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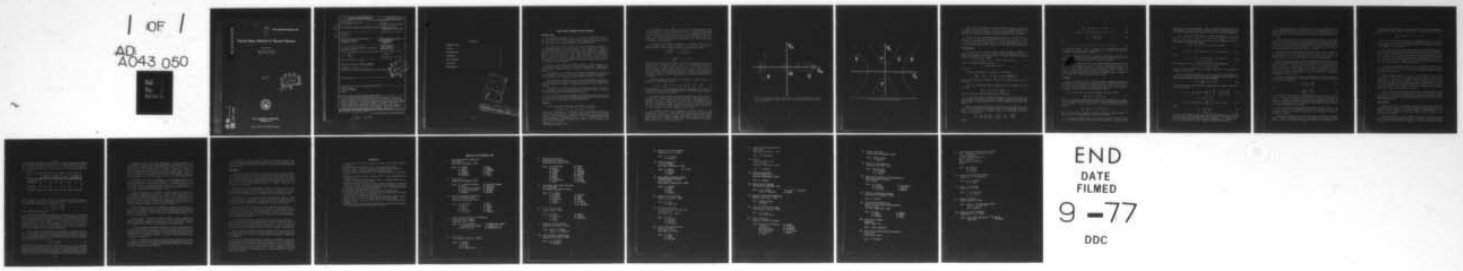

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Velocity-Space Methods for Reactor Plasmas

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Plasma Physics Division

June 1977

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In reactor systems, the reaction rates are calculated from a velocity space integral over the reaction cross-section times the distribution-functions of the reacting species. In a plasma, the shape of the distribution functions themselves is determined by solving a diffusion plus convection problem in velocity space. This review describes briefly the physical processes of central interest to such a description and the mathematical formulation of the problem. It presents the numerical methods which have been used in such calculations by various authors. Optimization on a vector computer (the Texas Instruments ASC) is described. Finally, some indication is given of what may be expected as reactor systems are treated in more detail.

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CONTENTS

Velocity-Space Methods for Reactor Plasmas

INTRODUCTION

The Fokker-Planck Equation is one of a number of specialized tools that can give you a handle on the dynamics of reactor plasmas. Like any tool, it is more useful in some cases than in others. The emphasis here is on providing the potential user with the orientation he needs to determine the usefulness of these methods for him.

This paper presents a brief review of the analytic and numerical development of the Fokker-Planck equation, and an introduction to recent work, much of which is presently unpublished. SI units are used throughout. The presentation is reasonably self-contained, but no attempt is made to present the details of derivations or applications. References are provided for those who wish to examine these details. The succeeding sections are organized by discipline, so that the reader with particular interests may quickly find that part of the material which interests him most. The section on physics also provides motivation for the present development of these methods, and the reactor section indicates the needs for future development.

The physics section emphasizes the difference between the gaseous and plasma state. It explains why velocity-space methods are necessary for plasma reactors. It depicts several practical physical problems, and explains how they are represented in the Fokker-Planck Equation.

The mathematics section shows how the Fokker-Planck Equation is derived, and what assumptions are made. It presents the general form of the equation, and described the initial and boundary values needed. Some analytic methods for solving it are presented.

The numerics section describes several ways to reduce the computational complexity of the general Fokker-Planck Equation. Numerical stability is discussed. Direct and transform methods of solution are presented. The speed gains available on vector computers are illustrated.

The applications section presents methods for adapting the Fokker-Planck Equation to experiments. Mirror and pinch systems are described. Tokamak systems are discussed, and the difficulties peculiar to toroidal systems are explored.

The reactor section describes the differences between the present-day environment and future reactor systems. It presents some of the problems associated with heating by energetic reaction products. It suggests some areas for future development.

PHYSICS

In reactor systems, the nuclear reaction rate depends on an integral

$$R = \int |\mathbf{V} - \mathbf{V}'| \sigma (|\mathbf{V} - \mathbf{V}'|) f_a(\mathbf{V}) f_b(\mathbf{V}') d^3V d^3V',$$

whose main contributions are from opposite ends of the distribution functions f_a, f_b of the reacting species a,b. This is because the thermal velocities of a and b, and thus the main bulk of their distributions, are normally at energies far below the peak of the nuclear reaction cross-section. The distribution functions f_a and f_b are rapidly changing at these energies, and the integral R is very sensitive to such variations. Thus it is necessary to calculate f_a and f_b quite accurately.

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This situation is quite different from that of conventional fluid mechanics. There only the first few moments of the distribution functions (mass, momentum, energy) are needed, and they may be calculated directly. The critical hypothesis from which fluid equations are derived is an assumption about the shape of the distribution function, usually that it is nearly Maxwellian. This, in turn, depends on the presence of some collisional process that drives the distribution function toward this state. In conventional fluids, this assumption is quite reasonable.

In the physics of hot plasmas, this assumption is not reasonable. Collisions become relatively infrequent. In fact, a material is considered to change from its gaseous state to its plasma state when its plasma frequency exceeds its collision frequency, namely,

$$\frac{1}{2\pi}\left[\frac{nq^2}{\epsilon_o m}\right]^{1/2} > \frac{n\,q^4\,ln\,\Lambda}{4\pi\epsilon_o^2\,m^2\,(kT/m)^{3/2}}, \tag{1}$$

or

$$T^3\,n^{-1} > \frac{q^6\,ln\Lambda}{4\epsilon_o^3\,K^3} \simeq 5\times 10^{-10}\ {}^{o}K^3\,m^3,$$

where m, q, n and T are the mass, charge, number density and temperature, respectively, of the material, k is Boltzmann's constant, and ln Λ is a dimensionless variable of order 10 [see Eq. (11)]. One cannot assume that the shape of the distribution function is known in a plasma. Fortunately, one can assume that its evolution is slow, since the collisions are infrequent. When a charged particle finds itself in a plasma, the other charged particles around it "shield" it from electrostatic encounters. An immediate consequence of this is that collisional changes in a particle trajectory take place in many small steps, rather than a few large ones.

These facts suggest that the evolution of the distribution is a diffusion process. The general form of this diffusion equation is

$$m_a\frac{\partial f_a}{\partial t} + q_a\frac{\partial}{\partial \mathbf{V}}\cdot f_a\,(\mathbf{E} + \mathbf{V}\times\mathbf{B}) = \frac{\partial}{\partial \mathbf{V}}\cdot\left[\underline{\mathbf{K}}_a\cdot\frac{\partial f_a}{\partial\,\mathbf{V}} - \mathbf{L}_a f_a\right], \tag{2}$$

where **E** and **B** are the electric and magnetic fields present in the plasma, and K and **L** are the diffusion and drag terms. Several specific properties of this diffusion process distinguish it from more conventional cases, like heat flow.

For example, $\underline{\mathbf{K}}$ and **L** depend on all the species present in the plasma, and may be anisotropic. Two instances of this anisotropy are illustrated schematically in Figure 1, where two hot species of dissimilar mass are injected into a colder background. For a light species (e.g. electrons), colliding with a colder, heavy species (e.g. hydrogen), the dominant process is pitch-angle scattering. Conversely when a heavy species (e.g. uranium) cools due to collisions with a lighter species, the dominant process is loss of energy. Figure 1 shows how these two cases result in different time evolutions for f.

Two examples of another physical phenomenon are depicted in Figure 2. Here velocity-space is divided into 3 regions, labeled E, C and T. Region E contains escaping particles (as in a mirror machine or pinch), C contains confined particles, and T contains trapped particles (as in a tokamak). Boundaries such as these are present in the velocity space of many systems of interest in reactor studies. Such boundaries are also usually in different locations in $V_{||}, V_{\perp}$ space for each species.

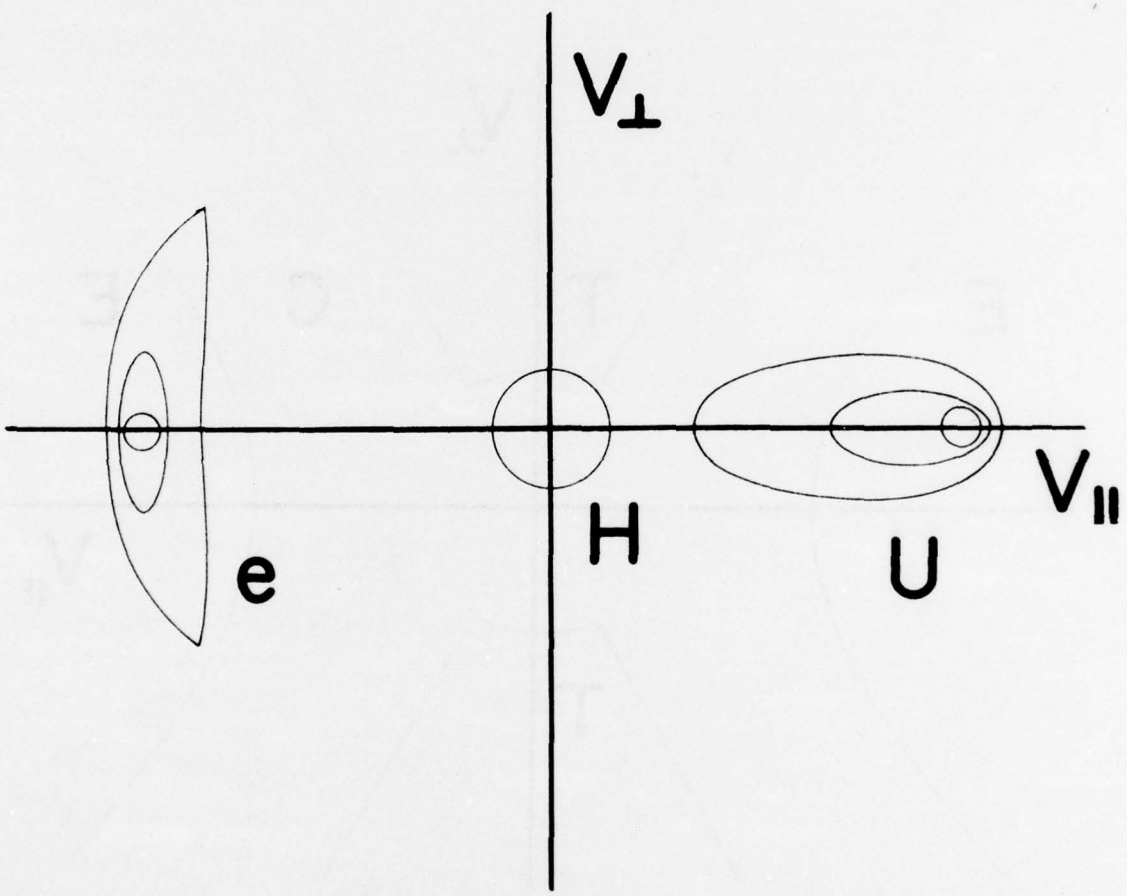

Fig. 1 — Velocity-space diffusion of light (E = electron) and heavy (U = uranium) ions under the influence of collisions with a cold background (H = hydrogen). The concentric contours represent successive stages of evolution of number density contours.

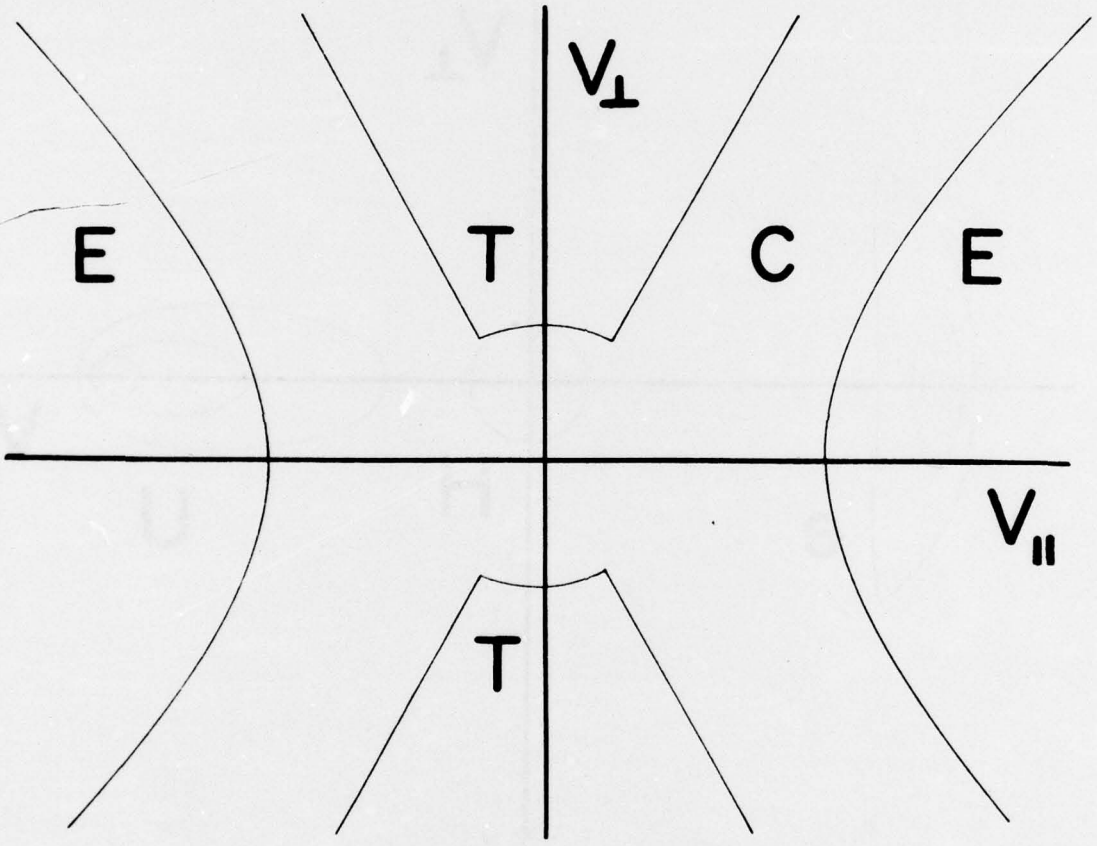

Fig. 2 — Velocity-space regions and boundaries. The figure shows typical escape regions (E) and toroidal trapping (T) regions, with the remainder occupied by confined (C) matter.

Finally, observe that Equation 2 is in conservative form. Energy losses, such as radiation, appear as "convection" terms (in **L**), since they conserve the number of particles. On the other hand, particles may be added or removed by nuclear reactions, or injection or loss processes. Such features are treated by including source and sink terms in the equation.

In brief, this is the physics the Fokker-Planck Equation can be used to study. It concentrates on the velocity-space dynamics of the various species in a plasma. it follows the evolution of the distribution function of each species. It can represent the loss cones of mirror machines and the trapped particles of tokamaks. Injection, losses, and nuclear reactions can be included. How these things are done is described in the following sections.

MATHEMATICS

The evolution of the distribution function under the influence of collisions is a Markov process. That is, it is a random process in which each step depends only on the result of the immediately preceding step. Thus f is a random variable of the coordinates $(\mathbf{V},t)$ which satisfies the evolution equation [1]

$$f(\mathbf{V},t) = \int f(\mathbf{V} - \Delta\mathbf{V}, t - \Delta t)\, p(\mathbf{V} - \Delta\mathbf{V}, t; \Delta\mathbf{V})\, d^3\Delta\mathbf{V} \tag{3}$$

where p is a probability distribution satisfying

$$\int p\,(\mathbf{V},t;\Delta\mathbf{V})\, d^3\Delta\mathbf{V} = 1 \tag{4}$$

for all $(\mathbf{V},t)$.

To make use of Equation 3, we expand it in a Taylor series,

$$f = \int \left\{ fp - \Delta t \frac{\partial f}{\partial t} p - \Delta\mathbf{V} \cdot \frac{\partial fp}{\partial \mathbf{V}} + \frac{1}{2} \Delta\mathbf{V}\,\Delta\mathbf{V} : \frac{\partial^2 fp}{\partial\mathbf{V}\,\partial\mathbf{V}} \right\} d^3\Delta\mathbf{V} \tag{5}$$

where we have taken Δt and ΔV sufficiently small that we can neglect higher order terms in the expansion.

Since f and the differential operators are independent of $\mathbf{V}$, the integrals can be carried out, yielding

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \mathbf{V}} \cdot f\mathbf{Q} + \frac{1}{2} \frac{\partial^2}{\partial\mathbf{V}\,\partial\mathbf{V}} : f\mathbf{P} \tag{6}$$

where Eq. (4) has been used, and P and Q remain to be determined.

In an elegant but tedious calculation, the properties of Coulomb collisions are used to obtain P and Q.[2] It is important to note that this calculation assumes both a minimum and a maximum range for the Coulomb interaction, to obtain the frequently used "Coulomb logarithm" $\ln\Lambda$.[3]

Before writing down general expressions for P and **Q**, it is useful to recall that the systems we are concerned with are multispecies plasmas. It is thus appropriate to rewrite Eq. (6) in a form which includes the collisions of each species with all others, including self-collisions. It can be written

$$\frac{df_a}{dt} = \sum_b \Gamma_{ab} \left\{ \frac{1}{2} \frac{\partial^2}{\partial\mathbf{V}\,\partial\mathbf{V}} : f_a \frac{\partial^2 g_b}{\partial\mathbf{V}\,\partial\mathbf{V}} - \frac{\partial}{\partial\mathbf{V}} \cdot f_a \frac{\partial h_{ab}}{\partial\mathbf{V}} \right\}, \tag{7}$$

where

$$g_b(\mathbf{V}) = \int f_b(\mathbf{V}') \mid \mathbf{V} - \mathbf{V}' \mid d^3\mathbf{V}', \tag{8}$$

$$h_{ab}(\mathbf{V}) = \frac{m_a + m_b}{m_b} \int f_b(\mathbf{V}') / \mid \mathbf{V} - \mathbf{V}' \mid d^3\mathbf{V}', \tag{9}$$

$$\Gamma_{ab} = \frac{q_a^2 \, q_b^2 \, ln \, \Lambda_{ab}}{4\pi \, \epsilon_o^2 \, M_a^2} \tag{10}$$

and

$$ln \, \Lambda_{ab} \simeq 22 - ln \, (n_e^{1/2} \, T_e^{-1}), \qquad \frac{m_e}{m_a} T_a < 10^5 \, {}^oK < T_e \tag{11}$$

The "Rosenbluth potentials" g and h are integrals over the distribution functions which in principle (see next section) need to be calculated once for each species, then summed in various ways to determine the effects of collisions.

The basic Fokker-Planck equation (7) is parabolic. Thus Dirichlet or Neumann conditions on a closed boundary, plus initial conditions within the boundary, constitute a well-posed problem. This is what is normally solved in cases of physical interest.

The integral operators in g and h cause sufficient difficulty in dealing with these equations that analytic solutions of the system (7-10) have been attempted only for special cases. One such case assumes each species is Maxwellian, with boundary conditions $f = 0$ at $v = \infty$. A more realistic model is obtained by expanding the distribution function and integrals in Legendre polynomials [Eqs. (18-20)] and retaining only the $n = 0$ and $n = 1$ terms. This model has yielded important advances in the transport properties of plasmas.[4,5] In less special cases, numerical assistance is usually required.

NUMERICS

There are two basically different numerical algorithms needed to solve the Fokker-Planck Equation. One is the relatively conventional solution of the diffusion equation [Eq. (7)]. The other is the calculation of the collision integrals [Eqs. (8,9)]. A cursory examination of Eqs. (7-9) in finite-difference form suggests that most of the work is involved in calculating the integrals. Let us see how that work can be reduced.[6]

First, it is not necessary to calculate both integrals, since they are related by

$$\frac{\partial}{\partial \mathbf{V}} \cdot \frac{\partial}{\partial \mathbf{V}} g_b = \frac{2m_b}{m_a + m_b} h_{ab} \tag{12}$$

Thus, once g is calculated, h can be obtained by differencing it. Alternatively, if h is calculated, g is the solution of a Poisson equation. In fact, if a method for solving a Poisson equation is readily available, it can also be used to obtain h, since the Laplacian of Eq. (9) can be written

$$\frac{\partial}{\partial \mathbf{V}} \cdot \frac{\partial}{\partial \mathbf{V}} h_{ab} = -4\pi \frac{m_a + m_b}{m_b} f_b \tag{13}$$

Before choosing between computing the integrals and solving the Poisson equations, it is necessary to select the coordinate system to be used in velocity space.

The required coordinate system creates much of the trouble. The spherical symmetry of the Coulomb interaction, and of the most common solutions (Maxwellian), strongly

suggests the use of spherical coordinates. Most of the problems of physical interest have axial symmetry, with the magnetic and electric fields (if any) in the axial direction. Note that most of the complexity of what follows is due to this choice of spherical coordinates.

Let us represent our velocity space in a spherical coordinate system (r,θ,ϕ) with metric $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$. We shall assume that $\frac{\partial f}{\partial \phi} = 0$. Then individual elements of $f(r,\theta)$ represent toroids in velocity space, and the integrals (8,9) can be expressed in terms of complete elliptic integrals

$$g_b(V,\theta) = \int u\, K(w)\, f_b(V',\theta')\, (V')^2 \sin\theta'\, dV'\, d\theta' \tag{14}$$

and

$$h_{ab}(V,\theta) = \frac{m_a + m_b}{m_b} \int E(W)\, f_b(V',\theta')\, (V')^2 \sin\theta'\, dV'\, d\theta', \tag{15}$$

where

$$U = 4V^2 + (V')^2 - 2VV' (\cos\theta \cos\theta' - \sin\theta \sin\theta')^{1/2}, \tag{16}$$

$$W = 16\, V V' \sin\theta \sin\theta' / U \tag{17}$$

If f is represented on an $N_V \times N_\theta$ grid, then the kernel of these integrals, e.g.

$$K_g(V,\theta;\, V',\theta') = UK(W)\, V' \sin\theta',$$

is an $N_V^2 \times N_\theta^2$ table of values for each pair of species, requiring an $N_V \times N_\theta$ dot product for each point and pair of species.

There is an alternative which appears to reduce the work involved in computing these integrals.[2,7] It is based on methods which have been developed for the Poisson equation. Suppose we expand the distribution function in Legendre polynomials,

$$f_b(V,\theta) = \sum_{n=0}^{\infty} \hat{f}_b(V,n)\, P_n(\cos\theta) \tag{18}$$

Then the radial and azimuthal integrals are separable, and the azimuthal integrals have a direct quadrature, giving

$$\hat{g}_b(V,n) = \frac{4\pi}{4n^2-1} \int_0^\infty (VV')^{1/2} \left(\frac{V_<}{V_>}\right)^{n-\frac{1}{2}} \left[1 - \frac{n-\frac{1}{2}}{n+\frac{3}{2}} \left(\frac{V_<}{V_>}\right)^2\right] \tag{19}$$

$$x\, \hat{f}_b(V',n)\, (V')^2\, dV'$$

$$\hat{h}_{ab}(V,n) = \frac{4\pi}{2n+1} \int_0^\infty (VV')^{-\frac{1}{2}} \left(\frac{V_<}{V_>}\right)^{n+\frac{1}{2}} \hat{f}_b(V',n)\, (V')^2\, dV' \tag{20}$$

where

$$V_< = \min(V, V') \tag{21}$$

$$V_> = \max(V, V') \tag{22}$$

In this way, the calculation of g and h has been reduced from complete $N_V \times N_\theta$ integrals to one N_V integral for each azimuthal mode, and the kernels of the integrals are (fractional) polynomials, rather than elliptic functions. Incidentally, the n=0 equations for g and h are the one-dimensional (radial) integrals for the Fokker-Planck Equation.

The Legendre expansion is a great improvement if only a few azimuthal modes are needed to describe the distribution of interest. However, if localized distributions occur, for example when intense beam injectors are present, a number of modes $\hat{N} \approx N_\theta$ may be required. Then the work can be comparable to direct integration (elliptic integrals are easy to calculate).

There is no consensus on this choice. Some workers solve Eqs. (14-17) while others solve Eqs. (19-22). The choice is to some extent dependent on the physical problem being addressed. At the present time, the direct solution appears to have an advantage for computational reasons. It is a very highly ordered operation on a lot of data, and as such its computational cost is reduced up to two orders of magnitude on a vector computer. (See Table I.) In any case, calculating these integrals is only half of the problem.

The other part of the Fokker-Planck quadrature is solving the diffusion equation [Eq. (7)]. In spherical coordinates, expansion of the operations in Eq. (7) is tedious, but straightforward. If Eq. (7) is transformed into the Legendre representation of f, the differential operators in θ are replaced by coupling terms between the various modes.

Since both f and the integrals g and h have azimuthal dependences, the spectrum of the operator on the right-hand side of Eq. (7) is not closed; that is, if f contains modes only up to $\hat{N}$, Equation (7) can yield modes in $\partial f/\partial t$ with $n > \hat{N}$. Some computational method must then be prescribed to truncate the spectrum of the operators. That is, the part of f which is "lost" to higher modes must be returned to the lower modes, in a way which conserves mass, etc. as far as possible. This is another problem with the Legendre representation. Because of these difficulties, the following remarks will be restricted to the difference formulation of Eq. (7).

Equation (7) contains both a convective and a diffusive part, so several numerical stability criteria must be satisfied by any explicit algorithm for solving it. In terms of the Q and P of Eq. (6), they are approximately

$$\delta t < \frac{1}{2} \min \left\{ \frac{\delta V}{Q_V}, \frac{V \delta \theta}{Q_\theta} \right\} \tag{23}$$

and

$$\delta t < \min \left\{ \frac{\delta V^2}{p_{VV}}, \frac{V^2 \delta \theta^2}{p_{\theta\theta}} \right\} \tag{24}$$

plus some obscure and nasty ones on the off-diagonal terms. The usual case is that the last term, $V^2 \delta \theta^2 / p_{\theta\theta}$ is the most restrictive, in the cells near the origin of the coordinate system. As is typical in computational fluid dynamics, the stability considerations for an algorithm are dominated by a region in which very little interesting physics is going on. In most cases of physical interest, the δt required for explicit stability is too small to be practical, and an implicit algorithm must be used.

It is clear from these considerations that the grid should not contain a cell at $V = 0$. The most natural choice of finite-difference cells uses the origin and the axis as cell boundaries. Then the axial boundary conditions take care of themselves: no flux flows through these boundaries.

When the differential operators in Eq. (7) are written out in components, a lot of terms appear.[2] Reduction to difference form requires a lot of algebra, but is otherwise straightforward. The speed of present-day computers makes practical what was prohibitive a few

years ago. Replacing the operator on the right-hand side of Eq. (7) by centered differences results in a 9-point spatial difference operator (5-point if second-order accuracy is not retained)

$$\sum_{\substack{l=-1,0,1 \\ m=-1,0,1}} L(i,j,l,m)\, f(i+l, j+m) = R(i,j) \tag{25}$$

Advancing f in time then requires solving a $(N_v \times N_\theta)^2$ sparse matrix equation. Typical grid resolution is $N_v = 50$, $N_\theta = 20$, so in principle a 1000x1000 matrix inversion is involved. In fact, methods are now available for solving such banded matrix equations by rapidly convergent iterations,[8], or cyclic reduction.[9]

Table I illustrates the gains available with such methods. It presents speeds obtained on the NRL (2-pipe) Texas Instruments "Advanced Scientific Computer." The two most expensive elements of the Fokker-Planck solution are the integral of Eq. (8) and the solution of the matrix equation [Eq. (25)]. Table I presents the scalar time, vector time, the ratio of these times, and the vector speed in millions of floating-point operations per second (MFLOPS) for these tasks. To put this speed in perspective, performing the integrals of Eq. (8) for each point of a 20x50 grid requires 2 million operations, and takes 40 milliseconds, about the time it takes to write one word on a disk (disk rotation time).

Initial and (exterior) boundary conditions are usually fairly simple. The initial conditions usually rapidly disappear behind the dynamics of the diffusion. There are no wave-like solutions for them to excite. They are *typically Maxwellian distributions, possibly* with some delta-function beams.

Boundary conditions require more care. At large velocities, collisions become less frequent, so particles which get there, can stay there. In particular, if an electric field is present, there is a velocity (for each species) above which a particle "runs away." This means that the final state of such a calculation is one in which all particles have run away; but for practical problems, the time it takes this to happen is long compared to experimental times. Specific values may be prescribed at the (exterior) boundary, or it may be taken impermeable. Whatever choice is made, the parameters of the calculation can usually be chosen so the results of interest are insensitive to them.

Finally, it should be mentioned that there are many "collisional" processes that are *not simple Coulomb processes.* For example, the particles may be scattered by electrostatic plasma waves. Such coherent processes are not described by the Coulomb integrals [Eqs. (8,9)], but the diffusion and drag resulting from them can readily be included in the Fokker-Planck formalism.

APPLICATIONS

Slight modifications of the Fokker-Planck Equation are often needed to apply it to specific experimental configurations. This section indicates what changes are required to study typical mirror, pinch, tokamak, or laser-plasma problems. These applications are illustrative of the adaptability of velocity-space methods, and show how details of an experiment can be represented.

The dominant feature of a mirror machine is the presence of loss cones in velocity space.[6] This means that particles in a region such as E of Figure 1 leave the machine in a very short time. Thus the boundary condition $f = 0$ should be applied on the surface

Table I

Table I presents measured performance data for the most expensive computational elements of the Fokker-Planck calculation. Equation (8) is readily vectorizable, while Eq. (25), which involves recursive calculations, is not. The "scalar" and "vector" speeds were measured on a 2-pipe Texas Instruments ASC, and are expressed in terms of "millions of floating-point operations per second."

	SCALAR SEC/FLOP	VECTOR SEC/FLOP	RATIO	VECTOR MFLOP/SEC
INTEGRATE Eq. (8)	5.76	0.0206	280	49
INVERT Eq. (25)	2.09	0.112	19	9

between regions C and E. What is unlike other fluid problems is that the boundary surface C-E is different for each q/m. For example, a "square well" magnetic mirror with mirror ratio B_{max}/B_{min} and maximum potential Φ has loss cones

$$\sin^2 \theta = \frac{B_{max}}{B_{min}} \left(1 + \frac{2q\Phi}{mV^2} \right) \tag{26}$$

which are different for each particle q/m.

The methods used to solve these boundary value problems are those of conventional fluid mechanics, there are just more boundaries than usual. The boundaries may be treated properly, solving Eq. (7) [that is, Eq. (25)] subject to them. A quick and dirty alternative is to solve the problem with boundary conditions at $V = \infty$ (or some V_{max}), and then set $f = 0$ in region E at each time step. For a diffusive problem, this will not result in an instability, just a first-order error in δt; but if an implicit method is in use which is first order in δt, such terms are already present. Typical computational results include the loss-rate of ions, and the potential Φ necessary to obtain charge neutrality.

A pinch experiment is characterized by a rapid change of the magnetic field B in time.[10] The appropriate Fokker-Planck Equation is obtained by replacing the left-hand side of Eq. (7) by the left-hand side of Eq. (2). The change with time of B produces a convection of f in the perpendicular direction. (Recall that B is aligned along the axis of our coordinate system.) A fluid element convects at a rate

$$\frac{dV_\perp}{dt} = \frac{1}{2} \frac{V_\perp}{B} \frac{dB}{dt} \tag{27}$$

The factor of 1/2 is present because there are two components of V in the perpendicular direction. The physical process involved is constancy of the magnetic moment. In typical pinch experiments, this compression heats the ions, The Fokker-Planck Equation is used to obtain the resulting shape of f. This determines the hot ion lifetimes and their rate of isotropization; i.e., the rate at which the ions acquire parallel velocity and are lost from the pinch.

Tokamaks have four features which are important to a velocity-space model. First, an electric field is present, parallel to the magnetic field; the plasma carries a net current. Second, there are magnetic mirrors within each magnetic surface, and they can trap some particles. Third, there is a significant variation of plasma conditions with spatial position; velocity-space modelling of a tokamak is viable because individual magnetic surfaces are fairly well isolated from each other, and conditions on each magnetic surface are fairly uniform. Fourth, there is a relatively high level of impurities in most tokamaks. Various workers have examined one or more of these features. They have not yet all been treated together.

The electric field results in distortion of the distribution function, and rapid convection along the axis. This can produce difficulties for the algorithm which solves Eq. (25), particularly around the origin. Physical problems of interest are the distortion of f and the resultant resistivity: the ratio of the current it carries to E.[11] The calculation of the resistivity has been explored thoroughly when no toroidal effects are present. Present work thus involves inclusion of some of the other effects as well.

Trapped particle effects can be included by treating separately the trapped and untrapped particles. (Regions T and C respectively of Figure 1) The particles in C satisfy a conventional Fokker-Planck Equation, while those in T satisfy a Fokker-Planck equation with no electric field, since they are not free to accelerate in that field, and the energy gain in one bounce is negligible compared to their average energy. This can be most simply modeled by making E a function of $V_{\parallel}$ which is the tokamak value in C, and zero in T. Calculations with this model yield the toroidal corrections to the resistivity.[12]

The magnetic surfaces are really not independent, since ion orbits (e.g., banana orbits) extend over different surfaces. Convection and loss mechanisms also couple surfaces. Solutions of a coupled radial transport equation plus a Fokker-Planck equation on several surfaces have been attempted,[13] however a large number of approximations are required. This is still a very difficult problem.

Finally, impurities introduce two important effects. First, they greatly increase pitch-angle scattering of trapped particles, beams, etc. Second, if they have high atomic numbers, they radiate, resulting in an electron energy loss. The first effect is important when studying beam injection. The second is important in overall energy balance.

As a last application, the Fokker-Planck Equation has several interesting areas of the laser-plasma interaction to investigate. In the absorption region, where the laser electric field is largest, the electron distribution function is swept back and forth in velocity space. From the point of view of the ions, the electron distribution function acquires an elliptical distortion. The Fokker-Planck Equation can be used to quantitatively calculate the heating rate due to this process.

Both in the tokamak and laser plasma environments, beams are formed in one region and transported to (injected into) another. The Fokker-Planck Equation can calculate the interaction between these beams and the background through which they are propagating. The beams are simply sources in the equation.

These applications indicate both the versatility of the Fokker-Planck equation, and the amount of effort which is required to solve such problems. The computational cost of these solutions is rapidly decreasing, and more examples of such applications are appearing. Will yours be the next?

REACTORS

It may appear from the previous section that nuclear reactions have been neglected in this velocity-space work. That is not so. The applications section showed how velocity-space methods are adaptable to the physical characteristics of various devices. In most of the experiments described above, reactor studies have been carried out with the resulting model. The nuclear reaction calculation is usually the most conventional part of such calculations. It uses the calculated distribution functions to compute reaction rates from the appropriate cross-sections.

Once the heating and loss processes of the experiment are modeled, it is easy to inject the reacting species, and calculate their dynamics. One part of that dynamics is the nuclear reactions they undergo. The reaction integral determines two things; the rate of production of nuclear energy, and the reaction constituent sources and sinks in velocity space.

At this point, you may take your choice and pay your money. There is no argument about the neutrons. They give their energy to the external environment. The fate of the reaction products is something else. The inexpensive choice is to deposit the energy of the charged reaction products where it will probably go first; namely, into the electrons.

The expensive choice is to follow the velocity-space behavior of these products. It is expensive for two reasons. First, the velocities with which the reaction products are born is very large compared to the original plasma velocities. Thus the required velocity-coordinate space becomes larger, and finer gridding is required. Second, collision integrals are required for each pair of species, so much more work is needed for each additional reaction product which is treated.

But it is the expensive choice that will be needed in the coming years. In practice, all the proposed plasma reactors have loss regions, and it is important to know how much energy is returned to the reactor plasma before products are either lost, or slow down. This will also allow a quantitative assessment of nuclear reactions in which the participants themselves participate. Such calculations are now on the drawing-boards, waiting their turn.

This is presently a wide-open area of research, and even a factor of two improvement in yield for some experimental design would be important news. Designs such as the "two-component torus" are presently on the edge of scientific breakeven, and looking for any available help.

This survey has indicated what kinds of problems can be addressed by velocity-space methods, and how they are modeled. The basic conclusion is that all of the major plasma reactor designs are amenable to such treatment, and each has had some important aspect analyzed in this way. It is also clear that much remains to be done. The cost of velocity-space computations has been large in the past, but is now coming down. Scientists who have an acquaintance with distribution-function methods should find a friend in the Fokker-Planck Equation.

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