number. There is an option to read the kill probability of a bomber by a fighter instead of calculating it. The penetration probability is defined as the probability that a bomber is not killed before delivery multiplied by its ASM inflight launch reliability. This last is unity for gravity bombers. The last point here is the calculation of NWP which is the number of reliable independently targetable weapons.

Subroutine SSKALC calculates the single-shot kill probabilities and modifies them to include any restrictions placed on the weapons such as not sending bombers against alert missiles. This is done using the single-shot kill models discussed in section II.

Subroutine MINNUM gives the minimum number of weapons of a given type required to obtain a payoff at a target. The bomber and missile terminal penetration models discussed in section II are used for this purpose.

This completes the description of the game model and how the specific game models of section II fit together. To recapitulate, this section of the program feeds to the inside a list of numbers of reliable weapons, kill probabilities, number of weapons required for a payoff, a target list, and numbers of weapon and target types. At this point the weapon-to-target allocation routine takes over. Then after the allocation is complete, and a summary result is printed, control returns to Code 50 where the attacking and defending sides are exchanged, the strike number is increased by unity, and the sequence starts again.

THE WEAPON ALLOCATION SEGMENT

The calling sequence for the weapon allocation segment is handled by subroutine ASSIGN. Its flow chart is shown in figures 20-23. First, it calculates several quantities including the total and terminally defended target system values. Next it calls an initializing routine (INITIAL), and then performs the steepest ascent initial laydown allocation. As discussed in section II in the Weapon Target Allocation Model, this allocation is a compormise in that it neither lays down all of a given weapon type at once, nor does it lay the weapons down individually. Rather it makes the laydown using one-fifth of each weapon type on the best targets. The best target for a given weapon type is found in subroutine MAXPAY. As some number of targets of a given type have weapons assigned to them, new target classes are created for the targets of that type which have not as yet been touched. This is done using subroutine SPLIT. When the laydown is completed, one more weapon of each type is then assigned to the best target for that weapon. The bare payoff or best alternative use for a single weapon is taken to be the starting value of the LaGrange multiplier. Having obtained a set of multipliers, the next step is to evaluate the allocation just made. To do this, one must have in some sense an optimum allocation. This optimum is obtained using subroutines MAXLAG and EVAL. The first of these finds the maximum LaGrangian payoff for each target if the optimum number of the best weapon type is used at each target. Subroutine EVAL then compares the total payoff using these optimal LaGrangian payoffs with the actual payoff obtained using the laydown allocation. Generally, the fractional error will be over the preset limit so a new allocation must be made. This new allocation is made on the most critical targets first. These are the targets on which the largest differences are obtained in the LaGrangian payoff if the optimum weapon were out of stock. This allocation proceeds until seventy percent of the total shadow value of the weapons has been expended. The remaining thirty percent is allocated using the steepest ascent laydown algorithm. Again the allocation is evaluated. If it is not good enough, the LaGrange multipliers are adjusted using subroutine ADJLAM. This subroutine first calls OPLAG which finds for the given types of weapons in stock the optimum number of each weapon to have. The difference between the optimum number and the actual number is the criterion by which the lambdas are adjusted. This adjusting involves a number of heuristic considerations and is both complicated and time-consuming. After it is completed, another allocation is made. Experience has indicated that this allocation is almost always very close to the optimum, although additional allocations may need to be made in some cases using the loop mechanism shown in figures 20-22. This completes the weapon-to-target allocation. If there is no assured destruction required, ASSIGN calls CONSOLID to find any repetitions in the allocation list and to combine such repetitions into a single target type subclass. ASSIGN then makes a final call on EVAL, calculates damage and calls on TARGSUM to print out the damage. Then ASSIGN calls WPSUM to print out a summary of the attacking side weapons. Finally, PRALLOC is called to print out a summary of the allocation. Control then returns to Code 50.

On the other hand, if there is some assured destruction requirement, a somewhat longer calculation is made. A check is first made to see if the assured destruction was obtained on the optimum allocation. If so, the allocation is complete. If not, the value of all the non-preferred targets is set to zero, and we start over. On returning to this point, a check is made to see if it is ever possible to obtain the assured destruction on the preferred targets. If not, a message is printed and the allocation is complete. If the assured damage level was obtained with the non-preferred target values set to 0.3 percent of their values, an iterative technique is used as explained in section II to find the factor by which the non-preferred target values are to be multiplied to obtain as nearly as possible the exact assured damage level. The allocation is then returned to Code 50 through MINNUN, TRANS and STRIKE, as shown in figure 19.

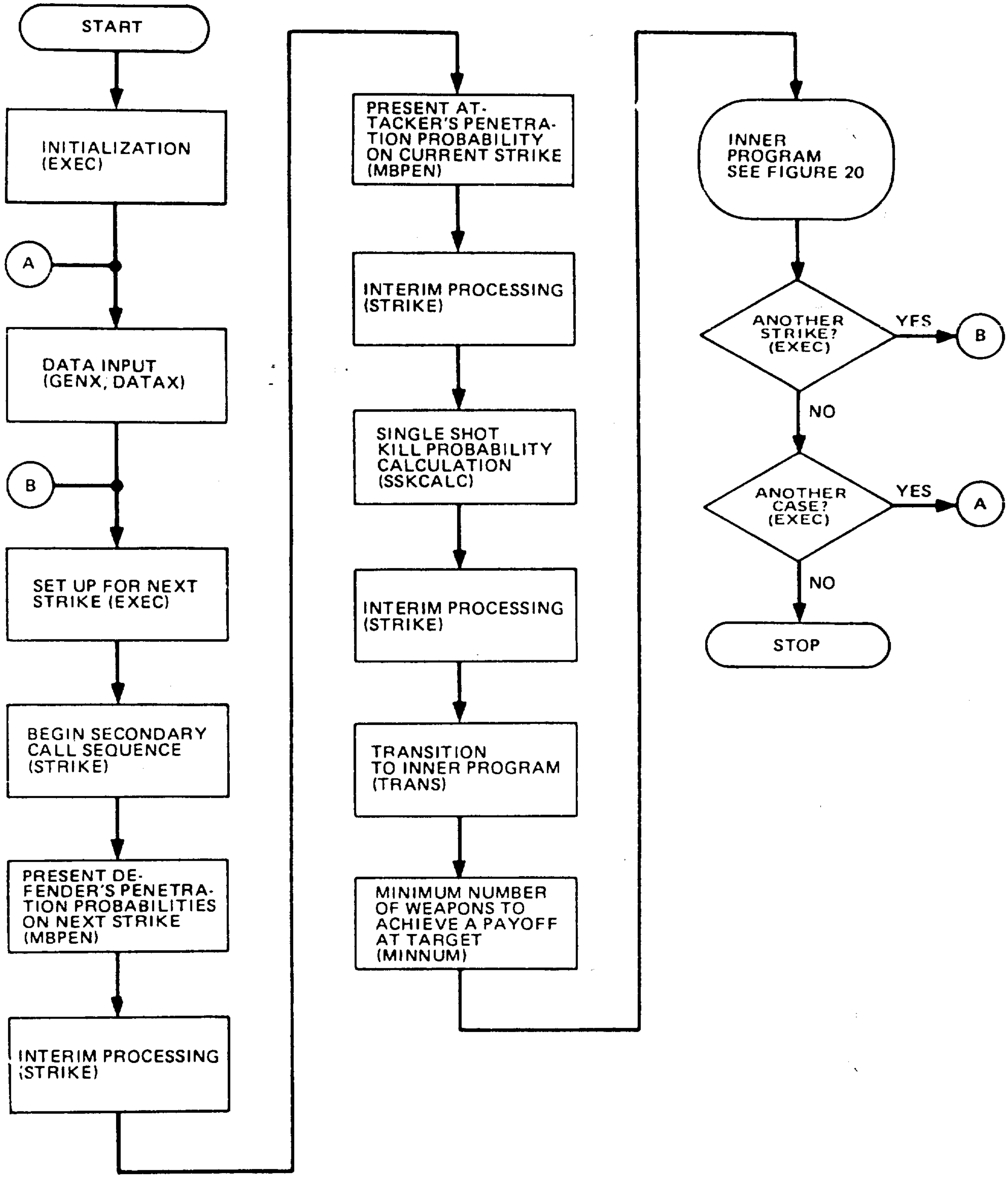

FIG. 19: OUTLINE OF CNA 50 OUTER CALCULATION

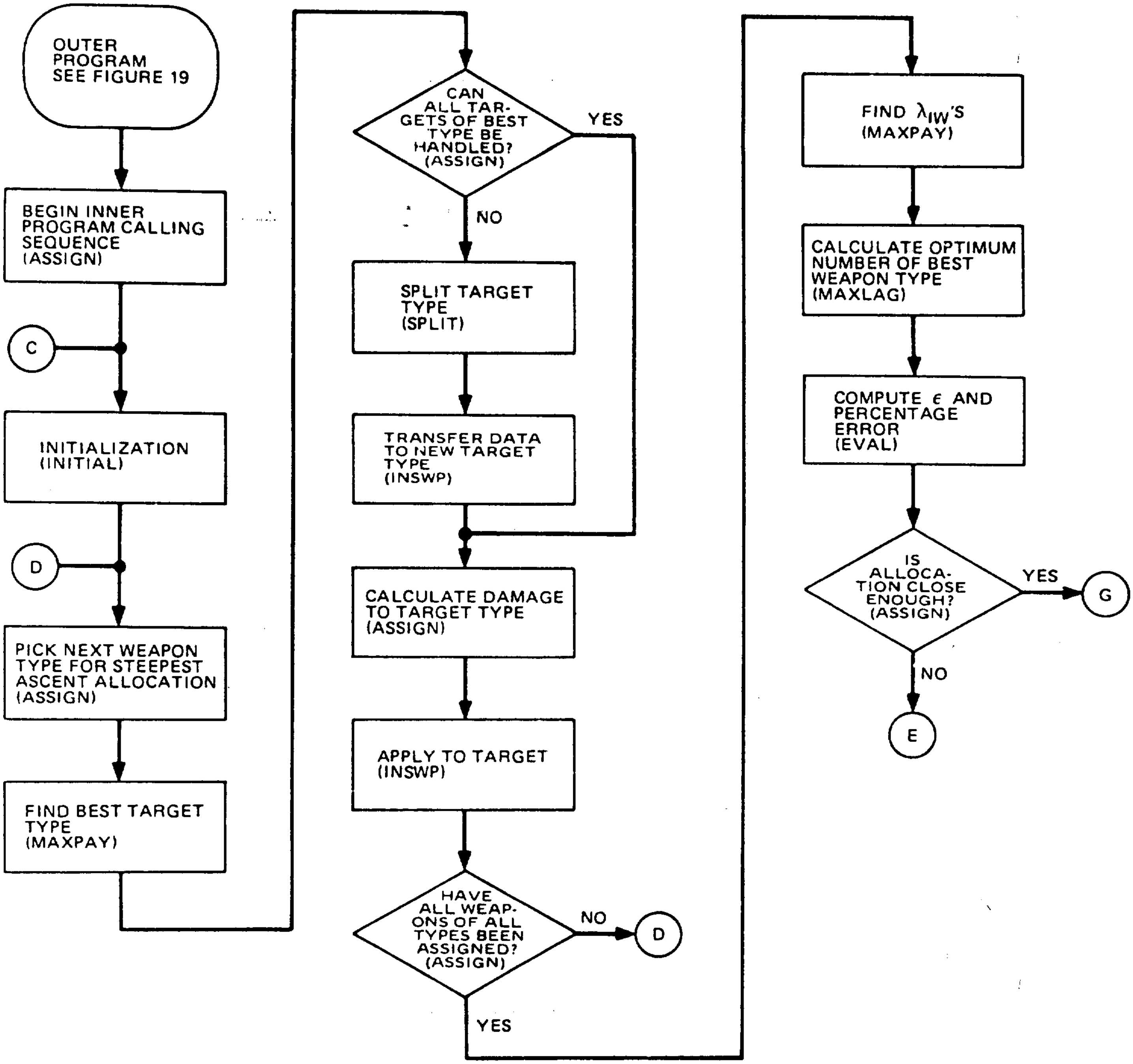

FIG. 20: OUTLINE OF INNER CALCULATION: FIRST ALLOCATION CALCULATION

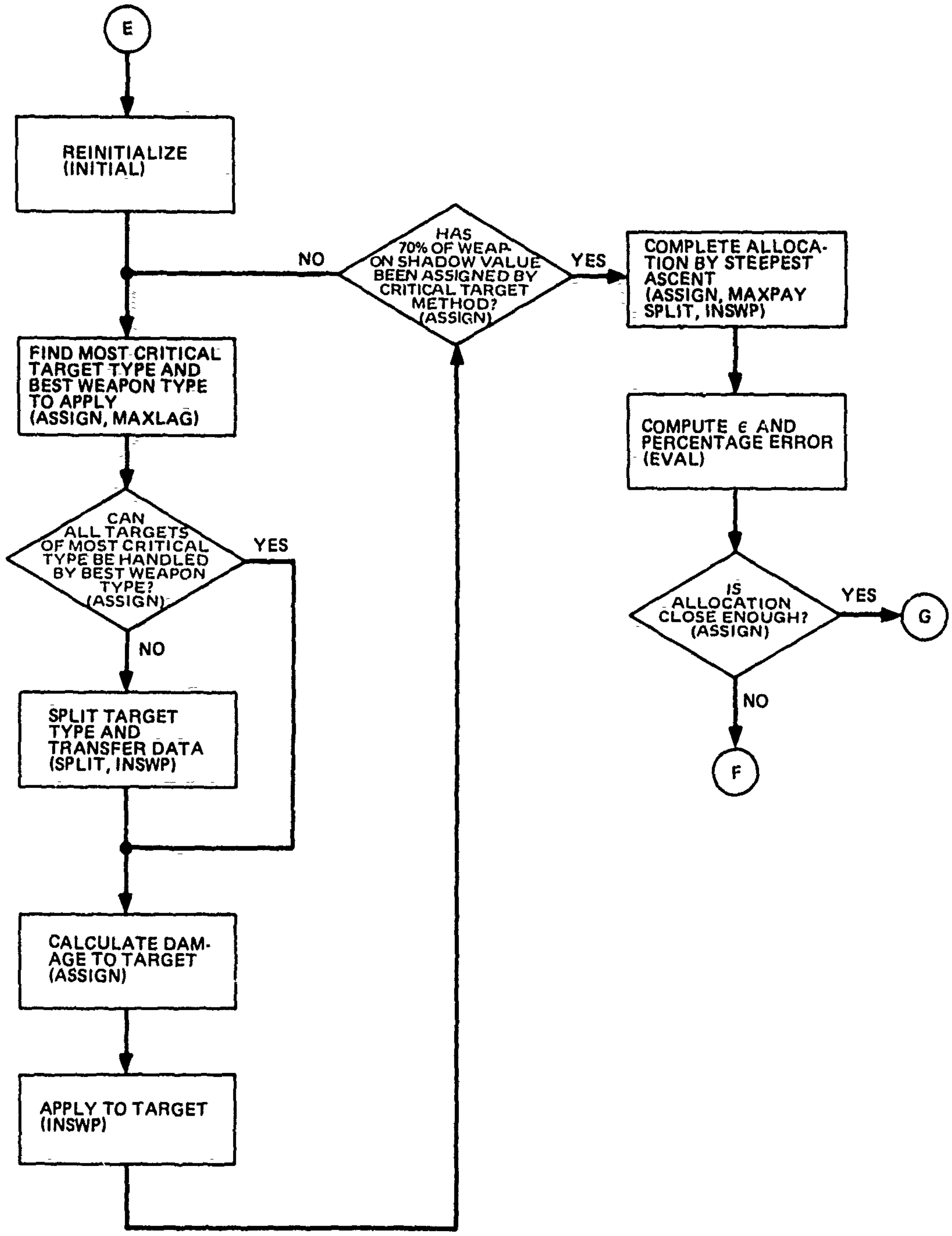

FIG. 21: OUTLINE OF INNER CALCULATION: LATER ALLOCATION CALCULATION

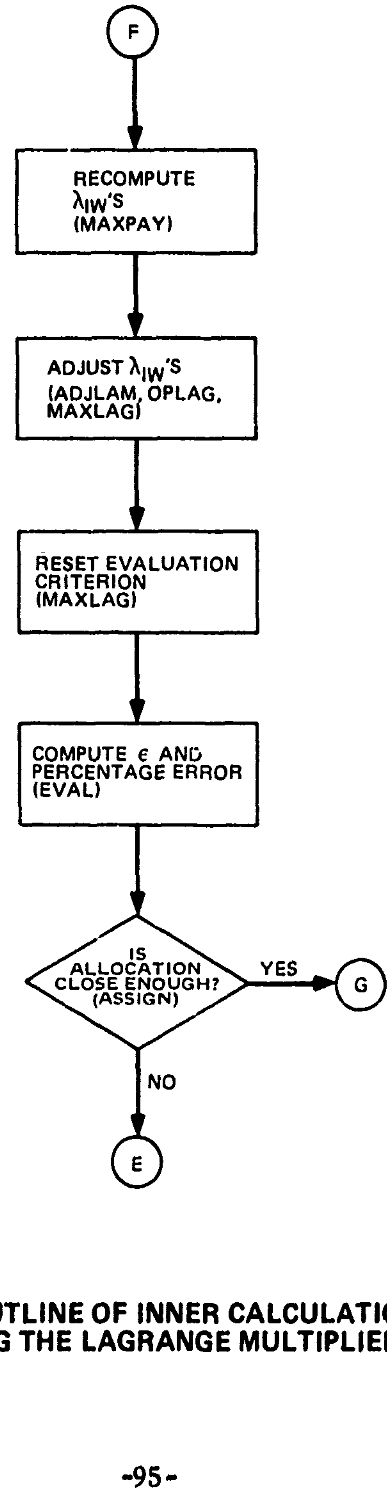

**FIG. 22: OUTLINE OF INNER CALCULATION:
ADJUSTING THE LAGRANGE MULTIPLIERS**

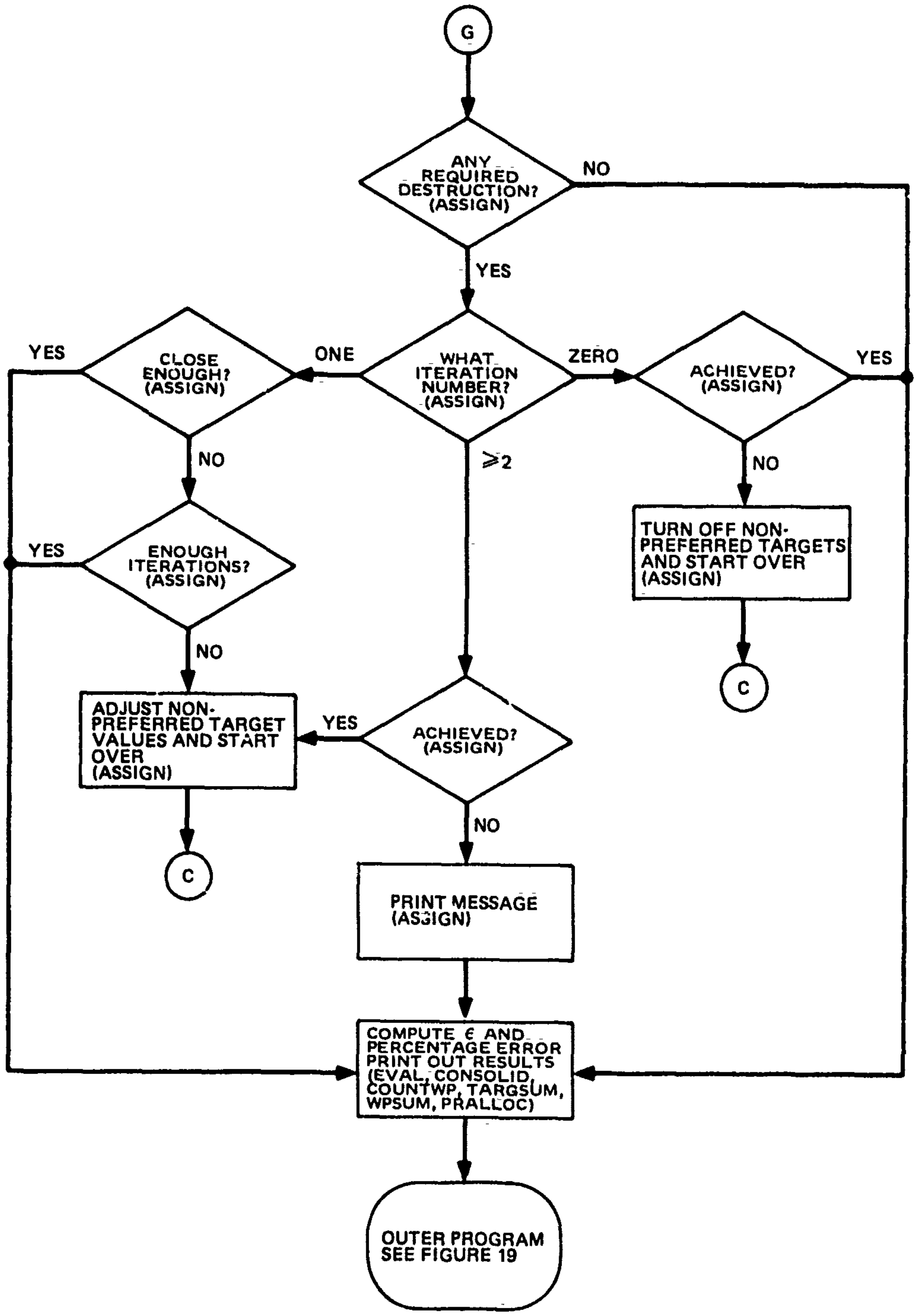

FIG. 23: OUTLINE OF INNER CALCULATION: REQUIRED DESTRUCTION CALCULATIONS

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13. ABSTRACT

The Code 50 Nuclear Exchange Model is a war game model produced by the LAMBDA Corporation. This Research Contribution derives and explains the basic mathematical models used in the computer programs of that model, including models of missile and bomber penetration as well as damage calculation, weapon allocation, and kill probability models. Model implementation and integration into the Code 50 program are also demonstrated.

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