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ORTHOGONAL POLYNOMIALS

Paul G. Nevai

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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MATHEMATICS RESEARCH CENTER

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ABSTRACT

The purpose of the present paper is to improve some results of R. Askey, P. Erdös, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegö and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem will (positively) be answered: is there any weight w with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to w diverge in L_w^p for some continuous function f ? Most of the paper deals with Christoffel functions and their applications. Many limit relations for orthogonal polynomials are found in the assumption that the coefficients in the recursion formula behave nicely.

AMS(MOS) Subject Classification - 42A52, 33A65, 41A05, 41A10, 41A20, 41A25, 41A35, 41A55, 41A60, 42A04, 42A56, 42A62, 40A25, 26A75, 26A78, 26A82, 26A84, 30A06, 30A84, 30A86, 65D30, 65F15, 65F35.

Key Words - Orthogonal polynomials, Quadrature processes, Fourier series, Interpolation, Positive operators, Toeplitz matrices, Christoffel functions.

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This work is dedicated

to

Richard Askey

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ORTHOGONAL POLYNOMIALS

Paul G. Nevai

The purpose of the present paper is to improve some results of R. Askey, P. Erdős, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegö and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, Turan's problem [1] will be answered: is there any weight w with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to w diverge in L_w^p for some continuous function f ? R. Askey [1] conjectured that the answer was yes and the solution was given by the Pollaczek weight because the logarithm of the Pollaczek weight is not integrable. We shall show that Askey's conjecture is right but for different reasons. In fact, there are many weights solving Turan's problem; some of them do have integrable logarithm, some of them do not.

Most of this paper deals with investigation of Christoffel functions and its generalization. The results and the methods are stronger than those of the above authors. The Christoffel functions play a very important role in the theory of orthogonal polynomials. Many results in orthogonal Fourier series and interpolation are based on estimates and asymptotics of Christoffel functions. We shall show how successfully Christoffel functions can be applied in finding necessary conditions for weighted mean convergence of orthogonal Fourier series and Lagrange interpolation processes. Introducing generalized Christoffel functions we shall find a connection between different weighted L^p norms of polynomials. Especially interesting is the case when $0 < p < 1$. We shall investigate a new kind of quadrature process which helps to find asymptotics for Christoffel functions and orthogonal polynomials outside the support of the weight function. Taking the recursion formula as a starting point and assuming properties on the coefficients in the recursion formula we shall obtain results on orthogonal polynomals. In certain cases we shall be able to calculate the weight function using the coefficients in the recursion formula. We shall also find the (C, 1) limit of Turan type determinants under rather weak *conditions*.

I learned the theory of orthogonal polynomials from G. Freud who has been supervising me for several years. Many of the methods I use in this paper can be found in his book on orthogonal polynomials which is a rich source of methods and unsolved research problems. I wish to express my deep feeling of gratitude to G. Freud as well as to R. Askey, L. Bers, M. Cwikel, J. Landin, G. G. Lorentz and W. Proxmire without the help of which this paper would never have been written. I am grateful to the American Mathematical Society, to the National Science Foundation and to the United States Army for sponsoring my research.

This whole work was born from the attempts to solve problems mentioned in R. Askey's paper [1]. I discussed my results with R. Askey several times. He read a draft version of the manuscript and made various

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suggestions to improve the presentation. I dedicate this work to Richard Askey because his generous support and help made it possible for me to carry out the research which led to this paper.

2. Notations

The function $\alpha: \mathbb{R} \to \mathbb{R}$ is called a weight function if it is *nondecreasing*, it has infinitely many points of increase and all the moments

$$\int_{-\infty}^{\infty} x^{2n} d\alpha(x) \qquad (n = 0, 1, \ldots)$$

are finite. For a given weight α the corresponding system of orthogonal polynomials $\{p_n(d\alpha)\}_{n=0}^{\infty}$ is defined by $p_n(d\alpha, x) = \gamma_n(d\alpha) x^n + \ldots,\ \gamma_n(d\alpha) > 0$ and

$$\int_{-\infty}^{\infty} p_n(d\alpha, x)\, p_m(d\alpha, x)\, d\alpha(x) = \delta_{nm} .$$

If α happens to be absolutely continuous then we shall usually write w and $p_n(w, x)$ instead of α' and $p_n(d\alpha, x)$ respectively. In the general case α can be written in the form

$$\alpha = \alpha_{ac} + \alpha_s + \alpha_j$$

where α_{ac} is absolutely continuous, α_s is singular and α_j is a jump function.

One of the basic properties of a system of orthogonal polynomials $\{p_n(d\alpha)\}$ is that the polynomials $p_n(d\alpha)$ satisfy the three term recurrence relation

$$xp_n(d\alpha, x) = \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)}\, p_{n+1}(d\alpha, x) +$$

$$+ \alpha_n(d\alpha)\, p_n(d\alpha, x) + \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_{n-1}(d\alpha, x)$$

$(n = 0, 1, \ldots)$ *where* $p_{-1} \equiv 0$ and

$$\alpha_n(d\alpha) = \int_{-\infty}^{\infty} t\, p_n^2(d\alpha, t)\, d\alpha(t) .$$

This recurrence relation will be one of our main points of interest. By a famous result of J. Favard if a system of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfies the recurrence formula

$$x\, p_n(x) = \frac{\gamma_n}{\gamma_{n+1}} p_{n+1}(x) + \alpha_n p_n(x) + \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x)$$

for $n = 0, 1, \ldots$ with $p_{-1} \equiv 0$, $p_0 \equiv \gamma_0$, $\gamma_n > 0$ and $\alpha_n \in \mathbb{R}$ then $\{p_n(x)\}$ is orthogonal with respect to *some weight* α *which may not uniquely be* determined (see Freud, §II.1)*. In this paper we are going to deal with such cases when both $\{|\alpha_n|\}$ and $\{\gamma_{n-1}/\gamma_n\}$ are bounded and then α is uniquely defined.

The zeros of $p_n(d\alpha)$, which are real and distinct, will be denoted by $x_{kn}(d\alpha)$: $x_{1n}(d\alpha) > x_{2n}(d\alpha) > \ldots > x_{nn}(d\alpha)$. The Christoffel function $\lambda_n(d\alpha)$ corresponding to α is defined by

$$\lambda_n(d\alpha, z) = \min_{\substack{\pi \in \mathbb{P}_{n-1} \\ \pi(t)=1}} \int_{-\infty}^{\infty} |\pi(t)|^2\, d\alpha(t)$$

for $z \in C$, $n = 1, 2, \ldots$ where $\mathbb{P}_n$ is the set of polynomials of degree at most n. It is rather easy to see that

$$\lambda_n(d\alpha, z)^{-1} = \sum_{k=0}^{n-1} |p_k(d\alpha, z)|^2$$

The numbers $\lambda_n(d\alpha, x_{kn}(d\alpha))$ are called Christoffel numbers and are usually

* In the following books listed in the references will be referred by mentioning the name of their authors.

denoted by $\lambda_{kn}(d\alpha)$. There are two important results involving Christoffel numbers which will often be used. The first of them is the Gauss-Jacobi mechanical quadrature formula:

$$\int_{-\infty}^{\infty} \pi(t)\, d\alpha(t) = \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \pi(x_{kn}(d\alpha))$$

for each $\pi \in \mathbb{P}_{2n-1}$, the second one is the Markov-Stieltjes inequalities which can be expressed as

$$\sum_{k=i+1}^{n} \lambda_{kn}(d\alpha) \le \int_{-\infty}^{x_{in}(d\alpha)} d\alpha(t) \le \sum_{k=i}^{n} \lambda_{kn}(d\alpha)$$

$(i = 1, 2, \ldots, n)$. The support of α, that is $\text{supp}(d\alpha)$, is the set of points of increase of α. If $\text{supp}(d\alpha)$ is bounded $\Delta(d\alpha)$ will denote the *smallest closed interval containing* $\text{supp}(d\alpha)$. The symbols Δ and τ will always mean closed interval, the interior part of Δ is denoted by Δ^0. For a given τ the Tschebyshev weight corresponding to τ will be written as v_τ. If $\tau = [-1, 1]$ then we write v instead of v_τ. If $\tau = [a-b, a+b]$ then

$$v_\tau(x) = [b^2 - (x-a)^2]^{-\frac{1}{2}} .$$

For $f \in L^1_{d\alpha}$ (supp($d\alpha$) is bounded) $S_n(d\alpha, f)$ denotes the n-th partial sum of the orthogonal Fourier series of f. Hence for $x \in \mathbb{R}$

$$S_n(d\alpha, f, x) = \int_{-\infty}^{\infty} f(t)\, K_n(d\alpha, x, t)\, d\alpha(t)$$

where

$$K_n(d\alpha, x, t) = \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_k(d\alpha, t)$$

or by the Christoffel-Darboux formula

$$K_n(d\alpha, x, t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha, t)\, p_n(d\alpha, x) - p_{n-1}(d\alpha, x)\, p_n(d\alpha, t)}{x-t} .$$

For a given function f the Lagrange interpolation polynomial $L_n(d\alpha, f)$ corresponding to α is defined to be the unique polynomial of degree at most $n-1$ which agrees with f at the nodes $x_{kn}(d\alpha)$ $(k = 1, 2, \ldots, n)$. If we denote by $\ell_{kn}(d\alpha)$ the fundamental polynomials of Lagrange interpolation then $L_n(d\alpha, f)$ can be written as

$$L_n(d\alpha, f, x) = \sum_{k=1}^{n} f(x_{kn}(d\alpha))\, \ell_{kn}(d\alpha, x) .$$

It will be useful to remember that

$$\ell_{kn}(d\alpha, x) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha)\, p_{n-1}(d\alpha, x_{kn}) \frac{p_n(d\alpha, x)}{x - x_{kn}}$$

$(x_{kn} \equiv x_{kn}(d\alpha))$.

The Tschebyshev polynomials $\cos n\theta$ $(x = \cos\theta)$ will always be denoted by $T_n(x)$. For a given set $\mathbb{M}$ the characteristic function of $\mathbb{M}$ is $1_{\mathbb{M}}$ and $\mathbb{M}(\varepsilon)$ $(\varepsilon > 0)$ means the ε-neighborhood of $\mathbb{M}$. π_n and P_n denote polynomials belonging to $\mathbb{P}_n$. The letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural integers, real numbers and complex numbers respectively. $\mathbb{R}^+$ is the set of positive real numbers.

For $0 < p < \infty$ $\| \ \|_{d\alpha, p}$ is defined by

$$\|f\|^p_{d\alpha, p} = \int_{-\infty}^{\infty} |f(t)|^p \, d\alpha(t) .$$

(Of course, for $0 < p < 1$ this is not a norm.)

Sometimes we shall omit unnecessary parameters in the formulas. (E.g. $x_k \equiv x_{kn}(d\alpha)$).

We assume that the reader is familiar with methods in the theory of one-sided approximation and positive operators.

For the convenience of the reader we give an index where to find the definition of symbols used frequently. D. 3.1.4 below means that see Definition 4 in Chapter 3.1.

a_n	- D. 7.6
A^ω_x, A^ω_τ, B^ω_x, B^ω_τ	- D. 6.2.37
α_g	- D. 6.1.3
α_τ, β_τ	- D. 6.2.42
$\alpha_{nk}(d\alpha)$	- D. 4.2.11
$C^{a,b}_k(d\alpha)$	- D. 3.1.4
$D(d\alpha, z)$	- D. 6.1.16
δ_t	- D. 7.14
GJ	- D. 9.28
$\Gamma(\theta)$	- D. 4.2.4
JS	- D. 10.17
$\lambda_n(d\alpha, p, x)$	- D. 6.3.1
$\lambda^*_n(d\alpha, x)$	- Formula 4.1(5)

$M(a,b)$	- D.3.1.6
Pollaczek weight	- D.6.2.12
$\rho(z)$	- D.4.1.8
S	- D.4.2.1
$u \equiv u^{(a,b)}$	- D.6.2.7
u_n	- D.6.3.4
w_n	- D.9.28
~	- D.6.3.3

3. Basic Facts

3.1. The Generalized Recurrence Formula

Let $U_n(x)$ denote the Tschebyshev polynomial of second kind, that is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta$$

$(0 \le \theta \le \pi,\ -1 \le x \le 1)$. The polynomials $U_n(x)$ satisfy the recurrence formula

(1) $$2x\, U_{n-1}(x) = U_n(x) + U_{n-2}(x), \qquad n = 1, 2, \ldots$$

where $U_{-1}(x) \equiv 0$ and $U_0(x) \equiv 1$.

Theorem 1. Let $0 \le k \le n$. For an arbitrary weight α $p_n(d\alpha, x)$ can be expressed as

(2) $$p_n(d\alpha, x) = U_{n-k}(x)\, p_k(d\alpha, x) - U_{n-k-1}(x)\, p_{k-1}(d\alpha, x) + R_{n,k}(d\alpha, x)$$

where

(3) $$R_{n,k}(d\alpha, x) = \sum_{j=k+1}^{n} U_{n-j}(x) \Big\{\Big[1 - 2\frac{\gamma_{j-1}(d\alpha)}{\gamma_{j-1}(d\alpha)}\Big] p_j(d\alpha, x) - 2\alpha_{j-1}(d\alpha)\, p_{j-1}(d\alpha, x) + \Big[1 - 2\frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}\Big] p_{j-2}(d\alpha, x)\Big\} .$$

Proof. We shall prove (2) by induction. If $k = n$ then (2) and (3) give $p_n(d\alpha, x) = p_n(d\alpha, x)$. If $n \ge 1$ and $k + 1 = n$ then (2) and (3) coincide with the recurrence formula. Now fix n and let $n - 1 > k \ge 0$. Suppose that (2) and (3) hold if we replace these k by $k+1$, that is

$$p_n = U_{n-k-1} p_{k+1} - U_{n-k-2} p_k + R_{n,k+1} .$$

Applying (2) and (3) to the case $n = k+1 \geq 1$ we obtain

$$p_{k+1} = U_1 p_k - U_0 p_{k-1} + R_{k+1,k} .$$

Thus by (i)

$$p_n = (U_1 U_{n-k-1} - U_0 U_{n-k-2}) p_k - U_{n-k-1} p_{k-1} +$$

$$+ R_{n,k+1} + U_{n-k-1} R_{k+1,k} =$$

$$= U_{n-k} p_k - U_{n-k-1} p_{k-1} + R_{n,k} ,$$

that is (2) and (3) hold also for k.

Remark 2. Putting $k = 0$ and $d\alpha$ = Tschebyshev weight we obtain from Theorem 1

$$T_n(x) = x U_{n-1}(x) - U_{n-2}(x) = T_1(x) U_{n-1}(x) - T_0(x) U_{n-2}(x) .$$

For $k = 1$ (2) and (3) give the same as for $k = 0$. When $2 \leq k \leq n$ we get

$$T_n(x) = T_k(x) U_{n-k}(x) - T_{k-1}(x) U_{n-k-1}(x) .$$

This formula may easily be checked directly. In fact, the above formula suggested (2) and (3).

Theorem 3. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \leq k \leq n$. Then

(4) $$p_n(d\alpha, x) = U_{n-k}\left(\frac{x-a}{b}\right) p_k(d\alpha, x) -$$

$$- U_{n-k-1}\left(\frac{x-a}{b}\right) p_{k-1}(d\alpha, x) + R^{a,b}_{n,k}(d\alpha, x)$$

where

(5) $$R^{a,b}_{n,k}(d\alpha, x) = \sum_{j=k+1}^{n} U_{n-j}\left(\frac{x-a}{b}\right) \cdot$$

$$\{[1 - \frac{2}{b} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}] p_j(d\alpha, x) + \frac{2}{b}[a - \alpha_{j-1}(d\alpha)] p_{j-1}(d\alpha, x) +$$

$$+ [1 - \frac{2}{b} \frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}] p_{j-2}(d\alpha, x)\} .$$

Proof. Let α^* be defined by $\alpha^*(t) = \alpha(bt + a)$. Then $p_n(d\alpha, x) = p_n(d\alpha^*, \frac{x-a}{b})$, $\alpha_n(d\alpha^*) = \frac{1}{b}\{\alpha_n(d\alpha) - a\}$ and $\gamma_{n-1}(d\alpha^*)/\gamma_n(d\alpha^*) = \frac{1}{b}[\gamma_{n-1}(d\alpha)/\gamma_n(d\alpha)]$. Apply now Theorem 1 to α^* and then return to α.

We shall call (4) and (5) the generalized recurrence formula. It will help us to prove many properties of orthogonal polynomials in case the coefficients of the recurrence formula are convergent.

Definition 4. Let $a \in \mathbb{R}$, $b \geq 0$. Then

$$C^{a,b}_k(d\alpha) = |\alpha_k(d\alpha) - a| + \left|\frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} - \frac{b}{2}\right| + \left|\frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)} - \frac{b}{2}\right| .$$

Corollary 5. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \leq k \leq n$. Then

(6) $$p_n(d\alpha, x) = U_{n-k}\left(\frac{x-a}{b}\right) p_k(d\alpha, x) - U_{n-k-1}\left(\frac{x-a}{b}\right) p_{k-1}(d\alpha, x) +$$

$$+ O(1) \frac{1}{\sqrt{b^2 - (x-a)^2}} \sum_{j=k-1}^{n} C^{a,b}_j(d\alpha) |p_j(d\alpha, x)|$$

for $x \in (a-b, a+b)$ where $|O(1)| \leq 2$.

Definition 6. Let $a \in \mathbb{R}$, $b \geq 0$. Then $\alpha \in M(a,b)$ if

$$\lim_{k\to\infty} C_k^{a,b}(d\alpha) = 0 .$$

Remark 7. When considering $M(a,b)$ we can always assume without loss of generality that either $a = 0$, $b = 1$ or $a = 0$, $b = 0$. For, if $\alpha \in M(a,b)$ with $b > 0$ then $\alpha^* \in M(0,1)$ where $\alpha^*(t) = \alpha(bt + a)$. If $\alpha \in M(a,0)$ then $\alpha^{**} \in M(0,0)$ where $\alpha^{**}(t) = \alpha(t+a)$.

Theorem 8. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 < \varepsilon < 1$, $x \in [a-b, a+b]$. Then

$$[b^2 - (x-a)^2]\, \lambda_{n+1}(d\alpha, x)\, p_n^2(d\alpha, x) \leq$$

$$\leq 6\{\frac{1}{\varepsilon n} + 2 \sum_{n(1-\varepsilon)\leq j\leq n} [C_j^{a,b}(d\alpha)]^2\}$$

for $n = 1, 2, \ldots$

Proof. We obtain from (6) that for $[(1-\varepsilon)n] + 1 \leq k \leq n$

$$[b^2 - (x-a)^2]^{\frac{1}{2}}\, |p_n(x)| \leq |p_k(x)| + |p_{k-1}(x)| +$$

$$+ 2\{\sum_{j=[(1-\varepsilon)n]}^{n} (C_j)^2\, \lambda_{n+1}^{-1}(x)\}^{\frac{1}{2}} ,$$

that is

$$[b^2 - (x-a)^2]\, p_n^2(x) \leq 3\, p_k^2(x) + 3\, p_{k-1}^2(x) + 12\, \lambda_{n+1}^{-1}(x) \sum_{j=[(1-\varepsilon)n]}^{n} (C_j)^2 .$$

Thus

$$\sum_{k=[(1-\varepsilon)n]+1}^{n} [b^2 - (x-a)^2]\, p_n^2(x) \leq 6\, \lambda_{n+1}^{-1}(x) + 12\, \lambda_{n+1}^{-1}(x) \cdot$$

$$\cdot \sum_{j=[(1-\varepsilon)n]}^{n} (C_j)^2 \sum_{k=[(1-\varepsilon)n]+1}^{n} 1 .$$

The theorem follows from this inequality.

Theorem 9. Let $\alpha \in M(a,b)$ with $b > 0$. Then

$$\lim_{n\to\infty} [b^2 - (x-a)^2]\, \lambda_{n+1}(d\alpha, x)\, p_n^2(d\alpha, x) = 0$$

uniformly for $x \in [a-b, a+b]$.

Proof. By Theorem 8 we have to show that we can choose $\varepsilon = \varepsilon_n \in (0,1)$ so that

$$\lim_{n\to\infty} \{\frac{1}{n\varepsilon_n} + \sum_{(1-\varepsilon_n)n\leq j\leq n} [C_j^{a,b}(d\alpha)]^2\} = 0 .$$

Let

$$\varepsilon_n = \frac{1}{n}[\sup_{j\geq\frac{n}{2}}([C_j]^2 + j^{-2})]^{\frac{1}{3}} .$$

Then $\varepsilon_n \leq \frac{1}{2}$. Thus

$$\frac{1}{n\varepsilon_n} + \sum_{(1-\varepsilon_n)n\leq j\leq n} [C_j]^2 \leq \frac{1}{n\varepsilon_n} + (\varepsilon_n n + 1) \sup_{j\geq\frac{n}{2}} [C_j]^2 \leq$$

$$\leq 2 \sup_{j\geq\frac{n}{2}} \{[C_j]^2 + j^{-2}\}^{\frac{1}{3}} + \sup_{j\geq\frac{n}{2}} [C_j]^2 \xrightarrow[n\to\infty]{} 0 .$$

Remark 10. It is useful to remember that if $\mathrm{supp}(d\alpha) = [-1, 1]$ and $v \log \alpha' \in L^1$ then $\alpha \in M(0,1)$ (See e.g. Freud).

It might be interesting to compare Theorem 9 with (weaker) results of Geronimus (See Chapter III in his book.)

Theorem 11. Let $\alpha \in M(a,b)$ with $b > 0$. Suppose that

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j\, C_j^{a,b}(d\alpha)^2 < \infty .$$

Let $\mathfrak{M} = \overline{\mathfrak{M}} \subset (a-b, a+b)$. Then the following two statements are equivalent.

(i) $\{p_n^2(d\alpha, x)\}$ is uniformly bounded for $x \in \mathfrak{M}$.

(ii) $\{n^{-1} \lambda_{n+1}^{-1}(d\alpha, x)\}$ is uniformly bounded for $x \in \mathfrak{M}$.

Proof. (i) $\Rightarrow$ (ii): $\{(n+1)^{-1} \lambda_{n+1}^{-1}(d\alpha, x)\}$ is the arithmetical mean of $\{p_n^2(d\alpha, x)\}$.

(ii) $\Rightarrow$ (i): Use Theorem 8.

Let us remark that if $C_j^{(a,b)}(d\alpha) = O(j^{-1})$ then the conditions of the theorem are satisfied. Example: Jacobi, Pollaczek polynomials.

Theorem 12. Let $\alpha \in M(a,b)$ with $b > 0$ and let

$$\sum_{j=0}^{\infty} C_j^{a,b}(d\alpha) < \infty .$$

If $\Delta \subset (a-b, a+b)$ then the sequence $\{|p_n(d\alpha, x)|\}$ is uniformly bounded for $x \in \Delta$.

Proof. Let, for simplicity, $\alpha \in M(0,1)$, $\Delta = [-\varepsilon, \varepsilon]$, $0 < \varepsilon < 1$, $c_j^{0,1}(d\alpha) = c_j$. Let us fix $k = k(\varepsilon)$ such that

$$\frac{2}{\sqrt{1-\varepsilon^2}} \sum_{j=k-1}^{\infty} c_j < \frac{1}{2} .$$

If $m > k$ then by Corollary 5

$$|p_m(x)| \le \Big[\frac{1}{\sqrt{1-\varepsilon^2}} + 2\Big]\big[|p_k(x)| + |p_{k-1}(x)|\big] + \frac{2}{\sqrt{1-\varepsilon^2}} \sum_{j=k+1}^{m} c_j |p_j(x)| .$$

Hence

$$|p_m(x)| \le \Big[\frac{1}{\sqrt{1-\varepsilon^2}} + \frac{1}{2}\Big]\big[|p_k(x)| + |p_{k-1}(x)|\big] + \frac{1}{2} \max_{k+1\le j\le m} |p_j(x)| .$$

Thus for $n > k$

$$\max_{k+1\le m\le n} |p_m(x)| \le \Big[\frac{1}{\sqrt{1-\varepsilon^2}} + \frac{1}{2}\Big]\big[|p_k(x)| + |p_{k-1}(x)|\big] + \frac{1}{2} \max_{k+1\le m\le n}\ \max_{k+1\le j\le m} |p_j(x)| =$$

$$= \Big[\frac{1}{\sqrt{1-\varepsilon^2}} + \frac{1}{2}\Big]\big[|p_k(x)| + |p_{k-1}(x)|\big] + \frac{1}{2} \max_{k+1\le m\le n} |p_m(x)| ,$$

in particular, for $n > k$

$$|p_n(x)| \le \Big[\frac{2}{\sqrt{1-\varepsilon^2}} + 1\Big]\big[|p_k(x)| + |p_{k-1}(x)|\big] .$$

Now remember that k does not depend on n.

Sometimes instead of Theorem 3 we shall use the following generalization of the recurrence formula.

<u>Theorem 13.</u> Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, $0 \le n \le k$. Then

$$p_n(d\alpha, x) = U_{k-n}\left(\frac{x-a}{b}\right)p_k(d\alpha, x) - U_{k-n-1}\left(\frac{x-a}{b}\right)p_{k+1}(d\alpha, x) + \bar{R}^{a,b}_{n,k}(d\alpha, x)$$

where

$$\bar{R}^{a,b}_{n,k}(d\alpha, x) = \sum_{j=n}^{k-1} U_{j-n}\left(\frac{x-a}{b}\right) \cdot$$

$$\cdot \left\{\left[1 - \frac{2}{b}\frac{\gamma_j(d\alpha)}{\gamma_{j+1}(d\alpha)}\right]p_j(d\alpha, x) + \frac{2}{b}\left[a - \alpha_{j+1}(d\alpha)\right]p_{j+1}(d\alpha, x) + \left[1 - \frac{2}{b}\frac{\gamma_{j+1}(d\alpha)}{\gamma_{j+2}(d\alpha)}\right]p_{j+2}(d\alpha, x)\right\}.$$

<u>Proof.</u> The theorem can be proved by induction in exactly the same way as Theorem 3.

<u>Corollary 14.</u> Let $k \ge 0$. Then

$$U_{k-1}(x)\, p_{k+1}(d\alpha, x) = U_k(x)\, p_k(x) + \sum_{j=0}^{k-1} U_j(x)\left\{\left[1 - 2\frac{\gamma_j(d\alpha)}{\gamma_{j+1}(d\alpha)}\right]p_j(d\alpha, x) - 2\alpha_{j+1}(d\alpha)\, p_{j+1}(d\alpha, x) + \left[1 - 2\frac{\gamma_{j+1}(d\alpha)}{\gamma_{j+2}(d\alpha)}\right]p_{j+2}(d\alpha, x)\right\} - \gamma_0(d\alpha).$$

<u>Theorem 15.</u> Let $\theta = \theta_1 + i\theta_2$ with $\theta_2 \le 0$. Then $p_n(d\alpha, x)$ $(x = \cos\theta)$ can be represented as

$$\text{(7)} \qquad \sin\theta\, p_n(d\alpha, \cos\theta) = |\phi_{2n}(d\alpha, e^{i\theta})| \cdot \sin[(n+1)\theta - \arg\phi_{2n}(d\alpha, e^{i\theta})]$$

or

$$\text{(8)} \qquad 2i\sin\theta\, p_n(d\alpha, \cos\theta) = e^{i(n+1)\theta}\phi_{2n}(d\alpha, e^{-i\theta}) - e^{-i(n+1)\theta}\phi_{2n}(d\alpha, e^{i\theta})$$

for $n = 0, 1, \ldots$ where

$$\phi_{2n}(d\alpha, e^{i\theta}) = \sum_{j=0}^{n} a_j(d\alpha, \cos\theta)\, e^{ij\theta},$$

$$a_j(d\alpha, x) = \left[1 - 2\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\right]p_j(d\alpha, x) - 2\alpha_{j-1}(d\alpha)\, p_{j-1}(d\alpha, x) + \left[1 - \frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)}\right]p_{j-2}(d\alpha, x)$$

$(j = 0, 1, \ldots)$. Consequently $\phi_{2n}(d\alpha)$ is a polynomial of degree at most $2n$ with $\phi_{2n}(d\alpha, 0) = 2^{-n}\gamma_n(d\alpha)$ and $\phi_{2n}(d\alpha, z^{-1}) = \overline{\phi_{2n}(d\alpha, z)}$ if $|z| = 1$.

<u>Proof.</u> Let us write (7) in the form

$$p_n(d\alpha, \cos\theta) = \operatorname{Re}\phi_{2n}(d\alpha, e^{i\theta})\, U_n(x) - \operatorname{Im}\phi_{2n}(d\alpha, e^{i\theta})\, T_{n+1}(x)\,(1 - x^2)^{-\frac{1}{2}}.$$

Thus (7) means that

$$p_n(d\alpha, x) = R_{n,-1}(d\alpha, x) \equiv U_n(x)\, p_0(d\alpha, x) + R_{n,0}(d\alpha, x)$$

which is equivalent to (2) applied with $k = 0$. (8) obviously follows from (7).

Corollary 16. Let k be a nonnegative integer and let

$$\frac{\gamma_{j-2}(d\alpha)}{\gamma_{j-1}(d\alpha)} = \frac{1}{2}, \quad \alpha_{j-1}(d\alpha) = 0$$

for $j > k$. Then for each $n > k$

$$\sin\theta\, p_n(d\alpha, \cos\theta) =$$

$$= |\phi_{2k}^{*}(d\alpha, e^{i\theta})| \sin[(n+1)\theta - \arg \phi_{2k}(d\alpha, e^{i\theta})] ,$$

that is

$$p_n(d\alpha, x) = U_{n-k}(x)\, p_k(d\alpha, x) - U_{n-k-1}(x)\, p_{k-1}(d\alpha, x) .$$

Let us note that the conditions of corollary 16 are satisfied if $\mathrm{supp}(d\alpha) = [-1, 1]$ and

$$\alpha(x) = \int_{-1}^{x} \frac{\sqrt{1-t^2}}{\pi(t)}\, dt \qquad (-1 \le x \le 1)$$

where π is a polynomial which is positive on $[-1, 1]$. In this case

$$\frac{2}{\pi} |\phi_{2k}(d\pi, e^{i\theta})|^2 = \pi(\cos\theta)$$

$(\theta \in \mathbb{R})$. (See Szegő, Chapter II.)

3.2. Modified Quadrature Processes

Lemma 1. Let $\alpha \in M(a, b)$ with $b > 0$. Let m be a nonnegative integer. If $n > m - 1$ then $x^m p_{n-1}(d\alpha, x)$ may be written in the form

$$(1) \qquad x^m p_{n-1}(d\alpha, x) = R^{d\alpha}_{m-1,n}(x)\, p_n(d\alpha, x) + a^{d\alpha}_{n-1,m}\, p_{n-1}(d\alpha, x) + \pi^{d\alpha}_{n-2,m}(x)$$

where $R^{d\alpha}_{m-1,n}$ and $\pi^{d\alpha}_{n-2,m}$ are polynomials of degree $m-1$ and $n-2$ respectively. Further

$$(2) \qquad \lim_{n\to\infty} a^{d\alpha}_{n-1,m} = \frac{2}{\pi b^2} \int_{a-b}^{a+b} t^m \sqrt{b^2 - (t-a)^2}\, dt .$$

Proof. Let, for simplicity, $\alpha \in M(0, 1)$. For $m = 0$ the lemma is certainly true. Suppose that for $m > 1$ we have

$$x^{m-1} p_{n-1}(d\alpha, x) = R^{d\alpha}_{m-2,n}(x)\, p_n(d\alpha, x) + \sum_{k=n-m}^{n-1} a^{d\alpha,n}_{k,m-1}\, p_k(d\alpha, x)$$

with existing

$$\lim_{n\to\infty} a^{d\alpha,n}_{k,m-1} \qquad (k = n-m, n-m+1, \ldots, n-1)$$

which depends only on $M(0, 1)$ and is independent of the particular $\alpha \in M(0, 1)$. Using the recursion formula we see that

$$x^m p_{n-1}(d\alpha, x) = [x R^{d\alpha}_{m-2,n}(x) + a^{d\alpha,n}_{n-1,m-1} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}] p_n(d\alpha, x) +$$

$$[a^{d\alpha,n}_{n-2,m-1} \frac{\gamma_{n-2}(d\alpha)}{\gamma_{n-1}(d\alpha)} + a^{d\alpha,n}_{n-1,m-1}\, \alpha_{n-1}(d\alpha)]\, p_{n-1}(d\alpha, x) +$$

$$\sum_{k=n-m+1}^{n-1} \left[a^{d\alpha,n}_{k-1,m-1} \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} + a^{d\alpha,n}_{k,m-1}\,\alpha_k(d\alpha) + a^{d\alpha,n}_{k+1,m-1} \frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)}\right] p_k(d\alpha, x) +$$

$$+ \left[a^{d\alpha,n}_{n-m,m-1}\,\alpha_{n-m}(d\alpha) + a^{d\alpha,n}_{n-m+1,m-1} \frac{\gamma_{n-m}(d\alpha)}{\gamma_{n-m+1}(d\alpha)}\right] p_{n-m}(d\alpha, x) +$$

$$+ a^{d\alpha,n}_{n-m,m-1} \frac{\gamma_{n-m-1}(d\alpha)}{\gamma_{n-m}(d\alpha)}\, p_{n-m-1}(d\alpha, x) .$$

This formula proves (1) and shows that $\lim_{n\to\infty} a^{d\alpha}_{n-1,m}$ exists and depends only on $M(0,1)$. To compute (2) we put in (1) α = Tschebyshev weight. We have in this case $\frac{\pi}{2} p^2_{n-1}(v, x_{kn}(v)) = 1 - x^2_{kn}(v)$. Thus by the Gauss-Jacobi mechanical quadrature formula and by (1) we have for $m + 3 < 2n$

$$\frac{2}{\pi}\int_{-1}^{1} t^m \sqrt{1-t^2}\, dt = \frac{2}{\pi}\sum_{k=1}^{n} \lambda_{kn}(v)\,[1 - x^2_{kn}(v)]\, x^m_{kn}(v) =$$

$$= \sum_{k=1}^{n} \lambda_{kn}(v)\, p^2_{n-1}(v, x_{kn}(v))\, x^m_{kn}(v) =$$

$$= a^{v}_{n-1,m} \sum_{k=1}^{n} \lambda_{kn}(v)\, p^2_{n-1}(v, x_{kn}(v)) + \sum_{k=1}^{n} \lambda_{kn}(v)\, p_{n-1}(v, x_{kn}(v)) \cdot$$

$$\cdot\, \pi^{v}_{n-2,m}(x_{kn}(v)) = a^{v}_{n-1,m} .$$

<u>Lemma 2.</u> Let $\alpha \in M(a,b)$ with $b > 0$. Then $[a-b, a+b] \subset \Delta(d\alpha)$.

<u>Proof.</u> It follows from Lemma 1 that if f is continuous on $\mathbb{R}$ and has compact support then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p^2_{n-1}(d\alpha, x_{kn}) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt .$$

If $[a-b, a+b] \not\subset \Delta(d\alpha)$, then we can choose f so that $f(x_{kn}) = 0$ for every n and $k = 1, 2, \ldots, n$ and

$$\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt < 0$$

which contradicts to the above limit relation.

<u>Theorem 3.</u> Let $\alpha \in M(a,b)$ with $b > 0$. Let f be a complex valued, bounded function on $\Delta(d\alpha)$. If f is Riemann integrable on $[a-b, a+b]$ then

$$(3)\quad \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p^2_{n-1}(d\pi, x_{kn}) = \frac{2}{\pi b^2}\int_{a-b}^{a+b} f(t)\sqrt{b^2-(t-a)^2}\, dt .$$

<u>Proof.</u> If f is a polynomial then the theorem follows immediately from Lemma 1. Otherwise we write

$$f = \mathrm{Re}(f)\,1_{\Delta} + \mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta} + i\,\mathrm{Im}(f)\,1_{\Delta} + i\,\mathrm{Im}(f)\,1_{\Delta(d\alpha)\setminus\Delta}$$

where $\Delta = [a-b, a+b]$. Let, for simplicity, $\Lambda_n(g) = \sum_{k=1}^{n} \lambda_k\, g(x_k)\, p^2_{n-1}(d\alpha, x_k)$. Fix $\varepsilon > 0$. We construct two polynomials π_1 and π_2 such that

$$\pi_1(x) \le \mathrm{Re}(f)(x)\,1_{\Delta}(x) \le \pi_2(x)$$

for $x \in \Delta(d\alpha)$ and

$$\frac{2}{\pi b^2}\int_{\Delta(d\alpha)} [\pi_2(t) - \pi_1(t)]\,dt < \varepsilon .$$

We can do this because $\mathrm{Re}(f)\,1_\Delta$ is Riemann integrable on $\Delta(d\alpha)$ (See e.g. Szegö, 1.5). Hence

$$\lim_{n\to\infty} \Lambda_n(\mathrm{Re}(f)\,1_\Delta) = \frac{2}{\pi b^2} \int_{a-b}^{a+b} \mathrm{Re}(f)(t)\sqrt{b^2-(t-a)^2}\,dt \ .$$

We have, further,

$$|\Lambda_n(\mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta})| \le \sup_{t\in\Delta(d\alpha)} |f(t)|\,\Lambda_n(1_{\Delta(d\alpha)\setminus\Delta})$$

and we can find a polynomial π such that

$$1_{\Delta(d\alpha)\setminus\Delta}(x) \le \pi(x) \quad (x \in \Delta(d\alpha))$$

and

$$\frac{2}{\pi b} \int_{a-b}^{a+b} \pi(t)dt < \varepsilon \ .$$

Thus

$$\lim_{n\to\infty} \Lambda_n(\mathrm{Re}(f)\,1_{\Delta(d\alpha)\setminus\Delta}) = 0 \ .$$

The limit of $\Lambda_n(\mathrm{Im}(f))$ can be found in the same way.

<u>Theorem 4.</u> Let $a \in \mathbb{R}$, $b > 0$. If for every polynomial π

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,\pi(x_{kn})\,p_{n-1}^2(d\alpha, x_{kn}) = \frac{2}{\pi b^2} \int_{a-b}^{a+b} \pi(t)\sqrt{b^2-(t-a)^2}\,dt$$

then $\alpha \in M(a,b)$.

<u>Proof.</u> We have by the recursion formula

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,x_{kn}\,p_{n-1}^2(d\alpha, x_{kn}) = a_{n-1}(d\alpha)$$

and

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,x_{kn}^2\,p_{n-1}^2(d\alpha, x_{kn}) = a_{n-1}^2(d\alpha) + \frac{\gamma_{n-2}^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)} \ .$$

The theorem follows immediately from the above identities.

3.3. The Support of $d\alpha$

$\operatorname{Supp}(d\alpha)$, that is the set of points of increase of α, is always closed. *Hence* $\operatorname{supp}(d\alpha)$ *is compact iff it is bounded.*

Lemma 1. The following three statements are equivalent. (i) $\operatorname{supp}(d\alpha)$ is compact. (ii) $\sup_{n\in\mathbb{N},\ k=1,\dots,n} |x_{kn}(d\alpha)| < \infty$. (iii) $\sup_{k\in\mathbb{N}} c_k^{0,0}(d\alpha) < \infty$.

Proof. Easy computation. Let us prove e.g. (iii) $\Rightarrow$ (ii). We have the following important identity by the Gauss-Jacobi mechanical quadrature formula:

$$(1)\qquad x_{kn}(d\alpha) = \lambda_{kn}(d\alpha)\int_{-\infty}^{\infty} x\, K_n^2(d\alpha, x, x_{kn})\,d\alpha(x) =$$

$$= \lambda_{kn}(d\alpha)\sum_{j=0}^{n-1} \alpha_j(d\alpha)\, p_j^2(d\alpha, x_{kn}) +$$

$$+ \lambda_{kn}(d\alpha)\sum_{j=1}^{n-1} 2\,\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\, p_{j-1}(d\alpha, x_{kn})\, p_j(d\alpha, x_{kn}) .$$

Thus

$$|x_{kn}(d\alpha)| \le \max_{0\le j\le n-1} |\alpha_j(d\alpha)| + 2 \max_{1\le j\le n-1} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} .$$

Lemma 2. If $\operatorname{supp}(d\alpha)$ is compact, $x \in \operatorname{supp}(d\alpha)$ and $\varepsilon > 0$ then there exists a number $N = N(\varepsilon, x)$ such that for every $n \ge N$ $p_n(d\alpha, t)$ has at least one zero in $[x-\varepsilon, x+\varepsilon]$, in particular,

$$\Delta(d\alpha) = [\lim_{n\to\infty} x_{nn}(d\alpha),\ \lim_{n\to\infty} x_{1n}(d\alpha)] .$$

Further, if α is constant on an interval Δ, then for every n $p_n(d\alpha, t)$ has no more than one zero in Δ.

Proof. *See Szegö and Freud.*

Note, that if $\alpha(x) + \alpha(-x) = \text{const}$, $\alpha(x) = \text{const}$ on $(-1,1)$ then for every n $p_{2n+1}(d\alpha, 0) = 0$ but $p_{2n}(d\alpha, t)$ has no zero in $(-1,1)$.

Lemma 3. Let $\operatorname{supp}(d\alpha)$ be compact. Then

$$\Delta(d\alpha) \subset [\inf_{j\ge 0} \alpha_j - 2\sup_{j\ge 0} \frac{\gamma_j}{\gamma_{j+1}},\ \sup_{j\ge 0} \alpha_j + 2\sup_{j\ge 0} \frac{\gamma_j}{\gamma_{j+1}}]$$

where $\alpha_j = \alpha_j(d\alpha)$ and $\gamma_j = \gamma_j(d\alpha)$.

Proof. Let $A = \inf_{j\ge 0} \alpha_j$, $B = \sup_{j\ge 0} \alpha_j$. Then by (1)

$$x_{kn} - \frac{A+B}{2} = \lambda_{kn} \sum_{j=0}^{n-1} [\alpha_j - \frac{A+B}{2}]\, p_j^2(d\alpha, x_{kn}) +$$

$$+ 2\lambda_{kn} \sum_{j=1}^{n-1} \frac{\gamma_{j-1}}{\gamma_j}\, p_{j-1}(d\alpha, x_{kn})\, p_j(d\alpha, x_{kn}) .$$

Hence

$$(2)\qquad |x_{kn} - \frac{A+B}{2}| \le \frac{B-A}{2} + 2\sup_{j\ge 0} \frac{\gamma_j}{\gamma_{j+1}} .$$

Put here $k = 1$ and let $n \to \infty$. By Lemma 2 we obtain

$$\Delta(d\alpha) \subset (-\infty, B + 2\sup_{j\ge 0} \frac{\gamma_j}{\gamma_{j+1}}] .$$

If we put $k = n$ in (2) and let $n \to \infty$ then we get

$$\Delta(d\alpha) \subset [A - 2 \sup_{j \geq 0} \frac{\gamma_j}{\gamma_{j+1}}, \infty) .$$

Lemma 4. Let supp$(d\alpha)$ be compact and let x be fixed. If for every $\varepsilon > 0$ α takes infinitely many values in $(x - \varepsilon, x + \varepsilon)$ then there exists a sequence of natural integers $\{k_n\}_{n=1}^{\infty}$ such that $1 \leq k_n \leq n$ and

(3) $$\lim_{n\to\infty} x_{k_n, n}(d\alpha) = x, \quad \lim_{n\to\infty} \lambda_{k_n, n}(d\alpha) = 0 .$$

Proof. Suppose, without loss of generality, that for every $\varepsilon > 0$ α takes infinitely many values in $(x-\varepsilon, x)$. Let for every n j_n be defined by $j_n = \{k: x_{kn}(d\alpha) < x \leq x_{k-1,n}(d\alpha)\}$ with $x_{0n} = +\infty$. Let $k_n = j_n + 1$. We shall show that $\{k_n\}_{n=N}^{\infty}$ satisfies the requirements of the lemma. Because of Lemma 2 $k_n < n$ for n large. If we can show that

(4) $$\lim_{n\to\infty} x_{k_n+1, n} = x$$

then

$$\lim_{n\to\infty} x_{k_n, n} = \lim_{n\to\infty} x_{j_n, n} = x$$

and by the Markov-Stieltjes inequalities

$$\lambda_{k_n, n} \leq \int_{x_{k_n+1, n}}^{x_{k_n-1, n}} d\alpha(t) \xrightarrow[n\to\infty]{} \alpha(x-0) - \alpha(x-0) = 0 .$$

Suppose now that (4) does not hold. Then there exists an $\varepsilon > 0$ and a sequence $\{n_\ell\}$ such that $p_{n_\ell}(d\alpha, t)$ has no more than two zeros in $(x-\varepsilon, x)$ for $\ell = 1, 2, \ldots$. Because α takes infinitely many values in

$(x-\varepsilon, x)$ we can find three points $x_1, x_2, x_3 \in (x-\varepsilon, x) \cap$ supp$(d\alpha)$ and by Lemma 2 $p_{n_\ell}(d\alpha, t)$ must have zeros near each x_k for every ℓ large. Hence $p_{n_\ell}(\alpha, t)$ has at least three zeros in $(x-\varepsilon, x)$. This contradiction proves (4).

Lemma 5. Let supp$(d\alpha)$ be compact and let $x \in \mathbb{R}$ be fixed. Suppose that for every $\varepsilon > 0$ α takes infinitely many values in $(x-\varepsilon, x+\varepsilon)$. Then for every $c \in \mathbb{R}$

$$|x-c| \leq \limsup_{j\to\infty} |\alpha_j(d\alpha) - c| + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} ,$$

in particular, if $a = \lim_{j\to\infty} \alpha_j(d\alpha)$ exists, then

$$x \in [a - 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}}{\gamma_j}, \quad a + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}}{\gamma_j}] .$$

Proof. Let us take $\{k_n\}$ from Lemma 4. Let $M \in \mathbb{N}$. Then by (1)

$$|x_{k_n, n} - c| \leq \lambda_{k_n, n} \sum_{j=0}^{M-1} |\alpha_j - c| \, p_j^2(d\alpha, x_{k_n, n}) +$$

$$+ \sup_{j\geq M} |\alpha_j - c| + 2 \lambda_{k_n, n} \sum_{j=1}^{M-1} \frac{\gamma_{j-1}}{\gamma_j} |p_{j-1}(d\alpha, x_{k_n, n}) \, p_j(d\alpha, x_{k_n, n})| +$$

$$+ 2 \sup_{j\geq M} \frac{\gamma_{j-1}}{\gamma_j} .$$

First let $n \to \infty$, then $M \to \infty$.

Lemma 6. Let $\alpha \in M(a, b)$ with $b > 0$. Then $[a-b, a+b] \subset$ supp$(d\alpha)$.

Proof. If $[a-b, a+b] \not\subset \operatorname{supp}(d\alpha)$ then $[a-b, a+b] \cap [\mathbb{R} \setminus [a-b, a+b] \cap \operatorname{supp}(d\alpha)]$ contains an interval Δ_1. Let $\Delta \subset \Delta_1^0$. Then by Theorem 3.2.3

(5) $$\sum_{x_{kn} \in \Delta} \lambda_{kn} p_{n-1}^2(d\alpha, x_{kn}) \underset{n\to\infty}{\longrightarrow} \frac{2}{\pi b^2} \int_{t \in \Delta} \sqrt{b^2 - (t-a)^2}\, dt > 0 .$$

On the other hand by Lemma 2 Δ contains no more than one x_{kn} for every n since $\Delta \cap \operatorname{supp}(d\alpha) = \phi$. Further $\Delta \subset (a-b, a+b)$ and by Theorem 3.1.9

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_{n-1}^2(d\alpha, x) = 0$$

uniformly for $x \in \Delta$. Thus the left side of (5) converges to 0 when $n \to \infty$. This contradiction shows that $[a-b, a+b] \subset \operatorname{supp}(d\alpha)$.

Theorem 7. Let $\operatorname{supp}(d\alpha)$ be compact and let $\lim_{j\to\infty} \alpha_j(d\alpha) = a$ exist. Then

$$\operatorname{supp}(d\alpha) = A \cup B, \quad A \cap B = \phi$$

where A is closed and belongs to

(6) $$[a - 2 \limsup_{i\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}, \quad a + 2 \limsup_{j\to\infty} \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}] .$$

B is denumerable, isolated, the only two possible limit points of B are the two endpoints of (6), if $x \in B$ then $\alpha_{ac} + \alpha_s$ is constant near x and α has an isolated jump at x, furthermore,

$$B \subset [\inf_{j\ge 0} \alpha_j - 2 \sup_{j\ge 1} \frac{\gamma_{j-1}}{\gamma_j}, \quad \sup_{j\ge 0} \alpha_j + 2 \sup_{j \ge 1} \frac{\gamma_{j-1}}{\gamma_j}] .$$

If $\alpha \in M(a,b)$ then A is the interval (6).

Proof. The theorem follows immediately from Lemmas 1-6. *The only thing which we have to show is that if* $\alpha \in M(a,0)$ then $a \in \operatorname{supp}(d\alpha)$. If

$a \notin \operatorname{supp}(d\alpha)$ then $B = \operatorname{supp}(d\alpha)$ and hence B is closed. But B can be closed only if B is finite and then α has only finitely many points of increase, that is α is not a weight.

Theorem 8. Let $\alpha \in M(a,b)$. If $x \notin \operatorname{supp}(d\alpha)$ then there exist $\varepsilon > 0$ and $N \ge 0$ such that for every $n \ge N$ $p_n(d\alpha, t)$ has no zeros in $[x-\varepsilon, x+\varepsilon]$.

Proof. Let, for simplicity, $a = 0$. By Theorem 7 $x \notin \operatorname{supp}(d\alpha)$ implies $x \notin [-b, b]$ (or $x \ne 0$ if $b = 0$). Suppose, without loss of generality, that $b < x < \infty$. If $x \notin \Delta(d\alpha)$ then the theorem says nothing since $x_{kn}(d\alpha) \in \Delta(d\alpha)$ for every n and $1 \le k \le n$. Now let $x \in \Delta(d\alpha) \cap (b, \infty)$. By Theorem 7 $(\frac{b+x}{2}, \infty) \cap \operatorname{supp}(d\alpha)$ is finite and it is not empty since $x \in \Delta(d\alpha)$. Let $t_1 < t_2 < \dots < t_m$ denote those points of $\operatorname{supp}(d\alpha)$ which belong to (x, ∞). Let $\varepsilon > 0$ be such that

$$x + \varepsilon < t_1 - \varepsilon < t_1 + \varepsilon < t_2 - \varepsilon < \dots < t_{m-1} + \varepsilon < t_m - \varepsilon < t_m + \varepsilon$$

and $[x-\varepsilon, x] \cap \operatorname{supp}(d\alpha) = \phi$. By Lemma 2 we can find $N = N(\varepsilon, \{t_i\})$ such that for every $n \ge N(\varepsilon)$ $p_n(d\alpha, t)$ has zeros in each $[t_i - \varepsilon, t_i + \varepsilon]$ $(i = 1, 2, \dots, m)$. Thus for $n \ge N(\varepsilon)$ $p_n(d\alpha, t)$ has not less than m zeros in $[t_1 - \varepsilon, \infty)$. On the other hand α takes exactly $m + 1$ values in $(x-\varepsilon, \infty)$ if we do not count the values of $\alpha(t_i)$. Further, $[t_m, \infty)$ does not contain zeros of $p_n(d\alpha, t)$ since $\Delta(d\alpha) \cap (t_m, \infty) = \phi$. Thus by Lemma 2 for every $n \ge 0$ $p_n(d\alpha, t)$ has no more than m zeros in $(x-\varepsilon, \infty)$. Hence for $n > N$ both $(x-\varepsilon, \infty)$ and $[t_1 - \varepsilon, \infty)$ contain exactly m zeros of $p_n(d\alpha, t)$, that is $[x-\varepsilon, x+\varepsilon]$ contains no zeros of $p_n(d\alpha, t)$ if $n \ge N$.

Let us note that without the assumption $\alpha \in M(a, b)$ Theorem 8 does not necessarily hold. (See Remark 4.1.6.)

4. Limit Relations

4.1. Pointwise Limits

We begin with a simple result which we shall not apply in the followin[g] but which is worth recording.

Theorem 1. For every weight α and $x \in \mathbb{R}$

$$(1) \qquad \sum_{k=0}^{\infty} \lambda_{k+1}^2(d\alpha, x)\, p_k^2(d\alpha, x) \le [1 + \alpha(\infty) - \alpha(-\infty)]^2 ,$$

in particular, for every $x \in \mathbb{R}$

$$\lim_{n \to \infty} \lambda_{n+1}(d\alpha, x)\, p_n(d\alpha, x) = 0 .$$

Proof. Let x be fixed and let $\beta = \alpha + \delta_x$ where δ_x is the unit mass concentrated at x. Let us expand $p_n(d\alpha, t)$ in a Fourier series in $p_k(d\beta, t)$ We have

$$p_n(d\alpha, t) = \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)}\, p_n(d\beta, t) + K_{n+1}(d\beta, t, x)\, p_n(d\alpha, x) .$$

Putting $t = x$ we obtain

$$p_n(d\alpha, x) = \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)}\, p_n(d\beta, x) + \lambda_{n+1}^{-1}(d\alpha, x)\, p_n(d\alpha, x) .$$

By an easy computation $\lambda_{n+1}(d\beta, x) = \lambda_{n+1}(d\alpha, x) + 1$, $\gamma_n(d\beta) \le \gamma_n(d\alpha)$ and $\lambda_{n+1}(d\alpha, x) \le \alpha(\infty) - \alpha(-\infty)$. Thus for every $n = 0, 1, \ldots$

$$\lambda_{n+1}^2(d\alpha, x)\, p_n^2(d\alpha, x) \le p_n^2(d\beta, x)\, [1 + \alpha(\infty) - \alpha(-\infty)]^2$$

and

$$\sum_{k=0}^{n-1} \lambda_{k+1}^2(d\alpha, x)\, p_k^2(d\alpha, x) \le \lambda_n^{-1}(d\beta, x)\,[1 + \alpha(\infty) - \alpha(-\infty)]^2 \le$$

$$\le [1 + \alpha(\infty) - \alpha(-\infty)]^2 .$$

Letting $n \to \infty$ we obtain (1).

<u>Lemma 2.</u> Let $\alpha \in M(a,b)$ with $b > 0$. Then for every $x \in [a-b, a+b]$

(2) $$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} = +\infty .$$

<u>Proof.</u> Let, for simplicity, $a \le x \le a+b$. Let $0 < x \le a+b$. Let $0 < \varepsilon < b$. Then

$$\sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x-x_{kn})^2} \ge \varepsilon^{-2} \sum_{x-\varepsilon \le x_{kn} \le x} \lambda_{kn}\, p_{n-1}^2(d\alpha, x_{kn})$$

and by Theorem 3.2.3

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{(x - x_{kn})^2} \ge \frac{2}{\varepsilon^2 \pi b^2} \int_{x-\varepsilon}^{x} \sqrt{b^2 - (t-a)^2}\, dt .$$

If $x < a + b$ the right side is exactly of order ε^{-1}, if $x = a + b$ it is exactly of order $\varepsilon^{-\frac{1}{2}}$. Letting $\varepsilon \to 0$ we obtain (2).

<u>Theorem 3.</u> Let $\alpha \in M(a,b)$ with $b > 0$. Then for $x \in [a-b, a+b]$

(3) $$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) = 0$$

and the convergence is uniform for $x \in \Delta \subset (a-b, a+b)$.

<u>Proof.</u> To get (3) use Lemma 2 and the formula

$$\lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) = \frac{\gamma_n^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)} \left[\sum_{k=1}^{n} \frac{\lambda_{kn}(d\alpha)\, p_{n-1}^2(d\alpha, x_{kn})}{(x-x_{kn})^2}\right]^{-1}$$

which follows from $\lambda_n^{-1}(d\alpha, x) = \sum_{k=1}^{n} \frac{\ell_{kn}^2(d\alpha, x)}{\lambda_{kn}(d\alpha)}$. The uniform convergence inside $(a-b, a+b)$ follows from Theorem 3.1.9 and from

$$p_n^2(d\alpha, x) \le C[p_{n-1}^2(d\alpha, x) + p_{n-2}^2(d\alpha, x)]$$

for $x \in \Delta(d\alpha)$ which can easily be proved using the recurrence formula.

There are two possible ways to define the Christoffel functions for complex values of the argument. We can either put

$$\lambda_n(d\alpha, z) = \{\sum_{k=0}^{n-1} p_k^2(d\alpha, z)\}^{-1}$$

or

$$\lambda_n(d\alpha, z) = [\sum_{k=0}^{n-1} |p_k(d\alpha, z)|^2]^{-1} .$$

It is easy to see that the second definition coincides with

$$\lambda_n(d\alpha, z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |(1 + (z-t)\, \pi_{n-2}(t))|^2\, d\alpha(t) .$$

To avoid confusion we shall write $\lambda_n^*(d\alpha, z)$ when we mean the first definition:

$$\lambda_n^*(d\alpha, z) = [\sum_{k=0}^{n-1} p_k^2(d\alpha, z)]^{-1} .$$

Let for $z, u \in \mathbb{C}$

$$K_n(d\alpha, z, u) = \sum_{k=0}^{n-1} p_k(d\alpha, z)\, \overline{p_k(d\alpha, u)} ,$$

$$k_n(d\alpha, z, u) = \sum_{k=0}^{n-1} p_k(d\alpha, z)\, p_k(d\alpha, u) .$$

Properties 4. λ_n is real valued, monotonic in n and positive, λ_n^* is menomorphic with $2n-2$ poles.

$$\lambda_n^{-1}(z) = K_n(z,z),\ K_n(z,u) = \overline{K_n(u,z)},\ \lambda_n^*(t)^{-1} = k_n(z,z),\ k_n(z,u) = k_n(u,z) ,$$

$$K_n(d\alpha, z, u) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha, z)\, \overline{\ell_{kn}(d\alpha, u)}}{\lambda_{kn}(d\alpha)} ,$$

$$k_n(d\alpha, z, u) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha, z)\, \ell_{kn}(d\alpha, u)}{\lambda_{kn}(d\alpha)} ,$$

further

$$(4)\quad \lambda_n^{-1}(d\alpha, z) = |p_n(d\alpha, z)|^2 \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{|z - x_{kn}|^2}$$

and

$$(5)\quad \lambda_n^*(d\alpha, z)^{-1} = p_n^2(d\alpha, z) \frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) \frac{p_{n-1}^2(d\alpha, x_{kn})}{(z - x_{kn})^2} .$$

We obtain immediately from (4) and (5) the following

Theorem 5. Let $\text{supp}(d\alpha)$ be compact and let $z \notin \Delta(d\alpha)$. Then

$$\liminf_{n\to\infty} \lambda_n(d\alpha, z)\, |p_n^2(d\alpha, z)| > 0$$

and

$$\liminf_{n\to\infty} |\lambda_n^*(d\alpha, z)\, p_n^2(d\alpha, z)| > 0 .$$

Remark 6. It is not true that Theorem 5 holds for every $z \notin \text{supp}(d\alpha)$. For, if $\alpha(x) + \alpha(-x) = \text{const}$, $\text{supp}(d\alpha)$ is compact and $\alpha(t) = \text{const}$ for $t \in [-\varepsilon, \varepsilon]$ $(\varepsilon > 0)$ then $p_{2k+1}(d\alpha, 0) = 0$ although $0 \notin \text{supp}(d\alpha)$.

Theorem 7. Let $\alpha \in M(a, 0)$. Then

(i) for every $z \notin \text{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n^*(d\alpha, z)\, p_n^2(d\alpha, z) = \infty, \quad \lim_{n\to\infty} \lambda_n(d\alpha, z)\, p_n^2(d\alpha, z) = \infty .$$

(ii) for every $x \in \text{supp}(d\alpha)\backslash a$

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) = 0 .$$

(iii) there exist two weights $\hat{\alpha}$ and $\hat{\hat{\alpha}}$ in $M(a, 0)$ such that

$$\lim_{n\to\infty} \lambda_n(d\hat{\alpha}, a)\, p_n^2(d\hat{\alpha}, a) = 0$$

and

$$\liminf_{n\to\infty} \lambda_n(d\hat{\hat{\alpha}}, a)\, p_n^2(d\hat{\hat{\alpha}}, a) = 0 ,$$

$$\limsup_{n\to\infty} \lambda_n(d\hat{\hat{\alpha}}, a)\, p_n^2(d\hat{\hat{\alpha}}, a) = \infty .$$

Proof. (i) If z is complex use (4) and (5). Let now $z = x$ be real and $x \notin \text{supp}(d\alpha)$. Since $\text{supp}(d\alpha)$ is compact and α is constant in a neighborhood of x, we have $\lim_{n\to\infty} \lambda_n(d\alpha, x) = 0$. Suppose that there exists a sequence $n_1 < n_2 < \dots$ such that

$$\lim_{k\to\infty} \lambda_{n_k}(d\alpha, x)\, p^2_{n_k}(d\alpha, x) < \infty .$$

We have by the recurrence formula

$$x\,\lambda_n^{-1}(x) = \sum_{k=0}^{n-1} \alpha_k\, p_k^2(x) + 2\sum_{k=0}^{n-2} \frac{\gamma_k}{\gamma_{k+1}}\, p_k(x)\, p_{k+1}(x) + \frac{\gamma_{n-1}}{\gamma_n}\, p_{n-1}(x)\, p_n(x) .$$

with $\lambda_n(x) = \lambda_n(d\alpha, x)$, $\alpha_k = \alpha_k(d\alpha)$ etc. Let M be a natural integer.

Then

$$|x-a| \le \lambda_n(x) \sum_{k=0}^{M-1} |\alpha_k - a|\, p_k^2(x) + \sup_{k\ge M} |\alpha_k - a| + 2\lambda_n(x) \sum_{k=0}^{M-1} \frac{\gamma_k}{\gamma_{k+1}} |p_k(x)\, p_{k+1}(x)| + 2\sup_{k\ge M} \frac{\gamma_k}{\gamma_{k+1}} + \frac{\gamma_{n-1}}{\gamma_n} [\lambda_n(x)\, p_n^2(x)]^{\frac{1}{2}} .$$

Put $n = n_\ell$, first let $\ell \to \infty$ and then $M \to \infty$. We get $|x-a| \le 0$ that **is** $x = a$. By Theorem 3. 37 $x \in \operatorname{supp}(d\alpha)$. **This is** a contradiction.

(ii) If $x \in \operatorname{supp}(d\alpha)\setminus a$ then by Theorem 3. 3. 7 x is a jump of α and consequently (ii) is true.

(iii) Let $\hat{\alpha}$ be defined by $\alpha_i(d\hat{\alpha}) \equiv a$ and $\gamma_i(d\hat{\alpha})/\gamma_{i+1}(d\hat{\alpha}) = (i+1)^{-\frac{1}{4}}$ for $i = 0, 1, \ldots$. Then $\hat{\alpha}(a+x) + \hat{\alpha}(a-x) =$ const **and thus** $p_{2k+1}(d\hat{\alpha}, a) = 0$ for $k = 0, 1, \ldots$. Hence

$$\lim_{k\to\infty} \lambda_{2k+1}(d\hat{\alpha}, a)\, p^2_{2k+1}(d\hat{\alpha}, a) = 0 . \tag{6}$$

Let us consider now $p_{2k}(d\hat{\alpha}, a)$. By the recurrence formula

$$\frac{\gamma_{2k-1}}{\gamma_{2k}}\, p_{2k}(a) + \frac{\gamma_{2k-2}}{\gamma_{2k-1}}\, p_{2k-2}(a) = 0 .$$

Hence

$$p^2_{2k}(a) = \left(\frac{2k}{2k-1}\right)^{\frac{1}{2}} p^2_{2k-2}(a) . \tag{7}$$

By repeating application of (7) we obtain that for every $j = 1, 2, \ldots, k$

$$p^2_{2k}(a) = \left[\frac{2k}{2k-1}\,\frac{2k-2}{2k-3}\cdots\frac{2k-2(j-1)}{2k-2(j-1)-1}\right]^{\frac{1}{2}} p^2_{2k-2j}(a) .$$

Thus for $j = 0, 1, \ldots, k-1$

$$p^2_{2k}(a) \le \sqrt{2k}\; p^2_{2j}(a) ,$$

that is

$$p^2_{2k}(a) \le \sqrt{\frac{2}{k}} \sum_{j=0}^{k-1} p^2_{2j}(a) = \sqrt{\frac{2}{k}}\; \lambda^{-1}_{2k}(a)$$

which together with (6) proves the first part of (iii). Let $\hat{\hat{\alpha}}$ be defined by $\alpha_i(d\hat{\hat{\alpha}}) \equiv a$ and $\gamma_i(d\hat{\hat{\alpha}})/\gamma_{i+1}(d\hat{\hat{\alpha}}) = \exp(-(i+1)^2)$ for $i = 0, 1, \ldots$. Repeating the above argument we see that (6) holds if we replace $\hat{\alpha}$ by $\hat{\hat{\alpha}}$. Further, similarity to (7),

$$p^2_{2k}(d\hat{\hat{\alpha}}, a) = e^{8k-2}\, p^2_{2k-2}(d\hat{\hat{\alpha}}, a) .$$

Thus

$$p^2_{2k}(d\hat{\hat{\alpha}}, a) \geq \frac{e^{8k-2}}{k} \lambda^{-1}_{2k}(d\hat{\hat{\alpha}}, a) .$$

Let now $k \to \infty$.

Let us remark that both $\hat{\alpha}$ and $\hat{\hat{\alpha}}$ are continuous at a.

Definition 8. For $z \in \mathbb{C}\setminus[-1,1]$ we define $\rho(z)$ by

$$\rho(z) = z + \sqrt{z^2-1}$$

where we take that branch of $\sqrt{z^2-1}$ for which $|\rho(z)| > 1$ whenever $z \in \mathbb{C}\setminus[-1,1]$. We have

$$\lim_{z\to\infty} \rho(z) = \infty, \quad \lim_{z\to\infty} |z/\rho(z)| = \frac{1}{2} .$$

Lemma 9. Let $|z| > 1$. If

$$\lim_{k\to\infty} a_k z^{-k} = a$$

then

$$\lim_{n\to\infty} \frac{z-1}{z^{n+1}} \sum_{k=0}^{n} a_k = a .$$

Proof. The matrix $\mu = \{\mu_{kn}\}$ where $\mu_{kn} = (z-1)z^{k-n-1}$ satisfies the conditions of Toeplitz-Silverman's theorem.

Lemma 10. Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and let v_τ denote the Tschebyshev weight corresponding to $\tau = [a-b, a+b]$. Then for every $z \in \mathbb{C}\setminus[a-b, a+b]$

$$\lim_{n\to\infty} \lambda_n(v_\tau, t)|p^2_n(v_\tau, t)| = |\rho(\frac{z-a}{b})|^2 - 1$$

and

$$\lim_{n\to\infty} \lambda^*_n(v_\tau, z)\, p^2_n(v_\tau, z) = \rho(\frac{z-a}{b})^2 - 1 .$$

Proof. Use Lemma 9 and the formula

$$p_n(v_\tau, z) = \frac{1}{\sqrt{2\pi}} [\rho(\frac{z-a}{b})^n + \rho(\frac{z-a}{b})^{-n}]$$

for $n = 1, 2, \ldots$.

Theorem 11. Let $\alpha \in M(a,b)$ with $b > 0$. Then

(i) for every $z \notin \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n(d\alpha, z)\, |p^2_n(d\alpha, z)| = |\rho(\frac{z-a}{b})|^2 - 1$$

and

$$\lim_{n\to\infty} \lambda^*_n(d\alpha, z)\, p^2_n(d\alpha, z) = \rho(\frac{z-a}{b})^2 - 1 ,$$

(ii) for every $x \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \lambda_n(d\alpha, x)\, p^2_n(d\alpha, x) = 0 ,$$

(iii) the convergence in (ii) is uniform inside $(a-b, a+b)$.

Proof. (i) If $z \notin \Delta(d\alpha)$ then both $|z-t|^{-2}$ and $(z-t)^{-2}$ are continuous functions of $t \in \Delta(d\alpha)$. Thus by (4), (5) and Theorem 3.2.3

$$(8) \quad \lim_{n\to\infty} \lambda_n(d\alpha, z)\, |p^2_n(d\alpha, z)| = [\frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{|z-t|^2}\, dt]^{-1}$$

and

$$(9) \quad \lim_{n\to\infty} \lambda^*_n(d\alpha, z)\, p^2_n(d\alpha, z) = [\frac{1}{2\pi} \int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{(z-t)^2}\, dt]^{-1} .$$

If $z \in \Delta(d\alpha)$ but $z \notin \text{supp}(d\alpha)$ then we take ε from Theorem 3.3.8 and put

$$f_1(t) = \begin{cases} |z-t|^{-2} & \text{for} \quad |z-t| > \varepsilon \\ 0 & \text{for} \quad |z-t| \leq \varepsilon \end{cases},$$

$$f_2(t) = \begin{cases} (z-t)^{-2} & \text{for} \quad |z-t| > \varepsilon \\ 0 & \text{for} \quad |z-t| \leq \varepsilon \end{cases}.$$

Both f_1 and f_2 satisfy the conditions of Theorem 3.2.3. By Theorem 3.3.8 neither (4) nor (5) will change if we replace $|z-t|^{-2}$ and $(z-t)^2$ by $f_1(t)$ and $f_2(t)$ respectively for $n \geq N$. Thus (8) and (9) hold for every $z \notin \text{supp}(d\alpha)$. To calculate the integrals on the right sides of (8) and (9) let us remark that it is the same for every $\alpha \in M(a,b)$, in particular, for the Tschebyshev weight corresponding to $[a-b, a+b]$. Now we use Lemma 10.

(ii) If $x \in [a-b, a+b]$ then use Theorem 3. If $x \in \text{supp}(d\alpha) \setminus [a-b, a+b]$ then by Theorem 3.3.7 α has a jump at x which implies (ii) again.

(iii) See Theorem 3.

<u>Theorem 12.</u> Let $\text{supp}(d\alpha)$ be compact and let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$. If there exists a sequence $\{z_k\}_{k=1}^{\infty}$ such that $z_k \in \mathbb{C}$, $\lim_{k\to\infty} z_k = \infty$ and

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)} = \rho\left(\frac{z-a}{b}\right)^{-1}$$

for $k = 1, 2, \ldots$ then $\alpha \in M(a,b)$.

<u>Proof.</u> Suppose without loss of generality that $z_k \notin \Delta(d\alpha)$ for every k. We have

$$(10) \qquad \frac{z p_{n-1}(\alpha, z)}{p_n(d\alpha, z)} = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \left[1 + \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, x_{kn} \frac{p_{n-1}^2(d\alpha, x_{kn})}{z - x_{kn}}\right]$$

which can easily be checked. Let $d(z) = \text{dist}(z, \Delta(d\alpha))$. Then we get with $C = C(\text{supp}(d\alpha))$

$$\left|\frac{z_k\, p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right| \leq \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \left[1 + C d(z_k)^{-1}\right].$$

Letting first $n \to \infty$ and then $k \to \infty$ we obtain

$$\frac{b}{2} \leq \liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}.$$

On the other hand we have by the recurrence formula and Lemma 3.3.1

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \leq \left[|z_k| + C_1\right] \left|\frac{p_{n-1}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right| + C_2 \left|\frac{p_{n-2}(d\alpha, z_k)}{p_n(d\alpha, z_k)}\right|$$

where C_1 and C_2 depend on $\text{supp}(d\alpha)$. First let $n \to \infty$ and then $k \to \infty$. We get

$$\limsup_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \leq \frac{b}{2}.$$

Using again the recurrence formula we obtain

$$\alpha_n(d\alpha) = z_1 - \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \frac{p_{n+1}(d\alpha, z_1)}{p_n(d\alpha, z_1)} - \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha, z_1)}{p_n(d\alpha, z_1)}.$$

Thus $\alpha_n(d\alpha)$ is convergent and letting $n \to \infty$ we see that

$$\lim_{n\to\infty} \alpha_n(d\alpha) = z_1 - \frac{b}{2}\rho\left(\frac{z_1 - a}{b}\right) - \frac{b}{2}\rho\left(\frac{z_1-a}{b}\right)^{-1} = a.$$

<u>Theorem 13.</u> Let $\alpha \in M(a,b)$ and let $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)} = \begin{cases} 0 & \text{for } b = 0 \\ \rho(\frac{z-a}{b})^{-1} & \text{for } b > 0 . \end{cases}$$

Proof. If $b = 0$ then the theorem follows immediately from (10) and Theorem 3.3.8. If $b > 0$ then by (10) and Theorems 3.2.3 and 3.3.8

$$\lim \frac{z p_{n-1}(d\alpha, z)}{p_n(d\alpha, z)}$$

exists for $z \notin \text{supp}(d\alpha)$ and equals to

$$\frac{z}{\pi b} \int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{z-t} dt .$$

It can directly be calculated that the latter expression equals $z\rho(\frac{z-a}{b})^{-1}$ but it is easier if we remark that $z p_{n-1}(v_\tau, z)/p_n(v_\tau, z)$ converges to the same limit when v_τ is the Tschebyshev weight corresponding to $\tau = [a-b, a+b]$.

Using Theorems 7, 11 and 13 we can prove limit relations for $\lambda_{n+1}(d\alpha, z)|p_n^2(d\alpha, z)|$ and $\lambda^*_{n+1}(d\alpha, z)\, p_n^2(d\alpha, z)$ as well. *We will not go* into details, we only formulate one result which we shall use later.

Theorem 14. Let $\alpha \in M(a, b)$ with $b > 0$ and let $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \lambda^*_{n+1}(d\alpha, z)\, p_n^2(d\alpha, z) = 1 - \rho(\frac{z-a}{b})^{-2} .$$

Lemma 15. Let $\alpha \in M(a, b)$ and $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p'_n(d\alpha, z)}{n\, p_n(d\alpha, z)} = \begin{cases} \frac{1}{z-a} & \text{for } b = 0 \\ \frac{1}{\sqrt{(z-a)^2-b^2}} & \text{for } b > 0 . \end{cases}$$

where $\sqrt{z^2-1} > 0$ for $z > 1$.

Proof. We have for $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\frac{p'_n(d\alpha, z)}{p_n(d\alpha, z)} = \sum_{k=1}^{n} \frac{1}{z - x_{kn}(d\alpha)} .$$

Let $\varepsilon > 0$ and

$$f(t) = \begin{cases} 0 & \text{for } |z-t| < \varepsilon \\ \frac{1}{z-t} & \text{for } |z-t| \geq \varepsilon . \end{cases}$$

By Theorem 3.3.8 if $\varepsilon = \varepsilon(z)$ is small enough and $n \geq N(z)$ then

$$\frac{p'_n(d\alpha, z)}{n\, p_n(d\alpha, z)} = \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) .$$

Using Theorems 5.2 and 5.3 we obtain

$$\lim_{n\to\infty} \frac{p'_n(d\alpha, z)}{n\, p_n(d\alpha, z)} = \begin{cases} f(a) & \text{for } b = 0 \\ \frac{1}{\pi} \int_{a-b}^{a+b} \frac{f(t)}{\sqrt{b^2-(t-a)^2}} dt & \text{for } b > 0 \end{cases}$$

and $f(a) = (z-a)^{-1}$ *for* $b = 0$ *and* $f(t) = (z-t)^{-1}$ for $b > 0$, $t \in [a-b, a+b]$. The calculation of the above integral is simple: put α = Tschebyshev weight corresponding to $[a-b, a+b]$.

Theorem 16. Let $\alpha \in M(a,b)$ and $z \in \mathbb{C}\setminus \mathrm{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} n\left[\frac{p_{n-1}(d\alpha,z)}{p_n(d\alpha,z)} - \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)}\right] = \begin{cases} 0 & \text{for } b=0 \\ \rho(\frac{z-a}{b})^{-1} & \text{for } b>0 . \end{cases}$$

Proof. From

$$p_{n-1}(d\alpha,z) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_n(d\alpha,z) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \frac{p^2_{n-1}(d\alpha,x_{kn})}{z-x_{kn}}$$

follows

$$n\left[\frac{p_{n-1}(d\alpha,z)}{p_n(d\alpha,z)} - \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)}\right] =$$

$$= n\frac{p_n(d\alpha,z)}{p'_n(d\alpha,z)} \cdot \frac{\gamma_{n-1}}{\gamma_n} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \frac{p^2_{n-1}(d\alpha,x_{kn})}{(z-x_{kn})^2} .$$

If $b = 0$ then use Theorem 3.3.8 and Lemma 15. If $b > 0$ then use Theorems 3.2.3, 3.3.8 and Lemma 15. For $b > 0$ we get

$$\lim_{n\to\infty} n\left[\frac{p_{n-1}(d\alpha,z)}{p_n(d\alpha,z)} - \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)}\right] =$$

$$= \sqrt{(\frac{z-a}{b})^2 - 1} \cdot \frac{1}{\pi}\int_{a-b}^{a+b} \frac{\sqrt{b^2-(t-a)^2}}{(z-t)^2}\, dt$$

and this integral has been calculated in the course of proof of Theorem 11.

From Theorems 13 and 16 we obtain

Theorem 17. Let $\alpha \in M(a,b)$ and $z \in \mathbb{C}\setminus \mathrm{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} \frac{p'_{n-1}(d\alpha,z)}{p'_n(d\alpha,z)} = \begin{cases} 0 & \text{for } b=0 \\ \rho(\frac{z-a}{b})^{-1} & \text{for } b>0 . \end{cases}$$

The following result is rather surprising if we compare it with Theorem 13.

Theorem 18. Let $\alpha \in M(a,b)$ with $b > 0$. Then for every $x \in \mathrm{supp}(d\alpha)\setminus[a-b,a+b]$

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha,x)}{p_n(d\alpha,x)} = \rho(\frac{x-a}{b}) .$$

Proof. We have

$$p_{n-1}(d\alpha,x) = \int_{-\infty}^{\infty} p_{n-1}(d\alpha,t)\, K_n(d\alpha,x,t)\, d\alpha(t) .$$

If $x \in \mathrm{supp}(d\alpha)\setminus[a-b,a+b]$ then by Theorem 3.3.7 x is an isolated point of $\mathrm{supp}(d\alpha)$. Hence we can find $\varepsilon > 0$ such that

$$p_{n-1}(d\alpha,x) = \int_{|x-t|>\varepsilon} p_{n-1}(d\alpha,t)\, K_n(d\alpha,x,t)\, d\alpha(t) +$$

$$+ \frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(d\alpha,x)}\, p_{n-1}(d\alpha,x) .$$

Using

$$K_n(d\alpha,x,t) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, \frac{p_{n-1}(d\alpha,t)\,p_n(d\alpha,x) - p_n(d\alpha,t)\,p_{n-1}(d\alpha,x)}{x-t}$$

we obtain

$$p_{n-1}(d\alpha, x)\,[1 - \frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(d\alpha, x)} + \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} .$$

$$\int_{|x-t|>\varepsilon} \frac{p_{n-1}(d\alpha, t) p\ (d\alpha, t)}{x-t}\, d\alpha(t)] =$$

$$= p_n(d\alpha, x) \cdot \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \int_{|x-t|>\varepsilon} \frac{p^2_{n-1}(d\alpha, t)}{x-t}\, d\alpha(t) .$$

We have

$$\lim_{n\to\infty} \frac{\alpha(x+0)-\alpha(x-0)}{\lambda_n(d\alpha, x)} = 1 .$$

(See Freud, Section II.2, supp($d\alpha$) is compact!) Thus by Theorem 4.2.13 $p_n(d\alpha, x) \neq 0$ for n large and

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, x)}{p_n(d\alpha, x)} = \frac{\frac{1}{\pi}\int_{a-b}^{a+b} (x-t)^{-1}[b^2-(t-a)^2]^{-\frac{1}{2}}dt}{\frac{1}{\pi}\int_{a-b}^{a+b} t(x-t)^{-1}[b^2-(t-a)^2]^{-\frac{1}{2}}dt}$$

which equals $\rho(\frac{x-a}{b})$.

Theorem 19. Let $\alpha \in M(0,1)$ and ℓ be a fixed nonnegative integer. Then

$$\text{(11)} \qquad \lim_{n\to\infty} \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_{k+\ell}(d\alpha, x) = T_\ell(x)$$

for each $x \in [-1,1]$ provided that α is continuous at x; in particular, (11) holds for almost every $x \in$ supp($d\alpha$) . If α is continuous on $\tau \subset (-1,1)$ then (11) is satisfied uniformly for $x \in \tau$.

Proof. If $\ell = 1$ then the theorem follows from Theorem 11 and from the formula

$$x - \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_{k+1}(d\alpha, x) =$$

$$= \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} \alpha_k(d\alpha)\, p_k^2(d\alpha, x) +$$

$$+ \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} [2\frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)} - 1]\, p_k(d\alpha, x)\, p_{k+1}(d\alpha, x) -$$

$$- \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, p_{n-1}(d\alpha, x)\, p_n(d\alpha, x)\, \lambda_n(d\alpha, x)$$

which is a direct consequence of the recurrence formula. Now let $\ell > 1$. Then by Theorem 3.1.1

$$\lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_{k+\ell}(d\alpha, x) =$$

$$= U_{\ell-1}(x)\, \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, p_{k+1}(d\alpha, x) -$$

$$- U_{\ell-2}(x) + \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, R_{k+\ell,\, k+1}(d\alpha, x) .$$

Since $U_{\ell-1}(x)x - U_{\ell-2}(x) = T_\ell(x)$ we obtain that (11) holds at those points x where it holds with $\ell = 1$ and where

$$\lim_{n\to\infty} \lambda_n(d\alpha, x) \sum_{k=0}^{n-1} p_k(d\alpha, x)\, R_{k+\ell,\, k+1}(d\alpha, x) = 0$$

is also satisfied. To finish the proof we apply 3.1.(3) and Theorem 11. Recall that

$$\lim_{n\to\infty} \lambda_n(d\alpha, x) = 0$$

at every x where α is continuous and the convergence is uniform on every interval of continuity of α since $\text{supp}(d\alpha)$ is compact. (See Freud, Section II. 3.)

4.2. Weak Limits

Definition 1. We write $\alpha \in S$ if $\text{supp}(d\alpha) = [-1,1]$ and $v \log \alpha' \in L^1(-1,1)$.

Lemma 2. If $\alpha \in S$ then

$$\lim_{n\to\infty} \gamma_n(d\alpha) 2^{-n} = \frac{1}{\sqrt{\pi}} \exp\{-\frac{1}{2\pi} \int_{-1}^{1} v(t) \log \alpha'(t) dt\} .$$

Proof. See e.g. Freud, §V.6.

Theorem 3. Let $\alpha \in S$ and f be Riemann integrable on $[-1,1]$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha) f(x_{kn}) \frac{p_{n-1}^2(d\alpha, x_{kn})}{1 - x_{kn}^2} = \frac{2}{\pi} \int_{-1}^{1} f(t) \frac{dt}{\sqrt{1-t^2}} .$$

Proof. Let β be defined by $d\beta(x) = (1-x^2) d\alpha(x)$. Then $(1-x^2)[p_{n-1}^2(d\beta, x) - \gamma_{n-1}^2(d\beta) x^{2n-2}]$ is a polynomial of degree $2n-1$ and we have by the Gauss-Jacobi mechanical quadrature formula

$$\sum_{k=1}^{n} (1 - x_k^2) [p_{n-1}^2(d\beta, x_k) - \gamma_{n-1}^2(d\beta) x_k^{2n-2}] \lambda_k =$$

$$= 1 + \gamma_{n-1}^2(d\beta) \int_{-1}^{1} (t^2-1) t^{2n-2} d\alpha(t)$$

(here $x_k = x_{kn}(d\alpha)$ and $\lambda_k = \lambda_{kn}(d\alpha)$. Thus

$$\sum_{k=1}^{n} (1 - x_k^2) p_{n-1}^2(d\beta, x_k) \lambda_k = 1 + \gamma_{n-1}^2(d\beta) \cdot$$

$$\cdot [\int_{-\pi}^{1} t^{2n} d\alpha(t) - \sum_{k=1}^{n} x_k^{2n} \lambda_k] .$$

Further

$$\sum_{k=1}^{n} x_k^{2n} \lambda_k = \sum_{k=1}^{n} L_n(d\alpha, y^{2n}, x_k) \lambda_k = \int_{-1}^{1} L_n(d\alpha, y^{2n}, t) d\alpha(t) .$$

Hence we obtain

$$\sum_{k=1}^{n} (1 - x_k^2)\, p_{n-1}^2(d\beta, x_k)\lambda_k =$$

$$= 1 + \gamma_{n-1}^2(d\beta)\int_{-1}^{1} [t^{2n} - L_n(d\alpha, y^{2n}, t)]\,d\alpha(t) \;.$$

Since $t^{2n} - L_n(d\alpha, y^{2n}, t)$ is a polynomial of degree $2n$ which vanishes at the zeros of $p_n(d\alpha, x)$ we have

$$t^{2n} - L_n(d\alpha, y^{2n}, t) = p_n(d\alpha, t)[\,\Gamma\, p_n(d\alpha, x) + \Pi_{n-1}(x)] \;.$$

Comparing the leading coefficients we see that $\Gamma = \gamma_n(d\alpha)^{-2}$. Consequently

(1) $$\sum_{k=1}^{n} (1 - x_k^2)\, p_{n-1}^2(d\beta, x_k)\lambda_k = 1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)} \;.$$

Expand $(1-x^2)\, p_{n-1}(d\beta, x)$ in a Fourier series in $p_k(d\alpha, x)$.*) It is easy to see that

$$(1-x^2)\, p_{n-1}(d\beta, x) = \sum_{k=n-1}^{n+1} a_k\, p_k(d\alpha, x)$$

with $a_{n-1} = \gamma_{n-1}(d\alpha)/\gamma_{n-1}(d\beta)$ and $a_{n+1} = -\gamma_{n-1}(d\beta)/\gamma_{n+1}(d\alpha)$. Thus

$$(1-x_k^2)p_{n-1}(d\beta, x_k) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)}\, p_{n-1}(d\alpha, x_k) -$$

$$- \frac{\gamma_{n-1}(d\beta)}{\gamma_{n+1}(d\alpha)}\, p_{n+1}(d\alpha, x_k) \;.$$

*) This argument is due to Christoffel and is given in Szegö, Chapter 3. In the following this argument will be used several times.

We obtain from the recursion formula that

$$p_{n+1}(d\alpha, x_k) = - \frac{\gamma_{n-1}(d\alpha)\,\gamma_{n+1}(d\alpha)}{\gamma_n^2(d\alpha)}\, p_{n-1}(d\alpha, x_k) \;.$$

Hence

$$(1-x_k^2)p_{n-1}(d\beta, x_k) = \Big[\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta)\,\gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)}\Big] \cdot$$

$$\cdot\, p_{n-1}(d\alpha, x_k) \;.$$

Putting this into (1) we obtain

$$\sum_{k=1}^{n} \lambda_k \frac{p_{n-1}^2(d\alpha, x_k)}{1 - x_k^2} = \Big[1 + \frac{\gamma_{n-1}^2(d\beta)}{\gamma_n^2(d\alpha)}\Big] \;.$$

$$\cdot \Big[\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\beta)} + \frac{\gamma_{n-1}(d\beta)\,\gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)}\Big]^{-2}$$

From $\alpha \in S$ follows $\beta \in S$ and we can use Lemma 2 to show that the limit of the right hand side is 2 . Thus by Theorem 3.2.3 if $\varepsilon \in (0,1)$ then

$$\lim_{n\to\infty} \sum_{|x_{kn}|>1-\varepsilon} \lambda_{kn}(d\alpha)\, \frac{p_{n-1}^2(d\alpha, x_{kn})}{1 - x_{kn}^2} = \frac{4}{\pi}\int_{1-\varepsilon}^{1} v(t)dt \;.$$

Let f be Riemann integrable on $[-1,1]$. By Theorem 3.2.3 we have to show that

$$\lim_{\varepsilon\to +0}\ \limsup_{n\to\infty} \Big| \sum_{|x_{kn}|>1-\varepsilon} \lambda_{kn}(d\alpha)\, f(x_{kn})\, \frac{p_{n-1}^2(d\alpha, x_{kn})}{1 - x_{kn}^2} \Big| = 0 \;.$$

Since f must be bounded on $[-1,1]$ this holds by the previous formula.

Theorem 3 will be used to investigate some interpolation processes.

<u>Definition 4.</u> Let $\alpha \in S$. Then

$$\Gamma(\theta) = -\frac{1}{2\pi}\int_{-1}^{1} \frac{\log W(t) - \log W(x)}{t-x} \frac{\sqrt{1-x^2}}{\sqrt{1-t^2}}\, dt, \quad x = \cos\theta ,$$

where $W(x) = \alpha'(x)\sqrt{1-x^2}$, $-1 \le x \le 1$, $0 \le \theta \le \pi$.

<u>Lemma 5.</u> If $\alpha \in S$ then

$$\lim_{n\to\infty} \int_0^{\pi} \Big| p_n(d\alpha, \cos\theta)\sqrt{\alpha'(\cos\theta)\sin\theta} - \sqrt{\frac{2}{\pi}}\cos[n\theta - \Gamma(\theta)]\Big|^2 d\theta = 0$$

<u>Proof.</u> See Geronimus, Chapter IX.

In the following we shall apply Lemma 5 several times, here we give three applications of it.

<u>Theorem 6.</u> If $\alpha \in S$ then

$$\lim_{n\to\infty} \int_{-1}^{1} p_n^2(d\alpha, x)\, d[\alpha_s(x) + \alpha_j(x)] = 0 .$$

<u>Proof.</u> By the Riemann-Lebesgue lemma and Lemma 5 $\int_{-1}^{1} p_n^2(d\alpha, x)\alpha'_{ac}(x)dx \to 1$ when $n \to \infty$.

<u>Theorem 7.</u> Let $\alpha \in S$ and $f \in L^{\infty}_{d\alpha}$. Then

$$\lim_{n\to\infty} \int_{-1}^{1} f(x)\, p_n^2(d\alpha, x)\, d\alpha(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx .$$

<u>Proof.</u> Use Lemma 5, Theorem 6 and the Riemann-Lebesgue lemma.

<u>Theorem 8.</u> Let $\alpha \in S$, $0 < p < \infty$, $g(\ge 0) \in L^1_{d\alpha}$. If

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(d\alpha, t)|^p g(t)\, d\alpha(t) = 0$$

then $g(t) = 0$ for almost every $t \in [-1,1]$.

<u>Proof.</u> Let first $2 \le p < \infty$. Put for $M > 0$

$$g_M(t) = \min\{g(t), M\} .$$

Then $g_M \in L^{\infty}_{d\alpha}$. Further

$$\int_{-1}^{1} p_n^2(d\alpha, t)\, g_M(t)^{\frac{2}{p}}\, d\alpha(t) \le [\int_{-1}^{1} |p_n(d\alpha, t)|^p g(t) d\alpha(t)]^{\frac{2}{p}} \cdot [\alpha(1) - \alpha(-1)]^{\frac{p-2}{p}} .$$

By the hypothesis and Theorem 7

$$\frac{1}{\pi}\int_{-1}^{1} g_M(t)^{\frac{2}{p}} \frac{dt}{\sqrt{1-t^2}} = 0$$

for every $M > 0$. Hence $g = 0$. Let now $1 \le p < 2$. Let $g^*(t) = g(\cos t)$, $\varepsilon > 0$. Then

$$\int_{-1}^{1} |p_n(d\alpha, t)|^p g(t)\, d\alpha(t) \ge \int_{-1}^{1} |p_n(d\alpha, t)|^p g(t)\alpha'(t)dt =$$

$$= \int_0^{\pi} |p_n(d\alpha, \cos t)\sqrt{\alpha'(\cos t)\sin t}\,|^p [\alpha'(\cos t)\sin t]^{1-\frac{p}{2}} g^*(t)dt .$$

Let $g_1(t) = [\alpha'(\cos t)\sin t]^{1-\frac{p}{2}} g^*(t)$. Then

$$[\int_{-1}^{1} |p_n(d\alpha,t)|^p g(t)\, d\alpha(t)]^{\frac{1}{p}} \geq$$

$$\geq [\varepsilon \int_{g_1 \geq \varepsilon} |p_n(d\alpha, \cos t)\sqrt{\alpha'(\cos t)\sin t}\,|^p dt]^{\frac{1}{p}} \geq$$

$$\geq \varepsilon^{\frac{1}{p}} [\int_{g_1 \geq \varepsilon} |\sqrt{\tfrac{2}{\pi}} \cos(nt - \Gamma(t))|^p]^{\frac{1}{p}} -$$

$$- \varepsilon^{\frac{1}{p}} [\int_0^{\pi} |p_n(d\alpha, \cos t)\sqrt{\alpha'(\cos t)\sin t} - \sqrt{\tfrac{2}{\pi}} \cos(nt - \Gamma(t))|^p dt]^{\frac{1}{p}} .$$

Since $p < 2$ $|\cos|^p \geq |\cos|^2$ and the second integral here converges to 0 by Lemma 5. Thus by the hypothesis

$$\liminf_{n\to\infty} \int_{g_1 \geq \varepsilon} \cos^2(nt - \Gamma(t))dt = 0 ,$$

that is $\text{mes}(g_1 \geq \varepsilon) = 0$. Thus $g_1 = 0$ and consequently $g = 0$. If $0 < p < 1$ we can repeat the previous arguments, the only difference is that we consider $\int|\,|^p$ instead of $[\int|\,|^p]^{\frac{1}{p}}$.

Lemma 9. Let $\alpha \in M(a,0)$. Let $\{n_k\}$ and $\{m_k\}$ be two sequences of natural integers such that at least one of them converges to ∞ when $k \to \infty$. *If* f *is continuous on* $\Delta(d\alpha)$ *then*

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, p_{n_k}(d\alpha,t)\, p_{m_k}(d\alpha,t)\, d\alpha(t) = \begin{cases} f(a) & \text{if } \lim\limits_{k\to\infty}(n_k - m_k) = 0 \\ 0 & \text{if } \liminf\limits_{k\to\infty} |n_k - m_k| > 0 . \end{cases}$$

Proof. In the first case we can suppose without loss of generality that $n_k = m_k = k$ for every k. Because of continuity we can also suppose that

f is a polynomial. If f is constant the Lemma is certainly true. Otherwise

$$f(x) = f(a) + \sum_{j=1}^{\deg f} \frac{f^{(j)}(a)}{j!}(x-a)^j .$$

We shall show that for every $j \geq 1$

$$(2) \qquad \lim_{k\to\infty} \int_{-\infty}^{\infty} (x-a)^j\, p_k^2(d\alpha,x)\, d\alpha(x) = 0 .$$

If $j = 1$ then (2) means that $\lim\limits_{k\to\infty} \alpha_k(d\alpha) = a$. If $j = 2$ then

$$(x-a)^2 p_k^2(d\alpha,x) = (x-a)\, p_k(d\alpha,x)\, [\frac{\gamma_k(d\alpha)}{\gamma_{k+1}(d\alpha)} p_{k+1}(d\alpha,x) +$$

$$+ (\alpha_k(d\alpha) - a)\, p_k(d\alpha,x) + \frac{\gamma_{k-1}(d\alpha)}{\gamma_k(d\alpha)} p_{k-1}(d\alpha,x)] .$$

Hence

$$\int_{-\infty}^{\infty} (x-a)^2 p_k^2(d\alpha,x)\, d\alpha(x) = \frac{\gamma_k^2}{\gamma_{k+1}^2} + (\alpha_k - a)^2 + \frac{\gamma_{k-1}^2}{\gamma_k^2} \xrightarrow[k\to\infty]{} 0 .$$

Since $\text{supp}(d\alpha)$ is compact (2) holds also for $j > 2$ if it holds for $j = 2$. The second case can be obtained from the first one as follows. If k is large then $m_k \neq n_k$. Thus

$$\int_{-\infty}^{\infty} f(t)\, p_{n_k}(d\alpha,t)\, p_{m_k}(d\alpha,t)\, d\alpha(t) =$$

$$= \int_{-\infty}^{\infty} [f(t) - f(a)]\, p_{m_k}(d\alpha,t)\, p_{n_k}(d\alpha,t)\, d\alpha(t) ,$$

that is the absolute value of the left side is not greater then

$$\{\int_{-\infty}^{\infty} |f(t) - f(a)|\, p_{n_k}^2(d\alpha, t)\, d\alpha(t) \cdot$$

$$\cdot \int_{-\infty}^{\infty} |f(t) - f(a)|\, p_{m_k}^2(d\alpha, t)\, d\alpha(t)\}^{\frac{1}{2}} .$$

Here both factors are bounded and at least one of them tends to 0 when $k \to \infty$.

Theorem 10. Lemma 9 remains true if f, instead of being continuous on $\Delta(d\alpha)$, is merely bounded on $\mathrm{supp}(d\alpha)$, continuous at a and it is $d\alpha$ measurable.

Proof. Let $\varepsilon > 0$. Then

$$\int_{-\infty}^{\infty} g(t)\, p_k^2(d\alpha, t)\, d\alpha(t) \le \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t)\, d\alpha(t) \le 1$$

where g is continuous function vanishing outside $[a-\varepsilon, a+\varepsilon]$ with $g(a) = 1$ and $0 < g(t) \le 1$ for $|a-t| \le \varepsilon$. Thus by Lemma 9

$$\lim_{k\to\infty} \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t)\, d\alpha(t) = 1 .$$

We have

$$|\int_{-\infty}^{\infty} f(t)\, p_k^2(d\alpha, t)\, d\alpha(t) - f(a)| \le \sup_{|t-a|\le\varepsilon} |f(t) - f(a)| \cdot$$

$$\cdot \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t)\, d\alpha(t) + 2 \sup_{t\in \mathrm{supp}(d\alpha)} |f(t)|\, [1 - \int_{a-\varepsilon}^{a+\varepsilon} p_k^2(d\alpha, t)\, d\alpha(t)] .$$

Let $k \to \infty$ and then $\varepsilon \to 0$. The case when $\liminf_{n\to\infty} |m_k - n_k| > 0$ follows from the case when $\lim_{k\to\infty} (m_k - n_k) = 0$.

Definition 11. Let us define the numbers $\alpha_{nk}(d\alpha)$ for $n = 1, 2, \ldots$ and $k = n-1, n, n+1$ as follows

$$\alpha_{nk}(d\alpha) = \begin{cases} \gamma_{n-1}(d\alpha)/\gamma_n(d\alpha) & \text{for } k = n-1 \\ \alpha_n(d\alpha) & \text{for } k = n \\ \gamma_n(d\alpha)/\gamma_{n+1}(d\alpha) & \text{for } k = n+1 . \end{cases}$$

Lemma 12. Let m be a nonnegative integer and $n > m$. Then

$$x^m p_n(d\alpha, x) = \sum_{\substack{-1\le k_i\le 1 \\ i=1,2,\ldots,m}} \alpha_{n,n+k_1}(d\alpha)\, \alpha_{n+k_1, n+k_1+k_2}(d\alpha) \cdot$$

$$\cdots \alpha_{n+k_1+\ldots+k_{m-1}, n+k_1+\ldots+k_m}(d\alpha)\, p_{n+k_1+\ldots+k_m}(d\alpha, x) .$$

Proof. Apply the recursion formula repeatedly.

Theorem 13. Let $\alpha \in M(a, b)$ with $b > 0$. Let $\{m_k\}$ and $\{n_k\}$ be two sequences of natural integers such that at least one of them converges to ∞ when $k \to \infty$ and the finite or infinite $\lim_{k\to\infty} (m_k - n_k)$ exists. Let f be $d\alpha$ measurable, bounded on $\mathrm{supp}(d\alpha)$ and continuous on $[a-b, a+b]$.

Then

$$(3) \qquad \lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, p_{m_k}(d\alpha, t)\, p_{n_k}(d\alpha, t)\, d\alpha(t) =$$

$$= \lim_{k\to\infty} \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{T_{|m_k - n_k|}(\frac{t-a}{b})}{\sqrt{b^2 - (t-a)^2}}\, dt .$$

Proof. Let, without loss of generality, $\alpha \in M(0,1)$. First we shall prove (3) when f is continuous on $\Delta(d\alpha)$. If $\lim_{k\to\infty} (m_k - n_k) = \infty$ then the right side in (3) equals 0. If f is a polynomial then the integral on the left side of (3) equals 0 if k is big. Thus for every continuous function f the left side in (3) also equals 0. Now let $\lim_{k\to\infty} (m_k - n_k) < \infty$. Then we may assume that for every k $n_k = k$, $m_k = k+\ell$ where ℓ is a fixed nonnegative integer. Because of linearity and continuity arguments; that is, because of Banach-Steinhaus' theorem, we can suppose that f is of the form $f(t) = t^m$ where m is a fixed nonnegative integer. Thus we have to show that for $m = 0,1,2,\ldots$

(4) $$\lim_{k\to\infty} \int_{-\infty}^{\infty} t^m p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) = \frac{1}{\pi}\int_{-1}^{1} t^m \frac{T_\ell(t)}{\sqrt{1-t^2}}\, dt .$$

Let us remark that (4) is true if α is the Tschebyshev weight. For if $k \geq 1$

$$\int_{-\infty}^{\infty} t^m p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) =$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos^m\theta \cdot \frac{1}{2}[\cos\ell\theta + \cos(2k+\ell)\theta]d\theta$$

which equals to

(5) $$\frac{1}{\pi}\int_{-1}^{1} t^m \frac{T_\ell(t)}{\sqrt{1-t^2}}\, dt$$

if $2k+\ell > m$. If $\alpha \in M(0,1)$ then by Lemma 12 we have for $k > m$

$$\int_{-\infty}^{\infty} t^m p_k(d\alpha,t)\, p_{k+\ell}(d\alpha,t)\, d\alpha(t) =$$

$$= \sum_{\substack{-1\leq k_i \leq 1 \\ i=1,2,\ldots,m \\ \sum_{i=1}^{m} k_i = \ell}} a_{k,k+k_1}(d\alpha)\, a_{k+k_1,k+k_1+k_2}(d\alpha) \ldots a_{k+k_1+\ldots+k_{m-1},k+\ell}(d\alpha) .$$

The right side here is convergent when $k\to\infty$ since $\alpha \in M(0,1)$ and ℓ is fixed, its limit depends only on $\lim_{j\to\infty} a_j(d\alpha)$ and $\lim_{j\to\infty} \gamma_{j-1}(d\alpha)/\gamma_j(d\alpha)$, that is in our case this limit equals (5). Hence (4) holds if f is continuous on $\Delta(d\alpha)$. If f is continuous only on $\text{supp}(d\alpha)$ which is closed then f can be extended to a function which is continuous on $\Delta(d\alpha)$. If f *is a function satisfying the conditions of the theorem then we can write* $f = f_1 + f_2$ where f_1 is continuous on $\Delta(d\alpha)$, f_2 is bounded and $d\alpha$ measurable, further f_2 vanishes on $[a-b, a+b]$. Hence, if we can show that

(6) $$\lim_{k\to\infty} \int_{a\pm b}^{\pm\infty} p_k^2(d\alpha,t)\, d\alpha(t) = 0$$

then we finish the proof of the theorem. Let g be continuous function on $\Delta(d\alpha)$ such that $g(t) \geq 0$ for $t \in \Delta(d\alpha)$, $g(t) = 1$ for $t \in \Delta(d\alpha)\setminus[a-b, a+b]$ and

$$\frac{1}{\pi}\int_{a-b}^{a+b} g(t)\frac{1}{\sqrt{b^2-(t-a)^2}}\, dt < \varepsilon$$

where $\varepsilon > 0$ is given. Then

$$\int_{-\infty}^{a-b} + \int_{a+b}^{\infty} p_k^2(d\alpha, t)\,d\alpha(t) \le \int_{-\infty}^{\infty} g(t)\, p_k^2(d\alpha, t)\, d\alpha(t)$$

and using the fact that the theorem has already been proved for continuous functions we obtain (6) by first letting $k \to \infty$ and then $\varepsilon \to 0$.

Using one-sided approximation machinery we obtain immediately from Theorem 13 the following

Theorem 14. Let $\alpha \in M(a,b)$ with $b > 0$. If f is $d\alpha$ measurable, bounded on $\mathrm{supp}(d\alpha)$ and Riemann integrable on $[a-b, a+b]$ then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, p_n^2(d\alpha, t)\, d\alpha(t) = \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{dt}{\sqrt{b^2-(t-a)^2}} .$$

Compares this theorem with Theorem 7.

Corollary 15. Let $\alpha \in M(a,b)$, $b > 0$, $x \in (a-b, a+b)$ and let α be continuous in a neighborhood of x. Then

$$\lim_{h\to 0} \lim_{n\to\infty} \frac{1}{2h} \int_{x-h}^{x+h} p_n^2(d\alpha, t)\, d\alpha(t) = \frac{1}{\pi} \frac{1}{\sqrt{b^2-(t-a)^2}} .$$

Proof. The function $1_{(x-\varepsilon, x+\varepsilon)}$ is $d\alpha$ measurable for $\varepsilon > 0$ small.

Theorem 16. Let $\alpha \in M(a,b)$, $b > 0$. Let f be bounded on $\Delta(d\alpha)$ and Riemann integrable on $[a-b, a+b]$. Then for every fixed integer ℓ

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha, x_{kn})\, p_{n+\ell}(\alpha, x_{kn}) =$$

$$= -\operatorname{sign} \ell \, \frac{2}{\pi b^2} \int_{a-b}^{a+b} f(t)\, U_{|\ell|-1}\left(\frac{t-a}{b}\right) \sqrt{b^2-(t-a)^2}\, dt .$$

Proof. We obtain from Theorems 3.1.3 ($\ell > 0$) and 3.1.13 ($\ell < 0$) and from the recurrence formula that

$$p_{n+\ell}(d\alpha, x_{kn}) = -\operatorname{sign} \ell \;\, U_{|\ell|-1}\left(\frac{x_{kn}-a}{b}\right) p_{n-1}(d\alpha, x_{kn})$$

$$+ \sigma(1) \left[|p_{n-1}(d\alpha, x_{kn})| + |p_{n-2}(d\alpha, x_{kn})| \right]$$

where $\lim_{n\to\infty} \sigma(1) = 0$ uniformly for $1 \le k \le n$ if ℓ is fixed. Thus the theorem follows from Theorem 3.2.3.

Theorem 17. Let $\alpha \in S$, f be Riemann integrable on $[-1,1]$ and ℓ be a fixed integer. Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha, x_{kn}) \frac{p_{n+\ell}(d\alpha, x_{kn})}{1 - x_{kn}^2} =$$

$$= -\operatorname{sign} \ell \, \frac{1}{\pi} \int_{-1}^{1} f(t)\, U_{|\ell|-1}(t) \frac{dt}{\sqrt{1-t^2}} .$$

Proof. Repeat the proof of Theorem 16 and use Theorem 3 instead of Theorem 3.2.3.

5. Eigenvalues of Toepliz Matrices

In Grenander-Szegō the proof of Theorem 7.7 and the example 8.1(f) are not correct. In the first the Gauss-Jacobi mechanical quadrature formula is used for polynomials of degree more than $2n-1$, in the second an orthonormal system is constructed, but it is, in fact, only normed but not orthogonal. In this section we shall obtain results which are a little bit more general than those of Grenander-Szegō, we shall use some methods of the above book in a simplified form.

Lemma 1. Let $\operatorname{supp}(d\alpha)$ be compact and let f be continuous on $\Delta(d\alpha)$ with the modulus of continuity ω. Then

$$\left|\sum_{k=1}^{n} f(x_{kn}(d\alpha)) - \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} f(t)\, p_k^2(d\alpha,t)\, d\alpha(t)\right| \le$$

$$\le n\,\omega(n^{-1/3})[1+\tfrac{1}{2}|\Delta(d\alpha)|^3]$$

for $n > |\Delta(d\alpha)|^{-3}$.

Proof. Since $\sum_{k=0}^{n-1} p_k^2(t) = \sum_{k=1}^{n} \frac{\ell_k^2(t)}{\lambda_k}$ and $\lambda_k = \int_{-\infty}^{\infty} \ell_k^2(t)\, d\alpha(t)$ we have

$$\sum_{k=1}^{n} f(x_{kn}) - \sum_{k=0}^{n-1}\int_{-\infty}^{\infty} f(t)\, p_k^2(t)\, d\alpha(t) =$$

$$= \sum_{k=1}^{n}\int_{\Delta(d\alpha)} [f(x_{kn}) - f(t)]\frac{\ell_{kn}^2(d\alpha,t)}{\lambda_{kn}(d\alpha)}\, d\alpha(t) \equiv A .$$

Let $\varepsilon > 0$. Then

$$|A| \le n\,\omega(\varepsilon) + \sum_{k=1}^{n}\int_{|t-x_k|\ge\varepsilon} |f(x_k) - f(t)|\frac{\ell_k^2(t)}{\lambda_k}\, d\alpha(t) \le$$

$$\le n\,\omega(\varepsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2}\frac{\omega(|\Delta(d\alpha)|)}{\varepsilon^2}\sum_{k=1}^{n}\lambda_k\, p_{n-1}^2(x_k)\int_{-\infty}^{\infty} p_n^2(t)\, d\alpha(t) =$$

$$= n[\omega(\varepsilon) + \frac{\gamma_{n-1}^2}{\gamma_n^2}\frac{\omega(|\Delta(d\alpha)|)}{n\,\varepsilon^2}] .$$

By easy calculation $\gamma_{n-1}/\gamma_n \le |\Delta(d\alpha)|/2$. Thus

$$|A| \le n[\omega(\varepsilon) + \frac{1}{4}|\Delta(d\alpha)|^2\frac{\omega(|\Delta(d\alpha)|)}{n\varepsilon^2}] .$$

Let now $\varepsilon = n^{-\frac{1}{2}}$ and use the inequality

$$\frac{\omega(a)}{a} \le 2\frac{\omega(b)}{b} \qquad (0 < b \le a)$$

which holds for every modulus of continuity ω.

Theorem 2. Let $\alpha \in M(a,0)$. Let f be bounded on $\Delta(d\alpha)$ and continuous at a. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_{kn}(d\alpha)) = f(a) .$$

Proof. If f is continuous on $\Delta(d\alpha)$ then the theorem follows immediately from Lemma 1 and Lemma 4.2.9. If f is continuous only at a and it is bounded on $\Delta(d\alpha)$ then we fix $\varepsilon > 0$ and we construct two continuous functions f_1 and f_2 on $\Delta(d\alpha)$ so that

$$f_1(x) \le f(x) \le f_2(x)$$

for $x \in \Delta(d\alpha)$ and $f_2(a) - f_1(a) \leq \varepsilon$ and we apply standard arguments.

Theorem 3. Let $\alpha \in M(a,b)$ with $b > 0$. Let f be bounded on $\Delta(d\alpha)$ and Riemann integrable on $[a-b, a+b]$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{1}{\pi} \int_{a-b}^{a+b} f(t) \frac{1}{\sqrt{b^2-(t-a)^2}}\, dt ,$$

in particular, for every segment $\Delta \subset \Delta(d\alpha)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x_{kn}(d\alpha)\in\Delta} 1 = \frac{1}{\pi} \int_{\Delta \cap [a-b, a+b]} \frac{1}{\sqrt{b^2-(t-a)^2}}\, dt .$$

Proof. If f is continuous on $\Delta(d\alpha)$ then use Lemma 1 and Theorem 4.2.13, otherwise apply the one-sided approximation machinery.

Now we shall translate the previous results into a different language.

Let $\operatorname{supp}(d\alpha)$ be compact, $f \in L^1_{d\alpha}$ be real valued and let us consider the Toeplitz matrix $A(f, d\alpha)$ defined as

$$A(f, d\alpha) = [[\int_{-\infty}^{\infty} f(t)\, p_i(d\alpha, t)\, p_j(d\alpha, t)\, d\alpha(t)]]_{i,j=0}^{\infty} .$$

Let, further, $A_n(f, d\alpha)$ be the truncated matrix consisting of n^2 elements. The characteristic polynomial $h_n(f, d\alpha, x)$ is $\det[A_n(f, d\alpha) - xE]$, the zeros of $h_n(f, d\alpha, x)$, which we denote by $x_{kn}(f, d\alpha)$ $(k = 1, 2, \ldots, n)$ are called the eigenvalues of $A_n(f, d\alpha)$. Since $A_n^* = A_n$ all x_{kn} are real.

If $f(t) \equiv 1$ then $A(f, d\alpha) = E$ and $h_n(f, d\alpha, x) = (1-x)^n$, that is $x_{kn}(f, d\alpha) = 1$ for $k = 1, 2, \ldots, n$.

Lemma 4. Let $f(t) \equiv t$. Then for $n = 1, 2, \ldots$

$$h_n(f, d\alpha, x) = (-1)^n \gamma_n^{-1}(d\alpha)\, p_n(d\alpha, x) .$$

Proof. $(-1)^n h_n(f, d\alpha, x)$ satisfy the same recurrence formula as $\gamma_n^{-1}(d\alpha)\, p_n(d\alpha, x)$, and for $n = 1, 2$ the lemma can easily be checked.

Definition 5. $\{a_{kn}\}_{k=1}^{n}$ and $\{b_{kn}\}_{k=1}^{n}$ $(n = 1, 2, \ldots;\ a_{kn} \in \mathbb{R},\ b_{kn} \in \mathbb{R})$ are equally distributed if there exists an interval Δ such that $a_{kn} \in \Delta$ and $b_{kn} \in \Delta$ for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$, further for every continuous function f on Δ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} [f(a_{kn}) - f(b_{kn})] = 0 .$$

We obtain from Theorems 2 and 3 the following

Theorem 6. Let $f(t) \equiv t$, $a \in \mathbb{R}$, $b \geq 0$. Then for every pair of weights α_1 and α_2 from $M(a,b)$ the eigenvalues of $A_n(f, d\alpha_1)$ and $A_n(f, d\alpha_2)$ are equally distributed.

Definition 7. Let $A = [[a_{ik}]]_{i,k=1}^{n}$ be a real $n \times n$ matrix. Then

$$\operatorname{Tr} A = \sum_{k=1}^{n} a_{kk} ,$$

$$\|A\| = |\tfrac{1}{n} T_r A| ,$$

$$((A))^2 = \sup_u \frac{(Au, Au)}{(u, u)} ,$$

here $u = (u_1, \ldots, u_n)$, $(u, v) = \sum_{k=1}^{n} u_k v_k$, further

$$\sqrt{A} = [[\sqrt{a_{jk}}\,]]_{j,k=1}^{n} .$$

Properties 8. $\operatorname{Tr} AB = \operatorname{Tr} BA$, $\operatorname{Tr} A = \Sigma$(eigenvalues of A),

$$((A))^2 \geq \max_{k=1,2,\ldots,n} \sum_{j=1}^{n} a_{jk}^2, \quad ((AB)) \leq ((A))((B)) .$$

If $A^* = A$ then $((A)) = \max|\text{eigenvalues of } A|$.

Lemma 9. For every $n \in \mathbb{N}^+$

$$((A_n(f, d\alpha))) \leq \sup_{t \in \operatorname{supp}(d\alpha)} |f(t)| .$$

Proof. Let λ be an eigenvalue of $A_n(f, d\alpha)$ for which $((A_n f, d\alpha))) = |\lambda|$ and let u be the corresponding eigenvector with $(u, u) = 1$. Then

$$((A_n(f, d\alpha))) = |\lambda|\,(u, u) = |(\lambda u, u)| = |(A_n(f, d\alpha)u, u)| =$$

$$= \Big|\int_{-\infty}^{\infty} f(t)\Big(\sum_{k=0}^{n-1} p_k(d\alpha, t)u_k\Big)^2 d\alpha(t)\Big| \leq \sup_{t \in \operatorname{supp}(d\alpha)} |f(t)|\,(u, u)$$

where $u = (u_0, u_1, \ldots, u_{n-1})$.

Lemma 10. Let $m \in \mathbb{N}$ be fixed and let f_i $(i = 1, 2, \ldots, m)$ be given. Then for every $j \in \{1, 2, \ldots, m\}$

$$\limsup_{n\to\infty} \Big\|\prod_{i=1}^{m} A_n(f_i, d\alpha)\Big\| \leq$$

$$\leq \limsup_{n\to\infty} \Big\|\sqrt{A_n(f_j^2, d\alpha)}\Big\| \prod_{\substack{i=1\\ i\neq j}}^{m} \sup_{t \in \operatorname{supp}(d\alpha)} |f_i(t)| .$$

Proof. For $m = 1$ the lemma is certainly true. Let $m \geq 2$. By Properties 8 we can suppose that $j = 1$. Let

$$B = \prod_{i=2}^{m} (f_i, d\alpha) .$$

Then

$$\|A_n(f_1, d\alpha)B\| = \Big|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}(f_1, d\alpha)b_{jk}\Big| \leq$$

$$\leq \Big|\frac{1}{n}\sum_{k=1}^{n}\Big(\sum_{j=1}^{n} a_{kj}^2(f_1, d\alpha)\sum_{j=1}^{n} b_{jk}^2\Big)^{\frac{1}{2}}\Big| \leq$$

$$\leq \max_{k=1,2,\ldots,n}\Big(\sum_{j=1}^{n} b_{jk}^2\Big)^{\frac{1}{2}}\Big|\frac{1}{n}\sum_{k=1}^{n}\Big(\sum_{j=1}^{n} a_{kj}^2(f_1, d\alpha)\Big)^{\frac{1}{2}}\Big| .$$

By Bessel's inequality

$$\sum_{j=1}^{n} a_{kj}^2(f_1, d\alpha) = \sum_{j=0}^{n-1}\Big\{\int_{-\infty}^{\infty} p_{k-1}(d\alpha, t)\, p_j(d\alpha, t)\, f_1(t)d\alpha(t)\Big\}^2 \leq$$

$$\leq \int_{-\infty}^{\infty} p_{k-1}^2(d\alpha, t)\, f_1^2(t)\, d\alpha(t) = a_{kk}(f_1^2, d\alpha) .$$

Thus by Properties 8

$$\|A_n(f_1, d\alpha)B\| \leq \Big\|\sqrt{A_n(f_1^2, d\alpha)}\Big\|\,((B)) \leq$$

$$\leq \Big\|\sqrt{A_n(f_1, d\alpha)}\Big\| \prod_{i=2}^{m} ((A_n(f_i, d\alpha)))$$

and now use Lemma 9.

Lemma 11. Let $\operatorname{supp}(d\alpha)$ be compact, $s \in \mathbb{N}$ and π be a polynomial. Then

$$\lim_{n\to\infty} \| A_n^s(\pi, d\alpha) - A_n(\pi^s, d\alpha) \| = 0 .$$

Proof. See Grenander-Szegö, §8.1.

Let us remark that in the previous lemma it is sufficient to suppose that for $d\alpha$ the moment problem is well defined, that is for $x^k f$, $x^k g \in L^2 d\alpha$ $(k = 0, 1, \dots)$ $A(fg, d\alpha) = A(f, d\alpha)\, A(g, d\alpha)$.

Lemma 12. Let $\alpha \in M(a,b)$, f be $d\alpha$ measurable and bounded on $\mathrm{supp}(d\alpha)$. If $b = 0$ and f is continuous at a then

$$\lim_{n\to\infty} \| \sqrt{A_n(f^2, d\alpha)} \| = |f(a)| .$$

If $b > 0$ and f is Riemann integrable on $[a-b, a+b]$ then

$$\lim_{n\to\infty} \| \sqrt{A_n(f^2, d\alpha)} \| = \left[\frac{1}{\pi} \int_{a-b}^{a+b} f^2(t) \frac{dt}{\sqrt{b^2 - (ta)^2}} \right]^{\frac{1}{2}} .$$

Proof. See Theorems 4.2.10 and 4.2.14.

Let us recall that we consider Toeplitz matrices $A(f, d\alpha)$ for real valued functions f.

Theorem 13. Let $\alpha \in M(a,b)$, $s \in \mathbb{N}$, f be $d\alpha$ measurable and bounded on $\mathrm{supp}(d\alpha)$. Let for $b = 0$ f be continuous at a and for $b > 0$ f be Riemann integrable on $[a-b, a+b]$. Then

$$\lim_{n\to\infty} \| A_n^s(f, d\alpha) - A_n(f^s, d\alpha) \| = 0 .$$

Proof. Let π be a polynomial. Since $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$ we have

$$\| A_n^s(f, d\alpha) - A_n(f^s, d\alpha) \| = \| A_n^s(f - \pi + \pi, d\alpha) -$$

$$- A_n((f - \pi + \pi)^s, d\alpha) \| =$$

$$= \| [A_n^s(\pi, d\alpha) - A_n(\pi^s, d\alpha)] +$$

$$+ \sum_{j=1}^{s} \binom{s}{j} A_n^j(f - \pi, d\alpha)\, A_n^{s-j}(\pi, d\alpha) +$$

$$+ A_n\Big(- \sum_{j=1}^{s} \binom{s}{j} (f - \pi)^j \pi^{s-j}, d\alpha\Big) \| =$$

$$= \| A_I + A_{II} + A_{III} \| .$$

Let, for the simplicity, $b > 0$. If $b = 0$ then we shall see from the proof that we can put $\pi(t) \equiv f(a)$. By Lemma 11

$$\lim_{n\to\infty} \| A_I \| = 0 .$$

By Theorem 4.2.14

$$(1) \qquad \lim_{n\to\infty} \| A_{III} \| = \Big| \frac{1}{\pi} \int_{a-b}^{a+b} \Big[\sum_{j=1}^{s} \binom{s}{j} (f(t) - \pi(t))^j \cdot \pi(t)^{s-j} \Big] \frac{dt}{\sqrt{b^2 - (t-a)^2}} \Big| .$$

We have, further, by Lemma 10

$$\limsup_{n\to\infty} \| A_{II} \| \le \sum_{j=1}^{s} \binom{s}{j} \limsup_{n\to\infty} \| \sqrt{A_n((f-\pi)^2, d\alpha)} \| \cdot \sup_{t \in \mathrm{supp}(d\alpha)} |f(t) - \pi(t)|^{j-1} \sup_{t \in \mathrm{supp}(d\alpha)} |\pi(t)|$$

Hence by Lemma 12

$$\limsup_{n\to\infty} \|A_{II}\| \le \left[\frac{1}{\pi}\int_{a-b}^{a+b} \frac{[f(t)-\pi(t)]^2}{\sqrt{b^2-(t-a)^2}}\,dt\right]^{\frac{1}{2}} \cdot$$

$$\cdot\, 2^s \sup_{t\in \operatorname{supp}(d\alpha)} (|f(t)| + |\pi(t)|)^{s-1} \equiv R(f,\pi)\ ,$$

and from (1) we get the same estimate for $\lim_{n\to\infty} \|A_{III}\|$:

$$\lim_{n\to\infty} \|A_{III}\| \le R(f,\pi)\ .$$

The theorem will be proved if we show that for every $\varepsilon > 0$ one can find a polynomial π such that $R(f,\pi) < \varepsilon$. This latter can be shown easily. Let f_1 be a function on $\Delta(d\alpha)$ such that $f_1(t) = f(t)$ for $t \in [a-b, a+b]$, $|f(t)| \le |f_1(t)|$ for $t \in \Delta(d\alpha)$ and $f_1 \in L^\infty(\Delta(d\alpha))$. Let us send $\Delta(d\alpha)$ to $[-1,1]$ by a linear transformation and then to $[0,\pi]$ by $x = \cos\theta$ $(-1 \le x \le 1,\ 0 \le \theta \le \pi)$. Set $g(\theta) = f_1(t)$ $(t \in \Delta(d\alpha),\ \theta \in [0,\pi])$. Let g^* denote the even extension of g to $[-\pi,\pi]$. Consider the Fejér sums of g^*. They are cosine polynomials, they are bounded in maximum norm: $\|\sigma_n(g^*)\| \le \sup_{t\in\Delta(d\alpha)} |f_1(t)|$, they converge to g^* in e.g. $L^{10}[-\pi,\pi]$. Let us return now to $\Delta(d\alpha)$ and remark that

$$\int_{a-b}^{a+b} \frac{|f(t)-\pi(t)|^2}{\sqrt{b^2-(t-a)^2}}\,dt \le C\left[\int_{\Delta(d\alpha)} |f(t)-\pi(t)|^{10}\, v_{\Delta(d\alpha)}(t)\,dt\right]^{1/5}\ .$$

Theorem 14. Let $\alpha \in M(a,0)$, f be $d\alpha$ measurable, bounded on $\operatorname{supp}(d\alpha)$ and continuous at a. Let $\mathfrak{F}$ be bounded on $\Delta \subset f(\operatorname{supp}(d\alpha))$ and continuous at $f(a)$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathfrak{F}(x_{kn}(f,d\alpha)) = \mathfrak{F}(f(a))\ .$$

Proof. Observe that if $f(\operatorname{supp}(d\alpha)) \subset \Delta$ then $x_{kn}(f,d\alpha) \in \Delta$ for $n = 1,2,\ldots$ and $k = 1,2,\ldots,n$. This can be shown by exactly the same argument as in Lemma 9. Let first $\mathfrak{F}(u) = u^s$ $(s = 0,1,2,\ldots)$. For $s = 0$ the theorem is true. For $s = 1$

$$\sum_{k=1}^{n} x_{kn}(f,d\alpha) = \operatorname{Tr} A_n(f,d\alpha)$$

and we apply Theorem 4.2.10. If $s \ge 2$ then

$$\sum_{k=1}^{n} x_{kn}^s(f,d\alpha) = \operatorname{Tr} A_n^s(f,d\alpha)$$

as is well known and we use Theorems 13 and 4.2.10. Hence the theorem is true if $\mathfrak{F}$ is a polynomial, and consequently it is true if $\mathfrak{F}$ is continuous on Δ. Otherwise we use one-sided approximations.

Theorem 15. Let $\alpha \in M(a,b)$ with $b > 0$. Let f be $d\alpha$ measurable, bounded on $\operatorname{supp}(d\alpha)$ and Riemann integrable on $[a-b, a+b]$. Let $\mathfrak{F}$ be continuous on $\Delta \supset f(\operatorname{supp}(d\alpha))$. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathfrak{F}(x_{kn}(f,d\alpha)) = \frac{1}{\pi}\int_{a-b}^{a+b} \frac{\mathfrak{F}(f(t))}{\sqrt{b^2-(t-a)^2}}\,dt\ .$$

Proof. The same as that of Theorem 14.

Theorems 14 and 15 give us the following

Theorem 16. Let $a \in \mathbb{R}$, $b \ge 0$ and f be continuous on $\mathbb{R}$. Then for each pair of weights α_1 and α_2 belonging to $M(a,b)$ the eigenvalues of $A_n(f,d\alpha_1)$ and $A_n(f,d\alpha_2)$ are equally distributed.

6. Christoffel Functions

6.1. An Interpolation Process

The Hermite-Fejér interpolation polynomial $H_n(d\alpha, f, x)$ is the unique polynomial at degree at most $2n-1$ which satisfies the conditions

$$H_n(d\alpha, f, x_{kn}) = f(x_{kn}), \quad H_n'(d\alpha, f, x_{kn}) = 0$$

for $k = 1, 2, \ldots, n$. Here $x_{kn} = x_{kn}(d\alpha)$. Hence

$$H_n(d\alpha, f, x) = \sum_{k=1}^{n} f(x_{kn})[1 - 2\ell_{kn}'(d\alpha, x_{kn})(x - x_{kn})]\ell_{kn}^2(d\alpha, x) .$$

Let us compute $\ell_k'(x_k)$. We have

$$\lambda_n^{-1}(x) = \sum_{k=1}^{n} \frac{\ell_k^2(x)}{\lambda_{kn}} ,$$

that is

$$-\lambda_n'(x)\lambda_n^{-2}(x) = \sum_{k=1}^{n} \frac{2\ell_k'(x)\ell_k(x)}{\lambda_{kn}} .$$

Putting here $x = x_{kn}$ we obtain

(1) $$-2\ell_k'(x_k) = \lambda_n'(x_k)\lambda_{kn}^{-1} .$$

Thus

$$H_n(d\alpha, f, x) = \sum_{k=1}^{n} f(x_{kn})[\lambda_{kn}(d\alpha) + \lambda_n'(d\alpha, x_{kn})(x - x_{kn})] \cdot \frac{\ell_{kn}^2(d\alpha, x)}{\lambda_{kn}(d\alpha)} .$$

This is Freud's representation for $H_n(d\alpha, f, x)$. (See [6]). In the brackets here we find an approximate expression for $\lambda_n(d\alpha, x)$:

$$\lambda_n(x) = \lambda_{kn} + \lambda_n'(x_{kn})(x - x_{kn}) + \frac{\lambda_n''(\theta)}{2}(x - x_{kn})^2$$

where θ is between x and x_{kn}. Let us replace the expression in the brackets by $\lambda_n(d\alpha, x)$. Denote the resulting expression by $F_n(d\alpha, f, x)$:

$$F_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \sum_{k=1}^{n} f(x_{kn}) \frac{\ell_{kn}^2(d\alpha, x)}{\lambda_{kn}(d\alpha)} .$$

For $z \in \mathbb{C}$ put

$$F_n(d\alpha, f, z) = \lambda_n^*(d\alpha, z) \sum_{k=1}^{n} f(x_{kn}) \frac{\ell_{kn}^2(d\alpha, z)}{\lambda_{kn}(d\alpha)}$$

and

$$\tilde{F}_n(d\alpha, f, z) = \lambda_n(d\alpha, z) \sum_{k=1}^{n} f(x_{kn}) \frac{|\ell_{kn}^2(d\alpha, z)|}{\lambda_{kn}(d\alpha)} .$$

(See 4.1.)

Properties 1. (i) If $f(x) \equiv 1$ then $F_n(f, x) \equiv 1$. (ii) If $f(x) \geq 0$ for $x \in \Delta(d\alpha)$ then $F_n(f, x) \geq 0$ for $x \in \mathbb{R}$. (iii) $F_n(d\alpha, f, x_{kn}) = f(x_{kn})$ for $k = 1, 2, \ldots, n$. (iv) $F_n'(d\alpha, f, x_{kn}) = 0$ for $k = 1, 2, \ldots, n$ (use (1)). (v) F_n is a rational function at degree $(2n-2, 2n-2)$, only the numerator depends on f.

Because of (ii) we can expect that for many weights α $F_n(d\alpha, f)$ converges to f whenever f is continuous. The surprising result is that the above class of weights α is very large. We shall consider convergence of $F_n(d\alpha, f)$ for $\alpha \in M(a,b)$ with $b > 0$ since the case when $\alpha \in M(a,0)$ is less interesting. In order to avoid complicated formulas we shall assume, without loss of generality, that $\alpha \in M(0,1)$. Concerning $\rho(z)$ see Definition 4.1.8.

Theorem 2. Let $\alpha \in M(0,1)$. Let f be bounded on $\Delta(d\alpha)$. If f is continuous at some $x \in \text{supp}(d\alpha)$ then

$$\lim_{n\to\infty} F_n(d\alpha, f, x) = f(x) \quad (2)$$

If f is continuous on the segment $\Delta \subset (-1,1)$ then (2) is satisfied uniformly for $x \in \Delta$. If f is Riemann integrable on $[-1,1]$ and bounded on $\Delta(d\alpha)$ then for every $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\lim_{n\to\infty} F_n(d\alpha, f, z) = \frac{\rho^2(z)-1}{2\pi} \int_{-1}^{1} f(t) \frac{\sqrt{1-t^2}}{(z-t)^2}\, dt$$

and

$$\lim_{n\to\infty} \tilde{F}_n(d\alpha, f, z) = \frac{|\rho(z)|^2-1}{2\pi} \int_{-1}^{1} f(t) \frac{\sqrt{1-t^2}}{|z-t|^2}\, dt .$$

Proof. The theorem follows from Theorems 3.2.3, 3.3.8, 4.1.11 and Properties 1. Let us prove e.g. the first part of the theorem. Let $\varepsilon > 0$. Then

$$|F_n(f,x) - f(x)| \le \sup_{|x-t|\le\varepsilon} |f(t) - f(x)| + 2\frac{\gamma_{n-1}^2}{\gamma_n^2} \varepsilon^{-2} \lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) \sup_{t\in\Delta(d\alpha)} |f(t)| .$$

First let $n \to \infty$ and then $\varepsilon \to 0$.

Definition 3. Let $g(\ge 0) \in L^1_{d\alpha}$. Then α_g is defined by

$$\alpha_g(t) = \int_{-\infty}^{t} g(u)\, d\alpha(u) .$$

Let us remark that α_g may not be a weight, it can happen that either α_g has only a finite number of points of increase or not each moment of α_g is finite. If g is a polynomial then α_g certainly is a weight. If $\text{supp}(d\alpha)$ is compact and $g^{-1} \in L^1_{d\alpha}$ then also α_g is a weight.

Lemma 4. Let g be a linear function, nonnegative on $\text{supp}(d\alpha)$ $(g(t) = c_1 t + c_2,\ c_1 \ne 0)$. Then

$$\lambda_n^{-1}(d\alpha_g, x) = \sum_{k=1}^{n} \frac{\ell_{kn}^2(d\alpha, x)}{g(x_{kn}(d\alpha))\lambda_{kn}(d\alpha)} . \quad (3)$$

Proof. (By Freud [7]). Let us denote the right hand side of (3) by A. We have to show that for every π_{n-1}

$$\pi_{n-1}^2(x) \le A \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha_g(t) \quad (4)$$

and for every $x \in \mathbb{R}$ there exists a π_{n-1}^* which turns (4) into equality. We have $\pi_{n-1} \equiv L_n(d\alpha, \pi_{n-1})$. Hence

$$\pi_{n-1}^2(x) \le \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, \pi_{n-1}^2(x_{kn})\, g(x_{kn}) A .$$

Since $\deg \pi_{n-1}^2 g \le 2n-1$ we can use the Gauss-Jacobi mechanical quadrature to obtain

$$\pi_{n-1}^2(x) \le A\int_{-\infty}^{\infty}\pi_{n-1}^2(t)\,g(t)\,d\alpha(t) = A\int_{-\infty}^{\infty}\pi_{n-1}^2(t)\,d\alpha_g(t) .$$

On the other hand we can define π_{n-1}^* by

$$\pi_{n-1}^*(t) = \sum_{k=1}^{n} \frac{\ell_{kn}(d\alpha, t)\,\ell_{kn}(d\alpha, x)}{g(x_{kn}(d\alpha))\,\lambda_{kn}(d\alpha)} .$$

<u>Lemma 5.</u> Let $\operatorname{supp}(d\alpha)$ be compact and let c be one of the endpoints of $\Delta(d\alpha)$. Then

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,\frac{p_{n-1}^2(d\alpha, x_{kn})}{|c-x_{kn}|} \le \frac{\gamma_n^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)}\,|c-\alpha_n(d\alpha)| .$$

<u>Proof.</u> (By Freud [8]). Since $p_n(d\alpha, c) \ne 0$

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\,\frac{p_{n-1}^2(d\alpha, x_{kn})}{c-x_{kn}} = \frac{\gamma_n}{\gamma_{n-1}}\,\frac{p_{n-1}(d\alpha, c)}{p_n(d\alpha, c)} .$$

Further $\operatorname{sign} p_{n+1}(d\alpha, c) = \operatorname{sign} p_{n-1}(d\alpha, c)$. Thus by the recurrence formula

$$|c-\alpha_n|\,|p_n(d\alpha, c)| \ge \frac{\gamma_{n-1}}{\gamma_n}\,|p_{n-1}(d\alpha, c)| .$$

The lemma follows from the above two formulas.

<u>Theorem 6.</u> Let $\alpha \in M(0,1)$. Let $g(t) = c_1 t + c_2$ $(c_1 \ne 0)$ be nonnegative on $\operatorname{supp}(d\alpha)$. Then for every $x \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} = g(x)$$

and the convergence is uniform for $x \in \Delta \subset (-1,1)$.

<u>Proof.</u> We have by Lemma 4

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha_g, x)} = F_n(d\alpha, g^{-1}, x) .$$

If g is positive on $\Delta(d\alpha)$ we can directly apply Theorem 2. Next let g vanish at one of the endpoints of $\Delta(d\alpha)$ which we denote by c. First we show that

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, c)}{\lambda_n(d\alpha, c)} = 0 \quad (= g(c)) .$$

Let $\varepsilon > 0$. Then $g+\varepsilon$ is positive on $\Delta(d\alpha)$. Hence

$$0 \le \frac{\lambda_n(d\alpha_g, c)}{\lambda_n(d\alpha, c)} \le \frac{\lambda_n(d\alpha_{g+\varepsilon}, c)}{\lambda_n(d\alpha, c)} \xrightarrow[n\to\infty]{} g(c) + \varepsilon = \varepsilon .$$

Now let $\varepsilon \to 0$. If $x \in \operatorname{supp}(d\alpha)\setminus c$ then for small $\delta > 0$ g^{-1} is bounded in $[x-\delta, x+\delta]$ (and g^{-1} is uniformly bounded in $\Delta \subset (-1,1)$). Writing $g(t) = A(t-c)$ we have

$$|F_n(d\alpha, g^{-1}, x) - g^{-1}(x)| = \Big|\sum_{|x-x_k|\le\delta} + \sum_{|x-x_k|>\delta}\Big|$$

$$\le \frac{A\delta}{g(x)} \max_{|t-x|\le\delta} g^{-1}(t) + \delta^{-2}\,\frac{\gamma_{n-1}^2}{\gamma_n^2}\,\lambda_n(x)\,p_n^2(x)\,\cdot$$

$$\cdot \sum_{|x-x_k|>\delta} |g^{-1}(x) - g^{-1}(x_k)|\,\lambda_k\,p_{n-1}^2(x_k) \le$$

$$\le \frac{A\delta}{g(x)} \max_{|t-x|\le\delta} g^{-1}(t) + \delta^{-2}\,\frac{\gamma_{n-1}^2}{\gamma_n^2}\,\lambda_n(x)\,p_n^2(x)\,g^{-1}(x) +$$

$$+ A^{-1}\delta^{-2}\,\frac{\gamma_{n-1}^2}{\gamma_n^2}\,\lambda_n(x)\,p_n^2(x)\sum_{k=1}^{n}\lambda_k\,\frac{p_{n-1}^2(x_k)}{|x_k-c|} .$$

By Lemma 5 we obtain

$$|F_n(d\alpha, g^{-1}, x) - g^{-1}(x)| \leq A\delta g^{-1}(x) \max_{|x-t|\leq\delta} g^{-1}(t) +$$

$$+ \delta^{-2} \frac{\gamma_{n-1}^2}{\gamma_n^2} \lambda_n(x)\, p_n^2(x)\, g^{-1}(x) + A^{-1}\delta^{-2} \lambda_n(x)\, p_n^2(x)\, |c - \alpha_n| .$$

This estimate and Theorem 4.1.11 shows that

$$\lim_{n\to\infty} F_n(d\alpha, g^{-1}, x) = g^{-1}(x)$$

for $x \in \mathrm{supp}(d\alpha)\backslash c$ and the convergence is uniform for $x \in \Delta \subset (-1,1)$.

Lemma 7. Let $u \in \mathbb{C}\backslash(-1,1)$ and $z \in \mathbb{C}\backslash[0l,1]$. Then

$$\frac{\rho^2(z)-1}{2\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(t-u)(t-z)^2}\, dt = \frac{1}{z-u} - \frac{\sqrt{z^2-1}\,\rho(z)}{(z-u)^2} [\rho^{-1}(u) - \rho^{-1}(z)]$$

where $\sqrt{z^2-1} > 0$ for $z > 1$.

Proof. Because of continuity arguments we can suppose that $z \neq u$ and $u \in \mathbb{C}\backslash[-1,1]$. Since

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{z-t}\, dt = \frac{1}{\rho(z)}$$

(See the proof of Theorem 4.1.13) we have

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}}{(u-t)(z-t)}\, dt = \frac{1}{(z-u)} [\rho^{-1}(u) - \rho^{-1}(z)] .$$

Differentiating this identity with respect to z we obtain the lemma.

Theorem 8. Let $\alpha \in M(0,1)$. Let $g(t) = A(t-B)$ be positive on $\Delta(d\alpha)$. Then for every $z \in \mathbb{C}\backslash\mathrm{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n^*(d\alpha, z)}{\lambda_n^*(d\alpha_g, z)} = g^{-1}(z) + \sqrt{z^2-1}\, \rho(z) \frac{d}{dz} g^{-1}(z) \cdot$$

$$\cdot [\rho^{-1}(B) - \rho^{-1}(z)] ,$$

in particular,

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha, B)}{\lambda_n(d\alpha_g, B)} = \frac{\rho(B)}{2(B^2-1)A} .$$

Proof. Use Theorem 2, Lemmas 4, 7, Theorem 3.3.8 and Theorem 4.1.11.

Lemma 9. Let $g(t) = A(t-B)$ $(A \neq 0)$ be nonnegative on $\mathrm{supp}(d\alpha)$. Then

$$\frac{\gamma_{n-1}^2(d\alpha_g)}{\gamma_{n-1}^2(d\alpha)} = -\frac{1}{A} \frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} \frac{p_{n-1}(d\alpha, B)}{p_n(d\alpha, B)} .$$

Proof. We have

$$\frac{\gamma_{n-1}^2(d\alpha_g)}{\gamma_{n-1}^2(d\alpha)} = \lim_{x\to\infty} \frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha_g, x)}$$

which equals by Lemma 4

$$\lim_{x\to\infty} \frac{\sum_{k=1}^{n} \frac{\ell_k^2(d\alpha, x)x^2}{\lambda_k(d\alpha)g(x_k)}}{\sum_{k=1}^{n} \frac{\ell_k^2(d\alpha, x)x^2}{\lambda_k(d\alpha)}} = \frac{\sum_{k=1}^{n} \lambda_k(d\alpha)\, p_{n-1}^2(d\alpha, x_k) g^{-1}(x_k)}{\sum_{k=1}^{n} \lambda_k(d\alpha)\, p_{n-1}^2(d\alpha, x_k)}$$

$$= -\frac{1}{A} \sum_{k=1}^{n} \lambda_k(d\alpha)\, p_{n-1}^2(d\alpha, x_k) \frac{1}{B-x_k} =$$

$$= -\frac{1}{A} \frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} L_n(d\alpha, p_{n-1}(d\alpha), B) \cdot \frac{1}{p_n(d\alpha, B)} =$$

$$= -\frac{1}{A} \frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} \frac{p_{n-1}(d\alpha, B)}{p_n(d\alpha, B)} .$$

Lemma 10. Let $\alpha \in M(0,1)$. Let $g(t) = A(t-B)$ be positive on $\Delta(d\alpha)$. Then $\alpha_g \in M(0,1)$ and

$$(5) \quad \lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_{n-1}(d\alpha)} = \left|\frac{2}{A\rho(B)}\right|^{\frac{1}{2}} = \exp\{-\frac{1}{2\pi}\int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} .$$

Proof. If g is positive on $\Delta(d\alpha)$ then B is outside $\Delta(d\alpha)$. Hence by Theorem 4.1.13

$$(6) \qquad \lim_{n\to\infty} \frac{p_{n-1}(d\alpha, B)}{p_n(d\alpha, B)} = \rho^{-1}(B) < \infty .$$

Applying Lemma 9 we see that the equality on the left side of (5) holds and consequently

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} = \frac{1}{2} .$$

Putting α = Tschebyshev weight we have $\alpha \in S$ and $\alpha_g \in S$ (Let us recall that $[-1,1] \subset \Delta(d\beta)$ for $\beta \in M(0,1)$ and hence g is positive on $[-1,1]$). Using Lemma 4.2.2 we obtain the right side equality in (5).

Now we have to show that

$$\lim_{n\to\infty} \alpha_n(d\alpha_g) = 0 .$$

Let us develop $g\, p_n(d\alpha_g)$ into a Fourier series in $p_k(d\alpha)$. It is easy to see that

$$g(x)\, p_n(d\alpha_g, x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha, x) + A \frac{\gamma_n(d\alpha_g)}{\gamma_{n+1}(d\alpha)} p_{n+1}(d\alpha, x) .$$

Hence

$$\int_{-\infty}^{\infty} g^2(x)\, p_n^2(d\alpha_g, x)\, d\alpha(x) = \frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} + A^2 \frac{\gamma_n^2(d\alpha_g)}{\gamma_{n+1}^2(d\alpha)} .$$

The left side equals

$$\int_{-\infty}^{\infty} A(x-B)\, p_n^2(d\alpha_g, x)\, d\alpha_g(x) = A\alpha_n(d\alpha_g) - AB .$$

Thus by Lemma 9

$$\alpha_n(d\alpha_g) = B - \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \left\{ \frac{p_{n+1}(d\alpha, B)}{p_n(d\alpha, B)} + \frac{p_n(d\alpha, B)}{p_{n+1}(d\alpha, B)} \right\} .$$

By (6) $\lim_{n\to\infty} \alpha_n(d\alpha_g)$ exists and equals $B - \frac{1}{2}[\rho(B) + \rho^{-1}(B)] = 0$. Consequently $\alpha_g \in M(0,1)$.

Remark 11. Lemma 9 and the proof of Lemma 10 show that if $g(t) = A(t-B)$ is positive on $\Delta(d\alpha)$ and $\alpha \in M(0,1)$ then

$$\lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{A}{\sqrt{2}} \left| \frac{1}{A\rho(B)} \right|^{\frac{1}{2}} [\rho(z) - \rho(B)]$$

for $z \in \mathbb{C}\setminus \operatorname{supp}(d\alpha)\setminus\{B\} = \mathbb{C}\setminus \operatorname{supp}(d\alpha_g)\setminus\{B\}$.

This remark and Theorem 4.1.11 give a new proof of Theorem 8.

Lemma 12. Let $\alpha \in M(0,1)$. Let $g(x) = (x-A)^2 + B^2$ with $A \in \mathbb{R}$, $B^2 > 0$. Then $\alpha_g \in M(0,1)$ and

$$(7) \qquad \lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2\left| \frac{1}{\rho(A+iB)\rho(A-iB)} \right|^{\frac{1}{2}} = \exp\{-\frac{1}{2\pi}\int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} .$$

Proof. Let us develop $g p_n(d\alpha_g)$ in a Fourier series in $p_k(d\alpha)$. We have

$$(8) \qquad g(x)\, p_n(d\alpha_g, x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha, x) + d_{n+1}\, p_{n+1}(d\alpha, x) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p_{n+2}(d\alpha, x)$$

Unfortunately we cannot directly calculate d_{n+1} . Let us note that $g(A \pm iB) = 0$. Hence

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha, A \pm iB) + d_{n+1}\, p_{n+1}(d\alpha, A \pm iB) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p_{n+2}(d\alpha, A \pm iB) = 0 .$$

Consequently

$$(9) \qquad -d_{n+1} = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} \frac{p_n(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} \frac{p_{n+2}(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)}$$

and

$$\frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{\gamma_n(d\alpha)}{\gamma_{n+2}(d\alpha)} \left[\frac{p_{n+2}(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} - \frac{p_{n+2}(d\alpha, A-iB)}{p_{n+1}(d\alpha, A-iB)} \right] \cdot \left[\frac{p_n(d\alpha, A-iB)}{p_{n+1}(d\alpha, A-iB)} - \frac{p_n(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} \right]^{-1} .$$

Letting $n\to\infty$ and using Theorem 4.1.13 we obtain

$$\lim_{n\to\infty} \frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{1}{4}[\rho(A+iB) - \rho(A-iB)] \cdot [\rho^{-1}(A-iB) - \rho^{-1}(A+iB)]^{-1} = \frac{1}{4}\rho(A+iB)\rho(A-iB)$$

which proves the left side equality in (7). The right side equality in (7) follows from Lemma 4.2.2. Now we shall show that for every $z \in \mathbb{C}\setminus\operatorname{supp}(d\alpha) = \mathbb{C}\setminus\operatorname{supp}(d\alpha_g)$ $(z \neq A \pm iB)$

$$(10) \qquad \lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_{n+1}(d\alpha_g, z)} = \rho(z)^{-1} .$$

If (10) holds then by Theorem 4.1.12 $\alpha_g \in M(0,1)$. We obtain from (8) and (9)

$$\frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} \left[\frac{\gamma_{n+2}(d\alpha)\, \gamma_n(d\alpha)}{\gamma_n^2(d\alpha_g)} \cdot \right.$$

$$\left(\frac{\gamma_{n+2}(d\alpha)\, \gamma_n(d\alpha)}{\gamma_n^2(d\alpha_g)} \frac{p_n(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} + \frac{p_{n+2}(d\alpha, A+iB)}{p_{n+1}(d\alpha, A+iB)} \right) \cdot$$

$$\left. \cdot \frac{p_{n+1}(d\alpha, z)}{p_n(d\alpha, z)} + \frac{p_{n+2}(d\alpha, z)}{p_n(d\alpha, z)} \right] .$$

By (7) and Theorem 4.1.13 for $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha) \setminus \{A \pm iB\}$

$$\text{(11)} \qquad \lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A+iB)\rho(A-iB)} \right|^{\frac{1}{2}} \cdot$$

$$\cdot [\rho(z) - \rho(A+iB)][\rho(z) - \rho(A-iB)] .$$

Now (10) follows from Theorem 4.1.13. Hence $\alpha_g \in M(0,1)$.

Let us remark that by Theorem 4.1.13 (10) holds also for $z = A \pm iB$.

<u>Lemma 13</u>. Let $\alpha \in M(0,1)$. Let $g(x) = (x-A)(x-B)$ $(A \neq B)$ be positive on $\operatorname{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2 \left| \frac{1}{\rho(A)\,\rho(B)} \right|^{\frac{1}{2}} =$$

$$= \exp\{-\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} ,$$

further for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha) \setminus \{A, B\}$

$$\lim_{n\to\infty} \frac{g(z)\, p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A)\,\rho(B)} \right|^{\frac{1}{2}} \cdot$$

$$\cdot [\rho(z) - \rho(A)][\rho(z) - \rho(B)] .$$

<u>Proof.</u> The proof of Lemma 12 can be repeated. Note that $A, B \notin [-1,1]$ but may belong to $\Delta(d\alpha)$.

<u>Lemma 14.</u> Let $\alpha \in M(0,1)$ and let $g(x) = (x-A)^2$ where $A \in \mathbb{R} \setminus \operatorname{supp}(d\alpha)$, that is A may belong to $\Delta(d\alpha)$. Then $\alpha_g \in M(0,1)$ and

$$\text{(12)} \qquad \lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = 2 \left| \frac{1}{\rho(A)} \right| = \exp\{-\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\}$$

<u>Proof.</u> The proof of Lemma 12 has to be modified. We have (8) and $g(A) = g'(A) = 0$. Hence

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p_n(d\alpha, A) + d_{n+1} p_{n+1}(d\alpha, A) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p_{n+2}(d\alpha, A) = 0$$

and

$$\frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} p'_n(d\alpha, A) + d_{n+1} p'_{n+1}(d\alpha, A) + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} p'_{n+2}(d\alpha, A) = 0 .$$

From here

$$\text{(13)} \qquad -d_{n+1} = \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} \frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} + \frac{\gamma_n(d\alpha_g)}{\gamma_{n+2}(d\alpha)} \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)}$$

and

$$\frac{\gamma_n^2(d\alpha)}{\gamma_n^2(d\alpha_g)} = \frac{\gamma_n(d\alpha)}{\gamma_{n+2}(d\alpha)} \left[\frac{p'_{n+2}(d\alpha, A)}{p'_{n+1}(d\alpha, A)} - \frac{p_{n+2}(d\alpha, A)}{p_{n+1}(d\alpha, A)} \right] .$$

$$\cdot \left[\frac{p_n(d\alpha, A)}{p_{n+1}(d\alpha, A)} - \frac{p'_n(d\alpha, A)}{p'_{n+1}(d\alpha, A)} \right]^{-1} .$$

Now (12) follows from Theorems 4.1.13, 4.1.16 and 4.1.17, further from Lemma 4.2.2. Using (8), (12), (13) and Theorem 4.1.13 we obtain by the same way as we did in the proof of Lemma 12 that

(14) $$\lim_{n\to\infty} \frac{g(z) p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2} \left| \frac{1}{\rho(A)} \right| [\rho(z) - \rho(A)]^2$$

for $z \in \mathbb{C} \setminus \mathrm{supp}(d\alpha) \setminus \{A\}$ which together with Theorems 4.1.12 and 4.1.13 shows that $\alpha_g \in M(0,1)$.

<u>Theorem 15.</u> Let $\alpha \in M(0,1)$. Let

$$g(x) = A \prod_{k=1}^{N} (x - B_k)$$

be positive on $\mathrm{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha_g)}{\gamma_n(d\alpha)} = \exp\{-\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\}$$

and for every $z \in \mathbb{C} \setminus \mathrm{supp}(d\alpha) \setminus \{B_k\}$

$$\lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = \frac{1}{2^N} \exp\{-\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} \cdot$$

$$\cdot \prod_{k=1}^{N} \frac{\rho(z) - \rho(B_k)}{z - B_k} .$$

<u>Proof.</u> Repeated application of Lemmas 10, 12, 13, 14, of Remark 11 and of formulas (11) and (14).

<u>Definition 16.</u> Let $\beta \in S$. Then the Szegö function $D(d\beta, z)$ is defined by

$$D(d\beta, z) = \exp\{\frac{1}{4\pi} \int_{-\pi}^{\pi} \log\beta'(\cos t) \cdot \frac{1 + ze^{-it}}{1 - ze^{-it}} dt\}$$

for $|z| < 1$.

<u>Properties 17.</u> $D \in H_2(|z| < 1)$, for almost every $t \in [-\pi, \pi]$

$$\lim_{r \to 1-0} D(d\beta, re^{it}) = D(d\beta, e^{it})$$

exists and $|D(d\beta, e^{it})|^2 = \beta'(\cos t)$ for almost every $t \in [-\pi, \pi]$, $D(d\beta, z) \neq 0$ for $|z| < 1$, $D(d\beta, 0) > 0$. (See e.g. Freud, Chapter V.)

<u>Lemma 18.</u> Let $\beta \in S$, $z \in \mathbb{C} \setminus [-1,1]$. Then

$$\lim_{n\to\infty} p_n(d\beta, z)\, \rho(z)^{-n} = \frac{1}{\sqrt{2\pi}} D(vd\beta, \rho(z)^{-1})^{-1}$$

and the convergence is uniform for $|\rho(z)| \geq R > 1$.

<u>Proof.</u> See e.g. Freud.

<u>Lemma 19.</u> Let $g(x) = A \prod_{k=1}^{N} (x - B_k)$ be positive on $[-1,1]$. Then for $z \in \mathbb{C} \setminus [-1,1]$

(15) $$D(g, \rho(z)^{-1}) = 2^N \exp\{\frac{1}{2\pi} \int_{-1}^{1} \log g(t) \frac{dt}{\sqrt{1-t^2}}\} \cdot$$

$$\cdot \prod_{k=1}^{N} \frac{z - B_k}{\rho(z) - \rho(B_k)} .$$

<u>Proof.</u> Put in Theorem 15 α = Tschebyshev weight and use Lemma 18. Then for $z \in \mathbb{C} \setminus [-1,1] \setminus \{B_k\}$ (15) holds and consequently it holds for every $z \in \mathbb{C} \setminus [-1,1]$.

Let us note that – because of continuity arguments – (15) holds if $g(\pm 1) = 0$, further (15) holds for $z = \infty$ if g is only nonnegative on $[-1, 1]$.

<u>Theorem 20.</u> Let $\alpha \in M(0,1)$ and let g be a polynomial which is positive on $\operatorname{supp}(d\alpha)$. Then $\alpha_g \in M(0,1)$ and

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} = D(g, 0) ,$$

$$\lim_{n\to\infty} \frac{p_n(d\alpha, z)}{p_n(d\alpha_g, z)} = D(g, \rho(z)^{-1})$$

for $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$.

<u>Proof.</u> By Theorem 15 and Lemma 19 the only thing which we have to show is that if $g(B) = 0$ then

$$\lim_{n\to\infty} \frac{p_n(d\alpha, B)}{p_n(d\alpha_g, B)} = D(g, \rho(B)^{-1}) . \tag{16}$$

Let $\delta > 0$ be small enough. Then for $|z-B| = \delta$

$$\lim_{n\to\infty} \frac{p_n(d\alpha, z)}{p_n(d\alpha_g, z)} = D(g, \rho(z)^{-1}) ,$$

that is by Theorem 4.1.13

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha_g, z)} = \rho(z)^{-1} D(g, \rho(z)^{-1})$$

for $|z - B| = \delta$. Since $p_{n-1}(d\alpha, z) = L_n(d\alpha_g, p_{n-1}(d\alpha), z)$ we have

$$\left|\frac{p_{n-1}(d\alpha, z)}{p_n(d\alpha_g, z)}\right| \le \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\varepsilon} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) |p_{n-1}(d\alpha, x_{kn})| \cdot$$

$$\cdot |p_{n-1}(d\alpha_g, x_{kn})| \le \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\varepsilon} \{\int_{-\infty}^{\infty} p_{n-1}^2(d\alpha, t)\, d\alpha_g(t)\}^{\frac{1}{2}}$$

$$\le \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} \frac{1}{\varepsilon} \max_{t \in \operatorname{supp}(d\alpha)} [g(t)]^{\frac{1}{2}}$$

for $n \ge N$ where ε and N are defined by Theorem 3.3.8. Since both $p_{n-1}(d\alpha, z)/p_n(d\alpha_g, z)$ and $\rho(z)^{-1} D(g, \rho(z)^{-1})$ are analytic in $|z-B| \le \delta$ if $\delta \le \varepsilon$ and $n \ge N$ we can apply Cauchy's integral formula and Lebesgue's theorem about $\lim \int f_n = \int \lim f_n$ and we obtain

$$\lim_{n\to\infty} \frac{p_{n-1}(d\alpha, B)}{p_n(d\alpha_g, B)} = \rho(B)^{-1} D(g, \rho(B)^{-1}) .$$

Thus by Theorem 4.1.13 (16) holds.

Now we can easily generalize Theorem 8.

<u>Theorem 21.</u> Let $\alpha \in M(0,1)$. Let g be a polynomial which is positive on $\operatorname{supp}(d\alpha)$. Then for every $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n^*(d\alpha_g, z)}{\lambda_n^*(d\alpha, z)} = D(g, \rho(z)^{-1})^2$$

and

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} = |D(g, \rho(z)^{-1})|^2 . \tag{17}$$

<u>Proof.</u> Apply Theorems 20 and 4.1.11.

Remark 22. Let us put $\rho(z)^{-1} = re^{i\theta}$ $(0 < r < 1)$ in (17). Then $z = \frac{1}{2}(re^{i\theta} + r^{-1}e^{-i\theta}) \xrightarrow[r\to 1-0]{} \cos\theta$. By Properties 17 for almost every $\theta \in [-\pi, \pi]$

$$\lim_{r\to 1-0} \lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} = g(x) \quad (x = \cos\theta)$$

which suggests

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} = g(x) \qquad (-1 \le x \le 1) .$$

Property 23. Let $z = re^{i\theta}$ $0 \le r < 1$. Then

$$|D(d\beta, z)|^2 = \exp\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log\beta'(\cos t) \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} dt\} .$$

(See e.g. Freud, Chapter V.)

Lemma 24. Let f be Riemann integrable on $\Delta \supset [-1,1]$ and let $f(x) \ge C > 0$ for $x \in \Delta$. Let $0 < R < 1$ be fixed. Then for every $\varepsilon > 0$ there exist two polynomials π_1 and π_2 such that

$$\frac{C}{2} \le \pi_1(x) \le f(x) \le \pi_2(x)$$

for $x \in \Delta$ and

$$|D(\pi_2, z)|^2 (1-\varepsilon) \le |D(f, z)|^2 \le |D(\pi_1, z)|^2 (1+\varepsilon)$$

for $|z| \le R$.

Proof. Let $\varepsilon > 0$. We construct a polynomial π_2 such that

$$f(x) \le \pi_2(x) \quad (x \in \Delta)$$

and

$$\int_{-\pi}^{\pi} [\pi_2(\cos t) - f(\cos t)] dt < \varepsilon .$$

(See Szegö, 1.5.) Then by Property 23

$$|D(\pi_2, z)|^2 = |D(\pi_2 f^{-1}, z)|^2 \, |D(f, z)|^2$$

and by Jensen's inequality

$$|D(\pi_2 f^{-1}, z)|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(\cos t)}{f(\cos t)} \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} dt$$

$(z = re^{i\theta})$. Hence

$$|D(\pi_2 f^{-1}, z)|^2 \le 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\pi_2(\cos t) - f(\cos t)}{f(\cos t)} \cdot$$

$$\cdot \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} dt \le 1 + \text{const} \cdot \varepsilon$$

for $|z| \le R$. The second part of the lemma can be proved in the same way. Since $f(x) \ge c > 0$ for $x \in \Delta$ we can choose π_1 so that $\pi_1(x) \ge \frac{c}{2}$ for $x \in \Delta$. (See Szegö, 1.5.)

Theorem 25. Let $\alpha \in M(0,1)$. Let $g(\ge 0)$ be $d\alpha$ measurable, $g^{\pm 1}$ be bounded on $\text{supp}(d\alpha)$ and g be Riemann integrable on $[-1,1]$. Then for every $z \in \mathbb{C} \setminus \text{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} = |D(g, \rho(z)^{-1})|^2 .$$

Proof. By the assumptions α_g is a weight, since $g^{-1} \in L^1 d\alpha$. Recall that

$$\lambda_n(d\alpha, z) = \min_{\pi_{n-2}} \int_{-\infty}^{\infty} |(1+(z-t)\pi_{n-2}(t))|^2 \, d\alpha(t) .$$

Then from $d\alpha \le d\beta$ it follows that $\lambda_n(d\alpha, z) \le \lambda_n(d\beta, z)$ for every $z \in \mathbb{C}$. Let us construct two functions f_1 and f_2 such that both f_1 and f_2 are Riemann integrable on $\Delta(d\alpha)$, $f_1(x) = f_2(x) = g(x)$ for $x \in [-1,1]$ and

$$0 < c_1 \le f_1(x) \le g(x) \le f_2(x) \le c_2 < \infty$$

for $x \in \Delta(d\alpha)$. We can do this by Theorem 3.3.7. Let $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha)$. Then $|\rho(z)^{-1}| < 1$. Let $\varepsilon > 0$. Then by Lemma 24 we can find two polynomials π_1 and π_2 such that $\pi_1(x) \ge c_1/2$, $\pi_2(x) \ge c_1/2$ for $x \in \Delta(d\alpha)$,

$$\lambda_n(d\alpha_{\pi_1}, z) \le \lambda_n(d\alpha_g, z) \le \lambda_n(d\alpha_{\pi_2}, z)$$

and

$$(1-\varepsilon)|D(\pi_2, \rho(z)^{-1})|^2 \le |D(g, \rho(z)^{-1})|^2 \le$$

$$\le (1+\varepsilon)\, |D(\pi_1, \rho(z)^{-1})|^2 .$$

Thus by Theorem 21

$$|D(g, \rho(z)^{-1})|^2 \frac{1}{1+\varepsilon} \le \liminf_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} \le \limsup_{n\to\infty} \frac{\lambda_n(d\alpha_g, z)}{\lambda_n(d\alpha, z)} \le \frac{1}{1-\varepsilon} |D(g, \rho(z)^{-1})|^2 .$$

Now let $\varepsilon \to 0$.

Theorem 26. Let $\alpha \in M(0,1)$ and g be as in Theorem 25. Then

$$\lim_{n\to\infty} \frac{\gamma_n(d\alpha)}{\gamma_n(d\alpha_g)} = D(g, 0) ,$$

and so

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(d\alpha_g)}{\gamma_n(d\alpha_g)} = \frac{1}{2} .$$

Proof. Recall that from $d\alpha \le d\beta$ follows $\gamma_n(d\beta) \le \gamma_n(d\alpha)$. Now we can repeat the proof of Theorem 25, the only difference is that this time we apply Theorem 20 instead of Theorem 21.

Theorem 27. Let $\alpha \in M(0,1)$ and g be as in Theorem 25. Then $\alpha_g \in M(0,1)$

Proof. If we could directly calculate $\alpha_n(d\alpha_g)$ the proof would probably be nice. Unfortunately we cannot do this. Let $x \notin \operatorname{supp}(d\alpha)$. Then by Theorem 25

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_{n+1}(d\alpha_g, x)} \cdot \frac{\lambda_{n+1}(d\alpha, x)}{\lambda_n(d\alpha, x)} = 1 ,$$

that is

$$\lim_{n\to\infty} \frac{1 + p_n^2(d\alpha_g, x)\,\lambda_n(d\alpha_g, x)}{1 + p_n^2(d\alpha, x)\,\lambda_n(d\alpha, x)} = 1 .$$

Thus by Theorem 4.1.11

$$\lim_{n\to\infty} p_n^2(d\alpha_g, x)\,\lambda_n(d\alpha_g, x) = \rho(x)^2 - 1 .$$

Using Theorem 26 we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) \frac{p_{n-1}^2(d\alpha_g, x_{kn})}{(x - x_{kn})^2} = \frac{4}{\rho(x)^2 - 1} = -2 \frac{d}{dx} \rho(x)^{-1}$$

for $x \notin \operatorname{supp}(d\alpha)$. By Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) \frac{p_{n-1}^2(d\alpha_g, x_{kn})}{x - x_{kn}} = \frac{2}{\rho(x)} = \frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{x-t}\, dt$$

for $x \notin \Delta(d\alpha)$. If f is continuous on $\Delta(d\alpha)$ then for every $\varepsilon > 0$ we can find a function F of the form

$$F(t) = \sum_{j=1}^{N} a_j \frac{1}{x_j - t}$$

where $a_j \in \mathbb{C}$ and $x_j \in \mathbb{R} \setminus \Delta(d\alpha)$ such that

$$\max_{t \in \Delta(d\alpha)} |F(t) - f(t)| \leq \varepsilon .$$

(See e.g. Achiezer, section of problems). Hence if f is continuous on $\Delta(d\alpha)$ then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha_g) f(x_{kn}) p_{n-1}^2(d\alpha_g, x_{kn}) = \frac{2}{\pi} \int_{-1}^{1} f(t) \sqrt{1-t^2}\, dt .$$

Consequently by Theorem 3.2.4 $\alpha_g \in M(0,1)$.

Remark 2.8. Later we shall show (with the aid of the Pollaczek polynomials), that if w is defined by

$$w(x) = \exp\{-(1-x^2)^{-\frac{1}{2}}\}$$

for $-1 \leq x \leq 1$ and $\operatorname{supp}(w) = [-1,1]$ then $w \in M(0,1)$. Consequently by

the previous theorem $gw \in M(0,1)$ if $g > 0$ is continuous on $[-1,1]$. Let us remark that the above w is the "nicest" weight which does not belong to S.

Theorem 29. Let $\alpha \in M(0,1)$ and let g satisfy the conditions of Theorem 2. Let $K \subset \mathbb{C} \cup \{\infty\} \setminus \operatorname{supp}(d\alpha)$ be an arbitrary closed set. Then

$$\lim_{n\to\infty} \frac{p_n(d\alpha_g, z)}{p_n(d\alpha, z)} = D(g, \rho(z)^{-1})^{-1} \tag{18}$$

uniformly for $z \in K$.

Proof. If $z \in \mathbb{R} \setminus \operatorname{supp}(d\alpha)$ then (18) follows immediately from Theorem 25, 27, 3.3.8 and 4.1.11. Let K^* be a region in $\mathbb{C} \cup \{\infty\}$ such that $K \subset K^*$, $\overline{K}^* \cap \operatorname{supp}(d\alpha) = \phi$ and $K^* \cap \mathbb{R} \neq \phi$. By Theorem 3.3.8 the functions $p_n(d\alpha_g, z)\, p_n(d\alpha, z)^{-1}$ are analytic in K^*. If we can show that

$$\frac{|p_n(d\alpha_g, z)|}{|p_n(d\alpha, z)|} \leq \text{const} \tag{19}$$

for $z \in \overline{K}^*$ and $n = N, N+1, \ldots$ where $N = N(\overline{K}^*)$ then the theorem will follow from Vitali's theorem. Let d_N be defined by $d_N = \operatorname{dist}(\overline{K}^*, \{x_{kn}(d\alpha)\}_{n=N,\,k=1}^{\infty,\,n})$. By Theorem 3.3.8 $d_N > 0$ for some $N \in \mathbb{N}$. Let $n \geq N$ and $z \in \overline{K}^*$. Then

$$|p_n(d\alpha_g, z)|^2 \leq \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha_g, t)\, d\alpha(t) \leq$$

$$\leq C \lambda_{n+1}(d\alpha, z)^{-1} \int_{-\infty}^{\infty} p_n^2(d\alpha_g, t)\, d\alpha_g(t)$$

where $C^{-1} = \inf_{t \in \operatorname{supp}(d\alpha)} g(t)$. Hence

$$|p_n(d\alpha_g, z)|^2 \le C|p_n(d\alpha, z)|^2 + C\,\lambda_n(d\alpha, z)^{-1} .$$

Further we have

$$\lambda_n(d\alpha, z)^{-1} = \sum_{k=1}^{n} \frac{|\ell_{kn}(d\alpha, z)|^2}{\lambda_{kn}(d\alpha)} \le$$

$$\le \frac{\gamma_{n-1}(d\alpha)^2}{\gamma_n(d\alpha)^2}\, |p_n(d\alpha, z)|^2\, d_N^{-2} .$$

Consequently (19) is satisfied with $const = [C(1 + d_N^{-2}\,|\Delta(d\alpha)|^2 \cdot 0.25)]^{\frac{1}{2}}$.

6.2. A Sequence of Positive Operators

Using the well known formula

$$\ell_{kn}(d\alpha, x) = \lambda_{kn}(d\alpha)\, K_n(d\alpha, x, x_{kn})$$

we obtain

$$F_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, K_n^2(d\alpha, x, x_{kn})$$

which is the Riemann-Stieltjes sum for

$$G_n(d\alpha, f, x) = \lambda_n(d\alpha, x) \int_{-\infty}^{\infty} f(t)\, K_n^2(d\alpha, x, t)\, d\alpha(t) .$$

For $z \in \mathbb{C}$ we put

$$G_n(d\alpha, f, z) = \lambda_n^*(d\alpha, z) \int_{-\infty}^{\infty} f(t)\, k_n^2(d\alpha, z, t)\, d\alpha(t) .$$

(See 4.1).

Properties 1. (i) If $f(x) \equiv 1$ then $G_n(f, x) \equiv 1$. (ii) If $f(x) \ge 0$ for $x \in supp(d\alpha)$ then $G_n(f, x) \ge 0$ for $x \in \mathbb{R}$. (iii) G_n is a rational function of degree $(2n-2, 2n-2)$ where the denominator does not depend on f.

Theorem 2. Let $\alpha \in M(0,1)$. Let f be $d\alpha$ measurable and bounded on $supp(d\alpha)$. Then for each $x \in supp(d\alpha) \setminus [-1,1]$

(1) $$\lim_{n\to\infty} G_n(d\alpha, f, x) = f(x) .$$

If $x \in [-1,1]$ and f is continuous at x then (1) holds. If f is continuous on $\Delta \subset (-1,1)$ then (1) holds uniformly for $x \in \Delta$. If f if continuous on $supp(d\alpha)$ and $z \in \mathbb{C} \setminus supp(d\alpha)$ then

$$\text{(2)} \qquad \lim_{n\to\infty} G_n(d\alpha, f, z) = \frac{\sqrt{z^2-1}}{\pi} \int_{-1}^{1} \frac{f(t)}{(z-t)\sqrt{1-t^2}}\, dt .$$

Here $\sqrt{z^2-1} > 0$ for $z > 1$.

Proof. (i) Let $x \in \operatorname{supp}(d\alpha)\setminus[-1,1]$. Then by Theorem 3.3.7 x is an isolated point of $\operatorname{supp}(d\alpha)$. Hence there exists $\varepsilon > 0$ such that

$$G_n(f, x) = f(x)\, \frac{\alpha(x+0) - \alpha(x-0)}{\lambda_n(x)} + \lambda_n(x) \cdot \int_{|x-t|\geq\varepsilon} f(t)\, K_n^2(x,t)\, d\alpha(t) .$$

Here the first term converges to $f(x)$ when $n\to\infty$. (See Freud, §II.2, $\operatorname{supp}(d\alpha)$ is compact !) Remembering that

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_{n-1}(t)\, p_n(x) - p_n(t)\, p_{n-1}(x)}{x-t}$$

and using Theorem 4.1.11 we see that

$$\text{(3)} \qquad \lim_{n\to\infty} \lambda_n(x) \int_{|x-t|>\varepsilon} f(t)\, K_n^2(x,t)\, d\alpha(t) = 0 .$$

(ii) Let $x \in [-1,1]$. Then for every $\varepsilon > 0$ (3) is satisfied and the convergence is uniform for $x \in \Delta \subset (-1,1)$. Thus by Properties 1 the usual machinery of positive operators can be applied. We do not go into details.

(iii) Let $z \in \mathbb{C}\setminus\operatorname{supp}(d\alpha)$. By Titze's theorem we can suppose that f is continuous on $\Delta(d\alpha)$. The function $(z-t)^{-2}$ restricted to $\operatorname{supp}(d\alpha)$ is continuous and we can extend it to a function g which is continuous on $\Delta(d\alpha)$. We have

$$G_n(f,z) = \lambda_n^*(z)\, \frac{\gamma_{n-1}^2}{\gamma_n^2} \int_{-\infty}^{\infty} f(t)\, g(t)\, [p_{n-1}(t)\, p_n(z) - p_n(t)\, p_{n-1}(z)]^2\, d\alpha(t) =$$

$$= \frac{\gamma_{n-1}^2}{\gamma_n^2} [\lambda_n^*(z)\, p_n^2(z) \int_{-\infty}^{\infty} f(t)\, g(t)\, p_{n-1}^2(t)\, d\alpha(t) + \lambda_n^*(z)\, p_n^2(z)\, \frac{p_{n-1}^2(z)}{p_n^2(z)} \int_{-\infty}^{\infty} f(t)\, g(t)\, p_n^2(t)\, d\alpha(t) - 2\lambda_n^*(z)\, p_n^2(z)\, \frac{p_{n-1}(z)}{p_n(z)} \int_{-\infty}^{\infty} f(t)\, g(t)\, p_{n-1}(t)\, p_n(t)\, d\alpha(t)] .$$

Now we apply Theorems 4.1.11, 4.1.13 and 4.2.13. We obtain

$$\lim_{n\to\infty} G_n(f,z) = \frac{1}{4\pi} [\rho^2(z) - 1][1 + \rho^{-2}(z)] \int_{-1}^{1} \frac{f(t)}{(z-t)^2\sqrt{1-t^2}}\, dt - \frac{1}{2\pi} [\rho^2(z) - 1]\, \rho^{-1}(z) \int_{-1}^{1} \frac{t f(t)}{(z-t)^2\sqrt{1-t^2}}\, dt .$$

But $[\rho^2(z) - 1][1+\rho^{-2}(z)] = 4z\sqrt{z^2-1}$ and $[\rho^2(z) - 1]\,\rho^{-1}(z) = 2\sqrt{z^2-1}$.

Hence

$$\lim_{n\to\infty} G_n(f,z) = \frac{\sqrt{z^2-1}}{\pi} \int_{-1}^{1} \frac{z-t}{(z-t)^2} \frac{f(t)}{\sqrt{1-t^2}}\, dt .$$

Let us note that once (2) holds for continuous functions then it also holds for Riemann integrable functions if $x \in \mathbb{R}\setminus\operatorname{supp}(d\alpha)$. We shall not go into details since in the following we shall concentrate on convergence of $G_n(d\alpha, f, x)$ for $x \in \operatorname{supp}(d\alpha)$. The following theorem explains why we introduced the operators $G_n(d\alpha, f)$ and why we should investigate them for as many weights α as possible.

<u>Theorem 3.</u> Let $g(\geq 0) \in L^1_{d\alpha}$. If α_g is a weight then

$$\frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \leq G_n(d\alpha, g, x) \tag{4}$$

for $x \in \mathbb{R}$ and if $g^{-1} \in L^1_{d\alpha}$ then

$$G_n^{-1}(d\alpha, g^{-1}, x) \leq \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \tag{5}$$

for $x \in \mathbb{R}$.

Before the proof let us remark that if $\operatorname{supp}(d\alpha)$ is compact and $g^{-1} \in L^1_{d\alpha}$ then α_g is a weight.

<u>Proof.</u> From

$$\lambda_n(d\alpha, x) = \min_{\pi_{n-1}} \pi_{n-1}^{-2}(x) \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha(t)$$

follows that

$$\lambda_n(d\alpha_g, x) \leq K_n^{-2}(d\alpha, x, x) \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t)\, d\alpha_g(t) =$$

$$= \lambda_n^2(d\alpha, x) \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t)\, g(t)\, d\alpha(t) = \lambda_n(d\alpha, x)\, G_n(d\alpha, g, x) .$$

On the other hand

$$\pi_{n-1}(x) = \int_{-\infty}^{\infty} K_n(d\alpha, x, t)\, \pi_{n-1}(t)\, d\alpha(t) .$$

Thus

$$\pi_{n-1}^2(x) \leq \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t)\, g^{-1}(t)\, d\alpha(t) \cdot$$

$$\cdot \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, g(t)\, d\alpha(t) = \lambda_n^{-1}(d\alpha, x)\, G_n(d\alpha, g^{-1}, x) \cdot$$

$$\cdot \int_{-\infty}^{\infty} \pi_{n-1}^2(t)\, d\alpha_g(t) ,$$

that is

$$\lambda_n^{-1}(d\alpha, g, x) \leq \lambda_n^{-1}(d\alpha, x)\, G_n(d\alpha, g^{-1}, x) .$$

From Theorems 2 and 3 we could immediately obtain limit relations for

$$\frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)}$$

when both g and g^{-1} are bounded on $\operatorname{supp}(d\alpha)$. This condition however may be weakened by using the following two results.

<u>Lemma 4.</u> Let $\alpha \in M(0,1)$. Let $\{k_n\}$ be a sequence of natural integers which is bounded: $k_n \leq k$ for every n. Then

$$\lim_{n\to\infty} \frac{\lambda_n(d\alpha, x)}{\lambda_{n+k_n}(d\alpha, x)} = 1$$

for every $x \in \operatorname{supp}(d\alpha)$ and the convergence is uniform for $x \in \Delta \subset (-1, 1)$.

<u>Proof.</u> Since

$$1 \leq \frac{\lambda_n(x)}{\lambda_{n+k_n}(x)} \leq \frac{\lambda_n(x)}{\lambda_{n+k}(x)} = \prod_{j=n}^{n+k-1} \frac{\lambda_j(x)}{\lambda_{j+1}(x)}$$

we have to consider only $\lambda_n(x)/\lambda_{n+1}(x)$ which equals

$$1 + \lambda_n(d\alpha, x)\, p_n^2(d\alpha, x) \ .$$

Now we apply Theorem 4.1.11.

<u>Theorem 5.</u> Let $g(\geq 0) \in L^1_{d\alpha}$. Let α_g be a weight. Then for every polynomial P_1 of degree m_1

$$(6) \qquad \frac{P_1^2(x)\, \lambda_n(d\alpha_g, x)}{\lambda_{n-m_1}(d\alpha, x)} \leq G_{n-m_1}(d\alpha, gP_1^2, x) \qquad (n > m_1) \ .$$

If P_2 is a polynomial of degree m_2 such that $P_2^2 g^{-1} \in L^1_{d\alpha}$ then

$$(7) \qquad P_2^2(x)\, G^{-1}_{n+m_2}(d\alpha, g^{-1} P_2^2, x) \leq \frac{\lambda_n(d\alpha_g, x)}{\lambda_{n+m_2}(d\alpha, x)} \ .$$

Let us note that if $\mathrm{supp}(d\alpha)$ is compact and $P_2^2 g^{-1} \in L^1_{d\alpha}$ for some polynomial P_2 then α_g is a weight.

<u>Proof.</u> (6) follows from

$$\lambda_n(d\alpha_g, x) \leq P_1^{-2}(x)\, K^{-2}_{n-m_1}(d\alpha, x, x) \int_{\infty}^{\infty} P_1^2(t)\, K^2_{n-m_1}(d\alpha, x, t) \cdot d\alpha_g(t)$$

whenever $n > m_1$. Further for every π_{n-1}

$$\pi_{n-1}(x)\, P_2(x) = \int_{-\infty}^{\infty} \pi_{n-1}(t)\, P_2(t)\, K_{n+m_2}(d\alpha, x, t)\, d\alpha(t) \ ,$$

that is

$$\pi^2_{n-1}(x)\, P_2^2(x) \leq \int_{-\infty}^{\infty} \pi^2_{n-1}(t)\, g(t)\, d\alpha(t) \cdot$$

$$\cdot \int_{-\infty}^{\infty} P_2^2(t)\, g^{-1}(t)\, K^2_{n+m_2}(d\alpha, x, t)\, d\alpha(t)$$

which implies (7).

<u>Theorem 6.</u> Let $\alpha \in M(0,1)$. Let $g(\geq 0) \in L^1_{d\alpha}$ and suppose that there exist two polynomials P_1 and P_2 such that gP_1^2 and $g^{-1} P_2^2$ are bounded on $\mathrm{supp}(d\alpha)$. Then

(i) for every $x \in \mathrm{supp}(d\alpha) \setminus [-1,1]$

$$(8) \qquad \lim_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} = g(x) \ .$$

(ii) if $x \in [-1,1]$ and g is continuous at x then (8) holds.

(iii) if g is continuous on $\Delta \subset (-1,1)$ and $g(t) > 0$ for $t \in \Delta$ then (8) is satisfied uniformly for $x \in \Delta$.

<u>Proof.</u> Let first $x \in \mathrm{supp}(d\alpha) \setminus [-1,1]$. Then by Theorem 3.3.7 x is an isolated point of $\mathrm{supp}(d\alpha)$. Hence g must be finite at x and then we can suppose that P_1 does not vanish at x. We obtain from Theorems 2, 5 and Lemma 4 that

$$(9) \qquad \limsup_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \leq g(x)$$

which implies (8) if $g(x) = 0$. If $g(x) > 0$ then we can assume that P_2 does not vanish at x. Then by the same argument

$$(10) \qquad \liminf_{n\to\infty} \frac{\lambda_n(d\alpha_g, x)}{\lambda_n(d\alpha, x)} \geq g(x) \ .$$

Let now $x \in [-1,1]$. If g is continuous at x then $g(x) < \infty$ and thus we can suppose that $P_1(x) > 0$. Hence (9) holds again. If $g(x) = 0$ then (8) follows from (9). If $g(x) > 0$ then we can suppose $P_2(x) > 0$ which implies (10). If g is continuous on $\Delta \subset (-1,1)$ then the above argument can be used if only g is positive on Δ.

In order to illustrate the strength of this theorem we give a few examples.

Definition 7. u denotes the Jacobi weight, that is $\text{supp}(u) = [-1,1]$ and

$$u(x) \equiv u^{(a,b)}(x) = (1-x)^a (1+x)^b$$

for $-1 \le x \le 1$ where $a, b > -1$. Hence $u^{(-\frac{1}{2}, -\frac{1}{2})} = v$.

In the following it will always be clear if a, b are related with $u^{(a,b)}$ or $M(a,b)$.

Example 8. Let α be the Tschebyshev weight $(d\alpha(x) = v(x)dx)$ and let $w(x)\sqrt{1-x^2} = g(x)$ be positive and continuous on $[-1,1]$. Then $w = v_g$. Further, by easy calculation,

$$\lambda_n^{-1}(v, x) = \frac{1}{\pi}[n - \frac{1}{2} + \frac{1}{2} U_{2n-2}(x)]$$

where U_n is the Tschebyshev polynomial of second kind. Hence for every $x \in [-1,1]$

$$\lim_{n\to\infty} [n - \frac{1}{2} + \frac{1}{2} U_{2n-2}(x)]\, \lambda_n(w, x) = \pi w(x)\sqrt{1-x^2} = \pi g(x)$$

(later we shall show that the convergence is uniform for $x \in [-1,1]$), in particular

$$\lim_{n\to\infty} n\, \lambda_n(w, \pm 1) = \frac{\pi}{2} g(\pm 1) .$$

Example 9. Let $w = \varphi u$ where $\varphi > 0$ is continuous on $[-1,1]$. Then

$$\lim_{n\to\infty} n\, \lambda_n(w, x) = \pi\sqrt{1-x^2}\; w(x)$$

uniformly for $x \in \Delta \subset (-1,1)$. This is, of course, not new.

Example 10. Let $b > 0$, $\text{supp}(w) = [-b,b]$ and $w > 0$ be continuous on $[-b,b]$. Then

$$w^b(x) = w(bx)$$

is a weight on $[-1,1]$. From the definition of Christoffel function we obtain

$$\lambda_n(w, x) = b\lambda_n(w^b, xb^{-1}) .$$

Hence

$$\lim_{n\to\infty} n\, \lambda_n(w, x) = \pi\sqrt{b^2 - x^2}\; w(x)$$

uniformly for $x \in \Delta \subset (-b, b)$.

Example 11. Let w be continuous on $[-1,1]$ and $w(x) > 0$ for $x \in (-1,1)$. Let $\varepsilon > 0$, $\delta > 0$ and $\Delta = [-1+\delta, 1-\delta]$. Then

$$1_\Delta(x)\, w(x) \le w(x) \le w(x) + \varepsilon$$

for $-1 \le x \le 1$. Hence

$$n\, \lambda_n(1_\Delta w, x) \le n\, \lambda_n(w, x) \le n\, \lambda_n(w+\varepsilon, x) ,$$

where $\text{supp}(w+\varepsilon) = \text{supp}(w)$ is assumed. Thus by the previous examples we have

$$w(x)\, \pi\sqrt{(1-\delta)^2 - x^2} \le \liminf_{n\to\infty} n\, \lambda_n(w, x) \le \limsup_{n\to\infty} n\, \lambda_n(w, x) \le \pi\sqrt{1-x^2}\, [w(x) + \varepsilon]$$

uniformly for $x \in \Delta_1 \subset \Delta^0$. Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary we obtain

$$\lim_{n\to\infty} n\,\lambda_n(w,x) = \pi\sqrt{1-x^2}\,w(x)$$

uniformly for $x \in \Delta_1 \subset (-1,1)$. Recall that $w(\pm 1)$ may vanish.

<u>Definition 12.</u> Let $a, b \in \mathbb{R}$ with $a > |b|$. The Pollaczek weight $w^{(a,b)}$ is defined by $\operatorname{supp}(w^{(a,b)}) = [-1,1]$ and

$$w^{(a,b)}(\cos\theta) =$$

$$= 2\exp\left\{\frac{\theta}{\sin\theta}(a\cos\theta + b)\right\}\left[1+\exp\left\{\frac{\pi}{\sin\theta}(a\cos\theta+b)\right\}\right]^{-1},$$

$\theta \in [0,\pi]$, $x = \cos\theta$.

<u>Properties 13.</u>

(i) We have

$$\alpha_n(w^{(a,b)}) = -\frac{b}{2n+1+a} = -\frac{b}{2n} + O\left(\frac{1}{n^2}\right)$$

for $n = 0, 1, 2, \ldots$ and

$$\frac{\gamma_{n-1}(w^{(a,b)})}{\gamma_n(w^{(a,b)})} = \frac{n}{2\sqrt{(n+\frac{a}{2})^2 - \frac{1}{4}}} = \frac{1}{2} - \frac{1}{4an} + O\left(\frac{1}{n^2}\right)$$

for $n = 1, 2, \ldots$ (See Szegö, Appendix)*)

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* Let us note that the formula (1.7) in the Appendix of Szegö's book is not quite correct, it should be written as

$$n P_n(x;a,b) =$$

$$= [(2n-1+a)x + b]\,P_{n-1}(x;a,b) -$$

$$-(n-1)\,P_{n-2}(x;a,b), \qquad n = 2, 3, 4, \ldots .$$

(ii) $w^{(a,b)} \in M(0,1)$ but $w^{(a,b)} \notin S$.

(iii) $\{p_n^2(w^{(a,b)}, x)\}$ is uniformly bounded for $x \in \Delta \subset (-1,1)$. To prove this use Theorem 3.1.11 and Example 11.

(iv) Let

$$\varphi(x) = w^{(a,b)}(x)\exp\left\{\frac{(a+b)\pi}{\sqrt{2}\sqrt{1-x}} + \frac{(a-b)\pi}{\sqrt{2}\sqrt{1+x}}\right\}.$$

Then φ is continuous and positive on $[-1,1]$.

(v) Let $w^{(a)} = w^{(a,0)}$. Then $w^{(a)}$ is even and

$$\lim_{n\to\infty} \frac{p_n^2(w^{(a)},1)}{\sqrt{n}\,\exp[4\sqrt{a}\sqrt{n}\,]} = \frac{1}{4\pi}\,e^{-a}\,\frac{1}{\sqrt{a}}\,.$$

(See Szegö, Appendix.)

(vi)

$$\lim_{n\to\infty} \frac{\gamma_n(w^{(a,b)})}{2^n n^{\frac{a}{2}}} = \Gamma\left(\frac{a+1}{2}\right)^{-1}$$

where Γ denotes the Γ function of Euler. (See Szegö, Appendix.)

(vii)

$$\lim_{n\to\infty} \lambda_n(w^{(a)},x)n = \pi\sqrt{1-x^2}\,w^{(a)}(x)$$

uniformly for $x \in \Delta \subset (-1,1)$ and

$$\lim_{n\to\infty} \lambda_n(w^{(a)},\pm 1)n\,\exp\{4\sqrt{a}\sqrt{n}\} = 2\pi e^a\,.$$

The first limit relation follows from Example 11, the second one from (v) by a standard calculation.

(viii) Let $z \in \mathbb{C}\setminus[-1,1]$. Then

$$\lim_{n\to\infty} p_n(w^{(a,b)}, z)\, \rho^{-n}(z)\, n^{-\frac{az+b}{2\sqrt{z^2-1}}} =$$

$$= \Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right)^{-1} \left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}} .$$

(See Szegö, Appendix.)

(ix) If $z \in \mathbb{C}\setminus[-1,1]$ then

$$\lim_{n\to\infty} \lambda_n^*(w^{(a,b)}, z)^{-1} \rho^{-2n}(z)\, n^{-\frac{az+b}{\sqrt{z^2-1}}} =$$

$$= \Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right)^{-2} \frac{1}{4(z^2-1)} \left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{\frac{az+b}{\sqrt{z^2-1}}}$$

and

$$\lim_{n\to\infty} \lambda_n(w^{(a,b)}, z)^{-1} |\rho(z)|^{-2n} \left|n^{\frac{az+b}{2\sqrt{z^2-1}}}\right|^{-2} =$$

$$= \frac{1}{|\rho(z)|^2-1} \left|\Gamma\left(\frac{1}{2} + \frac{az+b}{2\sqrt{z^2-1}}\right)\right|^{-2} \left|\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{az+b}{2\sqrt{z^2-1}}}\right|^2 .$$

These follow from (ii), (viii) and Theorem 4.1.11.

Example 14. Let w be defined by $\operatorname{supp}(w) = [-1,1]$ and

(11) $$w(x) = \exp\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

for $-1 \le x \le 1$. By Property 13(iv)

$$g(x) = w(x)\, w^{(\frac{1}{\pi})}(x)^{-1}$$

is positive and continuous on $[-1,1]$. We have $g(\pm 1) = \frac{1}{2}\exp(-\frac{1}{\pi})$.
Hence by Properties (ii), (vii) and Theorem 6

$$\lim_{n\to\infty} \lambda_n(w,x)\cdot n = \pi\sqrt{1-x^2}\, w(x)$$

uniformly for $x \in \Delta \subset (-1,1)$ and

$$\lim_{n\to\infty} \lambda_n(w, \pm 1)\, n \exp\{4\sqrt{\tfrac{n}{\pi}}\} = \pi .$$

By Theorem 6.1.27 $w \in M(0,1)$ since $w^{(\frac{1}{\pi})} \in M(0,1)$. By Property 13(vi) and Theorem 6.1.26

$$\lim_{n\to\infty} \frac{\gamma_n(w)}{2^n n^{\frac{1}{2\pi}}} = \Gamma\left(\frac{\pi+1}{2\pi}\right) D\left(\frac{w}{w^{(\frac{1}{\pi})}}, 0\right)^{-1}$$

and by Property 13(ix) and Theorem 6.1.25 for every $z \in \mathbb{C}\setminus[-1,1]$

$$\lim_{n\to\infty} \lambda_n(w,z)\, |\rho(z)|^{2n} \left|n^{\frac{z}{2\pi\sqrt{z^2-1}}}\right|^2 =$$

$$= (|\rho(z)|^2-1) \left|\Gamma\left(\frac{1}{2} + \frac{z}{2\pi\sqrt{z^2-1}}\right)\right|^2 \left|\left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{1-\frac{z}{\pi\sqrt{z^2-1}}}\right| \cdot$$

$$\cdot \left|D\left(\frac{w}{w^{(\frac{1}{\pi})}}, \rho(z)^{-1}\right)\right|^2 .$$

Further by Theorem 6.1.29 and Property 13(viii) for every $z \in \mathbb{C} \setminus [-1,1]$

$$\lim_{n\to\infty} p_n(w,z) \cdot \rho(z)^{-n} \cdot n^{\frac{-z}{2\pi\sqrt{1-z^2}}} =$$

$$= \Gamma\left(\frac{1}{2} + \frac{2\sqrt{z^2-1}}{2\pi\sqrt{z^2-1}}\right)^{-1} \cdot \left[\frac{2\sqrt{z^2-1}}{\rho(z)}\right]^{-\frac{1}{2}+\frac{z}{2\pi\sqrt{z^2-1}}} .$$

$$\cdot D\left(\frac{w}{w^{(\frac{1}{\pi})}}, \rho(z)^{-1}\right)^{-1} .$$

Using Example 14 and Theorems 6.1.25-27 we immediately obtain

Theorem 15. Let w be defined by (11). Let $g(\geq 0) \in L^1_w$ be equivalent to a strictly positive and Riemann integrable function. Then every result in Example 14 remains true if we replace w by $w_g = gw$, in particular, $w_g \in M(0,1)$.

Now we shall investigate $G_n(u,f)$ where $u = u^{(a,b)}$ is a Jacobi weight.

Lemma 16. There exists a constant $C = C(u)$ such that

$$p_n^2(u,x) \leq C[\sqrt{1-x} + \tfrac{1}{n}]^{-2a-1} [\sqrt{1+x} + \tfrac{1}{n}]^{-2b-1}$$

for $|x| \leq 1$, $n = 1,2,\ldots$.

Proof. See Szegö, §7.32.

Lemma 17. For $m = n-1, n$ and $n = 1,2,\ldots$

$$\max_{|x|\leq 1} \lambda_n(u,x)\, p_m^2(u,x) = O(\tfrac{1}{n}) .$$

Proof. Apply Lemmas 16 and 6.3.5.

Let us note that Theorem 3.1.11 and Lemma 4 give

$$\max_{x\in\Delta\subset(-1,1)} \lambda_n(u,x)\, p_m^2(u,x) = O(\tfrac{1}{n}) \quad (m = n-1, n) .$$

Lemma 18. Let $g \in L^1 \cap L^1_u$. Then

$$\lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} |g(t)|\, p_n^2(u,t)\, u(t)dt = 0 .$$

Proof. Let us consider $\int_0^1$. If $a < 0$ then use Lemma 16. Let $a \geq 0$. Fix $\varepsilon > 0$. Then

$$\frac{1}{n}\int_0^1 |g(t)|\, p_n^2(u,t)\, u(t)dt \leq \frac{1}{n} \max_{0\leq t\leq 1-\varepsilon} p_n^2(u,t) \cdot$$

$$\cdot \int_{-1}^{1} |g(t)|\, u(t)dt + C\int_{1-\varepsilon}^{1} |g(t)|dt$$

again by Lemma 16. First let $n \to \infty$ and then $\varepsilon \to 0$.

Theorem 19. Let $g \in L^1 \cap L^1_u$. If g is continuous on a closed set $\mathfrak{M} \subset [-1,1]$ then

$$\lim_{n\to\infty} G_n(u,g,x) = g(x)$$

uniformly for $x \in \mathfrak{M}$.

Proof. We have to show that

$$\lim_{n\to\infty} \max_{|x|\le 1} \{ \lambda_n(u,x) \int_{-1}^{1} (x-t)^2 K_n^2(u,x,t)|g(t)|\, u(t)dt \} = 0$$

and then the machinery of positive operators can be applied. But this is so by Lemmas 17 and 18.

<u>Theorem 20.</u> Let $g \in L^1 \cap L_u^1$. Let $x \in (-1,1)$ be a Lebesgue point of g. Then

$$\lim_{n\to\infty} G_n(u,g,x) = g(x) .$$

<u>Proof.</u> Let $x \in (-1,1)$ and $\varepsilon > 0$ be fixed. If ε is small enough then

$$|G_n(u,g,x) - g(x)| \le$$

$$\lambda_n(u,x) \left[\int_{|x-t|<\frac{1}{n}} + \int_{\frac{1}{n}\le|x-t|\le\varepsilon} + \int_{|x-t|>\varepsilon} \right] |g(t)-g(x)| \cdot$$

$$\cdot K_n^2(u,x,t)u(t)dt .$$

By Lemmas 16-18

$$(12) \qquad \lambda_n(u,x) \int_{|x-t|<\frac{1}{n}} < Cn \int_{|x-t|<\frac{1}{n}} |g(t)-g(x)|dt ,$$

$$(13) \qquad \lambda_n(u,x) \int_{\frac{1}{n}\le|x-t|\le\varepsilon} \le \frac{c}{n} \int_{\frac{1}{n}\le|x-t|\le\varepsilon} \frac{|g(t)-g(x)|}{(x-t)^2}\, dt$$

and the third term converges to 0 when $n \to \infty$. $\lim_{n\to\infty}$ (right side of (12)) $= 0$ because x is a Lebesgue point of g. To estimate the right side of (13) we integrate by parts and remember that $\varepsilon > 0$ is arbitrary.

<u>Lemma 21.</u> If $a, b > -1$ then

$$\lim_{n\to\infty} n^{2a+2} \lambda_n(u,1) = (a+1)\, 2^{a+b+1}\, \Gamma(a+1)^2$$

and

$$\lim_{n\to\infty} n^{2b+2} \lambda_n(u,-1) = (b+1)\, 2^{a+b+1}\, \Gamma(b+1)^2 .$$

<u>Proof.</u> Szegö, §4.5 and easy calculation.

<u>Theorem 22.</u> Let $\mathrm{supp}(w) = [-1,1]$ and suppose that there exists a polynomial P such that $w^{-1}P^2 \in L^1(-1,1)$. Then for almost every $x \in [-1,1]$

$$(14) \qquad \lim_{n\to\infty} n\, \lambda_n(w,x) = \pi\sqrt{1-x^2} .$$

If w is positive and continuous on a closed set $\mathfrak{M} \subset (-1,1)$ then (14) holds uniformly for $x \in \mathfrak{M}$. If w is continuous at $x \in (-1,1)$ then (14) holds. If there exists a Jacobi weight u such that w/u is positive and continuous on $\Delta \subset [-1,1]$ then

$$\lim_{n\to\infty} \frac{\lambda_n(w,x)}{\lambda_n(u,x)} = \frac{w(x)}{u(x)}$$

uniformly for $x \in \Delta$. If w/u in positive and continuous at 1 then

$$\lim_{n\to\infty} n^{2a+2} \lambda_n(w,1) = \frac{w(1)}{u(1)} (a+1) 2^{a+b+1}\, \Gamma(a+1)^2 .$$

If w/u is positive and continuous at -1 then

$$\lim_{n\to\infty} n^{2b+2} \lambda_n(w,-1) = \frac{w(-1)}{u(-1)} (b+1)\, 2^{a+b+1}\, \Gamma(b+1)^2 .$$

Proof. Put $g = w$, α = Legendre weight in the first part of the theorem and $g = w/u$, $\alpha = \int u$ in the second part and use **Theorems** 3, 5, 19, 20, Lemmas 4, 21 and Example 9.

Corollary 23. Let $\mathrm{supp}(w) \subset [-1,1]$. Then for almost every $x \in (-1,1)$

$$\lim_{n\to\infty} n\lambda_n(w,x) \le \pi\sqrt{1-x^2}\, w(x) .$$

Proof. Let $\varepsilon > 0$. Then $\lambda_n(w,x) \le \lambda_n(w+\varepsilon 1_{[-1,1]}, x)$ and $(w+\varepsilon 1_{[-1,1]})^{-1} \in L^1$.

Corollary 24. If $\frac{1}{\alpha'} \in L^1(\Delta)$ then

$$\limsup_{n\to\infty} \frac{1}{n\lambda_n(d\alpha, x)} < \infty$$

for almost every $x \in \Delta$.

Proof. $\lambda_n(d\alpha, x) \ge \lambda_n(\alpha', x) \ge \lambda_n(\alpha' 1_\Delta, x)$ and transform Δ to $[-1,1]$.

In the following we shall improve both corollaries. Corollary 24 is a very strong result. To see this compare Corollary 24 with Freud's result (See Freud, §IV. 6.)

Theorem. Let $\mathrm{supp}(w) \subset [-1,1]$ and

$$\int_0^\pi \frac{|w(\cos(\theta+h))\sin(\theta+h) - w(\cos\theta)\sin\theta|}{w(\cos\theta)\sin\theta}\, d\theta =$$

$$= O(\log^{-\varepsilon}\frac{1}{|h|})$$

for h small with $\varepsilon > 1$. Then for almost every $x \in [-1,1]$

$$\limsup_{n\to\infty} \frac{1}{n\lambda_n(w,x)} < \infty .$$

Let us mention two applications of Corollary 24:

Theorem 25. Let $\mathrm{supp}(d\alpha) \subset [-1,1]$ and $\frac{1}{\alpha'} \in L^1(\Delta)$ where $\Delta \subset [-1,1]$. If $f \in L^2_{d\alpha}$ then for almost every $x \in \Delta$ the Fourier series of f in $p_n(d\alpha, x)$ is strongly $(C,1)$ summable to $f(x)$.

Proof. Corollary 24 and Freud, §IV. 3.

Theorem 26. Let $\alpha \in M(a,b)$ with $b > 0$ and let $\frac{1}{\alpha'} \in L^1(\Delta)$ where $\Delta \subset [a-b, a+b]$. If

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j\, c_j^{a,b}(d\alpha)^2 < \infty$$

(See Definition 3.1.4.) then the sequence $\{p_n^2(d\alpha, x)\}$ is bounded for almost every $x \in \Delta$.

Proof. Theorem 3.1.11 and Corollary 24.

Later we shall see that in both Theorems 25 and 26 the condition $1/\alpha' \in L^1(\Delta)$ may be weakened to $[\alpha']^{-\varepsilon} \in L^1(\Delta)$ for some $\varepsilon > 0$.

Now we shall consider $G_n(d\alpha, f)$ for weights α which are less nice than the Jacobi weights. In the following τ, τ_1 etc. will denote closed intervals. Recall that τ^0 denotes the interior of τ.

Theorem 27. Let $\alpha \in M(0,1)$, $\tau \subset (-1,1)$. Let $\alpha'(t) \ge c > 0$ for almost every $t \in \tau$. Let the sequence $\{p_n^2(d\alpha, t)\}$ be uniformly bounded on every $\tau_1 \subset \tau^0$. Let $f \in L^1_{d\alpha}$ and π be a polynomial vanishing at the endpoints of τ. Let $|f(t)\,\pi(t)| \le M < \infty$ for $d\alpha$ almost every $t \in \mathrm{supp}(d\alpha) \setminus \tau$. Then for almost every $x \in \tau$

(15) $$\lim_{n\to\infty} G_n(d\alpha, f\pi, x) = f(x)\,\pi(x) .$$

Proof. Let $x \in \tau^0$ be a $d\alpha$ Lebesgue point of $f\pi$, that is let

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t)\,\pi(t) - f(x)\,\pi(x)|\,d\alpha(t) = 0 .$$

It is well known that almost every $x \in \tau^0$ is a $d\alpha$ Lebesgue point of $f\pi$. (See Freud, §IV. 2.) First we shall show that

(16) $$\exists\, \alpha'(x) < \infty \Rightarrow \limsup_{n\to\infty} n\,\lambda_n(d\alpha, x) < \infty .$$

Let v_Δ be the Tschebyshev weight corresponding to $\Delta(d\alpha)$. Then

$$\lambda_n(d\alpha, x) \le K_n^{-2}(v_\Delta, x, x) \int_{-\infty}^{\infty} K_n^2(v_\Delta, x, t)\,d\alpha(t) .$$

Since $x \in \tau^0 \in \tau \subset \tau \subset \Delta(d\alpha)^0$ we have by Example 8

$$n\,\lambda_n(d\alpha, x) \le C \cdot n[\alpha(x + \tfrac{1}{n}) - \alpha(x - \tfrac{1}{n})] +$$

$$+ \frac{C}{n} \int_{|x-t| \ge \frac{1}{n}} \frac{d\alpha(t)}{(x-t)^2}$$

and integrating the integral by part we obtain (16). Since $\alpha' \in L^1(-1,1)$ $\alpha'(x) < \infty$ for almost every $x \in \tau^0$. Let now $x \in \tau^0$ be a $d\alpha$ Lebesgue point of $f\pi$ and let $\alpha'(x) < \infty$. We shall prove (15) for such points x. Let $\varepsilon > 0$ be so small that $x \pm \varepsilon \in \tau^0$. If n is large then

$$G_n(d\alpha, f\pi, x) - f(x)\,\pi(x) =$$

$$= \lambda_n(d\alpha, x) \left[\int_{|x-t| \le \frac{1}{n}} + \int_{\frac{1}{n} < |x-t| \le \varepsilon} + \int_{\substack{|x-t| > \varepsilon \\ t \in \tau}} + \int_{t \notin \tau}\right]$$

$$[f(t)\,\pi(t) - f(x)\,\pi(x)]\, K_n^2(d\alpha, x, t)\,d\alpha(t) .$$

By (16)

$$\lim_{n\to\infty} \lambda_n(d\alpha, x) \int_{|x-t| \le \frac{1}{n}} = 0 .$$

We have further by (16)

$$\lambda_n(d\alpha, x) \left|\int_{\frac{1}{n} \le |x-t| \le \varepsilon}\right| \le \frac{C}{n} \int_{\frac{1}{n} \le |x-t| \le \varepsilon} |f(t)\,\pi(t) - f(x)\,\pi(x)| \cdot$$

$$\cdot \frac{d\alpha(t)}{(x-t)^2} .$$

Integrating by parts we obtain

$$\limsup_{n\to\infty} \lambda_n(d\alpha, x) \left|\int_{\frac{1}{n} \le |x-t| \le \varepsilon}\right| \le$$

$$\le C \sup_{|h| \le \varepsilon} \frac{1}{h} \int_x^{x+h} |f(t)\,\pi(t) - f(x)\,\pi(x)|\,d\alpha(t) .$$

Now consider $\int_{\substack{|x-t| > \varepsilon \\ t \in \tau}}$. We have

$$\lambda_n(d\alpha, x) \left|\int_{\substack{|x-t| > \varepsilon \\ t \in \tau}}\right| \le \frac{C}{n\varepsilon^2} \int_{t \in \tau} [p_n^2(d\alpha, t) + p_{n-1}^2(d\alpha, t)] \cdot$$

$$\cdot \, [|f(t)\,\pi(t)| + |f(x)\,\pi(x)|]\,d\alpha(t) \le$$

$$\le \frac{c}{n\varepsilon^2}\,|f(x)\,\pi(x)| + \sum_{k=n-1}^{n} \frac{c}{n\varepsilon^2} \int_{t\in\tau} p_k^2(d\alpha,t)\,|f(t)\,\pi(t)|\,d\alpha(t) \; .$$

Here we cannot use simple estimates since the sequence $\{p_k^2(d\alpha,t)\}$ is uniformly bounded only for $x \in \tau_1 \subset \tau^0$ but not for $x \in \tau$. By Theorem 4.1.11

$$\lim_{k\to\infty} \max_{t\in\tau} \lambda_k(d\alpha,t)\, p_k^2(d\alpha,t) = 0 \; .$$

Hence for $k = n-1, n$

$$\frac{1}{n}\int_{t\in\tau} p_k^2(d\alpha,t)\,|f(t)\,\pi(t)|\,d\alpha(t) =$$

$$= \sigma(1)\int_{t\in\tau} \frac{1}{k\,\lambda_k(d\alpha,t)}\,|f(t)\,\pi(t)|\,d\alpha(t) \; .$$

Let v_τ denote the Tschebyshev weight corresponding to τ. Then from $\alpha'(t) \ge c > 0$ for almost every $t \in \tau$ follows that

$$\lambda_n(d\alpha,t) \ge \frac{c}{n}\, v_\tau(t)^{-1}$$

for $t \in \tau$ (See Freud, §III. 3.) Hence for $k = n-1, n$

$$\frac{1}{n}\int_{t\in\tau} p_k^2(d\alpha,t)\;|f(t)\,\pi(t)|\;d\alpha(t) =$$

$$= \sigma(1)\int_{t\in\tau} v_\tau(t)\,|f(t)\,\pi(t)|\,d\alpha(t) \; .$$

Let us recall that π vanishes at the endpoints of τ. Thus

$$\lim_{n\to\infty} \lambda_n(d\alpha,x) \int_{\substack{|x-t|>\varepsilon \\ t\in\tau}} = 0 \; .$$

Finally, by the conditions and (16)

$$\lambda_n(d\alpha,x)\int_{t\notin\tau} \le \frac{c}{n} \; .$$

Consequently (15) holds for almost every $x \in \tau$.

Theorem 28. Let $\alpha \in M(0,1)$, $\tau \subset (-1,1)$, $\alpha'(t) \ge c > 0$ for almost every $t \in \tau$, $\{p_n^2(d\alpha,t)\}$ be uniformly bounded on each $\tau_1 \subset \tau^0$. Let $f \in L^1_{d\alpha}$, π be a polynomial vanishing at the endpoints of τ and let $|f(t)\,\pi(t)| \le M < \infty$ for $d\alpha$ almost every $t \in \operatorname{supp}(d\alpha)\setminus\tau$. If f is continuous at $x \in \tau^0$ and $|\alpha(x) - \alpha(t)| \le K|x-t|$ for $|x-t|$ small then (15) holds. If f is continuous on $\tau_1 \subset \tau^0$ and $\alpha \in \operatorname{Lip} 1$ on $\tau_2\,(\tau_1 \subset \tau_2^0)$ then (15) is satisfied uniformly for $x \in \tau_1$.

Proof. Repeat the proof of Theorem 27 with the necessary modifications.

Lemma 29. Let $\alpha \in S$. Let $\tau \subset (-1,1)$ and α be absolutely continuous on τ with $\alpha'(t) \equiv 1$ for $t \in \tau$. Then the sequence $\{p_n^2(d\alpha,x)\}$ is uniformly bounded on each $\tau_1 \subset t^0$ and

$$\lim_{n\to\infty} n\,\lambda_n(d\alpha,x) = \pi\sqrt{1-x^2}$$

uniformly for $x \in \tau_1 \subset \tau^0$. Moreover, $\alpha \in M(0,1)$.

Proof. See Geronimus, §5.4.

Lemma 30. Let $\beta \in S$. Let $\tau \subset (-1,1)$ and β be absolutely continuous on τ with $1/\beta' \in L^1(\tau)$. Then for almost every $x \in \tau$

$$\lim_{n\to\infty} n\,\lambda_n(d\beta, x) = \pi\beta'(x)\sqrt{1-x^2} .$$

Proof. Let us define α and g by

$$d\alpha(x) = \begin{cases} d\beta(x) & \text{for} \quad x \in [-1,1]\setminus\tau \\ dx & \text{for} \quad x \in \tau \end{cases}$$

with $\operatorname{supp}(d\alpha) = [-1,1]$ and

$$g(x) = \begin{cases} 1 & \text{for} \quad x \in [-1,1]\setminus\tau \\ \beta'(x) & \text{for} \quad x \in \tau . \end{cases}$$

Then $\alpha \in S$, $g(x)^{\pm 1} = 1 < \infty$ *for* $x \in [-1,1]\setminus\tau$ *and* $\beta = \alpha_g$. *Further* α satisfies the conditions of Lemma 29 and consequently α satisfies also the conditions of Theorem 27. Let us put $P_1 = v_\tau^{-2}$ where v_τ is the Tschebyshev weight corresponding to τ. Then by Theorem 5

$$\frac{v_\tau^{-4}(x)\,\lambda_n(d\beta, x)}{\lambda_{n-2}(d\alpha, x)} \le G_{n-2}(d\alpha, g\,v_\tau^{-4}, x) .$$

Since v_τ^{-4} vanishes at the endpoints of τ we obtain from Lemmas 4, 29 and Theorem 27

$$\limsup_{n\to\infty} n\,\lambda_n(d\beta, x) \le \pi\sqrt{1-x^2}\,\beta'(x)$$

for almost every $x \in \tau$.

On the other hand putting $P_2 = v_\tau^{-2}$ and using Theorem 5

$$v_\tau^{-4}(x)\,G_{n+2}^{-1}(d\alpha, g^{-1}v_\tau^{-4}) \le \frac{\lambda_n(d\beta, x)}{\lambda_{n+2}(d\alpha, x)} .$$

Thus by the same arguments $(g^{-1} \in L^1_{d\alpha})$

$$\pi\sqrt{1-x^2}\;\beta'(x) \le \liminf_{n\to\infty} \frac{\lambda_n(d\beta, x)}{\lambda_n(d\alpha, x)} .$$

Lemma 31. Let α be an arbitrary weight. Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \lambda_n(v, x)\int_{-1}^{1} K_n^2(v, x, t)\, d[\alpha_g(t) + \alpha_j(t)] = 0 .$$

Proof. We have for almost every $x \in [-1,1]$

$$\lim_{h\to 0} \frac{1}{h}\int_{s-h}^{x+h} |d[\alpha_s(t) + \alpha_j(t)]| =$$

$$= \lim_{h\to 0} \frac{1}{h}\int_{x-h}^{x+h} d[\alpha_s(t) + \alpha_j(t)] = 0$$

and we can use standard argument.

Lemma 32. Let α be an arbitrary weight. Then for almost every $x \in [-1,1]$

$$\lim_{n\to\infty} \lambda_n(v, x)\int_{-1}^{1} K_n^2(v, x, t)\, d\alpha(t) = \sqrt{1-x^2}\;\;\alpha'(x) .$$

Proof. Since $d\alpha(t) = \alpha_a'(t)\,dt + d[\alpha_s(t) + \alpha_j(t)] = g(t)\,v(t)dt + d[\alpha_s(t) + \alpha_j(t)]$ where $g = \alpha_a v^{-1}$ the lemma follows from Theorem 20 and Lemma 31.

Theorem 33. If $\operatorname{supp}(d\alpha) \subset [-1,1]$ then

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha, x) \le \pi\alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in [-1,1]$.

Proof. Use Lemma 32 and the inequality

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v,x)} \le \lambda_n(v,x)\int_{-1}^{1} K_n^2(v,x,t)\,d\alpha(t)$$

which follows from $\text{supp}(d\alpha) \subset [-1,1]$.

Now we can prove the following

Theorem 34. Let $\alpha \in S$, $\tau \subset [-1,1]$. If $1/\alpha' \in L^1(\tau)$ then

$$\lim_{n\to\infty} n\,\lambda_n(d\alpha, x) = \pi\alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$.

Proof. By Theorem 33 we have to show that

(17) $$\liminf_{n\to\infty} n\,\lambda_n(d\alpha, x) \ge \pi\alpha'(x)\sqrt{1-x^2}$$

for almost every $x \in \tau$. We can assume $\tau \subset (-1,1)$. Since

$$n\,\lambda_n(d\alpha, x) \ge n\,\lambda_n(\alpha', x)$$

and α' satisfies the conditions of Lemma 30 (17) follows from Lemma 30.

Theorem 35. Let $\alpha \in S$. Let $x \in (-1,1)$, let α be absolutely continuous near x and let α' be continuous at x. Then

(18) $$\lim_{n\to\infty} n\,\lambda_n(d\alpha, x) = \pi\,\alpha'(x)\sqrt{1-x^2}\,.$$

If α is absolutely continuous on $\tau \subset (-1,1)$, α' is continuous and positive on $\tau_1 \subset \tau^0$ then (18) holds uniformly for $x \in \tau_1$.

Proof. The proof is the same as that of Lemma 30. If in the first part of the theorem $\alpha'(x) = 0$ then we fix $\varepsilon > 0$ and we prove first that

$$\lim_{n\to\infty} n\,\lambda_n(d\beta, x) = \pi\beta'(x)\sqrt{1-x^2}$$

where $\beta(x) = \alpha(x) + \varepsilon x$ with $\text{supp}(d\beta) = [-1,1]$ and after we let $\varepsilon \to 0$.

Let us note that the first part of Theorem 35 easily follows from results of Geronimus and the second part of it has been obtained by Geronimus in his book. Further, we would not have obtained a stronger result in Theorem 34 if we had supposed the $\pi^2/\alpha' \in L^1(\tau)$ with a suitable polynomial π. Theorem 34 can be formulated as

Theorem 36. Let $\alpha \in S$, $\tau \subset [-1,1]$. If $1/\alpha' \in L^1(\tau)$ then

$$(C,1)\lim_{n\to\infty} \left[\alpha'(x)\sqrt{1-x^2}\; p_n^2(d\alpha, x) - \frac{2}{\pi}\cos^2(n\theta - \Gamma(\theta))\right] = 0$$

for almost every $x \in \tau$, where $x = \cos\theta$ $(0 \le \theta \le \pi)$ and $\Gamma(\theta)$ is defined in Definition 4.2.4.

Later we shall improve Theorem 36 for weights having nice coefficients in the recursion formula.

Definition 37. Let ω be a modulus of continuity. Then

(i) $f \in A_x^{\omega}$ iff $|f(x) - f(t)| \le C_x\,\omega(|x-t|)$

for $|x-t|$ small.

(ii) $f \in A_\tau^\omega$ if $|f(x) - f(t)| \le C_\tau \, \omega(|x-t|)$

for $x \in \tau$, $t \in \tau(\varepsilon)$ where $\tau(\varepsilon)$ is an ε neighborhood of τ.

(iii) $f \in B_x^\omega$ iff $f'(x)$ exists and

$$|f(t) - f(x) - f'(x)(t-x)| \le C_x \, \omega(|t-x|)\, |t-x|$$

for $|x-t|$ small.

(iv) $f \in B_\tau^\omega$ if $f'(x)$ exists for $x \in \tau$ and

$$|f(t) - f(x) - f'(x)(t-x)| \le C_\tau \, \omega(|t-s|)\, |t-x|$$

for $x \in \tau$ and $t \in \tau(\varepsilon)$.

Theorem 38. Let $\operatorname{supp}(d\alpha)$ be compact, $x \in \operatorname{supp}(d\alpha)$, α be absolutely continuous near x, $0 < c_1 \le \alpha'(t) \le c_2 < \infty$ for $|x-t|$ small and let the sequence $\{p_n^2(d\alpha, t)\}$ be uniformly bounded for $|x-t|$ small. Let f be $d\alpha$ measurable and bounded on $\operatorname{supp}(d\alpha)$. If $f \in A_x^\omega$ then

$$G_n(d\alpha, f, x) = f(x) + O(1)\,\frac{1}{n} \int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\, dt \,, \tag{19}$$

where $|O(1)| < C$ with C independent of n. If $\operatorname{supp}(d\alpha)$ is compact, α is absolutely continuous in $\tau(\varepsilon)$, $0 < c_1 \le \alpha'(t) \le c_2 < \infty$ in $\tau(\varepsilon)$, the sequence $\{p_n^2(d\alpha, t)\}$ is uniformly bounded in a neighborhood of τ, f is $d\alpha$ measurable and bounded on $\operatorname{supp}(d\alpha)$ and $f \in A_\tau^\omega$ then (19) holds uniformly for $x \in \tau$.

Proof. Let us prove, for simplicity, the first part of the theorem. Let $\varepsilon > 0$ be small. Then

$$|G_n(d\alpha, f, x) - f(x)| \le$$

$$\le \frac{c}{n} \left\{ \left| \int_{|x-t|<\varepsilon} \right| + \left| \int_{|x-t|\ge\varepsilon} \right| \right\} [f(t) - f(x)]\, K_n^2(d\alpha, x, t)\, d\alpha(t) \,.$$

The second integral is $O(1)$, the first one is

$$O(1) \int_{|x-t|<\varepsilon} \omega(|t-x|)\, \frac{n^2}{(1+n|t-x|)^2}\, dt$$

by the usual computation and the latter integral can easily be estimated.

Lemma 39. For $n = 1, 2, \ldots$

$$\lambda_n(d\alpha, x) \int_{-\infty}^{\infty} (t-x)\, K_n^2(d\alpha, x, t)\, d\alpha(t) =$$

$$= -\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\, \lambda_n(d\alpha, x)\, p_{n-1}(d\alpha, x)\, p_n(d\alpha, x) \,.$$

Proof. See Freud, §V.6.

Theorem 40. Let the conditions of the first part of Theorem 38 be satisfied with $f \in B_x^\omega$ instead of $f \in A_x^\omega$. Then

$$G_n(d\alpha, f, x) = f(x) + O(1)\,\frac{1}{n} \int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\, dt \tag{20}$$

where $|O(1)| \le C$ and C does not depend on n. If the conditions of the second part of Theorem 38 are satisfied with $f \in B_\tau^\omega$ instead of $f \in A_\tau^\omega$ then (20) holds uniformly for $x \in \tau$.

Proof. For simplicity let us consider the first part of the theorem. From the proof of Theorem 38 we see that we have to show

$$\left| \int_{|x-t|<\varepsilon} [f(t) - f(x)]\, K_n^2(d\alpha, x, t)\, d\alpha(t) \right| = O(1) \int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\, dt \,.$$

The left side here is not greater than

$$\int_{|x-t|<\varepsilon} |f(t) - f(x) - f'(x)(t-x)|\, K_n^2(d\alpha, x, t)\, d\alpha(t) +$$

$$+ |f'(x) \int_{|x-t|<\varepsilon} (t-x)\, K_n^2(d\alpha, x, t)\, d\alpha(t)| \ .$$

Since $f \in B_x^\omega$ the first integral here may easily be estimated. Further

$$\int_{|x-t|<\varepsilon} (t-x)\, K_n^2(d\alpha, x, t)\, d\alpha(t) = \left\{\int_{-\infty}^{\infty} - \int_{|x-t|\geq\varepsilon}\right\}$$

$$(t-x)\, K_n^2(d\alpha, x, t)\, d\alpha(t)$$

and we apply Lemma 39.

Remark 41. If $f \in A_x^\omega$ and $f(x) > 0$ then $f^{-1} \in A_x^\omega$. If $f \in A_\tau^\omega$ and $f(x) > 0$ for $x \in \tau$ then $f^{-1} \in A_\tau^\omega$. This is obvious. If $f \in B_x^\omega$ and $f(x) > 0$ then $f^{-1} \in B_x^\omega$ and if $f \in B_\tau^\omega$ and $f(x) > 0$ for $x \in \tau$ then $f^{-1} \in B_\tau^\omega$. Let us prove the latter. Because of continuity $f(t) > 0$ for $|x-t|$ small and $f(t) > 0$ for $t \in \tau(\varepsilon)$ with ε small respectively. Now we can use the identity

$$\frac{1}{f(t)} - \frac{1}{f(x)} + \frac{f'(x)}{f^2(x)}(t-x) = \frac{[f(x)-f(t)]^2}{f^2(x)\, f(t)} -$$

$$- \frac{1}{f^2(x)}[f(t) - f(x) - f'(x)(t-x)]$$

and

$$[f(x) - f(t)]^2 \leq C(t-x)^2 \leq C|t-x|\, \omega(|t-x|)$$

where C, of course, depends on f.

Definition 42. For a given weight β and interval τ the weight β_τ and the function $g \equiv g_{\beta,\tau}$ are defined by

$$d\beta_\tau(x) = \begin{cases} d\beta(x) & \text{for } x \notin \tau \\ dx & \text{for } x \in \tau \end{cases}$$

and

$$g(x) = \begin{cases} \beta'(x) & \text{for } x \in \tau \\ 1 & \text{for } x \notin \tau \ . \end{cases}$$

We have $d(\beta_\tau)_g \leq d\beta$ and equality holds iff β is absolutely continuous on τ.

Theorem 43. Let $\beta \in S$, $x \in (-1,1)$, β be absolutely continuous near x, $\beta' \in A_x^\omega$ (or $\beta' \in B_x^\omega$), $\beta'(x) > 0$, τ^0 be a sufficiently small neighborhood of x. Then

$$\frac{\lambda_n(d\beta, x)}{\lambda_n(d\beta_\tau, x)} = \beta'(x) + O(1) \begin{cases} \frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\, dt & (\beta' \in A_x^\omega) \\ \frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\, dt & (\beta' \in B_x^\omega) \ . \end{cases}$$

If $\tau_1 \subset (-1,1)$, β is absolutely continuous in $\tau_1(\varepsilon)$ with some $\varepsilon > 0$, $\beta' \in A_{\tau_1}^\omega$ (or $\beta' \in B_{t_1}^\omega$), τ^0 is a sufficiently small neighborhood of τ_1 then

$$\frac{\lambda_n(d\beta, x)}{\beta_n(d\beta_\tau, x)} = \beta'(x) + O(1) \begin{cases} \frac{1}{n} \int_{-1}^{1} \frac{\omega(t)}{t^2} dt & (\beta' \in A^{\omega}_{\tau_1}) \\ \frac{1}{n} \int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t} dt & (\beta' \in B^{\omega}_{\tau_1}) \end{cases}$$

uniformly for $x \in \tau_1$.

Proof. If τ is small then $d(\beta_\tau)_g = d\beta$, $g^{\pm 1}$ is bounded on $[-1,1]$, $g^{\pm 1} \in A^{\omega}_x(B^{\omega}_x)$ or $g^{\pm 1} \in A^{\omega}_{\tau_1}(B^{\omega}_{\tau_1})$ respectively by Remark 41. Further β_τ satisfies the conditions of Lemma 29 and consequently β_τ satisfies the conditions of Theorems 38 and 40. Finally, apply Theorem 3.

Lemma 44. If $\tau \subset [-1,1]$ then

$$\lambda_n(v,x) \int_{-1}^{1} K_n^2(v,x,t)\, d\beta_\tau(t) \leq \sqrt{1-x^2} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$ and consequently if $\operatorname{supp}(d\beta) \subset [-1,1]$ and

$$n\,\lambda_n(d\beta_\tau, x) \leq \sqrt{1-x^2} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$.

Proof. See Freud, §V.6.

Lemma 45. Let $\operatorname{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$. Let exist a polynomial π such that $\pi^2/\beta'_\tau \in L^1(-1,1)$. Then

$$\frac{1}{n\,\lambda_n(d\beta_\tau, x)} \leq \frac{1}{\pi\sqrt{1-x^2}} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$.

Proof. Let us consider (7) in Theorem 5. We put there α = Tschebyshev weight, $g = \beta'_\tau/v$ so that $d\alpha_g(x) = \beta'_\tau(x)dx$. Let $P_2 = v^{-2}\pi$. We obtain

$$\frac{v^{-4}(x)\,\pi^2(x)\,\lambda_{n+m}(v,x)}{\lambda_n(\beta'_\tau, x)} \leq G_{n+m}(v, \frac{v^{-3}\pi^2}{\beta'_\tau}, x)$$

where $m = \deg \pi + 2$ and $v^{-3}\pi^2/\beta'_\tau \in L^1_v$. Since $\beta'_\tau(t) = 1$ for $t \in \tau$ we can suppose that π has no zeros in τ^0. Hence for $x \in \tau^0$

$$\lambda_n^{-1}(d\beta_\tau, x) \leq \pi^{-2}(x)\, v^4(x)\, \lambda_{n+m}^{-1}(v,x)\, G_{n+m}\, v, \left(\frac{v^{-3}\pi^2}{\beta'_\tau}, x\right).$$

Now we should apply Theorem 40 with α = Tschebyshev weight, $f = v^{-3}\pi^2/\beta'_\tau \in B^{\omega}_{\tau_1}$ if $\tau_1 \subset t^0$ and $\omega(t) \equiv t$, but we cannot do this directly since in our case f is not bounded on $\operatorname{supp}(v)$. This small problem can be avoided by remarking that the Tschebyshev polynomials are uniformly bounded on $[-1,1]$ and thus

$$\int_{|x-t|>\varepsilon} |f(t)|\, K_n^2(v,x,t)\, v(t)dt \leq \frac{c}{\varepsilon^2} \int_{-1}^{1} |f(t)|\, v(t)dt$$

which in our case is finite. Hence

$$G_{n+m}\left(v, \frac{v^{-3}\pi^2}{\beta'_\tau}, x\right) = \frac{v^{-3}(x)\,\pi^2(x)}{\beta'_\tau(x)} + O(\tfrac{1}{n})$$

uniformly for $x \in \tau_1 \subset \tau^0$ which proves the lemma.

Lemmas 44 and 45 give us

Theorem 46. Let $\operatorname{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$ and let exist a polynomial π such that $\pi^2/\beta'_\tau \in L^1(-1,1)$. Then

$$n\,\lambda_n(d\beta_\tau, x) = \pi\sqrt{1-x^2} + O\left(\frac{1}{n}\right)$$

uniformly for $x \in \tau_1 \subset \tau^0$.

We obtain immediately from Theorems 43 and 46 the following

Theorem 47. Let $\operatorname{supp}(d\alpha) = [-1,1]$ and suppose that there exists a polynomial π such that $\pi^2/\alpha' \in L^1(-1,1)$. If $x \in (-1,1)$, α is absolutely continuous near x, $\alpha' \in A_x^\omega(B_x^\omega)$ and $\alpha'(x) > 0$ then

$$n\,\lambda_n(d\alpha, x) = \pi\alpha'(x)\sqrt{1-x^2} + O(1)\begin{cases} \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\,dt & (\alpha' \in A_x^\omega) \\ \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\,dt & (\alpha' \in B_x^\omega) \,. \end{cases}$$

If $\tau \subset (-1,1)$, α is absolutely continuous in a neighborhood of τ, $\alpha' \in A_\tau^\omega(B_\tau^\omega)$ and $\alpha'(t) > 0$ for $t \in \tau$ then

$$n\,\lambda_n(d\alpha, x) = \pi\alpha'(x)\sqrt{1-x^2} + O(1)\begin{cases} \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\,dt & (\alpha' \in A_\tau^\omega) \\ \dfrac{1}{n}\displaystyle\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t}\,dt & (\alpha' \in B_\tau^\omega) \,. \end{cases}$$

The reader should compare Theorem 47 with Freud's results where $\pi^2/\alpha' \in L^\infty$ has to be assumed. (See Freud, §V.6).

In Theorem 40 we have shown that $G_n(d\alpha, f)$ will converge to f with speed $1/n$ if f is good. On the other hand for $f \in \operatorname{Lip} 1$ we have only obtained $\log n/n$ as convergence speed for $G_n(d\alpha, f)$. (See Theorem 38.) We may ask two questions, namely, whether $\log n/n$ occurs because of our weak techniques and how to improve convergence.

Theorem 48. Let $f(x) \equiv |x|$. Then

$$G_n(v, f, 0) \geq C \cdot \frac{\log n}{n}$$

for $n \geq 3$.

Proof. Since

$$G_n(v, f, 0) = 2\,\lambda_n(v, 0)\int_0^1 t\,K_n^2(v, 0, t)\,v(t)dt$$

we have only to show that for k odd

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 kt}{\cos t}\,dt \geq C \log k \qquad (k \geq 3)\,.$$

The left side here equals

$$\int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{\sin t}\,dt \geq \int_0^{\frac{\pi}{2}} \frac{\sin^2 kt}{t}\,dt = \int_0^{\frac{k\pi}{2}} \frac{\sin^2 u}{u}\,du \geq C \log k \,.$$

By Theorem 48 if we want to improve the convergence properties of $G_n(d\alpha, f)$ then we have to modify these operators. Let us put for $n \leq m$

$$G_{n,m}(d\alpha, f, x) = \lambda_n(d\alpha, x)\int_{-\infty}^{\infty} f(t)\,K_n(d\alpha, x, t)\,K_m(d\alpha, x, t)\,d\alpha(t) \,.$$

For $z \in \mathbb{C}$ $G_{n,m}(d\alpha, f, z)$ can be defined by

$$G_{n,m}(d\alpha, f, z) = \lambda_n^*(d\alpha, z) \int_{-\infty}^{\infty} f(t)\, k_n(d\alpha, z, t)\, k_m(d\alpha, z, t)\, d\alpha(t) .$$

(See 4.1).

Properties 49. (i) $G_{n,m}(d\alpha, \pi_{m-n}) \equiv \pi_{m-n}$. (ii) $G_{n,m}$ is a rational function of degree $(n+m-2, 2n-2)$. (iii) The Lebesgue function $G^*_{n,m}(d\alpha, x)$ of the operator $G_{n,m}(d\alpha)$ is not greater than $[\lambda_n(d\alpha, x)\, \lambda_m^{-1}(d\alpha, x)]^{\frac{1}{2}}$.

Consequently if f is good globally and α is nice locally (near $x \in \operatorname{supp}(d\alpha)$) then for e.g. $m = 2n$ $G_n(d\alpha, f, x)$ may converge to $f(x)$ very rapidly. On the other hand if α is nice near x then the kernel function of $G_{n,2n}(d\alpha)$ has the same majorant only as that of $G_n(d\alpha) \equiv G_{n,n}(d\alpha)$, namely

$$\frac{Cn}{1+n^2(x-t)^2} , \tag{21}$$

which – as is well known – is too weak to assume good convergence properties for $G_{n,2n}(d\alpha, f, x)$ if f is nice only at x . For this reason we introduce another operator $G_N(d\alpha)$ $(N = (n_1, n_2, \ldots, n_k)$.

Let $k \geq 2$ be fixed and let $n_1 \leq n_2$, $n_1 - 1 + n_2 - 1 \leq n_3 - 1$ and, in gneral, $\sum_{j=1}^{i-1} (n_j - 1) \leq n_i - 1$ for $i = 2, \ldots, k$. We put

$$G_N(d\alpha, f, x) = \prod_{i=1}^{k-1} \lambda_{n_i}(d\alpha, x) \int_{-\infty}^{\infty} f(t) \prod_{i=1}^{k} K_{n_i}(d\alpha, x, t)\, d\alpha(t)$$

and for $z \in \mathbb{C}$ we define $G_N(d\alpha, f, z)$ in exactly the same way as we did when $k = 2$.

Let us note that if e.g. α = Tschebyshev weight then the kernel function of $G_N(d\alpha)$ may be majorated by

$$\frac{Cn}{1+n^k|x-t|^k} \tag{22}$$

($x, t \in [-1, 1]$ and all n_i are of order n), which differs very much from (21)! (22) implies that for $k \geq 3$ $G_N(v, f, x)$ converges to $f(x)$ with speed $1/n$ if all n_i are of order n and $f \in A_x^{\omega}$ with $\omega(t) \equiv t$.

It should be possible to improve most of the results of this section by using $G_N(d\alpha)$ instead of $G_n(d\alpha)$. At the present time we cannot do this. Let us mention a simple result which we shall need later.

Lemma 50. Let $x \in \mathbb{R}$. Then

$$|\pi_n(x)| \leq \frac{1}{2} \lambda_n(d\alpha, x) \int_{-\infty}^{\infty} |\pi_n(t)|\, [K_n^2(d\alpha, x, t) + K_{2n}^2(d\alpha, x, t)]\, d\alpha(t) .$$

Proof. Use Property 49(i).

Theorem 51. Let $\alpha \in M(0,1)$. Then

$$\limsup_{n\to\infty} n\, \lambda_n(d\alpha, x) \leq \pi \alpha'(x) \sqrt{1-x^2} \tag{23}$$

for almost every $x \in \operatorname{supp}(d\alpha)$.

Proof. By Theorem 3.3.7 it is enough to show that (23) holds for almost every $x \in [-1, 1]$. If $\operatorname{supp}(d\alpha) = [-1, 1]$ then (23) follows from Theorem 33. Let now $\Delta = \operatorname{supp}(d\alpha) \setminus [-1, 1]$ be not empty. Then there exists $\varepsilon_1 > 0$

such that for every $\varepsilon \in (0, \varepsilon_1)$ $\Delta_\varepsilon = \text{supp}(d\alpha) \setminus [-1-\varepsilon, 1+\varepsilon]$ is not empty. By Theorem 3.3.7 Δ_ε contains finitely many points. Let $\Delta_\varepsilon = \{a_k\}_{k=1}^m$ where $m = m(\varepsilon)$. Let π be defined by

$$\pi(x) = \prod_{k=1}^{m} (x - a_k) .$$

Then for $n > m$

$$\lambda_n(d\alpha, x) \leq \int_{-\infty}^{\infty} \frac{K_{n-m}^2(v_\varepsilon, x, t)\,\pi^2(t)}{K_{n-m}^2(v_\varepsilon, x, x)\,\pi^2(x)}\, d\alpha(t)$$

for $x \in [-1, 1]$ where v_2 *denotes the Tschebyshev weight corresponding* to $[-1-\varepsilon, 1+\varepsilon]$. Since π vanishes on Δ_ε we obtain

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v_\varepsilon, x)} \leq \frac{\lambda_{n-m}(v_\varepsilon, x)}{\lambda_n(v_\varepsilon, x)}\, \pi(x)^{-2} .$$

$$\cdot\ \lambda_{n-m}(v_\varepsilon, x) \int_{-1-\varepsilon}^{1+\varepsilon} K_{n-m}^2(v_\varepsilon, x, t)\, \pi^2(t)\, d\alpha(t) .$$

Transforming $[-1-\varepsilon, 1+\varepsilon]$ into $[-1, 1]$ we get

$$\frac{\lambda_n(d\alpha, x)}{\lambda_n(v, \frac{x}{1+\varepsilon})} \leq \frac{\lambda_{n-m}(v, \frac{x}{1+\varepsilon})}{\lambda_n(v, \frac{x}{1+\varepsilon})}\, \pi(x)^{-2} .$$

$$\cdot\ \lambda_{n-m}(v, \frac{x}{1+\varepsilon}) \int_{-1}^{1} K_{n-m}^2(v, \frac{x}{1+\varepsilon}, t)\, \pi^2((1+\varepsilon)t)\, d\alpha((1+\varepsilon)t) .$$

Let β be defined by $d\beta(t) = \pi^2((1+\varepsilon)t)\, d\alpha((1+\varepsilon)t)$. Then β is a weight on $[-1, 1]$. Hence by Lemmas 4 and 32

$$\limsup_{n\to\infty} \frac{\lambda_n(d\alpha, x)}{\lambda_n(v, \frac{x}{1+\varepsilon})} \leq \sqrt{1 - \frac{x^2}{(1+\varepsilon)^2}}\; \pi(x)^{-2}\, \beta'(\frac{x}{1+\varepsilon}) =$$

$$= \sqrt{(1+\varepsilon)^2 - x^2}\;\; \alpha'(x)$$

for almost every $x \in [-1, 1]$, that is by Example 9

$$\limsup_{n\to\infty} n\, \lambda_n(d\alpha, x) \leq \pi\, \alpha'(x) \sqrt{(1+\varepsilon)^2 - x^2}$$

for almost every $x \in [-1, 1]$. Now let $\varepsilon \to 0$.

Now we can prove the following theorem which is one of our main results.

<u>Theorem 52.</u> Let $\alpha \in M(0, 1)$, $\tau \subset (-1, 1)$, $1/\alpha' \in L^1(\tau)$. Let exist a sequence $\{\varepsilon_k (\geq 0)\}$ *with* $\lim_{k\to\infty} \varepsilon_k = 0$ *such that* $\log \alpha'(t)/\sqrt{(1-\varepsilon_k)^2 - x^2}$ $\in L^1(-1+\varepsilon_k, 1-\varepsilon_k)$ for every fixed k. Then

$$\lim_{n\to\infty} n\, \lambda_n(d\alpha, x) = \pi \alpha'(x) \sqrt{1 - x^2} \tag{24}$$

for almost every $x \in \tau$.

<u>Proof.</u> Because of Theorem 51 we only have to show that

$$\liminf_{n\to\infty} n\, \lambda_n(d\alpha, x) \geq \pi \alpha'(x) \sqrt{1 - x^2}$$

for almost every $x \in \tau$. We have $\lambda_n(d\alpha, x) \geq \lambda_n(\alpha', x)$. Let k be fixed and w be defined by $w(t) = \alpha'(1-\varepsilon_k)t)$ for $-1 \leq t \leq 1$ with $\text{supp}(w) = [-1, 1]$. By the conditions, $w \in S$ and $w^{-1} \in L^1(\tau_1)$, where $\tau_1 = [(1-\varepsilon_k)^{-1} c_1,$ $(1-\varepsilon_k)^{-1} c_2]$ if $\tau = [c_1, c_2]$. By Theorem 34

$$\lim_{n\to\infty} n\,\lambda_n(w,x) = \pi\sqrt{1-x^2}\; w(x)$$

for almost every $x \in \tau_1$. We have by Example 10

$$\lambda_n(w,x) = (1-\varepsilon_k)^{-1}\,\lambda_n(1_\Delta \alpha', (1-\varepsilon_k)x)$$

$$\leq (1-\varepsilon_k)^{-1}\,\lambda_n(\alpha', (1-\varepsilon_k)x)$$

where $\Delta = [-1+\varepsilon_k, 1-\varepsilon_k]$. Hence

$$\liminf_{n\to\infty} n\,\lambda_n(\alpha', (1-\varepsilon_k)x) \geq \pi\sqrt{(1-\varepsilon_k)^2 - (1-\varepsilon_k)^2x^2}\cdot \alpha'((1-\varepsilon_k)x)$$

for almost every $x \in \tau_1$, that is

$$\liminf_{n\to\infty} n\,\lambda_n(\alpha', x) \geq \pi\sqrt{(1-\varepsilon_k)^2 - x^2}\;\alpha'(x)$$

for almost every $x \in \tau$. Now let $k \to \infty$.

Later we shall show that if e.g.

$$\sum_{j=0}^{\infty}\left\{|\alpha_j(d\alpha)| + \left|\frac{\gamma_j(d\alpha)}{\gamma_{j+1}(d\alpha)} - \frac{1}{2}\right|\right\} < \infty$$

then for each $\tau \subset (-1,1)$ all the conditions of Theorem 52 are satisfied and thus (24) holds for almost every $x \in [-1,1]$. Let us note that Theorem 34 follows from Theorem 52.

Corollary 53. Let α and τ satisfy the conditions of Theorem 52 and let ℓ be a fixed nonnegative integer. Then for almost every $x \in \tau$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} p_k(d\alpha,x)\,p_{k+\ell}(d\alpha,x) = \frac{T_\ell(x)}{\pi\alpha'(x)\sqrt{1-x^2}}$$

where T_ℓ denotes the ℓ-th Tschebyshev polynomial.

Proof. Use Theorems 52 and 4.1.19.

Theorem 54. Let $\alpha \in M(0,1)$. Then for almost every $x \in \operatorname{supp}(d\alpha)$

(25) $$\limsup_{n\to\infty} n\,\lambda_n(d\alpha,x) = \pi\alpha'(x)\sqrt{1-x^2}\;.$$

Proof. Let $\mathfrak{M}$ be defined by $\mathfrak{M} = \{x: \alpha'(x) > 0\}$. By Theorem 33 we have to show that (25) holds for almost every $x \in \mathfrak{M}$.

Since α is almost everywhere continuous, for every $x \in (-1,1)$ we can find a sequence $\{\varepsilon_m\}$ such that $\varepsilon_m > 0$, $\lim_{m\to 0}\varepsilon_m = 0$, $[x-\varepsilon_m, x+\varepsilon_m] \subset (-1,1)$ and α is continuous at $x-\varepsilon_m$ and $x+\varepsilon_m$. By Theorem 4.2.14

$$\lim_{n\to\infty}\int_{x-\varepsilon_m}^{x+\varepsilon_m}\frac{1}{n\,\lambda_n(d\alpha,t)}\,d\alpha(t) = \frac{1}{\pi}\int_{x-\varepsilon_m}^{x+\varepsilon_m}\frac{dt}{\sqrt{1-t^2}}$$

for $m = 1,2,\ldots$. Thus by Fatou's lemma

$$\frac{1}{2\varepsilon_m}\int_{x-\varepsilon_m}^{x+\varepsilon_m}\liminf_{n\to\infty}\frac{1}{n\,\lambda_n(d\alpha,t)}\,\alpha'(t)dt \leq \frac{1}{2\pi\varepsilon_m}\int_{x-\varepsilon_m}^{x+\varepsilon}\frac{dt}{\sqrt{1-t^2}}\;.$$

Letting $m \to \infty$ and using Lebesgue's theorem we obtain

$$\liminf_{n\to\infty}\frac{1}{n\,\lambda_n(d\alpha,x)}\,\alpha'(x) \leq \frac{1}{\pi\sqrt{1-x^2}}$$

for almost every $x \in [-1,1]$. By Theorem 3.3.7 $\mathfrak{M} \subset [-1,1]$. Hence for almost every $x \in \mathfrak{M}$

$$\limsup_{n\to\infty} n\,\lambda_n(d\alpha, x) \geq \pi\alpha'(x)\sqrt{1-x^2} .$$

The converse inequality has been proved in Theorem 33.

Theorem 55. Let $\mathrm{supp}(d\alpha) = [-1,1]$ and $\alpha'(x) > 0$ for almost every $x \in [-1,1]$. Then (25) holds for almost every $x \in [-1,1]$.

Proof. If f is continuous on $[-1,1]$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_{kn}(d\alpha)) = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{\sqrt{1-t^2}} .$$

(See Freud, §III.9.) Hence by Lemma 5.1

$$\text{(26)} \qquad \lim_{n\to\infty} \int_{-1}^{1} f(t)\frac{1}{n\,\lambda_n(d\alpha, t)}\,d\alpha(t) = \frac{1}{\pi}\int_{-1}^{1} f(t)\frac{dt}{\sqrt{1-t^2}}$$

if f is continuous on $[-1,1]$. Using one-sided approximation we obtain that (26) remains valid if f is the characteristic function of a $d\alpha$ measurable interval $\Delta \subset (-1,1)$. Now we can repeat the proof of Theorem 54.

6.3. Generalized Christoffel Functions

Definition 1. Let $0 < p < \infty$. Then the generalized Christoffel function $\lambda_n(d\alpha, p, x)$ is defined by

$$\lambda_n(d\alpha, p, x) = \inf_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p\,d\alpha(t) .$$

Later we shall see why we do not introduce a normalization: $\lambda_n(d\alpha, p, x) = \inf[\quad]^{1/p}$.

Properties 2. (i) If $d\alpha \leq d\beta$ then $\lambda_n(d\alpha, p, x) \leq \lambda_n(d\beta, p, x)$. (ii) $\lambda_n(d\alpha, 2, x) \equiv \lambda_n(d\alpha, x)$. (iii) If $\mathrm{supp}(d\alpha)$ is compact then

$$\lambda_n(d\alpha, p, x) = \min_{\pi_{n-1}} \frac{1}{|\pi_{n-1}(x)|^p} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p\,d\alpha(t) .$$

Proof. Let us fix α, n, p and $\Delta \supset \Delta(d\alpha)$. Let us show that $\lambda_n(d\alpha, p, y) \geq \lambda > 0$ for $y \in \Delta$. Let m be an integer such that $m \geq p$. Let $y \in \Delta \supset \Delta(d\alpha)$. Then

$$\pi_{n-1}^m(y) = \int \pi_{n-1}^m(t)\,K_{mn}(d\alpha, y, t)\,d\alpha(t)$$

for $y \in \Delta$. Hence

$$\max_{y\in\Delta} |\pi_{n-1}(y)|^m \leq C \int_{\Delta} |\pi_{n-1}(t)|^m\,d\alpha(t)$$

where $C = C(n, m, d\alpha, \Delta)$ does not depend on π_{n-1}. Writing $|\pi_{n-1}(t)|^m = |\pi_{n-1}(t)|^p\,|\pi_{n-1}(t)|^{m-p}$ $(m-p \geq 0)$ we obtain

$$\max_{y\in\Delta} |\pi_{n-1}(y)|^p \le C \int_\Delta |\pi_{n-1}(t)|^p \, d\alpha(t) \quad ,$$

in particular $\lambda_n(d\alpha, p, y) \ge \lambda > 0$ for $y \in \Delta$. Thus

$$\int_{-\infty}^{\infty} |\pi_{n-1}(y)| \, d\alpha(y) \le \lambda^{-\frac{1}{p}} \left[\int_{-\infty}^{\infty} |\pi_{n-1}(t)\right]^{\frac{1}{p}} .$$

If we write $\pi_{n-1}(x) = \sum_{k=0}^{n-1} a_k \, p_k(d\alpha, x)$ then we obtain from the previous inequality that

$$a_k \le C \left[\int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p \, d\alpha(t)\right]^{\frac{1}{p}}$$

where $C = C(n, p, d\alpha)$ does not depend on π_{n-1}. Now (iii) follows from Bolzano-Weierstrass' theorem by standard arguments.

First of all we shall investigate the simplest case, that is when α is a Jacobi weight. Let us recall that the Jacobi weight is denoted by $u = u^{(a,b)}$. We shall find the exact order of $\lambda_n(u, p, x)$ on $[-1, 1]$ when $n \to \infty$.

Definition 3. We write $\varphi_n(x) \sim \psi_n(x)$ if for every n and for every x in consideration (usually for $-1 \le x \le 1$)

$$0 < c_1 \le \varphi_n(x)/\psi_n(x) \le c_2 < \infty .$$

$\varphi(x) \sim \psi(x)$, $n \sim m$ etc. have similar meanings.

Definition 4. Let $a, b \in \mathbb{R}$. Then u_n is defined by

$$u_n(x) \equiv u_n^{(a,b)}(x) = [\sqrt{1-x} + \tfrac{1}{n}]^{2a+1} [\sqrt{1+x} + \tfrac{1}{n}]^{2b+1}$$

for $x \in [-1, 1]$.

Let us remark that if $m \sim n$ then $u_n(x) \sim u_m(x)$.

Lemma 5. We have

$$\lambda_n(u^{(a,b)}, x) \sim \frac{1}{n} u_n^{(a,b)}(x)$$

for $-1 \le x \le 1$.

Proof. See [11].

Lemma 6. Let $\cos\theta_{kn} = x_{kn}(u)$ for $k = 0, 1, \ldots, n+1$ with $x_{0n} = 1$ and $x_{n+1,n} = -1$. Then

$$\theta_{kn} - \theta_{k-1,n} \sim \frac{1}{n}$$

for $k = 1, 2, \ldots, n+1$.

Proof. See e.g. [12].

Corollary 7. Let $x \in [x_{kn}(u), x_{k-1,n}(u)]$ $(k = 1, 2, \ldots, n+1)$. Then

$$u_n^{(a_1, b_1)}(x) \sim u_n^{(a_1, b_1)}(x_k) \sim u_n^{(a_1, b_1)}(x_{k-1})$$

for $a_1, b_1 \in \mathbb{R}$.

Let v be – as before – the Tschebyshev weight. Then

$$|K_n(v, x, t)| \le C \min\{n, \frac{1}{|x-t|}\}$$

for $x, t \in [-1, 1]$. This estimate is good inside $(-1, 1)$ but not for x and t close to the endpoints of $[-1, 1]$. We shall need a better estimate which we formulate as

Lemma 8. Let $x, t \in [-1,1]$. Then

$$|K_n(v,x,t)| \le C \min\left\{n, \frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\right\}.$$

Proof. The idea comes from Pollard probably:

$$T_n(x)\,T_{n-1}(t) - T_{n-1}(x)\,T_n(t) = [T_n(x) - T_{n-1}(x)]\,T_{n-1}(t) + T_{n-1}(x)\,[T_{n-1}(t) - T_n(t)].$$

Lemma 9. Let $0 < \delta < 1$ and $a, b, c \in \mathbb{R}$. Then we have uniformly in n and m such that $1 \le m \le \delta n$

$$\sum_{\substack{k=1\\k\ne m}}^{n} k^a |k+m|^b |k-m|^c \sim$$

$$\sim m^{b+c}\begin{Bmatrix} 1 & \text{if} & a<-1 \\ \log(m+2) & \text{if} & a=-1 \\ m^{a+1} & \text{if} & a>-1 \end{Bmatrix} + m^{a+b}\begin{Bmatrix} 1 & \text{if} & c<-1 \\ \log(m+2) & \text{if} & c=-1 \\ m^{c+1} & \text{if} & c>-1 \end{Bmatrix} +$$

$$+\begin{Bmatrix} m^{a+b+c+1} & \text{if} & a+b+c<-1 \\ \log(\frac{n}{m}+2) & \text{if} & a+b+c=-1 \\ n^{a+b+c+1} & \text{if} & a+b+c>-1 \end{Bmatrix}.$$

Proof. We write

$$\sum_{\substack{k=1\\k\ne m}}^{n} = \sum_{k=1}^{[\frac{m}{2}]} + \sum_{k=[\frac{m}{2}]+1}^{m-1} + \sum_{k=m+1}^{[\frac{m}{2}(\delta^{-1}+1)]} + \sum_{k=[\frac{m}{2}(\delta^{-1}+1)]+1}^{n}$$

and each sum can easily be estimated.

Now we can compute the following important

Theorem 10. Let $u = u^{(a,b)}$ be given. Then there exists a natural integer $N_1 = N_1(a,b)$ such that for every fixed $N \ge N_1$

$$(1) \qquad \int_{-1}^{1} \left|\frac{K_n(v,x,t)}{K_n(v,x,x)}\right|^N u(t)dt \le C\frac{1}{n}\, u_n^{(a,b)}(x)$$

for $-1 \le x \le 1$ and $n = 1, 2, \ldots$.

Proof. Let us note that if (1) holds for $N_1 = N$ then it holds also for every fixed $N > N_1$. Let N be an even natural integer. Then $n \sim nN \equiv M$ and by Definition 4 $u_n(x) \sim u_M(x)$ for $|x| \le 1$. Let us compute the integral on the left side of (1) by the Gauss-Jacobi mechanical quadrature formula. We can do this since N is even. The above integral equals to

$$(2) \qquad \sum_{k=1}^{M} \lambda_{kM}(u)\left[\frac{K_n(v,x,x_{kM}(u))}{K_n(v,x,x)}\right]^N$$

and we shall estimate this sum. We may suppose without loss of generality that $0 \le x \le 1$. Take a suitable value for N such that (1) holds. If $-1 \le x < 0$ then by similar arguments we obtain another value for N such that (1) is satisfied. Taking the maximum of these two values of N we get a new N which is good for every $x \in [-1,1]$. We have $K_n(v,x,x)^{-N} \sim M^{-N}$ for $-1 \le x \le 1$ by Example 6.2.8. Let m be defined by

$$|x-x_{kM}| \ge |x-x_{mM}| \quad \text{for } k = 1,2,\dots, M .$$

Then

$$(2) \sim M^{-N}\left\{\sum_{x_k \le -\frac{1}{2}} + \sum_{\substack{-\frac{1}{2}<x_k<1 \\ k\ne m}} + \sum_{k=m}\right\}\lambda_{kM}(u)\, K_n^N(v,x,x_{kM}) .$$

By Lemma 8 the first sum here is $O(1)$. By Lemma 5 and Corollary 7 the last sum is of order $M^{N-1} u_M(x)$. Hence

$$(2) \le C[M^{-N} + \frac{1}{M} u_M(x)] +$$

$$+ M^{-N} \sum_{\substack{-\frac{1}{2}\le x_k<1 \\ k\ne m}} \lambda_{kM}(u)\, K_n^N(v,x,x_{kM}) .$$

By Lemma 6 for $k \ne m$ $|x-x_{kM}| > C\frac{1}{M}\sqrt{1-x^2}$ and $|x-x_{kM}| \ge$ $\ge C\frac{1}{M^2}|k-m||k+m|$. Consequently by Lemmas 5, 6 and 8

$$M^{-N} \sum_{\substack{-\frac{1}{2}\le x_k<1 \\ k\ne m}} \lambda_{kM}(u)\, K_n^N(v,x,x_{kM}) \le$$

$$\le CM^{-N-1} \sum_{\substack{-\frac{1}{2}\le x_k<1 \\ k\ne m}} (1-x_{kM})^{a+\frac{1}{2}}\left(\frac{\sqrt{1-x^2}+\sqrt{1-x_{kM}^2}}{x-x_{kM}}\right)^N \le$$

$$\le CM^{-N-1}(1-x^2)^{\frac{N}{2}} \sum_{\substack{k=1 \\ k\ne m}}^{c_1 M} \left(\frac{k}{M}\right)^{2a+1}\left[\frac{(k-m)(k+m)}{M^2}\right]^{-N} +$$

$$+ CM^{-N-1} \sum_{\substack{k=1 \\ k\ne m}}^{c_1 M} \left(\frac{k}{M}\right)^{2a+1+N}\left[\frac{(k-m)(k+m)}{M^2}\right]^{-N} \equiv A+B$$

where $0 < c_1 < 1$. To estimate A and B we use Lemma 9. We obtain that if

$$N > 1 \quad\text{and}\quad 2a+1-2N < -1$$

then

$$A \sim (1-x^2)^{\frac{N}{2}} M^{N-2-2a} m^{2a+1-N} \sim (1-x^2)^{\frac{N}{2}} \frac{1}{M}\left[\sqrt{1-x} + \frac{1}{M}\right]^{2a+1-N} \le$$

$$\le C\frac{1}{M} u_M^{(a,b)}(x) ,$$

and if

$$N > 1 \quad\text{and}\quad 2a+1-N < -1$$

then

$$B \sim M^{-2-2a} m^{2a+1} \sim \frac{1}{M} u_M^{(a,b)}(x) .$$

Thus for $N > \max\{1, 2a+2\}$

$$(2) \le C[M^{-N} + \frac{1}{M} u_M^{(a,b)}(x)] \le C\frac{1}{M} u_M^{(a,b)}(x) \le$$

$$\le \frac{c}{n} u_n^{(a,b)}(x) .$$

To finish the proof of the theorem we choose $N > 0$ that $N > \max(1, 2a+2, 2b+2)$.

<u>Lemma 11.</u> Let $a \geq -\frac{1}{2}$, $0 < p < \infty$. Then

$$(3) \qquad |\pi_{n-1}(x)|^p \leq C n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a \, dt$$

for $\frac{1}{2} \leq x \leq 1$ and there exists a number $c_1 = c_1(a, p) > 0$ such that

$$(4) \qquad \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a dt \leq 2 \int_0^{1-c_1 n^{-2}} |\pi_{n-1}(t)|^p (1-t)^a dt .$$

<u>Proof.</u> Let k be a natural integer such that $2k \geq p$. Then $\deg \pi_{n-1}^k \sim n$ and by Lemma 5

$$\pi_{n-1}^{2k}(x) \leq C n^{2(a+1)} \int_{-1}^1 \pi_{n-1}^{2k}(t) (1-t^2)^a \, dt$$

for $-1 \leq x \leq 1$, that is

$$\max_{|x| \leq 1} |\pi_{n-1}(x)|^{2k} \leq \max_{|x| \leq 1} |\pi_{n-1}(x)|^{2k-p} \cdot$$

$$\cdot C n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t^2)^a \, dt .$$

Therefore

$$|\pi_{n-1}(x)|^p \leq C n^{2(a+1)} \int_{-1}^1 |\pi_{n-1}(t)|^p (1-t^2)^a \, dt$$

for $-1 \leq x \leq 1$. Let us put here $x^{2M} \pi_{n-1}(x^2)$ instead of $\pi_{n-1}(x)$. Then for M fixed

$$|x^{2M} \pi_{n-1}(x^2)|^p \leq C n^{2(a+1)} \int_0^1 |\pi_{n-1}(t)|^p t^{Mp-\frac{1}{2}} (1-t)^a \, dt$$

for $0 \leq x \leq 1$. Let now M be so large that $Mp - \frac{1}{2} \geq 0$. Hence (3) holds. We obtain from (3) that for $c_1 > 0$

$$\int_{1-\frac{c_1}{n^2}}^1 |\pi_{n-1}(x)|^p (1-x)^a dx \leq C n^{2(a+1)} \int_{1-\frac{c_1}{n^2}}^1 (1-x)^a \, dx .$$

$$\cdot \int_0^1 |\pi_{n-1}(t)|^p (1-t)^a \, dt$$

and if $c_1 > 0$ is small then

$$(5) \qquad C n^{2(a+1)} \int_{1-\frac{c_1}{n^2}}^1 (1-x)^a \, dx \leq \frac{1}{2} ,$$

which implies (4).

<u>Lemma 12.</u> Let $\min(a, b) \geq -\frac{1}{2}$, $0 < p < \infty$. Then

$$(6) \qquad \lambda_n(u, x) \leq C \lambda_n(u, p, x)$$

for $n = 1, 2, \ldots$ and $-1 \leq x \leq 1$.

<u>Proof.</u> Let k be an integer such that $2k \geq p$. Put $a_1 = (a+\frac{1}{2})\frac{2k}{p} - \frac{1}{2}$ and $b_1 = (b+\frac{1}{2})\frac{2k}{p} - \frac{1}{2}$. Then $a_1, b_1 \geq -\frac{1}{2}$ and by Lemma 5 and 11 there exists $\varepsilon > 0$ such that

$$u^{(a_1, b_1)}(x)\sqrt{1-x^2} \; \pi_{n-1}(x)^{2k} \leq$$

$$\leq C n \int_{-1+\frac{\varepsilon}{n^2}}^{1-\frac{\varepsilon}{n^2}} \pi_{n-1}(t)^{2k} u^{(a_1, b_1)}(t)\sqrt{1-t^2} \frac{1}{\sqrt{1-t^2}} \, dt$$

for $|x| \le 1 - \frac{\varepsilon}{n^2}$. Hence

$$|[u^{(a_1, b_1)}(x)\sqrt{1-x^2}]^{\frac{1}{2k}} \pi_{n-1}(x)|^p \le$$

$$\le C\, n \int_{-1+\frac{\varepsilon}{n^2}}^{1-\frac{\varepsilon}{n^2}} |[u^{(a_1, b_1)}(t)\sqrt{1-t^2}]^{\frac{1}{2k}} \pi_{n-1}(t)|^p \frac{1}{\sqrt{1-t^2}}\, dt ,$$

that is

$$|\pi_{n-1}(x)|^p \le n\, u_n^{(a,b)}(x)^{-1} \int_{-1}^{1} |\pi_{n-1}(t)|^p u(t)\, dt$$

for $|x| \le 1 - \frac{\varepsilon}{n^2}$. By Lemma 11 the latter inequality also holds for $1 - \frac{\varepsilon}{n^2} \le |x| \le 1$. Now (6) follows from Lemma 5.

<u>Theorem 13.</u> Let u be a Jacobi weight and $0 < p < \infty$. Then

$$\lambda_n(u, p, x) \sim \lambda_n(u, x)$$

for $-1 \le x \le 1$.

<u>Proof.</u> Let N be such that (1) holds. Let $m \in \mathbb{N}$ be so big that $mp \ge N$. If we put $n_1 = [\frac{n}{m}]$ then

$$\lambda_n(u, p, x) \le \int_{-1}^{1} \left| \frac{K_{n_1}(v, x, t)}{K_{n_1}(v, x, x)} \right|^{mp} u(t)dt \le C \frac{1}{n_1} u_{n_1}(x) \sim$$

$$\sim \frac{1}{n} u_n(x) \sim \lambda_n(u, x) \qquad (|x| \le 1)$$

by Lemma 5 and Theorem 10. Now we shall show that

$$\lambda_n(u, x) \le C\, \lambda_n(u, p, x) \tag{7}$$

for $|x| \le 1$. Let $u = u^{(a,b)}$, $u^* = u^{(\max\{a, -\frac{1}{2}\},\ \max\{b, -\frac{1}{2}\})}$ and $\hat{u} = u^{(\max\{-\frac{(a+\frac{1}{2})}{p} - \frac{1}{2}, -\frac{1}{2}\},\ \max\{-\frac{(b+\frac{1}{2})}{p} - \frac{1}{2}, -\frac{1}{2}\})}$. We have by Lemma 5 and 12

$$|\pi_{n-1}(x)\, K_n(\hat{u}, x, x)|^p \le C n [u_n^*(x)]^{-1} \cdot$$

$$\cdot \int_{-1}^{1} |\pi_{n-1}(t)\, K_n(\hat{u}, t, t)|^p u^*(t)\, dt \qquad (|x| \le 1) ,$$

$$|K_n(\hat{u}, t, t)|^p u^*(t) \le C n^p u(t) \qquad (|t| \le 1)$$

and

$$u_n^*(x) |K_n(\hat{u}, x, x)|^p \sim n^p u_n(x)$$

for $|x| \le 1$. Hence

$$|\pi_{n-1}(x)|^p \le u_n(x)^{-1} \int_{-1}^{1} |\pi_{n-1}(t)|^p u(t)dt \ .$$

From this inequality and Lemma 5 we obtain (7).

There is a very important consequence of Theorem 13 which we formulate as

<u>Theorem 14.</u> Let u be a Jacobi weight and $0 < p < \infty$. Then there exists a number $c_1 = c_1(u, p) > 0$ such that

$$\int_{-1}^{1} |\pi_n(t)|^p u(t)dt \le 2 \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)|^p u(t)dt$$

for every π_n .

Proof. Use Lemma 5, Theorem 13 and (5).

Corollary 15. Let u be a Jacobi weight, $0 < p < \infty$, $\varepsilon > 0$. Then for every π_n

$$\int_{-1}^{1} |\pi_n(t)|^p u(t)dt \le C n^{2\varepsilon} \int_{-1}^{1} |\pi_n(t)|^p u(t)(1-t^2)^{\varepsilon} dt$$

with $C = C(p, u, \varepsilon)$.

Corollary 16. Let $0 < q < p < \infty$. Let u^p be a Jacobi weight. Then

$$\left(\int_{-1}^{1} |\pi_n(t)u(t)|^p dt\right)^{\frac{1}{p}} \le C n^{2(\frac{1}{q}-\frac{1}{p})} \left(\int_{-1}^{1} |\pi_n(t)\, u(t)|^q dt\right)^{\frac{1}{p}}$$

for every π_n where $C = C(p, q, u)$.

Proof. By Lemma 5 and Theorems 13, 14

$$\int_{-1}^{1} |\pi_n(t)\, u(t)|^p dt \le 2 \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)\, u(t)|^{p-q+q} dt \le$$

$$\le C n^{2\frac{p-q}{p}} \left(\int_{-1}^{1} |\pi_n(t) u(t)|^p dt\right)^{\frac{p-q}{p}} \int_{-1+\frac{c_1}{n^2}}^{1-\frac{c_1}{n^2}} |\pi_n(t)\, u(t)|^q dt .$$

Let us note that Corollaries 15 and 16 for $1 \le p < \infty$ and $1 \le q < p < \infty$ are not new. (See Khalilova [9].)

Before we begin to investigate the generalized Christoffel functions for weights different from the Jacobi ones we shall need a few lemmas.

Lemma 17. Let $\alpha(x) + \alpha(-x) \equiv \text{const}$. Let α_1 and α_2 be defined by

$$\alpha_1(x) = \begin{cases} 0 & \text{for } x < 0 \\ \alpha(\sqrt{x}) - \alpha(0) & \text{for } x \ge 0 \end{cases}$$

and

$$\alpha_2(x) = \begin{cases} 0 & \text{for } x < 0 \\ \int_0^x t\, d\alpha(\sqrt{t}) & \text{for } x \ge 0 . \end{cases}$$

Then

$$\sum_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\, 2)}}^{n} p_k^2(d\alpha, x) = \begin{cases} \frac{1}{2}\, \lambda^{-1}_{\frac{n}{2}+1}(d\alpha_1, x^2) & \text{for } n \text{ even} \\ \frac{x^2}{2}\, \lambda^{-1}_{\frac{n+1}{2}}(d\alpha_2, x^2) & \text{for } n \text{ odd} \end{cases}$$

and

$$\sum_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\, 2)}}^{n} p_k^2(d\alpha, x) = \max_{\substack{\pi_n \equiv n (\mathrm{mod}\, 2) \\ \int_{-\infty}^{\infty} \pi_n^2(t)\, d\alpha(t) = 1}} \pi_n^2(x) ,$$

where $\pi_n \equiv n(\mathrm{mod}\, 2)$ means that π_n is even if n is even and π_n is odd if n is odd.

Proof. Easy calculation.

Having in mind later applications we shall prove the following

Lemma 18. Let $\alpha \in M(0,1)$ and let $\alpha(x) + \alpha(-x) \equiv \text{const}$. Let α_1 and α_2 be defined as in Lemma 17. Then $\alpha_1, \alpha_2 \in M(\frac{1}{2}, \frac{1}{2})$.

Proof. The lemma follows from the relations

$$\alpha_n(d\alpha_1) = \frac{\gamma_{2n}^2(d\alpha)}{\gamma_{2n+1}^2(d\alpha)} + \frac{\gamma_{2n-1}^2(d\alpha)}{\gamma_{2n}^2(d\alpha)},$$

$$\frac{\gamma_{n-1}(d\alpha_1)}{\gamma_n(d\alpha_1)} = \frac{\gamma_{2n-2}(d\alpha)}{\gamma_{2n}(d\alpha)}$$

and

$$\alpha_n(d\alpha_2) = \frac{\gamma_{2n+1}^2(d\alpha)}{\gamma_{2n+2}^2(d\alpha)} + \frac{\gamma_{2n}^2(d\alpha)}{\gamma_{2n+1}^2(d\alpha)},$$

$$\frac{\gamma_{n-1}(d\alpha_2)}{\gamma_n(d\alpha_2)} = \frac{\gamma_{2n-1}(d\alpha)}{\gamma_{2n+1}(d\alpha)}$$

which can easily be checked.

Lemma 19. Let $w(t) = |t|^\Gamma (1-t^2)^a$ for $-1 \le t \le 1$ with $\Gamma, a > -1$ and $\mathrm{supp}(w) = [-1,1]$. Then

$$\lambda_n(w,x) \sim \frac{1}{n}(|x| + \frac{1}{n})^\Gamma (\sqrt{1-x} + \frac{1}{n})^{2a+1} (\sqrt{1+x+\frac{1}{n}})^{2a+1}$$

for $-1 \le x \le 1$.

Proof. Apply Lemmas 5 and 17.

Theorem 20. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta \subset \mathrm{supp}(d\alpha)$, $t^* \in \Delta^0$, $\Gamma > -1$. Let α be absolutely continuous in Δ and let

$$\alpha'(t) \sim |t-t^*|^\Gamma \qquad (t \in \Delta).$$

Then

$$\lambda_n(d\alpha, x) \sim \frac{1}{n}(|x-t^*| + \frac{1}{n})^\Gamma$$

for $x \in \Delta_1 \subset \Delta^0$.

Proof. By Lemma 19 we have to show that

$$(8) \qquad \lambda_n(d\alpha, x) \le C\frac{1}{n}(|x-t^*| + \frac{1}{n})^\Gamma$$

for $x \in \Delta_1 \subset \Delta^0$. Let $\hat{v}$ denote the Tschebyshev weight corresponding to $\Delta(d\alpha)$ and m be a natural integer. Then

$$\lambda_n(d\alpha, x) \le \int_{-\infty}^{\infty} \left[\frac{K_{[\frac{n}{m}]}(\hat{v}, x, t)}{K_{[\frac{n}{m}]}(\hat{v}, x, x)}\right]^{2m} d\alpha(t)$$

and hence

$$\lambda_n(d\alpha, x) \le C\, n^{-2m} \int_{t \notin \Delta} d\alpha(t) + C \int_{t \in \Delta} |t-t^*|^\Gamma \frac{1}{1+n^{2m}(x-t)^{2m}}\, dt$$

uniformly for $x \in \Delta_1 \subset \Delta^0$ if m is fixed. The second integral on the right side of the latter inequality can be estimated by standard methods. Finally we obtain that (8) is satisfied if we choose m large enough.

Let us note that the calculation in Theorem 20 was simple because we needed estimates only for $x \in \Delta_1 \subset \Delta^0$ and not for $x \in \Delta$.

Lemma 21. Let $0 < p < \infty$, $\Gamma \ge 0$, $0 \in \Delta_1^0$, $\Delta_1 \subset \Delta^0$. Then for every π_{n-1}

$$|\pi_{n-1}(x)|^p \le C\, n^{\Gamma+1} \int_\Delta |\pi_{n-1}(t)|^p\, |t|^\Gamma\, dt$$

uniformly for $x \in \Delta_1$.

Proof. Let $\varepsilon > 0$ be such that $[-2\varepsilon, 2\varepsilon] \subset \Delta_1^0$. If $x \in \Delta_1 \setminus [-\varepsilon, \varepsilon]$ then by Lemma 5 and Theorem 13

$$|\pi_{n-1}(x)|^p \leq C\, n \int_{\Delta \setminus [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]} |\pi_{n-1}(t)|^p \, dt \leq$$

$$\leq C_1 n^{\Gamma+1} \int_\Delta |\pi_{n-1}(t)|^p \, |t|^\Gamma \, dt$$

since $\Gamma \geq 0$. Let now $x \in [-\varepsilon, \varepsilon]$. Let v_ε denote the Tschebyshev weight corresponding to $[-2\varepsilon, 2\varepsilon]$, M be an even natural integer and m be an integer such that $2m \geq p$. Then by Lemma 19

$$[v_\varepsilon(x)^{-M} \pi_{n-1}(x)]^{2m} v_\varepsilon(x)^{-1} \leq C\, n^{\Gamma+1} \int_{-2\varepsilon}^{2\varepsilon} [v_\varepsilon(t)^{-M} \pi_{n-1}(t)]^{2m} |t|^\Gamma \, dt$$

for $x \in [-2\varepsilon, 2\varepsilon]$. Hence

$$\left| v_\varepsilon(x)^{-M - \frac{1}{2m}} \pi_{n-1}(x) \right|^p \leq C\, n^{\Gamma+1} \int_{-2\varepsilon}^{2\varepsilon} |\pi_{n-1}(t)|^p \, v_\varepsilon(t)^{-Mp - \frac{p}{2m} + 1} \cdot$$

$$\cdot \, |t|^\Gamma \, dt$$

for $x \in [-2\varepsilon, 2\varepsilon]$. Let now M be so large that $-Mp - \frac{p}{2m} + 1 \leq 0$.

From Lemma 21 we obtain the following

Lemma 22. Let $0 < p < \infty$, $\Gamma \geq 0$, $0 \in \Delta^0$. Then there exists $c_1 = c_1(p, \Gamma, \Delta)$ such that

$$\int_\Delta |\pi_{n-1}(t)|^p \, |t|^\Gamma \, dt \leq 2 \int_{\substack{t \in \Delta \\ |t| \geq \frac{c_1}{n}}} |\pi_{n-1}(t)|^p \, |t|^\Gamma \, dt .$$

Lemma 23. Let $0 < p < \infty$, $\Gamma \geq 0$, $0 \in \Delta_1^0$, $\Delta_1 \subset \Delta^0$. Let $w(t) = |t|^\Gamma$ for $t \in \Delta$ with $\mathrm{supp}(w) = \Delta$. Then

$$\text{(9)} \qquad \frac{1}{n}\left(|x| + \frac{1}{n}\right)^\Gamma \leq C\, \lambda_n(w, p, x) \qquad (n = 1, 2, \ldots)$$

uniformly for $x \in \Delta_1$.

Proof. Let $\varepsilon > 0$ be such that $[-2\varepsilon, 2\varepsilon] \subset \Delta_1^0$. If $x \in \Delta_1 \setminus [-\varepsilon, \varepsilon]$ then (9) follows from Lemma 5 and Theorem 13. If $x \in [-\varepsilon, \varepsilon]$ then we find an integer m such that $2m \geq p$. We put $w^*(t) = |t|^{\frac{2m\Gamma}{p}} (4\varepsilon^2 - t^2)^{\frac{m}{p} - \frac{1}{2}}$ for $|t| \leq 2\varepsilon$ with $\mathrm{supp}(w^*) = [-2\varepsilon, 2\varepsilon]$. Then by Lemmas 19 and 22

$$\sqrt{4\varepsilon^2 - x^2} \; w^*(x) \, |\pi_{n-1}(x)|^{2m} \leq C\, n \int_{\frac{c_1}{n} \leq |t| \leq 2\varepsilon} |\pi_{n-1}(t)|^{2m} \, w^*(t) \, dt$$

for $\frac{c_1}{n} \leq |x| \leq 2\varepsilon$. Hence

$$\sqrt{4\varepsilon^2 - x^2} \; w(x) \, |\pi_{n-1}(x)|^p \leq C\, n \int_{\frac{c_1}{n} \leq |t| \leq 2\varepsilon} |\pi_{n-1}(t)|^p \, w(t) dt \leq$$

$$\leq C\, n \int_\Delta |\pi_{n-1}(t)|^p \, w(t) dt$$

for $\frac{c_1}{n} \leq |x| \leq 2\varepsilon$. Hence (9) holds also for $\frac{c_1}{n} \leq |x| \leq \varepsilon$. If $|x| \leq \frac{c_1}{n}$ we apply Lemma 21.

Lemma 24. Lemma 23 remains true if $-1 < \Gamma < 0$ instead of $\Gamma \geq 0$.

Proof. Let $\varepsilon > 0$ be such that $\Delta \subset (-\varepsilon, \varepsilon)$. Let $w^*(t) = |t|^{-\frac{\Gamma}{p}}$ for $|t| \leq \varepsilon$ with $\mathrm{supp}(w^*) = [-\varepsilon, \varepsilon]$. Then by Lemma 19

$$\lambda_n(w^*, x) \sim \frac{1}{n}(|x| + \frac{1}{n})^{-\frac{\Gamma}{p}}$$

for $x \in \Delta$. By Lemma 23

$$|\pi_{n-1}(x)\, \lambda_n^{-1}(w^*, x)|^p \leq C\, n \int_\Delta |\pi_{n-1}(t)\, \lambda_n^{-1}(w^*, t)|^p\, dt$$

for $x \in \Delta_1$. Hence

$$|\pi_{n-1}(x)|^p (|x| + \frac{1}{n})^\Gamma \leq C\, n \int_\Delta |\pi_{n-1}(t)|^p\, |t|^\Gamma\, dt$$

for $x \in \Delta_1$.

<u>Theorem 25</u>. Let $\text{supp}(d\alpha)$ be compact, $\Delta \subset \text{supp}(d\alpha)$, $t^* \in \Delta^0$, $\Gamma > -1$, $0 < p < \infty$. Let α be absolutely continuous in Δ and let

$$\alpha'(t) \sim |t - t^*|^\Gamma \qquad (t \in \Delta) .$$

Then

$$\lambda_n(d\alpha, p, x) \sim \lambda_n(d\alpha, x) \sim \frac{1}{n}(|x - t^*| + \frac{1}{n})^\Gamma$$

for $x \in \Delta_1 \subset \Delta^0$.

<u>Proof</u>. The inequality

$$\lambda_n(d\alpha, p, x) \leq \frac{c}{n}(|x - t^*| + \frac{1}{n})^\Gamma \qquad (x \in \Delta_1 \subset \Delta^0)$$

can be proved exactly by the same way as in Theorem 20 for $p = 2$. For the estimate from below we can suppose that $t^* \in \Delta_1^0$ and then we apply Lemmas 23 and 24.

<u>Corollary 26</u>. Lemma 22 remains valid for $-1 < \Gamma < 0$ and consequently if $\Gamma > -1$, $\varepsilon > 0$, $0 \in \Delta^0$ and $0 < p < \infty$ then for every π_n

$$\int_\Delta |\pi_n(t)|^p\, |t|^\Gamma\, dt \leq C\, n^\varepsilon \int_\Delta |\pi_n(t)|^p\, |t|^{\Gamma+\varepsilon}\, dt$$

where $C = C(p, \Gamma, \varepsilon, \Delta)$.

<u>Theorem 27.</u> Let $\text{supp}(d\alpha)$ be compact, $0 < p < \infty$, $a > -1$. Let $\Delta(d\alpha) = [c_1, c_2]$, $\delta > 0$ and let α be absolutely continuous in $[c_2 - \delta, c_2]$. Let

$$\alpha'(t) \sim (c_2 - t)^a \qquad (t \in [c_2 - \delta, c_2]) .$$

Then

$$\lambda_n(d\alpha, p, x) \sim \frac{1}{n}(\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for $x \in [c_2 - \frac{\delta}{2}, c_2]$.

<u>Proof</u>. We have by Theorem 10 and standard arguments

$$\lambda_n(d\alpha, p, x) \leq \frac{1}{n}(\sqrt{c_2 - x} + \frac{1}{n})^{2a+1}$$

for $x \in [c_2 - \frac{\delta}{2}, c_2]$. The converse inequality follows from Lemma 5 and Theorem 13.

From Theorems 25 and 27 we obtain

<u>Theorem 28.</u> Let $1 = t_1 > t_2 > \ldots > t_N = -1$, $\Gamma_k > -1$ for $k = 1, 2, \ldots, N$ and let w be defined by

$$w(t) = \prod_{k=1}^{N} |t - t_k|^{\Gamma_k} \qquad (-1 \leq t \leq 1)$$

with $\text{supp}(w) = [-1, 1]$. Let

$$\bar{w}_n(t) = (\sqrt{1-t} + \frac{1}{n})^{2\Gamma_1+1} \prod_{k=2}^{N-1} (|t-t_k| + \frac{1}{n})^{\Gamma_k} (\sqrt{1+t} + \frac{1}{n})^{2\Gamma_N+1}$$

for $-1 \leq t \leq 1$. Then for every $0 < p < \infty$

$$\lambda_n(w, p, x) \sim \lambda_n(w, x) \sim \frac{1}{n} \bar{w}_n(x)$$

for $|x| \leq 1$.

Corollary 29. We can establish inequalities similar to those in Theorem 14 and Corollaries 15 and 16.

The exact formulation of Corollary 29 is left to the reader.

Lemma 30. Let $p > 1$. Then

$$\int_{-1}^{1} |K_n(v, x, t)|^p v(t)dt \sim n^{p-1} \sim \lambda_n(v, x)^{1-p}$$

for $-1 \leq x \leq 1$.

Proof. The estimate

$$n^{p-1} \leq C \int_{-1}^{1} |K_n(v, x, t)|^p v(t)dt$$

follows immediately from Theorem 13. The converse estimate is obvious when $p \geq 2$ and can be obtained by a simple calculation from Lemma 8 when $1 < p < 2$.

Lemma 31. Let α be an arbitrary weight, $p > 1$. Then for almost every $x \in [-1, 1]$

$$\lim_{n \to \infty} \frac{\int_{-1}^{1} |K_n(v, x, t)|^p d\alpha(t)}{\int_{-1}^{1} |K_n(v, x, t)|^p v(t)dt} = \sqrt{1-x^2}\, \alpha'(x) .$$

Proof. Using Lemma 30 the lemma can be proved in almost the same way as Lemma 6.2.32. We shall not go into details.

Theorem 32. Let $\mathrm{supp}(d\alpha) \subset [-1, 1]$ and $0 < p < \infty$. Then

(10) $$\limsup_{n \to \infty} n \lambda_n(d\alpha, p, x) \leq C \alpha'(x) \sqrt{1-x^2}$$

for almost every $x \in [-1, 1]$ where $C = C(p)$.

Proof. Let m be a natural integer such that $mp > 1$. Then

$$\lambda_n(d\alpha, p, x) \leq \int_{-1}^{1} \left| \frac{K_{[\frac{n}{m}]}(v, x, t)}{K_{[\frac{n}{m}]}(v, x, x)} \right|^{mp} d\alpha(t) .$$

Now we apply Lemmas 30 and 31.

Theorem 33. Let $\alpha \in M(0, 1)$ and $0 < p < \infty$. Then for almost every $x \in \mathrm{supp}(d\alpha)$ (10) holds with $C = C(p)$.

Proof. Combine the arguments used in the proof of Theorems 32 and and 6.2.51.

Theorem 34. Let α be an arbitrary weight. Let Δ and $\varepsilon > 0$ be given and let v_Δ denote the Tschebyshev weight corresponding to Δ. Let $[\alpha']^{-\varepsilon} \in L^1(\Delta)$. Then for each $p \in (0, \infty)$

$$\liminf_{n\to\infty} n\,\lambda_n(d\alpha, p, x) \geq C\,\alpha'(x)\, v_{\Delta}(x)^{-1}$$

for almost every $x \in \Delta$ where $C = C(\varepsilon, \Delta, p)$.

Proof. Let $q = \varepsilon p(1+\varepsilon)^{-1}$, m and M be natural integers such that $mp\varepsilon > 1$ and $2\varepsilon pM > 1+\varepsilon$. Let $N = [\frac{n}{m}]$. We can suppose without loss of generality that $\Delta = [-1,1]$. Then by Theorem 13

$$|\lambda_N^{-m}(v,x)\, v(x)^{-2M} \pi_{n-1}(x)|^q \leq C\, n \int_{-1}^{1} |K_N^m(v,x,t)\, v(t)^{-2M} \cdot \pi_{n-1}(t)|^q\, v(t)dt .$$

Hence by Hölder's inequality

$$|\pi_{n-1}(x)|^p \leq C\, n^{\frac{p}{q}} \lambda_N(v,x)^{pm}\, v(x)^{2Mp} \cdot \int_{-1}^{1} |\pi_{n-1}(t)|^p \alpha'(t)dt \cdot \left\{ \int_{-1}^{1} |K_N(v,x,t)|^{\frac{mpq}{p-q}}\, v(t)^{\frac{p}{p-q} - \frac{2Mpq}{p-q}}\, \alpha'(t)^{-\frac{q}{p-q}} dt \right\}^{\frac{p-q}{q}} .$$

Using Lemma 30 we obtain

$$|\pi_{n-1}(x)|^p \leq C\, n\, v(x)^{2Mp} \int_{-\infty}^{\infty} |\pi_{n-1}(t)|^p\, d\alpha(t) \cdot \left\{ \frac{\int_{-1}^{1} |K_N(v,x,t)|^{m\varepsilon p}\, v(t)^{1+\varepsilon-2M\varepsilon p}\, \alpha'(t)^{-\varepsilon} dt}{\int_{-1}^{1} |K_N(v,x,t)|^{m\varepsilon p}\, v(t)dt} \right\}^{1/\varepsilon} .$$

Consequently

$$\frac{1}{n\lambda_n(d\alpha,p,x)} \leq C\, n\, v(x)^{2Mp} \cdot \left\{ \frac{\int_{-1}^{1} |K_N(v,x,t)|^{\varepsilon m p}\, v(t)^{1+\varepsilon-2M\varepsilon p}\, \alpha'(t)^{-\varepsilon} dt}{\int_{-1}^{1} |K_N(v,x,t)|^{m\varepsilon p}\, v(t)dt} \right\}^{1/\varepsilon} .$$

By the conditions $v^{1+\varepsilon-2M\varepsilon p}(\alpha')^{-\varepsilon}$ is a weight. Thus the theorem follows from Lemma 31.

In Theorem 34 the most important case is when $p = 2$. Let us formulate it separately as

Theorem 35. Let $\varepsilon > 0$. If $(\alpha')^{-\varepsilon} \in L^1(\Delta)$ then

$$\limsup_{n\to\infty} \frac{\alpha'(x)\, v_{\Delta}(x)^{-1}}{n\,\lambda_n(d\alpha, x)} \in L^{\infty}(\Delta) .$$

Let us note that Corollary 6.2.24 is contained in Theorem 35. Hence Theorems 6.2.25 and 6.2.26 remain valid if $[\alpha']^{-\varepsilon} \in L^1(\Delta)$ instead of $1/\alpha' \in L^1(\Delta)$.

7. The Coefficients in the Recurrence Formula.

Theorem 1. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$ and

$$\sum_{j=1}^{\infty} \left| \frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)} - \tfrac{1}{2} \right| < \infty .$$

Then $\alpha \in S$.

Proof. Let $0 \le k \le n$. Let us divide both sides in 3.1(2) by x^n and let $x \to \infty$. We obtain

$$2^{-n}\gamma_n(d\alpha) = 2^{-k}\gamma_k(d\alpha) + \sum_{j=k+1}^{n} 2^{-j}\gamma_j(d\alpha)\left[1 - 2\frac{\gamma_{j-1}(d\alpha)}{\gamma_j(d\alpha)}\right].$$

Let us fix k so that

$$\sum_{j=k+1}^{\infty} \left| 1 - 2\frac{\gamma_{j-1}}{\gamma_j} \right| < \tfrac{1}{2} .$$

Then for every $n \ge k+1$

$$2^{-n}\gamma_n \le 2^{-k}\gamma_k + \tfrac{1}{2} \max_{k+1 \le j \le n} 2^{-j}\gamma_j$$

and thus for every $m \ge k+1$

$$\max_{k+1 \le n \le m} 2^{-n}\gamma_n \le 2^{-k}\gamma_k + \tfrac{1}{2} \max_{k+1 \le j \le m} 2^{-j}\gamma_j ,$$

in particular for $m \ge k+1$

$$2^{-m}\gamma_m \le 2^{1-k}\gamma_k .$$

From this inequality and Lemma 4.2.2 we obtain by standard calculations that for every $\varepsilon > 0$

$$\frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1} v(t)\log[\alpha'(t)+\varepsilon]dt\right\} \le 2^{1-k}\gamma_k .$$

Letting $\varepsilon \to 0$ we see that

$$\int_{-1}^{1} v(t)\log\alpha'(t)dt > -\infty .$$

Hence $\alpha \in S$.

Lemma 2. Let w be of the form $w(x) = \varphi(x)|x|^{\varepsilon}$ $(\varepsilon > -1,\ -1 \le x \le 1)$ where φ is an even function of bounded variation with $\varphi(1) > 0$. Then for $n = 1,2,\dots$

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \tfrac{1}{2}\,\frac{n + \frac{\varepsilon}{2}[1+(-1)^{n+1}]}{[n+\frac{1}{2}(a_n+\varepsilon+1)]^{\frac{1}{2}}[n+\frac{1}{2}(a_{n-1}+\varepsilon-1)]^{\frac{1}{2}} - \frac{1}{2}b_n}$$

where

$$a_n = \int_{-1}^{1} p_n^2(w,x)x|x|^{\varepsilon}d\varphi(x)$$

and

$$b_n = \int_{-1}^{1} p_{n-1}(w,x)p_n(w,x)|x|^{\varepsilon}d\varphi(x) .$$

Proof. Since $\gamma_{n-1}(w)/\gamma_n(w) = \gamma_{n-1}(cw)/\gamma_n(cw)$ for every $c > 0$ we can suppose that $\varphi(\pm 1) = 1$.

We have

$$\int_{-1}^{1} x[p_n^2(w,x)]'w(x)dx = 2\int_{-1}^{1} p_n(w,x)[n\gamma_n(w)x^n + \dots]w(x)dx = 2n.$$

On the other hand since w is even

$$\int_{-1}^{1} x[p_n^2(w,x)]'w(x)dx = 2p_n^2(w,1) - \int_{-1}^{1} p_n^2(w,x)dxw(x) =$$

$$= 2p_n^2(w,1) - 1 - \int_{-1}^{1} p_n^2(w,x)x\,dw(x) .$$

Thus

$$2n+1 = 2p_n^2(w,1) - \int_{-1}^{1} p_n^2(w,x)x\,dw(x) .$$

Because $w(x) = \varphi(x)|x|^{\varepsilon}$ we obtain

$$\int_{-1}^{1} p_n^2(w,x)x\,dw(x) = \int_{-1}^{1} p_n^2(w,x)\,x|x|^{\varepsilon}\,d\varphi(x) +$$

$$+ \varepsilon\int_{-1}^{1} p_n^2(w,x)x\,\varphi(x)\,|x|^{\varepsilon-1}\operatorname{sign} x\,dx = \int_{-1}^{1} p_n^2(w,x)x|x|^{\varepsilon}\,d\varphi(x) + \varepsilon .$$

Consequently

$$(1) \qquad p_n^2(w,1) = n + \frac{1+\varepsilon}{2} + \tfrac{1}{2}\int_{-1}^{1} p_n^2(w,x)x\,|x|^{\varepsilon} d\,\varphi(x)$$

for $n = 0,1,\dots$. Now we shall consider another integral:

$$\int_{-1}^{1} [p_n(w,x)p_{n-1}(w,x)]'\, w(x)dx = \int_{-1}^{1} p_n'(w,x)p_{n-1}(w,x)w(x)dx =$$

$$= \int_{-1}^{1} p_{n-1}(w,x)[n\gamma_n(w)x^{n-1} + \dots]w(x)dx = n\frac{\gamma_n(w)}{\gamma_{n-1}(w)} .$$

But

$$\int_{-1}^{1} [p_n(w,x)p_{n-1}(w,x)]'w(x)dx = 2p_n(w,1)p_{n-1}(w,1) -$$

$$-\int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)dw(x)$$

and

$$\int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)dw(x) = \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)|x|^{\varepsilon}d\varphi(x) +$$

$$+ \varepsilon \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)\frac{w(x)}{x}dx .$$

If n is even then $p_{n-1}(w,x)x^{-1}$ is a polynomial of degree $n-2$ and consequently the latter integral equals 0 . If n is odd then $p_n(w,x)x^{-1}$ is a polynomial of degree $n-1$. Thus

$$\int_{-1}^{1} p_{n-1}(w,x)p_n(w,x)\frac{w(x)}{x}dx = \int_{-1}^{1} p_{n-1}(w,x)[\gamma_n x^{n-1} + \dots]w(x)dx =$$

$$= \frac{\gamma_n(w)}{\gamma_{n-1}(w)} .$$

Hence for $n = 1,2,\dots$

$$\int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)\frac{w(x)}{x}dx = \tfrac{1}{2}\frac{\gamma_n(w)}{\gamma_{n-1}(w)}[1 + (-1)^{n+1}] .$$

Thus we obtain

$$(2) \qquad \{n + \frac{\varepsilon}{2}[1 + (-1)^{n+1}]\}\frac{\gamma_n(w)}{\gamma_{n-1}(w)} =$$

$$= 2\,p_n(w,1)p_{n-1}(w,1) - \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)|x|^{\varepsilon}\,d\varphi(x) .$$

Putting (1) into (2) we finish the proof.

Theorem 3. Let $supp(w) \subset [-1,1]$, w be even and of bounded variation with $w(1) > 0$. Let

$$(3) \qquad \sup_{n\geq 1} \int_{-1}^{1} p_n^2(w,x)|dw(x)| < \infty .$$

If $w^{-\varepsilon} \in L^1(\tau)$ $(\varepsilon > 0,\ \tau \subset (-1,1))$ then the sequence $\{|p_n(w,t)|\}$ is bounded for almost every $t \in \tau$.

Proof. We apply Lemma 2 with $\varepsilon = 0$. Both $\{a_n\}$ and $\{b_n\}$ are bounded by (3). Thus

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \tfrac{1}{2} + O(\frac{1}{n}) .$$

Since w is even we obtain $c_j^{0,1}(w) = O(\frac{1}{n})$. Now we use Theorem 6.2.26.

Theorem 4. Let $w(x) = \varphi(x)|x|^{\varepsilon}$ $(\varepsilon > -1,\ -1 \leq x \leq 1)$. Let φ be even, continuous and positive and let φ' be also continuous. Then

$$\frac{\gamma_{n-1}(w)}{\gamma_n(w)} = \tfrac{1}{2} + (-1)^{n+1}\frac{\varepsilon}{4n} + o(\frac{1}{n})$$

for $n = 1,2,\dots$. If φ is constant then $o(\frac{1}{n})$ can be replaced by $O(\frac{1}{n^2})$.

Proof. Let us use Lemma 2. By the conditions $a_n = O(1)$ and $b_n = O(1)$. Thus $w \in M(0,1)$. Further

$$\{n + \frac{\varepsilon}{2}[1 + (-1)^{n+1}]\}\frac{\gamma_n}{\gamma_{n-1}} =$$

$$= 2n[1 + \frac{1}{2n}(a_n+\varepsilon+1)]^{\frac{1}{2}}[1 + \frac{1}{2n}(a_{n-1}+\varepsilon-1)]^{\frac{1}{2}} - b_n =$$

$$= 2n[1 + \frac{1}{4n}(a_n+\varepsilon+1) + O(\frac{1}{n^2})]\cdot[1 + \frac{1}{4n}(a_{n-1}+\varepsilon-1) + O(\frac{1}{n^2})] - b_n =$$

$$= 2n + \varepsilon + \tfrac{1}{2}(a_n + a_{n-1} - 2b_n) + O(\frac{1}{n}) .$$

Thus

$$\frac{\gamma_n}{\gamma_{n-1}} = 2 + \frac{\varepsilon(-1)^n}{n+\frac{\varepsilon}{2}[1+(-1)^{n+1}]} + O\left(\frac{|a_n + a_{n-1} - 2b_n|}{n}\right) + O\left(\frac{1}{n^2}\right).$$

Finally we obtain

$$\frac{\gamma_{n-1}}{\gamma_n} = \tfrac{1}{2} + (-1)^{n+1}\frac{\varepsilon}{4n} + O\left(\frac{|a_n + a_{n-1} - 2b_n|}{n}\right) + O\left(\frac{1}{n^2}\right).$$

If φ is constant then $a_n = a_{n-1} = b_n = 0$. If φ is not constant then we have to show that

(4) $$\lim_{n\to\infty} (a_n + a_{n-1} - 2b_n) = 0 .$$

By the recurrence formula

$$a_n = \frac{\gamma_n}{\gamma_{n+1}} b_{n+1} + \frac{\gamma_{n-1}}{\gamma_n} b_n .$$

Hence

$$a_n + a_{n-1} - 2b_n = \frac{\gamma_n}{\gamma_{n+1}} b_{n+1} + 2\left(\frac{\gamma_{n-1}}{\gamma_n} - 1\right)b_n + \frac{\gamma_{n-2}}{\gamma_{n-1}} b_{n-1} .$$

Since $w \in M(0,1)$ if $\lim_{n\to\infty} b_n$ exists and it is finite then (4) holds. But

$$b_n = \int_{-1}^{1} p_n(w,x)p_{n-1}(w,x)\frac{\varphi'(x)}{\varphi(x)} w(x)dx .$$

By the conditions φ'/φ is continuous on $[-1,1]$. Using Theorem 4.2.13 we obtain

$$\lim_{n\to\infty} b_n = \frac{1}{\pi}\int_{-1}^{1} \frac{t\,\varphi'(t)}{\varphi(t)\sqrt{1-t^2}} dt < \infty .$$

Consequently (4) is satisfied.

Theorem 5. Let α be such that either $\text{supp}(d\alpha) \subset [-1,1]$ or $\alpha \in M(0,1)$. Let $\tau \subset [-1,1]$ and φ be defined by

$$\varphi(x) = \sup_{n \geq 0} p_n^2(d\alpha,x) \qquad (x \in \tau).$$

Then for almost every $x \in \tau$

$$\alpha'(x)\sqrt{1-x^2} \geq \frac{1}{\pi\varphi(x)} ,$$

in particular if $\varphi(x)$ is finite for almost every $x \in \tau$ then $\alpha'(x) > 0$ for almost every $x \in \tau$ and if $\varphi(x) \leq K < \infty$ for almost every $x \in \mathfrak{M} \subset \tau$ then

(5) $$\alpha'(x)\sqrt{1-x^2} \geq \frac{1}{k\pi}$$

for almost every $x \in \mathfrak{M}$.

Proof. By the definition of φ, $n\lambda_n(d\alpha,x) \geq \varphi(x)^{-1}$ for $x \in \tau$ and we apply Theorems 6.2.33 and 6.2.51.

Let us note that putting α = Tschebyshev weight we see that the constant **K** in (5) is not exact.

Definition 6. Let $\text{supp}(d\alpha) \subset [-1,1]$. Let $\mu = \mu_\alpha$ be the weight on the unit circumference associated with α in the usual way:

$$\mu(\theta) = \begin{cases} \alpha(1) - \alpha(\cos\theta) & \text{for } 0 \leq \theta \leq \pi \\ \alpha(\cos\theta) - \alpha(1) & \text{for } -\pi \leq \theta \leq 0 . \end{cases}$$

Let $\phi_n(d\mu,z) = z^n + \ldots$ $(n = 0,1,\ldots)$ denote the corresponding system of orthogonal polynomials. It is known that the coefficients of $\phi_n(d\mu,z)$ are real. (See e.g. Freud, §V.1) We put

$$a_n = -\phi_{n+1}(d\mu,0) .$$

Lemma 7. $\alpha \in S$ iff $\sum_{n=0}^{\infty} a_n^2 < \infty$.

Proof. See Geronimus, §8.2.

Lemma 8. We have

$$\frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} = \tfrac{1}{2}\left[(1-a_{2n-1})(1-a_{2n}^2)(1+a_{2n+1})\right]^{\frac{1}{2}}$$

and

$$\alpha_n(d\alpha) = \tfrac{1}{2}[a_{2n-2}(1+a_{2n-1}) - a_{2n}(1-a_{2n-1})]$$

for $n = 1, 2, \ldots$.

Proof. Calculation. For the first relation see e.g. Geronimus, §9.1.

From Lemmas 7 and 8 we obtain

Theorem 9. Let $\alpha \in S$. Then

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha)^2 < \infty .$$

Remark 10. The converse of Theorem 9 is not true. Example: the Polleczek weight.

Let us note that Geronimus has proved that if $\sum_{n=0}^{\infty} |a_n| < \infty$ then μ is absolutely continuous, μ' is continuous and positive and $|\varphi_n(d\mu, z)| \le C < \infty$ for $n = 1, 2, \ldots$, $-\pi \le \theta \le \pi$, $z = e^{i\theta}$. Here $\varphi_n(d\mu, z)$ denotes the corresponding orthonormal polynomial. It is obvious that neither $\alpha'(x) > 0$ for $-1 \le x \le 1$ nor $|p_n(d\alpha, x)| \le C < \infty$ for $n = 1, 2, \ldots$; $-1 \le x \le 1$ follows from $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Later we shall show that $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ neither implies that $\mathrm{supp}(d\alpha) = [-1,1]$ nor that α is absolutely continuous but $\mathrm{supp}(\alpha') = [-1,1]$ follows from $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$.

Theorem 11. Let $\alpha \in S$. Then the series

$$\sum_{k=1}^{\infty} (1-x^2)\,\lambda_{2^k+1}(d\alpha, x)\,p^2_{2^k}(d\alpha, x)$$

converges uniformly for $x \in [-1,1]$.

Proof. By Theorem 3.1.8

$$(1-x^2)\lambda_{2^k+1}(d\alpha, x)p^2_{2^k}(d\alpha, x) \le C[2^{-k} + \sum_{j=2^{k-1}+1}^{2^k} c_j^{0,1}(d\alpha)^2]$$

and we apply Theorem 9.

Theorem 12. Let either $\alpha \in M(0,1)$ or $\mathrm{supp}(d\alpha) \subset [-1,1]$. Let $n_1 < n_2 < \ldots$ be such that $\sum_{k=1}^{\infty} n_k^{-1} < \infty$. Then the series

$$\sum_{k=1}^{\infty} \lambda_{n_k}(d\alpha, x)\,p^2_{n_k}(d\alpha, x)$$

converges for almost every $x \in \mathrm{supp}(d\alpha)$.

Proof. Use Theorems 3.3.7, 6.2.33, 6.2.51 and Beppo Levi's theorem.

Lemma 13. Let $p_n(d\alpha, x) = \gamma_n(d\alpha)x^n + \mu_n(d\alpha)x^{n-1} + \ldots$. Then

$$\sum_{j=0}^{n-1} \alpha_j(d\alpha) = -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)}$$

for $n = 1, 2, \ldots$, in particular,

$$\alpha_n(d\alpha) = \frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)} - \frac{\mu_{n+1}(d\alpha)}{\gamma_{n+1}(d\alpha)} .$$

Proof. We have

$$\sum_{k=0}^{n-1} \alpha_k(d\alpha) = \int_{-\infty}^{\infty} x\,\lambda_n^{-1}(d\alpha, x)\,d\alpha(x) =$$

$$= \sum_{k=1}^{n} \lambda_{kn}^{-1}(d\alpha) \int_{-\infty}^{\infty} x\,\ell^2_{kn}(d\alpha, x)\,d\alpha(x) = \sum_{k=1}^{n} x_{kn}(d\alpha) =$$

$$= -\frac{\mu_n(d\alpha)}{\gamma_n(d\alpha)} .$$

Definition 14. Let $t \in \mathbb{R}$. Then δ_t denotes the unit mass concentrated at t, that is

$$\delta_t(x) = \begin{cases} 0 & \text{for } x < t \\ 1 & \text{for } x \ge t . \end{cases}$$

Lemma 15. Let $\varepsilon > 0$, $t \in \mathbb{R}$, $\beta = \alpha + \varepsilon\delta_t$. Then

$$(6) \qquad p_n(d\beta, x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)}\left[p_n(d\alpha, x) - \frac{\varepsilon p_n(d\alpha, t)K_{n+1}(d\alpha, t, x)}{1 + \varepsilon K_{n+1}(d\alpha, t, t)}\right]$$

where

$$(7) \qquad \frac{\gamma_n^2(d\beta)}{\gamma_n^2(d\alpha)} = 1 - \frac{\varepsilon p_n^2(d\alpha, t)}{1 + \varepsilon K_{n+1}(d\alpha, t, t)} ,$$

further

$$(8) \qquad \alpha_n(d\beta) = \alpha_n(d\alpha) + \varepsilon \frac{\gamma_n(d\alpha)}{\gamma_{n+1}(d\alpha)} \frac{p_n(d\alpha,t)p_{n+1}(d\alpha,t)}{1+\varepsilon K_{n+1}(d\alpha,t,t)} - \varepsilon \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \frac{p_{n-1}(d\alpha,t)\,p_n(d\alpha,t)}{1+\varepsilon K_n(d\alpha,t,t)} .$$

If $\alpha(x) + \alpha(-x) = \text{const.}$ $t > 0$, $\varepsilon > 0$ and $\beta = \alpha + \varepsilon\delta_t + \varepsilon\delta_{-t}$ then

$$p_n(d\beta,x) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} [p_n(d\alpha,x) - \varepsilon\, p_n(d\alpha,t) \cdot \frac{K_{n+1}(d\alpha,t,x) + (-1)^n K_{n+1}(d\alpha,-t,x)}{1+\varepsilon[K_{n+1}(d\alpha,t,t) + (-1)^n K_{n+1}(d\alpha,-t,t)]}$$

and

$$\frac{\gamma_n^2(d\beta)}{\gamma_n^2(d\alpha)} = 1 - \frac{\varepsilon\, p_n^2(d\alpha,t)}{\frac{1}{2} + \varepsilon \sum\limits_{\substack{k=0 \\ k \equiv n (\mathrm{mod}\ 2)}}^{n} p_k^2(d\alpha,t)}$$

Proof. We shall prove only the first part of the Lemma, the second one can be shown exactly in the same way. Let us note that both (7) and (8) follow from (6). If we multiply both sides of (6) by $p_n(d\alpha,x)d\alpha(x)$ and we integrate over $\mathbb{R}$ then we get (7). Using Lemma 13 and comparing coefficients in (6) we obtain (8). Let us now prove (6). Develop $p_n(d\beta,x)$ into a Fourier series in $p_k(d\alpha,x)$. Then

$$p_n(d\beta,x) = \int_{-\infty}^{\infty} p_n(d\beta,u)K_{n+1}(d\alpha,x,u)d\alpha(u) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} p_n(d\alpha,x) - \varepsilon\, p_n(d\beta,t)K_{n+1}(d\alpha,x,t) .$$

Putting here $x = t$ we obtain

$$(9) \qquad p_n(d\beta,t) = \frac{\gamma_n(d\alpha)}{\gamma_n(d\beta)} p_n(d\alpha,t)[1+\varepsilon K_{n+1}(d\alpha,t,t)]^{-1} .$$

(6) follows from the above two formulas.

Lemma 16. Let $\alpha \in M(0,1)$, $\varepsilon > 0$, $t \in \mathbb{R}$, $\beta_t = \alpha + \varepsilon\delta_t$. Then

$$\lim_{n\to\infty} \frac{\gamma_n(d\beta_t)}{\gamma_n(d\alpha)} = \begin{cases} 1 & \text{for } t \in \operatorname{supp}(d\alpha) \\ |\rho(t)|^{-1} & \text{for } t \notin \operatorname{supp}(d\alpha) , \end{cases}$$

the convergence is uniform for $t \in \Delta \subset (-1,1)$, further

$$\lim_{n\to\infty} \alpha_n(d\beta_t) = 0$$

for every $t \in \mathbb{R}$ and the convergence is uniform for $t \in \Delta \subset (-1,1)$. If $z \in \mathbb{C} \setminus \operatorname{supp}(d\alpha) \setminus \{t\}$ then for $t \notin \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{p_n(d\beta_t,z)}{p_n(d\alpha,z)} = |\rho(t)| \left[1 - \frac{\sqrt{t^2-1}}{\rho(t)} \frac{\rho(z)-\rho(t)}{z-t} \right]$$

and for $t \in \operatorname{supp}(d\alpha)$

$$\lim_{n\to\infty} \frac{p_n(d\beta_t,z)}{p_n(d\alpha,z)} = 1$$

uniformly for $t \in \Delta \subset (-1,1)$. Furthermore

$$\lim_{n\to\infty} p_n(d\beta_t,t)p_n(d\alpha,t) = \begin{cases} 0 & \text{for } t \in \operatorname{supp}(d\alpha) \\ \frac{2}{\varepsilon}\sqrt{|t|^2-1} & \text{for } t \notin \operatorname{supp}(d\alpha). \end{cases}$$

If $x,t \in \operatorname{supp}(d\alpha)$, $x \neq t$ and the sequence $\{|p_k(d\alpha,x)|\}$ is bounded then

$$(10) \qquad \lim_{n\to\infty} [p_n(d\beta_t,x) - p_n(d\alpha,x)] = 0 .$$

(10) holds uniformly for $x \in \mathfrak{M} = \overline{\mathfrak{M}} \subset \operatorname{supp}(d\alpha)$ if $t \in \operatorname{supp}(d\alpha) \setminus \mathfrak{M}$ and $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \mathfrak{M}$. Finally, $\beta_t \in M(0,1)$ for each $t \in \mathbb{R}$.

Proof. The Lemma follows immediately from Lemma 15, Theorems 4.1.11, 4.1.13, 4.1.14 and (9).

Let us note that all the limits in Lemma 16 - except for one- are independent of ε.

Lemma 17. Let $\alpha \in M(0,1)$ and

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty .$$

Let for $0 \leq k \leq n$ $R_{n,k}(d\alpha,x)$ be as in 3.1 (2)-(3). Then

$$\lim_{n \geq k \to \infty} R_{n,k}(d\alpha,x) = 0$$

uniformly for $x \in \Delta \subset (-1,1)$.

Proof. By Theorem 3.1.12 the sequence $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \Delta \subset (-1,1)$. Now we apply Corollary 3.1.5.

Lemma 18. Let B be a function on $[0,\pi]$. Let

$$\varphi_n(x) = \cos[n\theta + B(\theta)] \qquad (x = \cos\theta)$$

for $n = 0,1,\ldots$. Then for $1 \leq k \leq n$

$$\varphi_n(x) = \varphi_k(x) U_{n-k}(x) - \varphi_{k-1}(x) U_{n-k-1}(x)$$

$(-1 \leq x \leq 1)$.

Proof. Apply Theorem 3.1.1.

Lemma 19. Let $\alpha \in M(0,1)$ and

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty .$$

Suppose that there exist three functions A,B and C on $[0,\pi]$ and a sequence $n_1 < n_2 < \ldots < n_k \underset{k\to\infty}{\to} \infty$ such that

(11) $$\lim_{k\to\infty} [C(\theta)\, p_{n_k}(d\alpha,x) - \varphi_{n_k}(x)] = 0$$

and

(12) $$\lim_{k\to\infty} [C(\theta) p_{n_k-1}(d\alpha,x) - \varphi_{n_k-1}(x)] = 0$$

where $x = \cos\theta$, $\varphi_n(x) = A(\theta)\cos[n\theta + B(\theta)]$, $-1 < x < 1$, $C(\theta) < \infty$. Then

(13) $$\lim_{n\to\infty} [C(\theta)\, p_n(d\alpha,x) - \varphi_n(x)] = 0 .$$

If the convergence in (11) and (12) is uniform for $x \in \mathfrak{M} \subset \Delta \subset (-1,1)$ and $C(\theta)$ is uniformly bounded for $x \in \mathfrak{M}$ then (13) holds uniformly for $x \in \mathfrak{M}$.

Proof. By Theorem 3.1.1 and Lemma 18

$$|C\,p_n - \varphi_n| \leq v(|C\,p_k - \varphi_k| + |C\,p_{k-1} - \varphi_{k-1}|) + C\,R_{n,k}$$

for $1 \leq k \leq n$ where v is the Tschebyshev weight. For a fixed n let $k = k(n)$ be defined by $k = \max\{n_j : n_j \leq n\}$. Then $\lim_{n\to\infty} k = \infty$. Now we use Lemma 17.

Recall that the function Γ has been defined in Definition 4.2.4.

Theorem 20. Let $\mathrm{supp}(d\alpha) \subset [-1,1]$ and

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty .$$

Then for almost every $x \in [-1,1]$

(14) $$\lim_{n\to\infty} [\sqrt{\alpha'(x)}\sqrt{1-x^2}\, p_n(d\alpha,x) - \sqrt{\tfrac{2}{\pi}} \cos(n\theta - \Gamma(\theta))] = 0$$

where $x = \cos\theta$.

Proof. *By Theorem 1 $\alpha \in S$. Thus the Theorem follows from Lemmas 19 and 4.2.5.*

Lemma 21. We have

$$\sum_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\ 2)}}^{n} p_k^2(v,x) = \frac{n+1}{2\pi} + \frac{1}{4\pi}\,\frac{U_{2n+1}(x)}{x}$$

and for $t > 1$

$$\frac{p_n^2(v,t)}{\frac{1}{2} + \sum\limits_{\substack{k=0 \\ k\equiv n (\mathrm{mod}\ 2)}}^{n} p_k^2(v,t)} = \frac{4t\sqrt{t^2-1}}{\rho(t)^2} + O(1)\, n\, \rho(t)^{-2n}$$

where $|O(1)| \leq C$ uniformly for $1 + \varepsilon \leq t < \infty$.

Proof. Calculation.

Theorem 22. Let α be the Tschebyshev weight, $t > 1$, $\beta = \alpha + \delta_t + \delta_{-t}$. Then

$$\gamma_n(d\beta) = \frac{2^{n-\frac{1}{2}}}{\sqrt{\pi}} [\rho(t)^{-2} + O(1)\, n\, \rho(t)^{-2n}]$$

for $n = 1,2,\ldots$.

Proof. Apply Lemmas 15 and 21.

Theorem 22 is interesting because of the following

Corollary 23. Let $C > 0$ and $\varepsilon > 0$. Then there exists a weight $\alpha \in M(0,1)$ such that $[-C,C] \subset \Delta(d\alpha)$ and $c_j^{0,1}(d\alpha) = O(\varepsilon^j)$, in particular $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$.

Proof. For the weight β constructed in Theorem 22 $\alpha_j(d\alpha) = 0$ for $j = 0,1,\ldots$.

Lemma 24. Let $\alpha \in M(0,1)$, $\varepsilon > 0$, $-1 < t < 1$, $\beta = \alpha + \varepsilon\delta_t$. If

$$\limsup_{n\to\infty} \sum_{j=n}^{2n} j\, c_j^{0,1}(d\alpha)^2 < \infty$$

then

$$\alpha_n(d\beta) = \alpha_n(d\alpha) + O(\tfrac{1}{n})$$

and

$$\frac{\gamma_{n-1}(d\beta)}{\gamma_n(d\beta)} = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} + O(\tfrac{1}{n})$$

for $n = 1,2,\ldots$.

Proof. The Lemma follows from Theorem 3.1.8 and Lemma 15. It might be interesting to remark that the estimates do not depend on ε.

By Lemma 16 from $\alpha \in M(0,1)$ follows that $\alpha + \varepsilon\delta_t \in M(0,1)$ for every $\varepsilon > 0$, $t \in \mathbb{R}$. Hence $\beta = \alpha + \sum_{k=1}^{N} \varepsilon_k \delta_{t_k}$ also belongs to $M(0,1)$ and by repeating application of the previous results we obtain asymptotics and estimates for $p_n(d\beta,z)$, $c_j^{0,1}(d\beta)$, $\lambda_n(d\beta,z)$ etc. *Let us mention two results.*

Theorem 25. Let $\alpha \in M(0,1)$, $t_k \in \mathbb{R}$, $\varepsilon_k > 0$ for $k = 1,2,\ldots,N$. Let $\beta = \alpha + \sum_{k=1}^{N} \varepsilon_k \delta_{t_k}$. Then $\beta \in M(0,1)$,

$$\lim_{n\to\infty} \frac{\gamma_n(d\beta)}{\gamma_n(d\alpha)} = \prod_{t_k \notin \operatorname{supp}(d\alpha)} |\rho(t_k)|^{-1}$$

and for every $z \in \mathbb{C}\setminus\operatorname{supp}(d\beta)$

$$\lim_{n\to\infty} \frac{p_n(d\beta,z)}{p_n(d\alpha,z)} = \prod_{t_k \notin \operatorname{supp}(d\alpha)} \{|\rho(t_k)|[1 - \frac{\sqrt{t_k^2-1}}{\rho(t_k)} \frac{\rho(z) - \rho(t_k)}{z - t_k}]\}.$$

Theorem 26. Let $\operatorname{supp}(d\alpha) \subset [-1,1]$, $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$, β be defined as in Theorem 25 with $t_k \in (-1,1)$ for $k = 1,2,\ldots,N$. Then for almost every $x \in [-1,1]$ the asymptotic formula (14) holds if we replace there $p_n(d\alpha,x)$ by $p_n(d\beta,x)$.

We suggest the reader combine these results with those of Sections 4.1, 4.2 and 6.1.

Theorem 27. Let

$$\sum_{j=0}^{\infty} j\, c_j^{(0,1)}(d\alpha) < \infty .$$

Then there exists a positive number $K = K(d\alpha)$ such that

$$\sqrt{1-x^2}\, |p_n(d\alpha,x)| \le K$$

for $-1 \le x \le 1$ and $n = 0,1,\ldots$. Further

$$\alpha'(x) \ge \frac{1}{K^2\pi} \sqrt{1-x^2}$$

for almost every $x \in [-1,1]$, in particular, $\alpha' \in S$.

Proof. By an inequality of S. Bernstein

$$\max_{|x|\le 1} |\pi_m(x)| \le m \max_{|x|\le 1} |\sqrt{1-x^2}\, \pi_m(x)|$$

for every π_m if $m \ge 1$. Let $2 \le k \le n$. Then we have by Corollary 3.5.1

$$\max_{|x|\le 1} |\sqrt{1-x^2}\, p_n(d\alpha,x)| \le \max_{|x|\le 1} [|p_k(d\alpha,x)| + |p_{k-1}(d\alpha,x)|] +$$

$$+ 2\sum_{j=k-1}^{n} j\, c_j^{(0,1)}(d\alpha) \max_{|x|\le 1} |\sqrt{1-x^2}\, p_j(d\alpha,x)| ,$$

or in short

$$A_n \le C(k) + 2\sum_{j=k+1}^{n} j\, c_j A_j$$

where $c_j = c_j^{0,1}(d\alpha)$, $A_j = \max_{|x|\le 1} |\sqrt{1-x^2}\, p_j(d\alpha,x)|$ and $C(k)$ is a constant depending on k and $d\alpha$. Let now k be so large that $\sum_{j=k+1}^{\infty} j\, c_j < \frac{1}{4}$.

Then

$$A_{k+1} \le 2\, C(k) .$$

Suppose that $A_m \le 2\, C(k)$ for $k < m < n$. If $A_n > 2\, C(k)$ then $A_m \le 2\, C(k) < A_n$, $m = k+1, \ldots, n-1$, so

$$A_n \le C(k) + 2\sum_{j=k+1}^{n} j c_j A_j < C(k) + A_n\, 2\sum_{j=k+1}^{n} j\, c_j < C(k) + \tfrac{1}{2}A_n ,$$

that is $A_n < 2C(k)$. This contradiction shows that $A_n \le 2\, C(k)$ for $n > k$.

To finish the proof we apply Theorem 5.

Remark 28. If we put $d\alpha(x) = \sqrt{1-x^2}\, dx$, $\operatorname{supp}(d\alpha) = [-1,1]$ then $c_j^{0,1}(d\alpha) = 0$ for every $j = 0,1,\ldots$, $\alpha'(x) = \sqrt{1-x^2}$ and for each $\tau \subset [-1,1]$

$$\max_{x\in\tau} |\sqrt{1-x^2}\; p_n(d\alpha,x)| = \sqrt{\tfrac{2}{\pi}}$$

if $n \ge n_1(\tau)$ $(n_1([-1,1]) = 0)$. Hence Theorem 27 - except for the constant- cannot be improved.

From Theorems 5 and 3.1.12 we obtain

Theorem 29. Let $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Then for each $\tau \subset (-1,1)$ there exists a number $K = K(\tau, d\alpha) > 0$ such that $\alpha'(x) \ge K$ for almost every $x \in \tau$. In particular if α' is continuous at $t \in (-1,1)$ and $\alpha'(t) = 0$ then $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) = \infty$.

Corollary 30. If $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ then $\operatorname{supp}(\alpha') = [-1,1]$.

Proof. Theorems 3.3.7 and 29.

Theorem 31. Let α be such that either $\operatorname{supp}(d\alpha) \subset [-1,1]$ or $\alpha \in M(0,1)$. Let $p \ge 2$ and $w(\ge 0) \in L^1(-1,1)$. Then from

$$\sup_{n\ge 0} \int_{-1}^{1} |p_n(d\alpha,x)|^p\, w(x)dx < \infty$$

follows

$$(15)\qquad \int_{-1}^{1} [\alpha'(x)\sqrt{1-x^2}]^{-\frac{p}{2}}\, w(x)dx < \infty .$$

Proof. Since $p/2 \ge 1$ we have

$$\|n^{-1}\lambda_n^{-1}(d\alpha)\|_{w,\frac{p}{2}} \le \frac{1}{n}\sum_{k=0}^{n-1} \|p_n^2(d\alpha)\|_{w,\frac{p}{2}} \le \sup_{n\ge 0} \|p_n^2(d\alpha)\|_{w,\frac{p}{2}} .$$

Because $\int \liminf |f_n| \le \limsup \int |f_n|$ the theorem follows from Theorems 6.2.33 and 6.2.51.

Theorem 32. Let $\alpha \in S$, $0 < p < \infty$, $w(\ge 0) \in L^1(-1,1)$. Then

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(d\alpha,x)|^p\, w(x)dx < \infty$$

implies (15).

Proof. First let $0 < p \le 2$. Then we shall use Lemma 4.2.5. Let $N > 0$ and φ_N be defined by

$$\varphi_N(t) = \min\{N,\, w(\cos t)\sin t[\sin t\,\alpha'(\cos t)]^{-\frac{p}{2}}\}$$

for $0 \le t \le \pi$. Then

$$[\int_0^{\pi} |\sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^2 \varphi_N(t)dt]^{\frac{1}{p}} \le [\int_0^{\pi} |\sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^p \varphi_N(t)dt]^{\frac{1}{p}} \le$$

$$\le 2^{\frac{1}{p}}[\int_0^{\pi} |p_n(d\alpha,\cos t)\sqrt{\alpha'(\cos t)\sin t} - \sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^p \varphi_N(t)dt]^{\frac{1}{p}} +$$

$$+ 2^{\frac{1}{p}}[\int_0^{\pi} |p_n(d\alpha,\cos t)\sqrt{\alpha'(\cos t)\sin t}|^p \varphi_N(t)dt]^{\frac{1}{p}} \le$$

$$\le 2^{\frac{1}{p}} N^{\frac{1}{p}} \pi^{\frac{2-p}{2}} [\int_0^{\pi} |p_n(d\alpha,\cos t)\sqrt{\alpha'(\cos t)\sin t} -$$

$$- \sqrt{\tfrac{2}{\pi}}\cos[nt-\Gamma(t)]|^2 dt]^{\frac{1}{2}} + 2^{\frac{1}{p}}[\int_{-1}^{1} |p_n(d\alpha,x)|^p w(x)dx]^{\frac{1}{p}} .$$

By Lemma 4.2.5 the first term in the right side converges to 0 when $n \to \infty$. The limit inferior of the second term is finite. By the Riemann-Lebesgue lemma the left side converges to

$$\left(\frac{1}{\pi}\int_0^\pi \varphi_N(t)dt\right)^{\frac{1}{p}}$$

when $n \to \infty$. Thus by the above inequalities

$$\frac{1}{\pi}\int_0^\pi \varphi_N(t)dt \le 2 \liminf_{N\to\infty} \int_{-1}^1 |p_n(d\alpha,x)|^p w(x)dx .$$

By Beppo Levi's theorem $\lim_{N\to+\infty} \varphi_N \in L^1(0,\pi)$ that is (15) holds. If $p > 2$ then for $N > 0$

$$\int_{-1}^1 p_n^2(d\alpha,x)[\alpha'(x)\sqrt{1-x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)\alpha'(x)\sqrt{1-x^2}dx \cdot$$

$$\cdot \left\{\int_{-1}^1[\alpha'(x)\sqrt{1-x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)dx\right\}^{\frac{2-p}{p}} \le$$

$$\le \left\{\int_{-1}^1 |p_n(d\alpha,x)|^p w(x)dx\right\}^{\frac{1}{p}}$$

where $w_N(x) = \min\{N, w(x)\}$. Letting $n \to \infty$ we obtain from Lemma 4.2.5 and from Riemann-Lebesgue's lemma that

$$\pi^{-p}\left\{\int_{-1}^1[\alpha'(x)\sqrt{1-x^2} + N^{-1}]^{-\frac{p}{2}} w_N(x)dx\right\}^2 \le$$

$$\le \liminf_{n\to\infty} \int_{-1}^1 |p_n(d\alpha,x)|^p w(x)dx .$$

Hence again by Beppo Levi's theorem (15) holds.

<u>Theorem 33.</u> *Let $w \in M(0,1)$, $supp(w) = [-1,1]$, w be Riemann integrable on $[-1,1]$. Let $g(\ge 0)$ be almost everywhere continuous on $[-1,1]$ and $p \ge 2$. Then*

$$\liminf_{n\to\infty} \int_{-1}^1 |p_n(w,x)|^p g(x)dx < \infty$$

implies

$$\int_{-1}^1 [w(x)\sqrt{1-x^2}]^{-\frac{p}{2}} g(x)dx < \infty$$

<u>Proof.</u> In the conditions the function φ_N defined by

$$\varphi_N(x) = [w(x)\sqrt{1-x^2} + N^{-1}]^{-\frac{p}{2}} \min\{N, g(x)\}\sqrt{1-x^2}$$

$(N > 0)$ is Riemann integrable for each $N > 0$. Now we can repeat the second part of the proof of Theorem 32. Applying Theorem 4.2.14 we obtain the theorem.

<u>Theorem 34.</u> *Let $\alpha \in M(0,1)$ and*

$$\sum_{j=0}^\infty c_j^{0,1}(d\alpha) < \infty .$$

Then

$$\lim_{k\to\infty} [p_k^2(d\alpha,x) - p_{k-1}(d\alpha,x)p_{k+1}(d\alpha,x)] = \frac{2\sqrt{1-x^2}}{\pi\,\alpha'(x)}$$

for almost every $x \in supp(d\alpha)$.

<u>Proof.</u> Let $0 \le k \le n$ and $\Delta \subset (-1,1)$ Then by Lemma 17

$$p_n(d\alpha,x) = U_{n-k}(x)p_k(d\alpha,x) - U_{n-k-1}(x)p_{k-1}(d\alpha,x) + \sigma(1)$$

where $\lim_{n\ge k\to\infty} \sigma(1) = 0$ uniformly for $x \in \Delta$. By Theorem 3.1.12 the sequence $\{|p_k(d\alpha,x)|\}$ is uniformly bounded for $x \in \Delta$. Hence

$$p_n^2(d\alpha,x) = U_{n-k}^2(x)p_k^2(d\alpha,x) + U_{n-k-1}^2(x)p_{k-1}^2(d\alpha,x) -$$

$$-2p_{k-1}(d\alpha,x)p_k(d\alpha,x)U_{n-k-1}(x)U_{n-k}(x) + \sigma(1).$$

Thus for $m > k$

$$\lambda_{m+1}^{-1}(d\alpha,x) = \sum_{j=0}^k p_k^2(d\alpha,x) + \sum_{j=1}^{m-k} U_j^2(x)p_k^2(d\alpha,x) +$$

$$+ \sum_{j=0}^{m-k-1} U_j^2(x)p_{k-1}^2(d\alpha,x) -$$

$$-2\sum_{j=0}^{m-k-1} U_j(x)U_{j+1}(x)p_{k-1}(d\alpha,x)p_k(d\alpha,x) + m\,\sigma(1) .$$

Let us divide this formula by m and let $m \to \infty$. We obtain from Theorems 29, 6.2.52 and Corollary 6.2.53 that

$$\frac{1}{\pi\alpha'(x)\sqrt{1-x^2}} = \frac{1}{2(1-x^2)}\left[p_k^2(d\alpha,x) + p_{k-1}^2(d\alpha,x)\right] - \frac{T_1(x)}{1-x^2}\, p_{k-1}(d\alpha,x)p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$, that is

(16) $$\frac{2\sqrt{1-x^2}}{\pi\alpha'(x)} = p_k^2(d\alpha,x) + p_{k-1}^2(d\alpha,x) - 2xp_{k-1}(d\alpha,x)p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$. By Theorem 3.1.12 and by the recurrence formula

$$\lim_{k\to\infty} \left| 2xp_{k-1}(d\alpha,x)p_k(d\alpha,x) - p_k^2(d\alpha,x) - p_{k-2}(d\alpha,x)p_k(d\alpha,x)\right| = 0$$

uniformly for $x \in \Delta$. Hence

(17) $$\frac{2\sqrt{1-x^2}}{\pi\alpha'(x)} = p_{k-1}^2(d\alpha,x) - p_{k-2}(d\alpha,x)p_k(d\alpha,x) + \sigma(1)$$

for almost every $x \in \Delta$ where $\lim_{k\to\infty} \sigma(1) = 0$ uniformly for $x \in \Delta$. Since $\Delta \subset (-1,1)$ is arbitrary the Theorem follows.

Let us note that the determinant

$$\begin{vmatrix} p_k(d\alpha,x) & p_{k-1}(d\alpha,x) \\ p_{k+1}(d\alpha,x) & p_k(d\alpha,x) \end{vmatrix} = D_k(d\alpha,x)$$

is a rather famous expression, its positivity has been investigated by several authors. (See Szegő, Problems and exercises.) So far $D_k(d\alpha,x)$ has been considered for the classical weights. From Theorem 29 and (17) we obtain the following

Corollary 35. *Let* $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ *and* $\Delta \subset (-1,1)$. *Then there exists a* number $N = N(\alpha,\Delta) > 0$ such that for each $k \geq N$ $D_k(d\alpha,x) > 0$ whenever $x \in \Delta$.

The example of the Tschebyshev polynomials shows that Δ cannot be replaced by $[-1,1]$ in Corollary 35.

Corollary 36. If $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ then

$$\limsup_{n\to\infty} \alpha'(x)\sqrt{1-x^2}\; p_n^2(d\alpha,x) = \frac{2}{\pi}$$

for almost every $x \in \mathrm{supp}(dx)$.

Proof. By Theorems 29 and 6.2.51

$$\limsup_{n\to\infty} \alpha'(x)\sqrt{1-x^2}\; p_n^2(d\alpha,x) \geq \frac{2}{\pi}$$

for almost every $x \in \mathrm{supp}(d\alpha)$. On the other hand by (16)

$$p_{k-1}(d\alpha,x) = x\,p_k(d\alpha,x) \pm \left[(x^2-1)p_k^2(d\alpha,x) + \frac{2\sqrt{1-x^2}}{\pi\alpha'(x)} + \sigma(1)\right]^{\frac{1}{2}}$$

for almost every $x \in [-1,1]$, that is by Theorem 3.3.7 for almost every $x \in \mathrm{supp}(d\alpha)$. Hence

$$(1-x^2)p_n^2(d\alpha,x) \leq \frac{2\sqrt{1-x^2}}{\pi\alpha'(x)} + \sigma(1).$$

Letting $n \to \infty$ and using Theorem 29 we obtain the Corollary.

Although the proof of the following Theorem is very simple it is one of our strongest results.

Theorem 37. Let $\alpha \in S$. Then

$$\lim_{n\to\infty} \frac{p_n^2(d\alpha,x)}{n} = 0$$

for almost every $x \in [-1,1]$.

Proof By Corollary 3.1.5.

(18) $$(1-x^2)p_n^2(d\alpha,x) \leq 2\{|p_k(d\alpha,x)| + |p_{k-1}(d\alpha,x)|\}^2 + 8n\sum_{j=k-1}^{\infty} c_j^{0,1}(d\alpha)^2\; p_j^2(d\alpha,x).$$

By Theorem 9 and Beppo Levi's theorem

$$\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha)^2 \, p_j^2(d\alpha,x) < \infty$$

for almost every $x \in [-1,1]$. Dividing both sides of (18) by n and first letting $n \to \infty$ and then $k \to \infty$ we finish the proof.

Lemma 38. Let $\phi_{2n}(dw,z)$ be defined as in Theorem 3.1.15. If $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$ then

$$(19) \qquad \phi(d\alpha,z) = \lim_{n\to\infty} \phi_{2n}(d\alpha,z)$$

$(z=e^{i\theta})$ exists for each $\theta \in (0,2\pi)\setminus\{\pi\}$ and the convergence is uniform for $\theta \in \tau \subset (0,2\pi)\setminus\{\pi\}$, in particular, $\phi(d\alpha,e^{i\theta})$ is continuous on $(0,2\pi)\setminus\{\pi\}$. If $\sum_{j=0}^{\infty} j\,c_j^{0,1}(d\alpha) < \infty$ then (19) exists for each $|z| \le 1$, the convergence is uniform in the unit disc, $\phi(d\alpha,z)$ is analytic for $|z| < 1$ and continuous for $|z| \le 1$.

Proof. Apply Theorems 3.1.12, 27 and Bernstein's inequality

$$\max_{|x|\le 1} |\pi_m(x)| \le m \max_{|x|\le 1} |\sqrt{1-x^2}\; \pi_m(x)| \qquad (m \ge 1).$$

Lemma 39. Let $\phi_{2n}(d\alpha,z)$ be as in Theorem 3.1.15. Let $\sum_{j=0}^{\infty} j\,c_j^{0,1}(d\alpha) < \infty$. Then

$$\lim_{n\to\infty} z^{2n}\, \phi_{2n}(d\alpha,z^{-1}) = 0$$

uniformly for $|z| \le 1-\varepsilon$ $\qquad (\varepsilon > 0)$.

Proof. We have

$$z^{2n}\, \phi_{2n}(d\alpha,z^{-1}) = \sum_{j=0}^{n} a_j(d\alpha, \frac{z+z^{-1}}{2})\, z^{2n-j} .$$

By Theorem 27

$$\sqrt{1-x^2}\; |p_j(d\alpha,x)| \le K$$

for $-1 \le x \le 1$, $j = 0,1,\ldots$. Hence

$$|(1-z^2)z^j\, p_j(d\alpha, \frac{z+z^{-1}}{2})| \le C$$

for $|z| \le 1$, $j = 0,1,\ldots$. Consequently

$$|z^{2n}\, \phi_{2n}(d\alpha,z^{-1})| \le C(1-|z|^2)^{-1} \cdot$$

$$\cdot \sum_{j=0}^{n} \Big\{ |1-2\frac{\gamma_{j-1}}{\gamma_j}|\,|z|^{2n-2j} + 2|a_{j-1}|\,|z|^{2n-2j+1} +$$

$$+ |1-2\frac{\gamma_{j-2}}{\gamma_{j-1}}|\,|z|^{2n-2j+2} \Big\}$$

which converges uniformly to 0 if $|z| \le 1-\varepsilon$.

Theorem 40. Let $\alpha \in M(0,1)$ and $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Then $d\alpha$ can be written in the form

$$d\alpha(t) = \alpha'(t)\,dt + d\alpha_j(t)$$

where α' is continuous and positive in $(-1,1)$, $\mathrm{supp}(\alpha') = [-1,1]$ and $\alpha_j(t)$ is constant for $-1 < t < 1$. Further

$$(20) \quad \lim_{n\to\infty} \Big\{ \sin\theta\, p_n(d\alpha,\cos\theta) - \Big[\frac{2}{\pi}\frac{\sin\theta}{\alpha'(\cos\theta)}\Big]^{\frac{1}{2}} \sin[(n+1)\theta - \varphi(\theta)] \Big\} = 0$$

uniformly for $\theta \in \tau \subset (0,\pi)$, where $\varphi(\theta) = \arg\phi(d\alpha,e^{i\theta})$ (See (19).) is continuous in $(0,\pi)$. α' can be calculated by the formula

$$\frac{2}{\pi}\frac{\sin\theta}{\alpha'(\cos\theta)} = |\phi(d\alpha,e^{i\theta})|^2 = \lim_{n\to\infty} p_n^2(d\alpha,x) - p_{n-1}(d\alpha,x)p_{n+1}(d\alpha,x)$$

$(x = \cos\theta)$. If $\sum_{j=0}^{\infty} j\,c_j^{0,1}(d\alpha) < \infty$ then

$$\lim_{n\to\infty} \{ \sin\theta\, p_n(d\alpha,\cos\theta) - \psi(\theta)\sin[(n+1)\theta - \varphi(\theta)] \} = 0$$

uniformly for $\theta \in [0,\pi]$. Here $\psi(\theta) = |\phi(d\alpha,e^{i\theta})|$ and $\varphi(\theta) = \arg\phi(d\alpha,e^{i\theta})$ are continuous functions on $[0,\pi]$.

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Proof. Let first $\sum_{j=0}^{\infty} c_j^{0,1}(d\alpha) < \infty$. Then by Lemma 38 and Theorem 3.1.15

$$\lim_{n\to\infty} \{\sin\theta\, p_n(d\alpha,\cos\theta) - |\phi(d\alpha,e^{i\theta})| \sin[(n+1)\theta - \arg\phi(d\alpha,e^{i\theta})]\} = 0$$

uniformly for $\theta \in \tau \subset (0,\pi)$. Now let us calculate $|\phi(d\alpha,e^{i\theta})|$. We have

$$(1-x^2)\frac{1}{n\lambda_n(d\alpha,x)} = |\phi(d\alpha,e^{i\theta})|^2 \cdot$$

$$\cdot \frac{1}{2n}\sum_{k=0}^{n}\{1-\cos[2(k+1)\theta - 2\arg\phi(d\alpha,e^{i\theta})]\} + o(1)$$

$(x = \cos\theta)$. Here the right side converges to $\frac{1}{2}|\phi(d\alpha,e^{i\theta})|^2$ when $n\to\infty$ and the convergence is uniform for $\theta \in \tau \subset (0,\pi)$. By Theorem 6.2.54

$$\liminf_{n\to\infty}(1-x)\frac{1}{n\lambda_n(d\alpha,x)} = \frac{\sqrt{1-x^2}}{\pi\,\alpha'(x)}$$

for almost every $x \in \operatorname{supp}(d\alpha)$. Hence

$$(21) \qquad |\phi(d\alpha,e^{i\theta})|^2 = \frac{2\sqrt{1-x^2}}{\pi\alpha'(x)}$$

for almost every $\theta \in \tau \subset (0,\pi)$. Consequently $[\alpha']^{-1}\sqrt{1-x^2}$ is equivalent to a continuous function. By Theorem 29 $\alpha'(x) \geq k > 0$ for almost every $\theta \in \tau \subset (-1,1)$. Thus $\alpha'(\cos\theta)$ and $|\phi(d\alpha,e^{i\theta})|$ are continuous and positive in $(0,\pi)$ and (21) holds for each $\theta \in (0,\pi)$. Hence (20) holds uniformly for $\theta \in \tau \subset (0,\pi)$. From (20) we obtain

$$(22) \qquad \lim_{n\to\infty}\frac{1}{n\lambda_n(d\alpha,x)} = \frac{1}{\pi\alpha'(x)\sqrt{1-x^2}}$$

uniformly for $x \in \Delta \subset (-1,1)$. Thus

$$\limsup_{n\to\infty} p_n^2(d\alpha,x) > 0$$

for every $x \in (-1,1)$, that is α_j must be constant in $(-1,1)$. Now we shall show that α has no singular component. Because α_j is constant in $(-1,1)$ for every $\Delta \subset (-1,1)$ $1_\Delta \alpha' v^{-1}$ is $d\alpha$ measurable. Consequently by

Theorem 4.2.14

$$\lim_{n\to\infty}\int_\Delta \alpha'(t)\sqrt{1-t^2}\,p_n^2(d\alpha,t)d\alpha(t) = \frac{1}{\pi}\int_\Delta \alpha'(t)dt$$

for every $\Delta \subset (-1,1)$. By (22) we obtain

$$\int_\Delta d\alpha(t) = \int_\Delta \alpha'(t)dt,$$

that is $\alpha_s(t) \equiv 0$ for $-1 < t < 1$. Finally we apply Theorem 3.3.7. If $\sum_{j=0}^{\infty} j\, c_j^{0,1}(d\alpha) < \infty$ then we use Lemma 38 and Theorem 3.1.15.

Remark 41. In general the function $\varphi(\theta)$ in (20) does not coincide with $\Gamma(\theta) + \theta - \frac{\pi}{2}$ where Γ is defined in Definition 4.2.4. For instance if β is the weight introduced in Theorem 22 then $\varphi(\theta) \neq \Gamma(\theta) + \theta - \frac{\pi}{2}$. If we know that $\operatorname{supp}(d\alpha) = [-1,1]$ then by Theorem 1 $\alpha \in S$ and by Theorem 20 $\varphi(\theta) = \Gamma(\theta) + \theta - \frac{\pi}{2}$. Thus by Theorem 41 (14) holds uniformly for $x \in \tau \subset$ $\subset (-1,1)$ if the conditions of Theorem 20 are satisfied.

Theorem 42. Let $\alpha \in M(0,1)$ and $\sum_{j=0}^{\infty} j\, c_j^{0,1}(d\alpha) < \infty$. Then

$$\lim_{n\to\infty} p_n(d\alpha,z)\,\rho(z)^{-n-1} = \frac{1}{2\sqrt{z^2-1}}\,\phi(d\alpha,\rho(z)^{-1})$$

uniformly for $|\rho(z)| \geq R > 1$ where ϕ is defined by (19). $\phi(d\alpha,\rho(z)^{-1})$ is analytic in the domain $|\rho(z)| > 1$ and vanishes for $z \in \operatorname{supp}(d\alpha)\setminus[-1,1]$.

Proof. Use Lemmas 38, 39 and Theorem 3.1.15. If $x \in \operatorname{supp}(d\alpha)\setminus[-1,1]$ then by Theorem 3.1.7 α has a jump at x. Hence $\lim_{n\to\infty} p_n(d\alpha,x) = 0$ that is $\phi(d\alpha,\rho(x)^{-1}) = 0$.

Remark 43. If $\operatorname{supp}(d\alpha) = [-1,1]$ then $\alpha \in S$ and by Lemma 6.1.18

$$\frac{\rho(z)}{2\sqrt{z^2-1}}\,\phi(d\alpha,\rho(z)^{-1}) = \frac{1}{\sqrt{2\pi}}\,D(v\,d\alpha,\rho(z)^{-1})^{-1}$$

for $|\rho(z)| > 1$.

Theorem 44. Let $\alpha \in M(0,1)$, $\sum_{j=0}^{\infty} j\, c_j^{0,1}(d\alpha) < \infty$. Let $g(\geq 0)$ be Riemann integrable on $[-1,1]$ and let $g^{\pm 1}$ be bounded on $\operatorname{supp}(d\alpha)$. Then

$$\lim_{n\to\infty} 2^{-n} \gamma_n(d\alpha_g) = \phi(d\alpha,0)\, D(g,0)^{-1},$$

$$\lim_{n\to\infty} p_n(d\alpha_g, z)\, \rho(z)^{-n-1} = \frac{1}{2\sqrt{z^2-1}}\, \phi(d\alpha,\rho(z)^{-1}) \cdot D(g,\rho(z)^{-1})^{-1}$$

for $|\rho(z)| > 1$ and

$$\lim_{n\to\infty} \lambda_n(d\alpha_g, z)^{-1} |\rho(z)|^{-2n-2} =$$

$$= \frac{1}{4(|\rho(z)|^2-1)|z^2-1|}\, |\phi(d\alpha,\rho(z)^{-1})|^2 |D(g,\rho(z)^{-1})|^{-2}$$

for $|\rho(z)| > 1$.

Proof. Use Theorems 42, 4.2.11, 6.1.25, 6.1.26 and 6.1.29.

Let us note that some results of Case [4] follows from the previous theorems. For example, Case proved Theorem 40 under the condition $c_j^{0,1}(d\alpha) = O(j^{-2})$.

8. Fourier Series

Lemma 1. Let $\operatorname{supp}(d\alpha)$ be compact, $g \geq 0$, $g^{\pm 1} \in L^1_{d\alpha}$. Let $f \in L^2_{d\alpha_g}$.

Then

(1) $$|S_n(d\alpha_g, f, x) - \lambda_n(d\alpha, x)\lambda_n^{-1}(d\alpha_g, x) S_n(d\alpha, fg, x)| \leq$$

$$\leq \|f\|_{d\alpha_g,2} \{\lambda_n^{-1}(d\alpha_g, x)[G_n(d\alpha, g^{-1}, x) G_n(d\alpha, g, x) - 1]\}^{\frac{1}{2}}$$

for $x \in \mathbb{R}$ and $n = 1,2,\ldots$.

Proof. Let us denote the left side in (1) by $R(x)$. Then

$$R(x) = \int_{-\infty}^{\infty} f(t) g(t) [K_n(d\alpha_g, x, t) - \frac{\lambda_n(d\alpha, x)}{\lambda_n(\alpha_g, x)} K_n(d\alpha, x, t)] d\alpha(t).$$

Hence

$$|R(x)|^2 \leq \|f\|^2_{d\alpha_g,2} \cdot K(x)$$

where

$$K(x) = \int_{-\infty}^{\infty} [K_n(d\alpha_g, x, t) - \frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha_g, x)} K_n(d\alpha, x, t)]^2 d\alpha_g(t).$$

Let us calculate $K(x)$. We have

$$K(x) = K_n(d\alpha_g, x, x) -$$

$$- 2\frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha_g, x)} \int_{-\infty}^{\infty} K_n(d\alpha_g, x, t) K_n(d\alpha, x, t) d\alpha_g(t) +$$

$$+ \frac{\lambda_n^2(d\alpha, x)}{\lambda_n^2(d\alpha_g, x)} \int_{-\infty}^{\infty} K_n^2(d\alpha, x, t) d\alpha_g(t) =$$

$$= \lambda_n^{-1}(d\alpha_g, x)\left[\frac{\lambda_n(d\alpha, x)}{\lambda_n(d\alpha_g, x)} G_n(d\alpha, g, x) - 1\right].$$

Now use Theorem 6.2.3.

Note that putting $f = p_{n-1}(d\alpha_g)$ in (1) we obtain an inequality which may help us derive asymptotics for $p_{n-1}(d\alpha_g, x)$.

Recall that α_τ, A_x^ω, B_x^ω, g etc. have been defined in 6.2.

Theorem 2. Let $\alpha \in S$, $f \in L^2_{d\alpha}$. Let $x \in (-1,1)$, α be absolutely continuous near x. Let $\alpha' \in B^{\omega}_x$ with $\omega(t)/t \in L^1$, $\alpha'(x) > 0$. If τ^o is sufficiently small neighborhood of x then

(2) $\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(d\alpha_\tau, f 1_\delta, x)] = 0$

where 1_δ denotes the characteristic function of an arbitrary but fixed neighborhood of x. If $\tau_1 \subset (-1,1)$, α is absolutely continuous in $\tau_1(\varepsilon)$, $\alpha' \in B^{\omega}_{\tau_1}$ with $\omega(t)/t \in L^1$, $\alpha'(x) > 0$ for $x \in \tau_1$ and τ^o is a sufficiently small neighborhood of τ_1 then (2) holds uniformly for $x \in \tau_1$ if 1_δ is the characteristic function of a neighborhood of τ_1.

Proof. Since $\alpha = (\alpha_\tau)_g$ we obtain from Theorems 6.2.40, 6.2.43, Remark 6.2.41 and Lemma 1 that

(3) $\left| S_n(d\alpha, f, x) - \frac{\lambda_n(d\alpha_\tau, x)}{\lambda_n(d\alpha, x)} s_n(d\alpha_\tau, fg, x) \right| \le C \|f\|_{d\alpha,2}$

for $n = 1, 2, \ldots$. Note that g is bounded, thus $fg \in L^2_{d\alpha_\tau}$. Let us consider now $S_n(d\alpha_\tau, fg, x)$. We have

(4) $S_n(d\alpha_\tau, fg, x) - g(x) S_n(d\alpha_\tau, f, x) =$

$$= \int_{-1}^{1} \frac{g(t)-g(x)}{t-x} f(t)(t-x) K_n(d\alpha_\tau, x, t) d\alpha_\tau(t).$$

Since the sequence $\{|p_n(d\alpha_\tau, x)|\}$ is uniformly bounded for $x \in \tau^* \subset \tau^o$ (See Lemma 6.2.29.) and

$$\int_{t \in \tau} \left[\frac{g(t)-g(x)}{t-x}\right]^2 |f(t)|^2 dt < \infty$$

we obtain from Bessel's inequality that the right side in (4) tends to 0 when $n \to \infty$. Further by Theorem 6.2.43

$$\frac{\lambda_n(d\alpha_\tau, x)}{\lambda_n(d\alpha, x)} = \frac{1}{\alpha'(x)} + O(\tfrac{1}{n}).$$

Hence

$$\frac{\lambda_n(d\alpha_\tau, x)}{\lambda_n(d\alpha, x)} S_n(d\alpha_\tau, fg, x) = S_n(d\alpha_\tau, f, x) + $$
$$+ O(\tfrac{1}{n}) S_n(d\alpha_\tau, f, x) + O(\tfrac{1}{n}) + \sigma(1).$$

We have further

$$|(S_n(d\alpha_\tau, f, x)| \le \|f\|_{d\alpha_\tau,2} \lambda_n^{-\frac{1}{2}}(d\alpha_\tau, x) = O(\sqrt{n})$$

since $d\alpha_\tau(t) = dt$ for $t \in \tau$. Thus

$$\frac{\lambda_n(d\alpha_\tau, x)}{\lambda_n(d\alpha, x)} = S_n(d\alpha_\tau, f, x) + \sigma(1).$$

To α_τ we can apply Freud's localization principle (Freud, §IV.5.), by which

$$S_n(d\alpha_\tau, f, x) = S_n(d\alpha_\tau, f 1_\delta, x) + \sigma(1).$$

Hence by (3)

(5) $\limsup_{n\to\infty} \max_{\substack{t=x \\ (\text{or } t \in \tau_1)}} |S_n(d\alpha, f, t) - S_n(d\alpha_\tau, f 1_\delta, t)| \le C \|f\|_{d\alpha,2}$

where C does not depend on f. Putting here $f - P$ instead of f where P is a polynomial with $\|f-P\|_{d\alpha,2} < \varepsilon$ $(\varepsilon > 0)$ and again using Freud's localization principle for α_τ we obtain that the left side in (5) is not greater than $C\varepsilon$. Now let $\varepsilon \to 0$.

Theorem 3. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\tau \subset (-1,1)$, α be absolutely continuous on τ, $\alpha'(t) = 1$ for $t \in \tau$, $\tau_1 \subset \tau^o$. Suppose that there exists a polynomial π such that $\pi^2/\alpha' \in L^1(-1,1)$. Let $f \in L^2_{d\alpha}$ and let 1_δ be the characteristic function of a sufficiently small neighborhood of τ_1. Then

$$\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(v, f 1_\delta, x)] = 0$$

uniformly for $x \in \tau_1$.

Proof. We could repeat Freud (§V.7.) but his proof can be simplified. He requires, moreover, that $\pi^2/\alpha' \in L^\infty$. First of all $f 1_\delta \in L_v^2$ for δ small. Further, $v^{-1} f 1_\delta \in L_v^2$ also and it is easy to see that

$$\lim_{n\to\infty} [S_n(v, f 1_\delta, x) - v(x) S_n(v, v^{-1} f 1_\delta, x)] = 0$$

uniformly for $x \in \tau_1$ since v is nice on τ. By Lemma 6.2.29 and by Freud's localization principle

$$\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(d\alpha, f 1_\delta, x) = 0$$

uniformly for $x \in \tau_1$. By Theorem 6.2.46

$$\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} = v(x) + O(\frac{1}{n}) \qquad (x \in \tau_1).$$

Hence

$$\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} S_n(v, v^{-1} f 1_\delta, x) = v(x) S_n(v, v^{-1} f 1_\delta, x) + O(\frac{1}{\sqrt{n}}) \| v^{-1} f 1_\delta \|_{v,2}$$

$(x \in \tau_1)$. Consequently we have to show that

$$|S_n(d\alpha, f 1_\delta, x) - \frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} S_n(v, v^{-1} f 1_\delta, x)|$$

converges to 0 uniformly for $x \in \tau_1$ when $n\to\infty$. But this expression equals

$$|\int_{t\in\delta} f(t)[K_n(d\alpha,x,t) - \frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} K_n(v,x,t)]dt| \le$$

$$\le \|f 1_\delta\|_{d\alpha,2} \left\{\lambda_n^{-1}(d\alpha,x)\left[\frac{\lambda_n(v,x)}{\lambda_n(d\alpha,x)} \cdot \lambda_n(v,x) \cdot \int_{-1}^{1} K_n^2(v,x,t)d\alpha(t) - 1\right]\right\}^{\frac{1}{2}}.$$

By Theorem 6.2.46 and Lemma 6.2.44 the latter expression is not greater than

$$C\|f 1_\delta\|_{d\alpha,2} \sqrt{n}\, \{(v(x) + O(\tfrac{1}{n}))(v(x)^{-1} + O(\tfrac{1}{n})) - 1\}^{\frac{1}{2}} = O(1) \|f 1_\delta\|_{d\alpha,2}$$

and this is enough for our purposes.

From Theorems 2 and 3 we obtain the following equiconvergence

Theorem 4. Let $\pi^2/\alpha' \in L^1(-1,1)$ with a suitable polynomial π. If the conditions of the first part of Theorem 2 are satisfied then

$$\lim_{n\to\infty} [S_n(d\alpha, f, x) - S_n(v, f 1_\delta, x)] = 0$$

and in the conditions of the second part of Theorem 2 the convergence is uniform for $x \in \tau_1$. Here $1_\delta = 1_{[x-\delta,x+\delta]}$ (or $= 1_{\tau_1(\delta)}$) and δ is sufficiently small.

Corollary 5. Let $\operatorname{supp}(d\alpha) = [-1,1]$, $\pi^2/\alpha' \in L^1$ with a suitable polynomial π. Let α be absolutely continuous in $\tau \subset (-1,1)$, $\alpha'(t) > C_1 > 0$ for $t \in \tau$, $\alpha' \in C^1(\tau)$, $\omega(\alpha'', t)/t \in L^1$ where ω is the modulus of continuity of α''. Let $f \in L_{d\alpha}^2$. Then

$$\lim_{n\to\infty} S_n(d\alpha, f, x) = f(x)$$

for almost every $x \in \tau$.

Proof. Use Theorem 4 and Carleson [3].

In the following we shall investigate the Lebesgue functions

$$K_n(d\alpha, x) = \int_{-\infty}^{\infty} |K_n(d\alpha, x, t)| d\alpha(t)$$

$n = 1, 2, \ldots$. One trivial thing is sure:

$$K_n^2(d\alpha, x) \le \lambda_n^{-1}(d\alpha, x)[\alpha(\infty) - \alpha(-\infty)].$$

Hence estimating λ_n^{-1} we obtain estimates for K_n. If e.g. $\alpha'(t) \ge C > 0$ for $t \in [x-\varepsilon, x+\varepsilon]$ then

$$K_n^2(d\alpha, x) \le C\, n.$$

It is rather surprising that nobody has tried to improve this estimate for weights satisfying weak conditions (e.g. for $\alpha \in S$). We shall see that $C\,n$ can be replaced by $\sigma(n)$ in many cases. First we shall find conditions for

$$(6) \qquad \lim_{n\to\infty} \lambda_n(d\alpha,x)\, K_n^2(d\alpha,x) = 0 .$$

If supp$(d\alpha)$ is compact and α has a jump at x then

$$\liminf_{n\to\infty} \lambda_n(d\alpha,x)\, K_n^2(d\alpha,x) \ge \alpha(x+0) - \alpha(x-0)$$

so that (6) cannot hold.

Lemma 6. Let $\varepsilon > 0$, $x \in \mathbb{R}$. Then

$$\lambda_n(d\alpha,x)\, K_n^2(d\alpha,x) \le 2[\alpha(x+\varepsilon)-\alpha(x-\varepsilon)] +$$

$$+ \frac{2}{\varepsilon^2}\lambda_n(d\alpha,x)\Big[\frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)} + 2(x-\alpha_{n-1}(d\alpha))^2\Big] p_{n-1}^2(d\alpha,x) +$$

$$+ 2\frac{\gamma_{n-2}^2(d\alpha)}{\gamma_{n-1}^2(d\alpha)} p_{n-2}^2(d\alpha,x)\Big] [\alpha(\infty)-\alpha(-\infty)].$$

Proof. We shall use the Christoffel-Darboux and the recurrence formulas.

We have

$$K_n(d\alpha,x) = \Big\{\int_{|x-t|\le\varepsilon}\Big\} + \Big\{\int_{|x-t|>\varepsilon}\Big\} |K_n(d\alpha,x,t)|\, d\alpha(t).$$

Hence

$$K_n^2(d\alpha,x) \le 2\Big\{\int_{|x-t|\le\varepsilon}\Big\}^2 + 2\Big\{\int_{|x-t|>\varepsilon}\Big\}^2 .$$

Here the first integral in the square is not greater than

$$[\alpha(x+\varepsilon)-\alpha(x-\varepsilon)]\int_{-\infty}^{\infty} K_n^2(d\alpha,x,t)\,d\alpha(t) = \lambda_n^{-1}(d\alpha,x)[\alpha(x+\varepsilon)-\alpha(x-\varepsilon)].$$

Further

$$\Big\{\int_{|x-t|\ge\varepsilon}\Big\}^2 \le \frac{1}{\varepsilon^2}\frac{\gamma_{n-1}^2(d\alpha)}{\gamma_n^2(d\alpha)}[p_{n-1}^2(d\alpha,x) + p_n^2(d\alpha,x)]\cdot[\alpha(\infty) - \alpha(-\infty)].$$

Now the final estimate follows from the recurrence formula.

Corollary 7. Let supp$(d\alpha)$ be compact, $\varepsilon > 0$ and Δ be fixed. Then

$$\lambda_n(d\alpha,x)\, K_n^2(d\alpha,x) \le 2[\alpha(x+\varepsilon)-\alpha(x-\varepsilon)] +$$

$$+ C\varepsilon^{-2}\lambda_n(d\alpha,x)[p_{n-1}^2(d\alpha,x) + p_n^2(d\alpha,x)]$$

for $x \in \Delta$, $n = 1,2,\ldots$ where $C = C(d\alpha,\Delta)$.

Proof. Lemmas 6 and 3.3.1.

Theorem 8. Let $\alpha \in M(0,1)$. If α is continuous at $x \in [-1,1]$ then

$$(7) \qquad \lim_{n\to\infty} \lambda_n(d\alpha,x)\, K_n^2(d\alpha,x) = 0 .$$

If α is continuous on the closed set $\mathfrak{M} \subset (-1,1)$ then (7) is satisfied uniformly for $x \in \mathfrak{M}$.

Proof. Use Corollary 7 and Theorem 4.1.11.

Theorem 9. Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$, $\varepsilon > 0$. If $[\alpha']^{-\varepsilon} \in L^1(\tau)$ then

$$(8) \qquad \lim_{n\to\infty} n^{-\frac{1}{2}} K_n(d\alpha,x) = 0$$

for almost every $x \in \tau$. If α is continuous on τ and $\alpha'(t) \ge C > 0$ for almost every $t \in \tau$ then (8) holds uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. Since α is almost everywhere continuous in $[-1,1]$ the first part of the Theorem follows from Theorems 8 and 6.3.35. The second part follows from Theorem 8 and Example 6.2.9.

From Theorem 9 one can easily obtain convergence theorems for Lip$\frac{1}{2}$ (and not for lip$\frac{1}{2}$). Let us leave the details to the reader.

Let us note that (8) is good for bad weights, for nice weights better estimates can be found.

Theorem 10. Let supp$(d\alpha)$ be compact, $\tau \subset$ supp$(d\alpha)$, $\varepsilon > 0$, $[\alpha']^{-\varepsilon} \in L^1(\tau)$.

Then

$$\liminf_{n\to\infty} n^{-\frac{1}{3}} K_n(d\alpha,x) < \infty$$

for almost every $x \in \tau$.

Proof. Let us put in Lemma 6 $\varepsilon = n^{-\frac{1}{3}}$. Using Lemma 3.3.1 we obtain

$$n^{-\frac{2}{3}} K_n^2(d\alpha, x) \le 2[n\,\lambda_n(d\alpha, x)]^{-1} \cdot$$
$$\cdot\, n^{\frac{1}{3}}[\alpha(x+n^{-\frac{1}{3}}) - \alpha(x-n^{-\frac{1}{3}})] +$$
$$+ C\,p_{n-1}^2(d\alpha, x) + C\,p_{n-2}^2(d\alpha, x)\,.$$

Summing for $n = 1, 2, \ldots, m$ we see that

$$\frac{1}{m}\sum_{n=1}^{m} n^{-\frac{2}{3}} K_n^2(d\alpha, x) \le$$

$$\le 2\,\frac{1}{m}\sum_{n=1}^{m}[n\lambda_n(d\alpha, x)]^{-1} n^{\frac{1}{3}}[\alpha(x+n^{-\frac{1}{3}}) - \alpha(x-n^{-\frac{1}{3}})] + C[m\,\lambda_m(d\alpha, x)]^{-1}.$$

Since α is almost everywhere differentiable we obtain from Theorem 6.3.35 that

$$\limsup_{n\to\infty} \frac{1}{m}\sum_{n=1}^{m} n^{-\frac{2}{3}} K_n^2(d\alpha, x) < \infty$$

for almost every $x \in \tau$. Hence the Theorem follows.

Now we shall be again in the situation $\alpha, \tau \to g, \alpha_\tau$. (See 6.2.)

<u>Lemma 11.</u> Let $\alpha \in S$, $\tau \subset (-1, 1)$. Then

$$K_n(d\alpha_\tau, x) \le C \log n \qquad (n \ge 3)$$

uniformly for $x \in \tau_1 \subset \tau^o$.

<u>Proof.</u> Lemma 6.2.29 and Freud, §IV,4.

<u>Theorem 12.</u> Let $\alpha \in S$. Let $x \in (-1, 1)$, α be absolutely continuous near x, $\alpha' \in A_x^\omega$, $\alpha'(x) > 0$. Then

(9) $$K_n(d\alpha, x) \le C\{\log n + [\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2} dt]^{\frac{1}{2}}\} \qquad (n \ge 3)$$

where C does not depend on n. If $\tau_1 \subset (-1, 1)$, α is absolutely continuous near τ_1, $\alpha' \in A_{\tau_1}^\omega$, $\alpha'(t) > 0$ for $t \in \tau_1$ then (9) holds uniformly for $x \in \tau_1$ with C independent of x and n.

<u>Proof.</u> Let us choose τ ($x \in \tau^o$ or $\tau_1 \subset \tau^o$) so small that the corresponding g is bounded from below and above in $[-1, 1]$. We have $\alpha = (\alpha_\tau)_g$. Hence by Lemma 1

$$K_n(d\alpha, x) \le C\,\lambda_n(d\alpha_\tau, x)\,\lambda_n^{-1}(d\alpha, x)\,K_n(d\alpha_\tau, x) +$$
$$+ C\{\lambda_n^{-1}(d\alpha, x)[G_n(d\alpha_\tau, g^{-1}, x)\,G_n(d\alpha_\tau, g, x) - 1]\}^{\frac{1}{2}}.$$

By Theorem 6.2.6

$$\lambda_n(d\alpha_\tau, x)\,\lambda_n^{-1}(d\alpha, x) \le C\,.$$

By Example 6.2.9 $\lambda_n^{-1}(d\alpha, x) \le Cn$. Now the Theorem follows from Lemma 11, Theorem 6.2.38, Remark 6.2.41 and Lemma 6.2.29.

Note that if $\omega(t) = t|\log t|$ then

$$\int_{\frac{1}{n}}^{1} \frac{\omega(t)}{t^2}\, dt \sim [\log n]^2.$$

A weaker version of Theorem 12 was obtained by Freud, §V.7.

<u>Theorem 13.</u> Let $\alpha \in S$, $u(\ge 0) \in L^1_{d\alpha}$, $w(\ge 0) \in L^1_{d\alpha}$, $\text{meas}(u > 0) > 0$, $\text{meas}(w > 0) > 0$, $1 < q \le \infty$, $u^{1/1-q} \in L^1_{d\alpha}$ $(q < \infty)$, $u^{-1} \in L^1_{d\alpha}$ $(q = \infty)$, $0 < p < \infty$, $p \le q$. If $q < \infty$ and for every $f \in L^1_{ud\alpha}$

$$\|S_n(d\alpha, f)\|_{wd\alpha, p} \le C\|f\|_{ud\alpha, q}$$

for $n = 1, 2, \ldots$ with C independent of n and f then

(10) $$\int_{-1}^{1}\left[\alpha'(t)\sqrt{1-t^2}\right]^{-\frac{p}{2}} w(t)\alpha'(t)dt < \infty$$

and

$$\int_{-1}^{1}\left[\alpha'(t)\sqrt{1-t^2}\right]^{\frac{q}{2(1-q)}} u(t)^{\frac{1}{1-q}} \alpha'(t)dt < \infty\,.$$

If $q = \infty$ and for $u\,f \in L^q_{dq}$

$$\|S_n(d\alpha, f)\|_{wd\alpha, p} \le C\|u\,f\|_{d\alpha, q}$$

with $C \neq C(n,f)$ for $n = 1,2,\dots$ then (10) and

$$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}\,]^{-\frac{1}{2}}\, u(t)^{-1}\, \alpha'(t)dt < \infty$$

hold.

__Proof.__ For simplicity we shall consider the case $1 < q < \infty$. By the conditions

$$\|S_n(d\alpha,f) - S_{n-1}(d\alpha,f)\|_{wd\alpha,p} \leq C\|f\|_{ud\alpha,q}.$$

This means that

$$\|p_n(d\alpha)\|_{wd\alpha,p} \cdot \|f\, p_n(d\alpha)\|_{d\alpha,1} \leq C\|f\|_{ud\alpha,q}.$$

By Hölder's inequality this is equivalent to

$$\sup_{n\geq 1} \{\|p_n(d\alpha)\|_{wd\alpha,p} \cdot \|p_n(d\alpha)u^{-1}\|_{ud\alpha,q'}\} < \infty$$

where $q' = q/(q-1)$. By Theorem 4.2.8 the latter condition is equivalent to

$$\sup_{n\geq 1} \|p_n(d\alpha)\|_{wd\alpha,p} < \infty$$

and

$$\sup_{n\geq 1} \|p_n(d\alpha)u^{-1}\|_{ud\alpha,q'} < \infty .$$

The Theorem follows now from Theorems 7.31 and 7.32.

Let us note that many special cases of Theorem 13 have been known. We refer to Badkov [2] and to the literature mentioned there. (In particular, to Askey, Muckenhaupt, Newman-Rudin, Pollard, Stein and Wainger.)

__Corollary 14.__ There exists an absolutely continuous $\alpha \in S$ such that from

$$\text{(11)} \qquad \sup_{n\geq 1} \|S_n(d\alpha)\|_{L^p_{d\alpha} \to L^p_{d\alpha}} < \infty$$

follows $p = 2$.

__Proof.__ Put $\alpha'(x) = \exp\{-(1-x^2)^{-\frac{1}{4}}\}$. If $1 < p < \infty$ then apply Theorem 13. For $p = 1,\infty$ (11) can never hold.

9. Inequalities

When investigating the Lebesgue functions of Lagrange interpolating processes we shall have to be able to estimate the expression

$$\text{(1)} \qquad \sum_{|x-x_{kn}(d\alpha)|<\varepsilon} \lambda_{kn}(d\alpha) .$$

The following result is very simple.

__Lemma 1.__ Let $\operatorname{supp}(d\alpha)$ be compact. Then

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) = \alpha(x+0) - \alpha(x-0)$$

for each $x \in \mathbb{R}$.

__Proof.__ For every $\varepsilon > 0$ we can find $y_1 \in (x-2\varepsilon, x-\varepsilon)$ and $y_2 \in (x+\varepsilon, x+2\varepsilon)$ such that α is continuous at y_1 and y_2. Thus by using convergence theorems for mechanical quadrature processes (See e.g. Freud, §III.1) we obtain

$$\limsup_{n\to\infty} \sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) \leq \int_{x-y_1}^{x+y_2} d\alpha(t) \leq \alpha(x+2\varepsilon) - \alpha(x-2\varepsilon).$$

Now let $\varepsilon \to 0$.

Unfortunately, it is not true that (1) is not greater than $\alpha(x+\varepsilon) - \alpha(x-\varepsilon)$ or $\alpha(x+2\varepsilon) - \alpha(x-2\varepsilon)$. From the Markov-Stieltjes inequalities we obtain that

$$\sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) \leq \alpha(x^1) - \alpha(x^2)$$

where $x^1 = \min_{x_{kn} \geq x+\varepsilon} x_{kn}$ if $\{k : x_{kn} \geq x+\varepsilon\}$ is not empty and otherwise $x^1 = +\infty$ and similarly $x^2 = \max_{x_{kn} \leq x-\varepsilon} x_{kn}$ or $x^2 = -\infty$. Suppose that neither $\{k : x_{kn} < x^1\}$ nor $\{k : x_{kn} > x^2\}$ is empty. Let $x_1^1 = \max_{x_{kn} < x^1} x_{kn}$ and $x_2^2 = \min_{x_{kn} > x^2} x_{kn}$. Then

(2) $$\sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) \le \alpha(x+\varepsilon + x^1 - x_1^1) - \alpha(x-\varepsilon + x^2 - x_2^2) .$$

Hence we see that to estimate (1) we have to know the behavior of $x_k(d\alpha) - x_{k+1,n}(d\alpha)$.

Lemma 2. Let $\beta \in S$. Then there exists a number $C = C(d\beta) > 1$ such that

$$p_n^2(d\beta, x) \le C^{\sqrt{n}}$$

for $x \in [-1,1]$ and $n = 1,2,\dots$.

Proof. See Geronimus, §8.2.

Lemma 3. Let α be an arbitrary weight, $\Delta \subset \operatorname{supp}(d\alpha)$. Let v_Δ denote the Tschebyshev weight corresponding to Δ. If $v_\Delta \log \alpha' \in L^1(\Delta)$ then

$$\max_{x \in \Delta} \pi_n^2(x) \le C^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_n^2(t)d\alpha(t) \qquad (n \ge 1)$$

for each π_n with a suitable $C = C(d\alpha) > 1$.

Proof. Let $\alpha^*(t) = \alpha(t)$ for $t \in \Delta$ and $\alpha^*(t)$ be constant otherwise. Let us transform Δ into $[-1,1]$. We get a weight α^{**} which satisfies the conditions of Lemma 2. Returning to Δ we obtain

$$\max_{x \in \Delta} \pi_n^2(x) \le \max_{x \in \Delta} \lambda_{n+1}^{-1}(d\alpha^*, x) \int_\Delta \pi_n^2(t)d\alpha^*(t) \le$$

$$\le n\, C^{\sqrt{n}} \int_\Delta \pi_n^2(t)d\alpha(t) \le C_1^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_n^2(t)d\alpha(t) .$$

Theorem 4. Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta \subset \operatorname{supp}(d\alpha)$, $v_\Delta \log \alpha' \in L^1(\Delta)$. Then

(3) $$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \le C \frac{1}{\sqrt{n}} \qquad (n \ge 1)$$

for $x_k, x_{k+1} \in \Delta$ with C independent of n and k. If $\Delta_1 \subset \Delta^o$ then (3) holds if either x_k or x_{k+1} belongs to Δ_1.

Proof. Let v^* denote the Tschebyshev weight corresponding to $\Delta(d\alpha)$. Let m be a natural integer and $N = [\frac{n}{m}]$. Then

$$\pi_x(t) = K_N^m(v^*, x, t)\, K_N^{-m}(v^*, x, x)$$

is a π_{n-1} with $\pi_x(x) = 1$. By Lemma 3

(4) $$1 \le C^{\sqrt{n}} \int_{-\infty}^{\infty} \pi_x^2(t)d\alpha(t)$$

for $x \in \Delta$. Let $x_k, x_{k+1} \in \Delta$ and $x = \frac{1}{2}(x_k + x_{k+1})$. Then $x \in \Delta$. Further

$$|\pi_x(x_{jn}(d\alpha))| \le [C_1 \tfrac{m}{n}(x_k - x_{k+1})^{-1}]^m$$

for $j = 1,2,\dots,n$ with $C_1 = C_1(\Delta(d\alpha))$. Calculating the integral on the right side of (4) by the Gauss-Jacobi mechanical quadrature formula we obtain

$$[x_k - x_{k+1}]^{2m} \le C^{\sqrt{n}} [C_1 \tfrac{m}{n}]^{2m} [\alpha(\infty) - \alpha(-\infty)],$$

that is

$$x_k - x_{k+1} \le C_2\, C^{\frac{\sqrt{n}}{2m}} \frac{m}{n} .$$

Putting here $m = [\sqrt{n}\,]$ the first part of the Theorem follows. Using Lemma 3.3.2 we obtain the second part of the Theorem.

Theorem 5. Let $\operatorname{supp}(d\alpha)$ be compact, $\Delta \subset \operatorname{supp}(d\alpha)$, $\varepsilon > 0$, $[\alpha']^{-\varepsilon} \in L^1(\Delta)$ Then

(5) $$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \le C \frac{\log n}{n} \qquad (n \ge 3)$$

for $x_k, x_{k+1} \in \Delta$ where C does not depend on n and k. If $\Delta_1 \subseteq \Delta^o$ ther (5) holds for either $x_k \in \Delta_1$ or $x_{k+1} \in \Delta_1$.

Proof. We obtain from Theorem 6.3.13 and from $[\alpha']^{-\varepsilon} \in L^1(\Delta)$ that

$$\max_{x \in \Delta} \pi_n^2(x) \le n^A \int_{-\infty}^{\infty} \pi_n^2(t)d\alpha(t) \qquad (n \ge 2)$$

for every π_n with a suitable constant $A > 1$. Now we repeat the proof of Theorem 4 and finally we put $m = [\log n]$.

Let us note that the proof of Theorems 4 and 5 is based on an idea of **Erdős**-Turán but our result is stronger than that of Erdős-Turán. (See Szegő, §6.11.)

<u>Theorem 6.</u> Let $\text{supp}(d\alpha)$ be compact, $\Delta \subset \text{supp}(d\alpha)$, $v_\Delta \log\alpha' \in L^1(\Delta)$, $\varepsilon_1 > 0$, $\tau \subset \tau(\varepsilon_1) \subset \Delta^o$. Then

$$\sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) \leq \alpha(x+\varepsilon+\frac{C}{\sqrt{n}}) - \alpha(x-\varepsilon-\frac{C}{\sqrt{n}})$$

uniformly for $n=1,2,\ldots$, $x \in \tau$, $0 \leq \varepsilon \leq \varepsilon_1$ where $C \neq C(n,x,\varepsilon)$, $C > 0$.

<u>Proof.</u> Use (2) and Theorem 4.

<u>Lemma 7.</u> Let $\text{supp}(d\alpha)$ be compact, $\varepsilon > 0$, $\tau \subset \tau(\varepsilon) \subset \Delta^o \subset \Delta \subset \text{supp}(d\alpha)$. Then there exists a number $N = N(\varepsilon, d\alpha, \Delta)$ such that

$$\sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) \leq \alpha(x+2\varepsilon) - \alpha(x-2\varepsilon)$$

for $x \in \tau$ and $n \geq N$.

<u>Proof.</u> Apply Lemma 3.2.2, (2) and the Heine-Borel theorem.

In the following we shall also need estimates for

$$(6) \qquad \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| .$$

It is obvious that (6) is not greater than $[\alpha(\infty) - \alpha(-\infty)]^{\frac{1}{2}}$. The question if (6) may converge to 0 when $n \to \infty$ seems to be more difficult.

<u>Example 8.</u> Let w be the Hermite weight, that is $w(x) = \exp(-x^2)$ for $x \in \mathbb{R}$. Then

$$|p_{n-1}(w,x_{kn})| \leq C\, n^{-\frac{1}{4}}\, w(x_{kn})^{-\frac{1}{2}}$$

for $k = 1,2,\ldots,n$. Hence by old theorems about quadrature sums

$$\sum_{k=1}^{n} \lambda_{kn}(w)|p_{n-1}(w,x_{kn})| \leq C\, n^{-\frac{1}{4}} \int_{-\infty}^{\infty} w(t)^{\frac{1}{2}}dt \underset{n\to\infty}{\to} 0 .$$

We shall show that this cannot happen if $\text{supp}(d\alpha)$ is compact and α is nice in a certain sense. For $D(d\alpha,0)$ se Definition 6.1.16.

<u>Lemma 9.</u> Let $\alpha \in S$. Then

$$\liminf_{n\to\infty} \sum_{k=1}^{n} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| \geq \frac{\sqrt{\pi}}{2} D(d\alpha,0) .$$

<u>Proof.</u> Let $n \geq 1$. Then $T_{n-1}(x) = L_n(d\alpha, T_{n-1}, x)$. Let us divide both sides by x^{n-1} and let $x \to \infty$. We obtain

$$2^{n-2} \leq \gamma_{n-1}(d\alpha) \sum_{k=1}^{n} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| .$$

Now apply Lemma 4.2.2.

The following result is a poor but very useful analogue of Theorem 4.2.8.

<u>Theorem 10.</u> Let $\alpha \in S$. Then there exists a number $\delta = \delta(d\alpha) > 0$ such that if $\Omega \subset [-1,1]$ is an arbitrary finite system of disjoint intervals with $|\Omega| \geq 2-\delta$ then

$$(7) \qquad \liminf_{n\to\infty} \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| > 0 .$$

<u>Proof.</u> Let $c\Omega = [-1,1]\setminus\Omega$. Then $1_{c\Omega}$ is Riemann integrable on $[-1,1]$. We have

$$\sum_{k=1}^{n} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| = \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| +$$

$$+ \sum_{k=1}^{n} 1_{c\Omega}(x_{kn})\, \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| \leq$$

$$\leq \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| +$$

$$+ [(\alpha(1)-\alpha(-1)) \sum_{k=1}^{n} 1_{c\Omega}(x_{kn})\lambda_{kn}(d\alpha)p_{n-1}^2(d\alpha,x_{kn})]^{\frac{1}{2}} .$$

Now let $n \to \infty$. By Lemma 9 and Theorem 3.2.3

$$\frac{\sqrt{\pi}}{2} D(d\alpha,0) \le \liminf_{n\to\infty} \sum_{x_{kn}\in\Omega} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| +$$

$$+ \left[(\alpha(1)-\alpha(-1)) \frac{2}{\pi}\int_{c\Omega} \sqrt{1-t^2}\,dt\right]^{\frac{1}{2}} .$$

Hence (7) holds if $|c\Omega|$ is small.

<u>Theorem 11.</u> Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$. Then

$$\begin{matrix}\limsup\\ \liminf\\ n\to\infty\end{matrix} \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| \ge \frac{2}{\pi}\int_\tau \sqrt{1-t^2}\,dt \cdot$$

$$\cdot \left\{ \begin{matrix}\liminf\\ \limsup\\ n\to\infty\end{matrix} \max_{x_{kn}\in\tau} |p_{n-1}(d\alpha,x_{kn})| \right\}^{-1} .$$

<u>Proof.</u> The Theorem follows from Theorem 3.2.3 and the inequality

$$\sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)p_{n-1}^2(d\alpha,x_{kn}) \le \max_{x_{kn}\in\tau} |p_{n-1}(d\alpha,x_{kn})| \cdot$$

$$\cdot \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| .$$

<u>Theorem 12.</u> Let $\mathrm{supp}(d\alpha) = [-1,1]$, $\alpha'(x) > 0$ for almost every $x \in (-1,1)$, $\Delta \subset [-1,1]$. Then

$$(8) \qquad \liminf_{n\to\infty} \{ \max_{x\in\Delta} |p_n(d\alpha,x)| \sum_{x_{kn}\in\Delta} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| \} \ge$$

$$\ge \frac{|\Delta|}{2\pi} \int_\Delta v_\Delta(t)^{-1} v(t)dt.$$

<u>Proof.</u> By Bernstein's inequality

$$|p_n'(d\alpha,t)| \le \frac{2n}{|\Delta|} v_\Delta(t) \max_{x\in\Delta} |p_n(d\alpha,x)|$$

for $x \in \Delta$. Further $\gamma_{n-1}(d\alpha) \le \gamma_n(d\alpha)$ and

$$\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn}(d\alpha)p_{n-1}(d\alpha,x_{kn}) = [p_n'(d\alpha,x_{kn})]^{-1} .$$

Hence the left side in (8) is not less than

$$\frac{|\Delta|}{2} \liminf_{n\to\infty} \frac{1}{n} \sum_{x_{kn}\in\Delta} v_\Delta(x_{kn})^{-1} .$$

But

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x_{kn}\in\Delta} v_\Delta(x_{kn})^{-1} = \frac{1}{\pi}\int_\Delta v_\Delta(t)^{-1} v(t)dt .$$

(See Freud, §III.9.)

<u>Corollary 13.</u> Let the conditions of Theorem 12 be satisfied. Let the sequence $\{|p_n(d\alpha,x)|\}$ be uniformly bounded for $x \in \Delta$. Then

$$\liminf_{n\to\infty} \sum_{x_{kn}\in\Delta} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| > 0 .$$

Let us note that the Pollaczek weight satisfies the conditions of Corollary 13.

Now we shall deal with weighted Bernstein-Markov inequalities. Our aim will be to generalize the following result of Khalilova [9].

<u>Lemma 14.</u> Let $1 \le p < \infty$, $a \in \mathbb{R}$, $b \in \mathbb{R}$ and u be a Jacobi weight. Then

$$\|\pi_n' v^{-1}\|_{u,p} \le C\, n\, \|\pi_n\|_{u,p}$$

and

$$\max_{|x|\le 1} \{ |\pi_n'(x)| (\sqrt{1-x}+\tfrac{1}{n})^{a+1} (\sqrt{1+x}+\tfrac{1}{n})^{b+1} \} \le$$

$$\le C\, n \max_{|x|\le 1} \{ |\pi_n(x)| (\sqrt{1-x}+\tfrac{1}{n})^{a} (\sqrt{1+x}+\tfrac{1}{n})^{b} \}$$

for every π_n where C does not depend on π_n and n.

Lemma 15. Let $a > -1$, $1 \le p \le \infty$. Then

$$(9)\qquad \int_{-1}^{1} |\pi_n'(t)|^p |t|^a dt \le C n^p \int_{-2}^{2} |\pi_n(t)|^p |t|^a dt$$

for every π_n.

Proof. Let first π_n be even, that is let $\pi_n(x) = G_n(x^2)$. Then $\pi_n'(x) = 2x\,G_n'(x^2)$ and we have to show that

$$\int_{-1}^{1} |x\,G_n'(x^2)|^p |x|^a dx \le C n^p \int_{-2}^{2} |G_n(x^2)|^p |x|^a dx$$

or

$$\int_0^1 |G_n'(x)|^p |x|^{\frac{p+a-1}{2}} dx \le C n^p \int_0^4 |G_n(x)|^p |x|^{\frac{a-1}{2}} dx .$$

But

$$\int_0^1 |G_n'(x)|^p |x|^{\frac{p+a-1}{2}} dx \le C \int_0^4 |G_n'(x)\sqrt{x}\sqrt{4-x}\,|^p |x|^{\frac{a-1}{2}} dx.$$

Hence (9) follows from Lemma 14 when π_n is even. Let now π_n be odd: $\pi_n(x) = x\,G_n(x^2)$. In this case $\pi_n'(x) = G_n(x^2) + 2x^2 G_n'(x^2)$ and we shall prove that

$$\int_{-1}^{1} |x^2 G_n'(x^2)|^p |x|^a dx \le C n^p \int_{-2}^{2} |x\,G_n(x^2)|^p |x|^a dx$$

and

$$\int_{-1}^{1} |G_n(x^2)|^p |x|^a dx \le C n^p \int_{-2}^{2} |x\,G_n(x^2)|^p |x|^a dx .$$

The first inequality here follows from the first part of the proof by putting there $p+a$ instead of a. The second inequality has been proved in Corollary 6.3.26. Hence (9) holds if π_n is either even or odd. But then it also holds for every π_n with a possibly bigger constant C.

From Lemmas 14 and 15 follows

Theorem 16. Let $1 = t_1 > t_2 > \ldots > t_N = -1$, $\Gamma_k > -1$ for $k = 1,2,\ldots,N$,

$$w(t) \sim \prod_{k=1}^{N} |t-t_k|^{\Gamma_k} \qquad (-1 \le t \le 1).$$

Let $1 \le p < \infty$. Then for every π_n

$$\|\pi_n' v^{-1}\|_{w,p} \le C n \|\pi_n\|_{w,p}$$

where C does not depend on n and π_n.

Lemma 17. Let $a \in \mathbb{R}$. Then there exists a number $\varepsilon = \varepsilon(a) > 0$ such that for every π_n

$$(10)\qquad \max_{|x| \le \frac{\varepsilon}{n}} |\pi_n(x)| \le C n^a \max_{\frac{\varepsilon}{n} \le |x| \le 1} \{|x|^a |\pi_n(x)|\}$$

with $C = C(a)$.

Proof. Let first $a = 0$. Then (10) follows from Lemmas 6.2.50, 6.3.5 and 6.3.22 applied for the Legendre weight. If (10) holds for $a = 0$ then it also holds for $a \ge 0$. If $a < 0$ then we remark that (10) with $a = 0$ implies that

$$(11)\qquad \max_{|x| \le \frac{\varepsilon}{n}} |\pi_n(x)| \le C \max_{\frac{\varepsilon}{n} \le |x| \le \frac{1}{2}} |\pi_n(x)|$$

with a possibly new $\varepsilon > 0$ and C. Let w be defined by $w(t) = |t|^{-a}$ for $-1 \le t \le 1$, $\operatorname{supp}(w) = [-1,1]$. Putting in (11) $K_n(w,x,x)\pi_n(x)$ instead of $\pi_n(x)$ we obtain from Lemma 6.3.19

$$\max_{|x| \le \frac{\varepsilon}{3n}} |\pi_n(x)| n^{1-a} \le C \max_{\frac{\varepsilon}{3n} \le |x| \le \frac{1}{2}} \{|\pi_n(x)||x|^a\} n .$$

Hence (1) follows for $a < 0$ also.

Lemma 18. Let $a \in \mathbb{R}$. Then

$$\max_{|x| \le 1} \{|\pi_n'(x)|(|x| + \tfrac{1}{n})^a\} \le C n \max_{|x| \le 2} \{|\pi_n(x)|(|x| + \tfrac{1}{n})^a\}$$

for every π_n.

Proof. Repeat the proof of Lemma 15 and use Lemmas 14 and 17.

By Lemmas 14 and 18 we obtain the following

Theorem 19. Let $A_k \in \mathbb{R}$ for $k = 1,2,\ldots,N$, $1 > t_2 > \ldots > t_{N-1} > -1$,

$$w_n(x) = \left(\sqrt{1-x} + \frac{1}{n}\right)^{2A_1} \prod_{k=2}^{N-1} \left(|x-t_k| + \frac{1}{n}\right)^{A_k} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2A_N}$$

for $|x| \le 1$. Then

$$\max_{|x|\le 1} \{|\pi_n'(x)| w_n(x)(\sqrt{1-x} + \tfrac{1}{n})(\sqrt{1+x} + \tfrac{1}{n})\} \le$$

$$< C\, n \max_{|x|\le 1} \{|\pi_n(x)| w_n(x)\}$$

for every π_n where C is independent of n and π_n.

Now we return to estimating the distance between two consecutive zeros of orthogonal polynomials.

Theorem 20. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta \subset \mathrm{supp}(d\alpha)$, $t^* \in \Delta^o$, $\Gamma > -1$. Let α be absolutely continuous in Δ with

$$\alpha'(t) \sim |t-t^*|^\Gamma \qquad \forall t \in \Delta.$$

Then

$$x_{kn}(d\alpha) - x_{k+1,n}(d\alpha) \sim \frac{1}{n}$$

for $x_{kn} \in \Delta_1 \subset \Delta^o$.

Proof. By Lemma 3.3.2 we can suppose that both x_{kn} and $x_{k+1,n}$ belong to Δ_1. First we shall show that

$$x_{kn} - x_{k+1,n} \le C n^{-1}.$$

We have the following possibilities.

(12) $t^* \le x_{k+1} < x_k \le t^* + \frac{1}{n}$ or $t^* - \frac{1}{n} \le x_{k+1} < x_k \le t^*$

or $t^* - \frac{1}{n} < x_{k+1} \le t^* < x_k < t^* + \frac{1}{n}$,

(13) $t^* \le x_{k+1} \le t^* + \frac{1}{n} < x_k$ or $x_{k+1} < t^* - \frac{1}{n} \le x_k \le t^*$,

(14) $t^* + \frac{1}{n} \le x_{k+1} < x_k$ or $x_{k+1} < x_k \le t^* - \frac{1}{n}$,

(15) $x_{k+1} \le t^* - \frac{1}{n} < t^* < x_k \le t^* + \frac{1}{n}$ or

$t^* - \frac{1}{n} \le x_{k+1} \le t^* < t^* + \frac{1}{n} \le x_k$

and

(16) $x_{k+1} \le t^* - \frac{1}{n} < t^* + \frac{1}{n} \le x_k$.

In all cases (12)-(16) we shall use Theorem 6.3.25 and the estimate

$$\int_{x_{k+1,n}}^{x_{kn}} d\alpha(t) \le \lambda_{kn}(d\alpha) + \lambda_{k+1,n}(d\alpha) \qquad (k = 1,2,\dots,n-1)$$

which follows from the Markov-Stieltjes inequalities. In the first case of (13) we obtain

$$(x_k - t^*)^{\Gamma+1} \le C[n^{-\Gamma-1} + \frac{1}{n}(x_k - t^*)^\Gamma].$$

Hence $x_k \le t^* + C\frac{1}{n}$. In the first case of (14) we have

$$(x_k-t^*)^{\Gamma+1} - (x_{k+1}-t^*)^{\Gamma+1} \le C\frac{1}{n}[(x_k-t^*)^\Gamma + (x_{k+1}-t^*)^\Gamma].$$

Thus

$$x_k - x_{k+1} \le C\frac{1}{n} \frac{[(x_k-t^*)-(x_{k+1}-t^*)][(x_k-t^*)^\Gamma + (x_{k+1}-t^*)^\Gamma]}{(x_k-t^*)^{\Gamma+1} - (x_{k+1}-t^*)^{\Gamma+1}}$$

and

$$x_k - x_{k+1} \le C\frac{1}{n} \sup_{1\le x<\infty} \frac{(x-1)(x^\Gamma+1)}{x^{\Gamma+1}-1} \le C\frac{1}{n}$$

since $\Gamma+1 > 0$. The other possibilities may be treated similarly. To estimate $x_k - x_{k+1}$ from below let us remark as Erdős-Turán did that

$$1 = (x_{kn} - x_{k+1,n}) \frac{d}{dx} \ell_{kn}^2(d\alpha, x^*)$$

where $x_{k+1} \le x^* \le x_k$. We have

$$\ell_{kn}^2(d\alpha, x) \le \lambda_{kn}(d\alpha)\, \lambda_n^{-1}(d\alpha, x).$$

Using Theorems 19 and 6.3.25 we obtain

$$\left|\frac{d}{dx}\, \ell^2_{kn}(d\alpha, x)\right| \le C\, n$$

uniformly for $x_{kn}, x \in \Delta_1 \subset \overset{o}{\Delta}$. Hence the estimate from below for $x_k - x_{k+1}$ follows.

Theorem 21. Let $\mathrm{supp}(d\alpha)$ be compact, $\Delta(d\alpha) = [c_1, c_2]$, $a > -1, \delta > 0$. Let α be absolutely continuous in $[c_2-\delta, c_2]$ and let

$$\alpha'(t) \sim (c_2 - t)^a$$

for $t \in [c_2-\delta, c_2]$. Let $x_{kn}(d\alpha) = \frac{1}{2}(c_1+c_2) + \frac{1}{2}(c_2-c_1)\cos\theta_k$ for $k = 0, 1, \ldots, n+1$ where $0 \le \theta_k \le \pi$ and $x_{0n} = c_2$, $x_{n+1,n} = c_1$. Then

$$\theta_{k+1} - \theta_k \sim \frac{1}{n}$$

for $x_{kn} \in \Delta \setminus (c_2-\delta, c_2]$.

Proof. We can assume without loss of generality that $\Delta(d\alpha) = [-1,1]$ and $\delta \le \frac{1}{2}$. (Concerning the second assumption see e.g. Freud, §III.5.) We obtain immediately from Theorem 6.3.27 and Markov-Stieltjes' inequalities that $\theta_1 = O(\frac{1}{n})$ and $\theta_2^{-1} = O(n)$. Now we shall show that $\theta_1^{-1} = O(n)$. Let $m \ge n$ be fixed. Then by the Gauss-Jacobi mechanical quadrature formula

$$(1-x_{1n})\,\lambda_{1n}(d\alpha) = \int_{-1}^{1} (1-t)\, \ell^2_{1n}(d\alpha, t)\, d\alpha(t) =$$

$$= \sum_{k=1}^{m} (1-x_{km})\, \ell^2_{1n}(d\alpha, x_{km})\, \lambda_{km}(d\alpha)\,.$$

Hence

$$(1-x_{1n})\lambda_{1n}(d\alpha) \ge (1-x_{2m}) \sum_{k=1}^{m} \ell^2_{1n}(d\alpha, x_{km})\, \lambda_{km}(d\alpha) -$$

$$-(1-x_{2m})\, \ell^2_{1n}(d\alpha, x_{1m})\lambda_{1m}(d\alpha) =$$

$$= (1-x_{2m})\, \lambda_{1n}(d\alpha)\left[1 - \frac{\ell^2_m(d\alpha, x_{1m})}{\lambda_{1n}(d\alpha)}\, \lambda_{1m}(d\alpha)\right]$$

and consequently

$$1 - x_{1n} \ge (1-x_{2m})\left[1 - \frac{\lambda_{1m}(d\alpha)}{\lambda_m(d\alpha, x_{1m})}\right].$$

Putting here $m = Nn$ *where* N *is big but fixed we obtain from Theorem 6.3.27* that

$$1 - x_{1n} \ge \tfrac{1}{2}(1-x_{2m})$$

if only n is big enough. Hence $\theta_1^{-1} = O(n)$. To prove $\theta_{k+1} - \theta_k = O(n^{-1})$ for $x_k \in \Delta \subset (1-\delta, 1]$ we shall use the inequalities

$$(17)\qquad \sum_{k=i+1}^{n} (1 \pm x_{kn})\, \lambda_{kn}(d\alpha) \le \int_{-1}^{x_{in}} (1 \pm t)\, d\alpha(t) \le$$

$$\le \sum_{k=i}^{n} (1 \pm x_{kn})\, \lambda_{kn}(d\alpha) \qquad (i = 1, 2, \ldots, n)$$

which is always true if $\Delta(d\alpha) \subset [-1,1]$. We shall not prove (17), it can be *proved in the same way that* Freud proves the Markov-Stieltjes inequalities in his book. From (17) and Theorem 6.3.27 we get for $x_k \in \Delta$

$$\theta_{k+1} - \theta_k \le \frac{C}{n} \sup_{0 \le x, y \le \frac{\pi}{2}} \frac{|x-y|\,[\sin^{2a+3} x + \sin^{2a+3} y]}{|\sin^{2a+4} x - \sin^{2a+4} y|}$$

which is of order $1/n$ since $2a+3 > 1$. The estimate $[\theta_{k+1} - \theta_k]^{-1} = O(n)$ for $x_k \in \Delta$ follows from Lemma 14 and Theorem 6.3.27 in the same way as we obtained the estimates from below in Theorem 20.

Theorem 22. Let w be as in Theorem 16, $x_{kn}(w) = \cos\theta_{kn}$ ($x_{0n} = 1$, $x_{n+1,n} = -1$, $0 \le \theta_{kn} \le \pi$). Then

$$\theta_{k+1,n} - \theta_{kn} \sim \frac{1}{n}$$

for $k = 0, 1, \ldots, n$.

Proof. Use Theorems 20 and 21.

The following inequalities will play a fundamental role in investigations of mean convergence of interpolation processes.

<u>Theorem 23.</u> Let $\alpha, \Delta, t^*, \Gamma$ and Δ_1 be as in Theorem 20. Let $1 \le p < \infty$ and $m \le c_1 n$. Then for each π_m and n

$$\sum_{x_{kn} \in \Delta_1} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \le C \int_\Delta |\pi_m(t)|^p d\alpha(t)$$

where $C = C(\alpha, \Delta_1, p, c_1)$.

<u>Proof.</u> Let $w = 1_{\Delta}\alpha'$ with $\mathrm{supp}(w) = \Delta$. Then by Theorem 6.3.25

$$\lambda_{kn}(d\alpha) \sim \lambda_n(w, x_{kn}(d\alpha)) \sim \lambda_n(w, p, x_{kn}(d\alpha)) \sim \lambda_{n+m}(w, p, x_{kn}(d\alpha))$$

for $x_{kn}(d\alpha) \in \Delta_1 \subset \Delta^o$ since $m \le c_1 n$. Hence

$$|\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \le C \int_\Delta |\pi_m(t)|^p d\alpha(t)$$

for $x_{kn}(d\alpha) \in \Delta_1$. Further we can suppose that $t^* \in \Delta_1$. Let $j = j(n)$ be defined by $x_{j+1,n} \le t^* < x_{jn}$. Consider

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) .$$

Observe that

$$|\pi_m(x_{kn})|^p \le |\pi_m(x)|^p + p \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1} |\pi_m'(t)| dt$$

for $x_{k+1,n} \le x \le x_{k-1,n}$. Thus by the Markov-Stieltjes inequalities we obtain

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \le 2 \int_\Delta |\pi_m(x)|^p d\alpha(x) +$$

$$+ p \sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} \lambda_{kn}(d\alpha) \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1} |\pi_m'(t)| dt .$$

By Theorems 20 and 6.3.25

$$\lambda_{kn}(d\alpha) \sim \frac{1}{n} |t - t^*|^\Gamma$$

for $x_{k+1,n} \le t \le x_{k-1,n}$ if $k < j-1$ and $x_{kn} \in \Delta_1$. Consequently

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} \lambda_{kn}(d\alpha) \int_{x_{k+1,n}}^{x_{k-1,n}} |\pi_m(t)|^{p-1} |\pi_m'(t)| dt \le$$

$$\le C \frac{1}{n} \{ \int_{\Delta_1(\varepsilon)} |\pi_m(t)|^p |t - t^*|^\Gamma dt \}^{\frac{p-1}{p}} \cdot \{ \int_{\Delta_1(\varepsilon)} |\pi_m'(t)|^p |t - t^*|^\Gamma dt \}^{\frac{1}{p}} ,$$

where $\varepsilon > 0$ is chosen so that $\Delta_1(\varepsilon) \subset \Delta^o$. By Lemma 15 we obtain

$$\sum_{\substack{x_{kn} \in \Delta_1 \\ k < j-1}} |\pi_m(x_{kn})|^p \lambda_{kn}(d\alpha) \le C \frac{m+n}{n} \int_\Delta |\pi_m(t)|^p d\alpha(t) .$$

The sum for $x_{kn} \in \Delta_1$, $k > j+1$ can be estimated similarly.

<u>Theorem 24.</u> Let α, c_2, a, δ and Δ be as in Theorem 21. Let $\Gamma > -1-a$, $1 \le p < \infty$, $m \le c^* n$. Then

$$\sum_{x_{kn} \in \Delta} |\pi_m(x_{kn})|^p (c_2 - x_{kn})^\Gamma \lambda_{kn}(d\alpha) \le$$

$$\le C \int_{c_2 - \delta}^{c_2} |\pi_m(t)|^p (c_2 - t)^\Gamma d\alpha(t)$$

for each π_m where $C = C(\alpha, p, \Delta, \Gamma, c^*)$.

<u>Theorem 25.</u> Let w be as in Theorem 16, $u \in L_w^1$ be a Jacobi weight, $1 \le p < \infty$ and $m \le c^* n$. Then

$$\sum_{k=1}^n |\pi_m(x_{kn})|^p u(x_{kn}) \lambda_{kn}(w) \le C \int_{-1}^1 |\pi_m(t)|^p u(t) w(t) dt$$

for every π_m where $C = C(w, u, p)$.

Theorems 24 and 25 can be proved by the same method as Theorem 23. As an application of the previous results we shall prove two theorems.

Theorem 26. Let $w = w^{(a,b)}$ be the Pollaczek weight defined in Definition 6.2.12. Let $1 \le p \le \infty$, $p \ne 2$. Then the sequence of operators $\{S_n(w)\}$ is not uniformly bounded in L^p_w.

Proof. By Corollary 13 and Theorem 23 ($\Gamma = 0$)

$$\liminf_{n\to\infty} \int_{-1}^{1} |p_n(w,t)|^q w(t)dt > 0$$

for $1 \le q < \infty$. Let $1 < p < \infty$, $p \ne 2$. Suppose that

$$\sup_{n\ge 1} \|S_n(w)\|_{L^p_w \to L^p_w} < \infty .$$

Then we obtain exactly in the same way as in Theorem 8.13 that

$$\sup_{n\ge 1} \|p_n(w)\|_{w,p} < \infty$$

and

$$\sup_{n\ge 1} \|p_n(w)\|_{w,p'} < \infty$$

where $p' = \frac{p}{p-1}$. Since either p or p' is greater than 2 this cannot happen by Theorem 7.31. For $p = 1$ or $p = \infty$ the Theorem follows from old results. (See e.g. Freud, remarks on Chapter IV.)

Theorem 27. Theorem 26 remains valid if we replace the Pollaczek weight there by the weight defined in Example 6.2.14.

Proof. By Korous' theorem (See Freud, §I.7.) the corresponding system is uniformly bounded on each $\Delta \subset (-1,1)$. Now we can repeat the proof of Theorem 26.

Definition 28. The weight w is a generalized Jacobi weight ($w \in GJ$) if $\mathrm{supp}(w) = [-1,1]$ and

$$w(t) = \varphi(t)(1-t)^{\Gamma_1} \prod_{k=2}^{N-1} |t_k - t|^{\Gamma_k} (1+t)^{\Gamma_N}$$

where $\Gamma_k > -1$ ($k = 1,2,\dots,N$), $1 > t_2 > \dots > t_{N-1} > -1$, $\varphi (> 0)$ is continuous

on $[-1,1]$ and $\omega(\delta)/\delta \in L^1(0,1)$ where ω is the modulus of continuity of φ. If $w^* \sim w$ where $w \in GJ$ then we write $w^* \sim GJ$. Hence if $w \in GJ$ then also $w \sim GJ$. For $w \sim GJ$ w_n is defined by

$$w_n(t) = (\sqrt{1-t} + \tfrac{1}{n})^{2\Gamma_1} \prod_{k=2}^{N-1} (|t_k - t| + \tfrac{1}{n})^{\Gamma_k} (\sqrt{1+t} + \tfrac{1}{n})^{2\Gamma_N}$$

$(-1 \le t \le 1 ; n = 1,2,\dots)$.

Lemma 29. Let $w \in GJ$. Then

$$(\sqrt{1-x} + \tfrac{1}{n})(\sqrt{1+x} + \tfrac{1}{n}) w_n(x)\, p_n^2(w,x) \le C$$

for $-1 \le x \le 1$, $n = 1,2,\dots$ where $C \ne C(n,x)$.

Proof. See Badkov [2].

Lemma 30. Let $\alpha \in S$, $g = v^{-2}$. Then

$$|p_{n-1}(d\alpha, x_k)| \sim (1-x_k^2) |p_{n-1}(d\alpha_g, x_k)|$$

where $x_k = x_{kn}(d\alpha)$.

Proof. In course of the proof of Theorem 4.2.3 we have shown that

$$(1-x_k^2) p_{n-1}(d\alpha_g, x_k) = p_{n-1}(d\alpha, x_k) \cdot$$

$$\cdot \left[\frac{\gamma_{n-1}(d\alpha)}{\gamma_{n-1}(d\alpha_g)} + \frac{\gamma_{n-1}(d\alpha_g)\gamma_{n-1}(d\alpha)}{\gamma_n^2(d\alpha)} \right].$$

The expression in the brackets does not depend on k and by Lemma 4.2.2 it converges to 1 when $n \to \infty$.

Theorem 31. Let $w \in GJ$. Then

$$w_n(x_{kn}) p_{n-1}^2(w, x_{kn}) \sim \sqrt{1-x_{kn}^2} \sim (\sqrt{1-x_{kn}} + \tfrac{1}{n})(\sqrt{1+x_{kn}} + \tfrac{1}{n}).$$

Proof. If $w \in GJ$ then $v^{-2} w \in GJ$. Hence by Lemmas 29 and 30

$$w_n(x_{kn}) p_{n-1}^2(w, x_{kn}) \le C(\sqrt{1-x_{kn}} + \tfrac{1}{n})(\sqrt{1+x_{kn}} + \tfrac{1}{n}).$$

By Theorem 22 the right side here is $\sim \sqrt{1-x_{kn}^2}$. The converse inequality follows from

$$p_{n-1}(w,x_{kn})^{-1} = \frac{\gamma_{n-1}(w)}{\gamma_n(w)} \lambda_{kn}(w) p_n'(w,x_{kn})$$

and from Theorems 6.3.28, 19, 22 and Lemma 29.

Lemma 32. Let α be an arbitrary weight. Let $x_{0n}(d\alpha) = \infty$, $x_{n+1,n}(d\alpha) = -\infty$, $\ell_{0n}(d\alpha,x) \equiv 0$ and $\ell_{n+1,n}(d\alpha,x) \equiv 0$. Let $x_{k+1,n}(d\alpha) \le x \le x_{kn}(d\alpha)$ ($k = 0,1,\ldots,n$). Then

$$\ell_{kn}(d\alpha,x) + \ell_{k+1,n}(d\alpha,x) \ge 1 .$$

Proof. See Erdős-Turán [5].

Theorem 33. Let $w \in GJ$. Then

$$(18) \qquad \lambda_n(w,x)p_n^2(w,x) \sim n(x-x_k)^2(\sqrt{1-x} + \tfrac{1}{n})^{-2}(\sqrt{1+x} + \tfrac{1}{n})^{-2}$$

for $-1 \le x \le 1$ where x_k is the zero of $p_n(w,x)$ which is closest to x.

Proof. By Theorems 22 and 6.3.28

$$\ell_{kn}^2(w,x) \le \lambda_{kn}(w)\lambda_n^{-1}(w,x) \le C$$

for $-1 \le x \le 1$ where k is the index of x_k in (18). Further, if $k = 1$ and $x_k \le x \le 1$ or $k = n$ and $-1 \le x \le x_k$ then by Lemma 32 $\ell_{kn}^2(w,x) \ge 1$. Otherwise x is between either x_{k-1} and x_k ($k > 1$) or x_k and x_{k+1} ($k < n$). Let for simplicity $x_k \le x \le x_{k-1}$. Then by Lemma 32

$$\ell_{k-1,n}(w,x) + \ell_{kn}(w,x) \ge 1 .$$

By Theorems 22, 31 and 6.3.28

$$|\lambda_{k-1,n}(w)p_{n-1}(w,x_{k-1,n})| \sim |\lambda_{kn}(w)p_{n-1}(w,x_{kn})|$$

and obviously $\ell_{k-1,n}(w,x) \ge 0$, $\ell_{kn}(w,x) > 0$ and $\operatorname{sign} p_{n-1}(w,x_{k-1,n}) = -\operatorname{sign} p_{n-1}(w,x_{kn})$. Hence

$$\ell_{k-1,n}(w,x) \le C\,\ell_{kn}(w,x) .$$

Consequently in all possible cases

$$\ell_{kn}^2(w,x) \sim 1.$$

The Theorem follows now from Theorems 22, 31 and 6.3.28.

Corollary 34. Let $w \in GJ$. Then

$$p_n(w,1) \sim n^{\Gamma_1+\frac{1}{2}}$$

and

$$|p_n(w,-1)| \sim n^{\Gamma_n+\frac{1}{2}}$$

Proof. In this case either $k = 1$ or $k = n$.

Remark 35. Let α, a and c_2 be as in Theorem 21. Then $p_n(d\alpha,c_2) \ge C n^a$ for $n = 1,2,\ldots$ where $C \ne C(n)$. This follows immediately from Theorems 21, 6.3.27 and from $\lambda_n^{-1}(d\alpha,x) = \sum_{k=1}^{n} \lambda_{kn}^{-1}(d\alpha)\ell_{kn}^2(d\alpha,x)$.

Lemma 36. Let supp($d\alpha$) be compact. Let there exist an interval τ such that the sequence $\{|p_n(d\alpha,x)|\}$ is uniformly bounded for $x \in \tau$. Then

$$\liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} > 0 .$$

Proof. By Theorem 7.5 $\tau \subset \operatorname{supp}(\alpha') \subset \operatorname{supp}(d\alpha)$. Let $\tau = [c_1,c_2]$ and $m_1 = m_1(n)$, $m_2 = m_2(n)$ be defined by $x_{m_1+1,n} < C_1 \le x_{m_1 n}$ and $x_{m_2,n} \le C_2 < x_{m_2-1,n}$ respectively where $x_{n+1,n} = -\infty$ $x_{0n} = \infty$. By Lemma 3.2.2 $\lim_{n\to\infty} x_{m_1,n} = c_1$ and $\lim_{n\to\infty} x_{m_2,n} = c_2$. We can suppose that α is continuous at c_1 and c_2. If not we can replace τ by a smaller interval. Let $m_2 < k \le m_1$. Then by Lemma 32

$$\ell_{k-1,n}(d\alpha, \frac{x_{k-1}+x_k}{2}) + \ell_{kn}(d\alpha, \frac{x_{k-1}+x_k}{2}) \ge 1 ,$$

that is

$$x_{k-1} - x_k \le 2\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} C^2[\lambda_{k-1,n}(d\alpha) + \lambda_{kn}(d\alpha)]$$

where $C = \sup_{n\geq 0} \max_{x\in\tau} |p_n(d\alpha,x)|$. Hence

$$x_{m_2} - x_{m_1} \le 4C^2 \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \sum_{k=m_2}^{m_1} \lambda_{kn}(d\alpha) .$$

Letting $n\to\infty$ we obtain

$$|\tau| \le 4C^2 \liminf_{n\to\infty} \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \int_\tau d\alpha(t) .$$

<u>Theorem 37.</u> Let $\mathrm{supp}(d\alpha) \subset [-1,1]$, $\mathrm{supp}(d\beta) \subset [-1,1]$, $\tau \subset (-1,1)$ and $c\tau = [-1,1]\setminus\tau$. Let $d\alpha(t) = d\beta(t)$ for $t\in\tau$, α be absolutely continuous in $c\tau$ and let there exist two polynomials π_1 and π_2 such that $\pi_1\alpha'/\beta' \in L^1_{d\alpha}(c\tau)$ and $|\pi_2(x)p_n(d\alpha,x)| \le K$ for $x\in c\tau$ and $n=1,2,\ldots$. Then

$$|p_n(d\beta,x)| \le C[|p_n(d\alpha,x)| + |p_{n-1}(d\alpha,x)|]$$

uniformly for $x\in\tau_1\subset\tau^o$ and $n=1,2,\ldots$.

<u>Proof.</u> We can suppose that neither π_1 nor π_2 has zeros in τ^o. Let $\rho = \pi_1\pi_2$, $\deg\rho = m$. Then

$$\rho(x)p_n(d\beta,x) = \int_{-1}^1 \rho(t)p_n(d\beta,t)K_{n+m+1}(d\alpha,x,t)d\alpha(t).$$

By the conditions $\int_{-1}^1(\ )d\alpha = \int_{-1}^1(\ )d\beta + \int_{c\tau}(\ )d\alpha - \int_{c\tau}(\ )d\beta$. Let $x\in\tau_1\subset\tau^o$. Then

$$\{\int_{c\tau}\rho(t)p_n(d\beta,t)K_{n+m+1}(d\alpha,x,t)d\beta(t)\}^2 \le$$

$$\le C[|p_{n+m}(d\alpha,x)| + |p_{n+m+1}(d\alpha,x)|]^2 \int_{-1}^1 \pi_1^2(t)d\beta(t)$$

and

$$\{\int_{c\tau}\rho(t)p_n(d\beta,t)K_{n+m+1}(d\alpha,x,t)d\alpha(t)\}^2 \le$$

$$\le C[|p_{n+m}(d\alpha,x)| + |p_{n+m+1}(d\alpha,x)|]^2 \cdot$$

$$\cdot \int_{c\tau} \rho^2(t)\frac{\alpha'(t)^2}{\beta'(t)}dt \int_{c\tau} p_n^2(d\beta,t)\beta'(t)dt .$$

Further for $n > m$

$$\int_{-1}^1 \rho(t)p_n(d\beta,t)K_{n+m+1}(d\alpha,x,t)d\beta(t) =$$

$$\int_{-1}^1 \rho(t)p_n(d\beta,t)\sum_{k=n-m}^{n+m}[p_k(d\alpha,x)p_k(d\alpha,t)]d\beta(t) .$$

Consequently

$$|\int_{-1}^1 \rho(t)p_n(d\beta,t)K_{n+m+1}(d\alpha,x,t)d\beta(t)| \le$$

$$\le C\sum_{k=n-m}^{n+m}|p_k(d\alpha,x)|\{\int_{-1}^1\rho(t)^2p_k^2(d\alpha,t)d\beta(t)\}^{\frac{1}{2}} .$$

But

$$\int_{-1}^1 \rho(t)^2p_k^2(d\alpha,t)d\beta(t) = \int_\tau \rho(t)^2p_k^2(d\alpha,t)d\alpha(t) +$$

$$+ \int_{c\tau}\rho(t)^2p_k^2(d\alpha,t)d\beta(t) \le C .$$

Thus we have proved that for $x\in\tau_1\subset\tau^o$

$$|p_n(d\beta,x)| \le C\sum_{k=n-m}^{n+m+1}|p_k(d\alpha,x)| .$$

By Lemma 36 the sequence $\left\{\frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)}\right\}$ is bounded from below by a positive constant. Thus by Lemma 3.3.1 and by the recurrence formula

$$|p_k(d\alpha,x)| \le C^*[|p_n(d\alpha,x)| + |p_{n-1}(d\alpha,x)|]$$

whenever $n\geq 1$, $n-k = O(1)$ and $x\in\Delta$ where $C^* = C^*(|n-k|,\alpha,\Delta)$.

<u>Remark 38.</u> Theorem 37 becomes useful if we combine it with Korous' theorem and with results in Freud and Geronimus.

Theorem 39. Theorem 37 remains valid if τ is of the form $[a,1]$ or $[-1,a]$ where $|a| < 1$ and $\tau_1 \subset (a,1]$ or $\tau_1 \subset [-1,a)$ respectively.

Proof. The same as that of Theorem 37.

10. Lagrange Interpolation.

First we shall consider the Lebesgue function $L_n(d\alpha,x)$ of Lagrange interpolation corresponding to α which is defined by

$$L_n(d\alpha,x) = \sum_{k=1}^{n} |\ell_{kn}(d\alpha,x)| .$$

The estimate

$$(1) \qquad L_n^2(d\alpha,x) \le \lambda_n^{-1}(d\alpha,x)[\alpha(\infty)-\alpha(-\infty)]$$

shows that if α has a jump at x then

$$L_n^2(d\alpha,x) \le \frac{\alpha(\infty)-\alpha(-\infty)}{a(x+0)-\alpha(x-0)}$$

but in general (1) is not a very strong result. Our aim will be to improve (1).

Lemma 1. Let $\mathrm{supp}(d\alpha)$ be compact and $\varepsilon > 0$. Then

$$\lambda_n(d\alpha,x)L_n^2(d\alpha,x) \le 2 \sum_{|x-x_{kn}|<\varepsilon} \lambda_{kn}(d\alpha) + C\,\varepsilon^{-2}\lambda_n(d\alpha,x)p_n^2(d\alpha,x)$$

for $x \in \Delta$, $n=1,2,\ldots$ where $C = C(\alpha,\Delta)$.

Proof. Repeat the proof of Lemma 8.6 and apply Lemma 3.3.1.

Theorem 2. Let $\alpha \in M(0,1)$. If α is continuous at $x \in [-1,1]$ then

$$(2) \qquad \lim_{n\to\infty} \lambda_n(d\alpha,x)\, L_n^2(d\alpha,x) = 0 .$$

If α is continuous on the closed set $\mathfrak{M} \subset (-1,1)$ then (2) holds uniformly for $x \in \mathfrak{M}$.

Proof. Apply Theorem 4.1.11, Lemmas 9.1, 9.7 and 1.

Theorem 3. Let $\alpha \in M(0,1)$, $\tau \subset [-1,1]$. If $[\alpha']^{-\varepsilon} \in L^1(\tau)$ with some $\varepsilon > 0$ then

(3) $$\lim_{n\to\infty} n^{-\frac{1}{2}} L_n(d\alpha,x) = 0$$

for almost every $x \in \tau$. If $\alpha'(t) \geq c > 0$ for almost every $t \in \tau$ and α is continuous on τ then (3) is satisfied uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. Repeat the reasoning in the proof of Theorem 8.9.

Corollary 4. Convergence theorems for $\mathrm{Lip}\frac{1}{2}$.

In the following we shall investigate $L_n^*(d\alpha,x)$ defined by

$$L_n^*(d\alpha,x)^2 = \frac{1}{n}\sum_{k=1}^{n} L_k^2(d\alpha,x).$$

If we can estimate $L_n^*(d\alpha,x)$ we can also estimate the Lebesgue function of the (C,1) means of Lagrange interpolating polynomials, which we denote by $\hat{L}_n(d\alpha,x)$, since obviously $\hat{L}_n(d\alpha,x) \leq L_n^*(d\alpha,x)$. Moreover, we can also estimate the convergence rate of the strong (C,1) means:

$$\sigma_n(d\alpha,f,x) = \frac{1}{n}\sum_{k=1}^{n} |f(x) - L_k(d\alpha,f,x)|.$$

Let $E_k(f)$ denote the best approximation of f by π_{k-1} in $C(\Delta(d\alpha))$. Then

$$\sigma_n(d\alpha,f,x) \leq \frac{1}{n}\sum_{k=1}^{n} [1 + L_k(d\alpha,x)]E_k(f).$$

Hence by Jackson's theorem

$$\sigma_n(f,d\alpha,x) \leq C \cdot L_n^*(d\alpha,x)\left\{\frac{1}{n}\int_{\frac{1}{n}}^{1} \frac{\omega_R(f,t)^2}{t^2}dt\right\}^{\frac{1}{2}}$$

where ω_R denotes the R-th modulus of smoothness of f.

Theorem 5. Let $\mathrm{supp}(d\alpha)$ be compact, $\tau \subset \mathrm{supp}(d\alpha)$, $\varepsilon > 0$ and $[\alpha']^{-\varepsilon} \in L^1(\tau)$. Then

(4) $$\limsup_{n\to\infty} n^{-\frac{1}{3}} L_n^*(d\alpha,x) < \infty$$

for almost every $x \in \tau$. If $\alpha \in \mathrm{Lip}_x 1$ and $\alpha'(t) \geq c > 0$ for $|x-t|$ small then (4) holds. If $\alpha \in \mathrm{Lip}\,1$ on τ and $\alpha'(t) \geq c > 0$ for $t \in \tau$ then (4) is satisfied uniformly for $x \in \tau_1 \subset \tau^o$.

Proof. For simplicity let us prove the first part of the Theorem. Let $x \in \tau_1 \subset \subset \tau^o$ and $\varepsilon = n^{-\frac{1}{3}}$. Then by Theorem 9.6 and Lemma 1

$$\lambda_n(d\alpha,x)L_n^2(d\alpha,x) \leq 2[\alpha(x+cn^{-\frac{1}{3}}) - \alpha(x-cn^{-\frac{1}{3}})] + $$
$$+ C_1 n^{\frac{2}{3}} \lambda_n(d\alpha,x)p_n^2(d\alpha,x).$$

Hence

$$L_n^*(d\alpha,x)^2 \leq$$
$$\leq 2\, n^{-\frac{1}{3}} \sum_{k=1}^{n} \frac{1}{k\,\lambda_k(d\alpha,x)} k^{\frac{1}{3}}[\alpha(x+ck^{-\frac{1}{3}}) - \alpha(x-ck^{-\frac{1}{3}})]$$
$$+ C_1 n^{-\frac{1}{3}} \lambda_{n+1}^{-1}(d\alpha,x).$$

Since α is almost everywhere differentiable we obtain from Theorem 6.3.25 that (3) holds for almost every $x \in \tau_1$. But $\tau_1 \subset \tau^o$ is arbitrary.

Let us note that the second part of Theorem 5 is not new. (See Freud, Some unsolved problems.)

Corollary 6. Convergence of (C,1) and strong (C,1) means of Lagrange interpolation polynomials for $f \in \mathrm{lip}\frac{1}{3}$.

Corollary 7. If $\mathrm{supp}(d\alpha)$ is compact and $[\alpha']^{-\varepsilon} \in L^1(\tau)$ with some $\varepsilon > 0$ then

$$\liminf_{n\to\infty} n^{-\frac{1}{3}} L_n(d\alpha,x) < \infty$$

for almost every $x \in \tau$.

Theorem 8. Let $\alpha \in S$ and $x \in [-1,1]$. Then

$$|p_n(d\alpha,x)| \leq 4\sqrt{\frac{2}{\pi}}$$ [illegible] $$(d\alpha,0)^{-1} L_n(d\alpha,x)$$

for $n=1,2,\ldots$.

Proof. We obtain from Lemma 6.1.19 and from the inequality between the arithmetic and geometric means that

$$\gamma_{n-1}(d\alpha) \le 2^{n-1}\sqrt{\tfrac{2}{\pi}}\; D(d\alpha,0)^{-1}.$$

Further

$$\frac{2^{n-2}}{\gamma_{n-1}(d\alpha)} = \int_{-1}^{1} T_{n-1}(t)p_{n-1}(d\alpha,t)d\alpha(t) \le$$

$$\le 2\sum_{k=1}^{n} \lambda_{kn}(d\alpha)\frac{|p_{n-1}(d\alpha,x_{kn})|}{|x-x_{kn}|}$$

for $-1 \le x \le 1$. Hence the Theorem follows.

Remark 9. In general Theorem 8 cannot be improved. Let $w \sim GJ$ with $\Gamma_1 > 0$. Then $w \in S$ and

$$\left\{\sum_{k=1}^{n} \lambda_{kn}(w)\frac{|p_{n-1}(w,x_{kn})|}{1-x_{kn}}\right\}^2 \le$$

$$\le 2\sum_{k=1}^{n}\lambda_{kn}(w)\frac{p_{n-1}^2(w,x_{kn})}{1-x_{kn}^2}\sum_{k=1}^{n}\lambda_{kn}(w)(1-x_{kn})^{-1}$$

which is bounded by Theorems 4.2.3, 6.3.28 and 9.22.

Definition 10. Let $x_{0n}(d\alpha) = \infty$, $x_{n+1,n}(d\alpha) = -\infty$, $\ell_{0n}(d\alpha,x) = \ell_{n+1,n}(d\alpha,x) = 0$. Let $m = m(n,x)$ be defined by $x_{m+1,n}(d\alpha) < x \le x_{mn}(d\alpha)$. Then we put

$$\tilde{L}_n(d\alpha,x) = \sum_{\substack{k=1\\ k\ne m,m+1}}^{n} |\ell_{kn}(d\alpha,x)|.$$

Lemma 11. Let α be an arbitrary weight. Then

$$L_n(d\alpha,x) - 1 \sim \tilde{L}_n(d\alpha,x)$$

for $x \in \mathbb{R}$ and $n=1,2,\ldots$.

Proof. By Lemma 9.32

$$\tilde{L}_n(d\alpha,x) + 1 \le L_n(d\alpha,x).$$

On the other hand, since $\ell_{mn}(d\alpha,x) \ge 0$ and $\ell_{m+1,n}(d\alpha,x) \ge 0$, we have

$$L_n(d\alpha,x) = \tilde{L}_n(d\alpha,x) + 1 - \sum_{\substack{k=1\\ k\ne m,m+1}}^{n} \ell_{kn}(d\alpha,x) \le 1 + 2\tilde{L}_n(d\alpha,x).$$

Recall that GJ and w_n have been defined in Definition 9.28.

Theorem 12. Let $w \in GJ$. Then

(5) $$L_n(w,x) - 1 \sim$$

$$\sim \frac{n|x-x_j|}{\sqrt{1-x^2}+\frac{1}{n}} \int_{n|x-t| > \sqrt{1-x^2}+\frac{1}{n}} \left\{\frac{w_n(t)(\sqrt{1-t^2}+\frac{1}{n})}{w_n(x)(\sqrt{1-x^2}+\frac{1}{n})}\right\}^{\frac{1}{2}}\frac{1}{|x-t|}\,dt$$

for $-1 \le x \le 1$ where x_j denotes the zero of $p_n(w)$ closest to x.

Proof. The Theorem follows by calculation from Theorems 6.3.28, 9.22, 9.31, 9.33 and Lemma 11.

For the case when w is a Jacobi weight Theorem 12 has been proved by Natanson [10]. The integral on the right hand side of (5) is a rather standard one, it can easily be estimated but the final formula is so complicated that we shall omit it. We shall formulate only one particular case as

Corollary 13. Let $w \in GJ$. Then

$$L_n(w,1) \sim \begin{cases} 1 & \text{for } -1 < \Gamma_1 < -\frac{1}{2} \\ \log n & \text{for } \Gamma_1 = -\frac{1}{2} \\ n^{\Gamma_1+\frac{1}{2}} & \text{for } \Gamma_1 > -\frac{1}{2}. \end{cases}$$

Before finding necessary conditions for mean boundedness of Lagrange interpolation processes let us make some remarks. If we define $L_n(d\alpha)$ by $L_n(d\alpha)f = L_n(d\alpha,f)$ then the norm of $L_n(d\alpha)$ as a mapping from some L^q $(0 < q < \infty)$ is never bounded, f must always be bounded in $\Delta(d\alpha)^{o}$. To avoid complication, which we cannot solve at the present time, we shall assume that f is bounded on $\Delta(d\alpha)$ and we shall write $f \in L^{\infty}(\Delta(d\alpha))$ where

$\|f\|_\infty = \sup_{t \in \Delta(d\alpha)} |f(t)|$. An important difference between Fourier sums and Lagrange interpolation polynomials is that $L_{n+1}(d\alpha, f) - L_n(d\alpha, f)$ is not proportional to $p_n(d\alpha)$. If we write

$$L_n(d\alpha, f, x) = \sum_{k=0}^{n-1} a_k\, p_k(d\alpha, x)$$

and introduce the notation

$$L_{n,k}(d\alpha, f, x) = \sum_{j=0}^{k-1} a_j\, p_j(d\alpha, x)$$

for $1 \le k \le n-1$ then

$$L_n(d\alpha, f, x) - L_{n,n-1}(d\alpha, f, x) = a_{n-1}\, p_{n-1}(d\alpha, x)$$

where obviously

$$a_{n-1} = \sum_{k=1}^{n} \lambda_{kn}(d\alpha)\, f(x_{kn})\, p_{n-1}(d\alpha, x_{kn}) .$$

Theorem 14. Let either $\alpha \in S$ or α satisfy the conditions of Corollary 9.13. Let β be an arbitrary weight. Let us consider the following three conditions.

(i) $\sup_{n \ge 1} \|L_n(d\alpha)\|_{L^\infty(\Delta(d\alpha)) \to L^p_{d\beta}} < \infty$,

(ii) $\sup_{n \ge 2} \|L_{n,n-1}(d\alpha)\|_{L^\infty(\Delta(d\alpha)) \to L^p_{d\beta}} < \infty$

and

(iii) $\sup_{n \ge 1} \|p_{n-1}(d\alpha)\|_{d\beta, p} < \infty$

where $p \in (0,\infty)$ is given. Then each pair of (i)-(iii) implies the third one.

Proof. Apply Lemma 9.9 and Corollary 9.13.

The following Theorem is one of our main results.

Theorem 15. Let $\alpha \in S$, $0 < p < \infty$ and $w(\ge 0) \in L^1(-1,1)$ Then from

$$\liminf_{n \to \infty} \|L_n(d\alpha)\|_{L^\infty(-1,1) \to L^p_w} < \infty$$

follows

(6) $$\int_{-1}^{1} [\alpha'(t)\sqrt{1-t^2}\,]^{-\frac{p}{2}}\, w(t)dt < \infty .$$

Proof. Let $\delta = \delta(d\alpha) > 0$ be defined by Theorem 9.10. Let $\tau \subset [-1,1]$ with $|\tau| = \frac{\delta}{2}$. Then we can find a system $\Omega = \{\tau_1, \tau_2\}$ with $\tau_1 \tau_2 = \phi$, $|\Omega| > 2-\delta$ and $\mathrm{dist}(\tau, \Omega) > 0$ such that (9.7) is satisfied. Let f be a function on $[-1,1]$ which satisfies the conditions $\|f\|_\infty = 1$ and

$$f(x_{kn}) = 1_\Omega(x_{kn})\,\mathrm{sign}[p_{n-1}(d\alpha, x_{kn})(x_{kn} - B)]$$

where B is the center of τ. Of course f depends on n, τ, Ω, and α.

We have

$$\|1_\tau L_n(d\alpha, f)\|_{w,p} \le \|L_n(d\alpha)\|_{L^\infty \to L^p_w} .$$

Since $|x - x_{kn}| \le 2$ for $x \in \tau$, $x_{kn} \in \Omega$ we obtain

$$\|1_\tau p_n(d\alpha)\|_{w,p} \sum_{x_{kn} \in \Omega} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha, x_{kn})| \le$$

$$\le 2 \frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} \|L_n(d\alpha)\|_{L^\infty \to L^p_w} .$$

Letting $n \to \infty$ we receive from Lemma 4.2.2 and Theorem 9.10 that

$$\liminf_{n \to \infty} \|1_\tau p_n(d\alpha)\|_{w,p} < \infty .$$

Hence by Theorem 7.32

$$\int_\tau [\alpha'(t)\sqrt{1-t^2}\,]^{-\frac{p}{2}}\, w(t)dt < \infty .$$

Since this inequality holds for every $\tau \subset [-1,1]$ with $|\tau| = \frac{\delta}{2}$ and $\delta = \delta(d\alpha) > 0$ it also holds if $\tau = [-1,1]$.

Using the results of sections 7 and 9 we can prove similar theorems when $\alpha \notin S$. We restrict ourselves to the following

Theorem 16. Let $\mathrm{supp}(d\alpha) = [-1,1]$, $\alpha'(x) > 0$ for almost every $x \in [-1,1]$ and let there exist an interval $\tau \subset [-1,1]$ such that the sequence $\{|p_n(d\alpha, x)|\}$

is uniformly bounded for $x \in \tau$. Let $w(\geq 0) \in L^1(-1,1)$. If $0 < p < \infty$ and

(7) $$\limsup_{n\to\infty} \|L_n(d\alpha)\|_{L^\infty(-1,1)\to L^p_w} < \infty$$

then

$$\limsup_{n\to\infty} \|p_n(d\alpha)\|_{w,p} < \infty .$$

If $p \geq 2$ and (7) holds then (6) is satisfied.

__Proof.__ By the conditions $\{\|1_\tau\, p_n(d\alpha)\|_{w,p}\}$ is bounded. Let $c\tau = [-1,1]\setminus\tau$. Let f be defined by $\|f\|_\infty = 1$ and

$$f(x_{kn}) = 1_\tau(x_{kn}) \operatorname{sign} p_{n-1}(d\alpha, x_{kn}) .$$

Then

$$\|1_{c\tau}\, p_n(d\alpha)\|_{w,p} \cdot \sum_{x_{kn}\in\tau} \lambda_{kn}(d\alpha)|p_{n-1}(d\alpha,x_{kn})| \leq$$

$$\leq 2\frac{\gamma_n(d\alpha)}{\gamma_{n-1}(d\alpha)} \|L_n(d\alpha)\|_{L^\infty\to L^p_w} .$$

Now the Theorem follows from Theorem 7.31, Corollary 9.13 and Lemma 9.36.

__Definition 17.__ Let us say that α just belongs to S ($\alpha \in$ JS) if $\alpha \in S$ but for every $\varepsilon > 0$ $[\alpha']^{-\varepsilon} v \notin L^1$. Example: $\alpha'(x) = \exp\{-(1-x^2)^{-\delta}\}$ $(0 < \delta < \frac{1}{2})$ or $\alpha'(x) = \exp\{-|x|^{-\delta}\}$ $(0 < \delta < 1)$.

__Corollary 18.__ Let either $\alpha \in$ JS or α be a Pollaczek weight or α be defined by

$$\alpha'(t) = \varphi(t)\exp\{-(1-t^2)^{-\frac{1}{2}}\},$$

$\varphi(>0) \in$ Lip1, $\operatorname{supp}(d\alpha) = [-1,1]$ and α is absolutely continuous. Then for every $p > 2$ there exists a function $f \in C[-1,1]$ such that

$$\limsup_{n\to\infty} \int_{-1}^{1} |f(t) - L_n(d\alpha,f,t)|^p\, \alpha'(t)dt > 0 .$$

__Proof.__ Theorems 15, 16 and Banach-Steinhaus' theorem.

Let us remark that Corollary 18 gives a more or less complete answer to Turán's problem and solves Askey's conjecture ([1]). Let us recall that Turán asked if there exists a weight α with $\operatorname{supp}(d\alpha) = [-1,1]$ such that the conclusion of Corollary 18 holds and Askey conjectured that the Pollaczek weight solves Turán's problem.

__Theorem 19.__ Let $\operatorname{supp}(d\alpha) \subset [-1,1]$, $0 < p < \infty$, $w(\geq 0) \in L^1(-1,1)$. Let the sequence $n_1 < n_2 ..$ be given. If for every $f \in C[-1,1]$

$$\lim_{k\to\infty} \int_{-1}^{1} |L_{n_k}(d\alpha,f,x) - f(x)|^p w(x)dx = 0$$

then

(8) $$\limsup_{k\to\infty} \|L_{n_k}(d\alpha)\|_{L^\infty\to L^p_w} < \infty$$

__Proof.__ If $p \geq 1$ then the Theorem follows from Banach-Steinhaus' theorem. Now let $0 < p < 1$. Let us define the functionals $\varphi_k : C[-1,1] \to \mathbb{R}$ by

$$\varphi_k(f) = \int_{-1}^{1} |L_{n_k}(d\alpha,f,x) - f(x)|^p w(x)dx .$$

Then $\varphi_k(f+g) \leq \varphi_k(f) + \varphi_k(g)$, $\varphi_k(\lambda f) = |\lambda|^p \varphi_k(f)$, $\varphi_k(f) \geq 0$ and $\lim_{k\to\infty} \varphi_k(f) = 0$ for every $f, g \in C$. Suppose there exists a subsequence $k_1 < k_2 < \ldots$ such that

$$c_j = \sup_{\|f\|^p_C \leq 1} \varphi_{k_j}(f) \underset{j\to\infty}{\to} \infty$$

Let us put $j_1 = 1$ and find a function $f_1 \in C$ such that $\|f_1\|_C \leq 1$ and $\varphi_{k_1}(f_1) \geq \frac{1}{2}c_1$. Then there exists a number $j_2 > j_1$ such that for each $j \geq j_2$ $\varphi_{k_j}(f_1) \leq 1$. Now we find a function $f_2 \in C$ such that $\varphi_{k_{j_2}}(f_2) \geq \frac{1}{2}c_{j_2}$ and $\|f_2\|_C \leq 1$. After we choose $j_3 > j_2$ so that $\varphi_{k_j}(f_2) < 1$ for every $j \geq j_3$. Continuing this process we build up two sequences $\{j_\ell\}_{\ell=1}^{\infty}$ and $\{f_\ell\}_{\ell=1}^{\infty}$

so that $f_\ell \in C$, $\|f_\ell\|_C \le 1$, $\varphi_{k_{j_\ell}}(f_\ell) \ge \frac{1}{2} c_{j_\ell}$ and $\varphi_{k_{j_\ell}}(f_m) \le 1$ for $m = 1, 2, \ldots, \ell - 1$. Let us chouse a subsequence $j_{\ell_1} < j_{\ell_2} < \ldots$ such that $\sum_{\nu=1}^{\infty} c_{j_\ell}^{-p} \le 1$ and $c_{j_{\ell_m}} \left\{ \sum_{\nu=m+1}^{\infty} c_{j_{\ell_\nu}}^{-1} \right\}^p \le 1$ for $m = 1, 2, \ldots$. Let $f = \sum_{\nu=1}^{\infty} c_{j_{\ell_\nu}}^{-1} f_{\ell_\nu}$. Then $f \in C$. Further for $m \ge 1$

$$\varphi_{k_{j_{\ell_m}}}(f) \ge c_{j_{\ell_m}}^{-p} \varphi_{k_{j_{\ell_m}}}(f_{\ell_m}) - \sum_{\nu=1}^{m-1} c_{j_{\ell_\nu}}^{-p} \varphi_{k_{j_{\ell_\nu}}}(f_{\ell_\nu}) - c_{j_{\ell_m}} \left\{ \sum_{\nu=m+1}^{\infty} c_{j_{\ell_\nu}}^{-1} \right\}^p .$$

Hence

$$\varphi_{k_{j_{\ell_m}}}(f) \ge \tfrac{1}{2} c_{j_{\ell_m}}^{1-p} - 2 .$$

Letting $m \to \infty$ we obtain

$$\limsup_{k \to \infty} \varphi_k(f) = \infty .$$

The contradiction show that

$$\limsup_{k \to \infty} \sup_{\|f\|_C^p \le 1} \varphi_k(f) < \infty$$

which is equivalent to (8) .

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Department of Mathematics and
Mathematics Research Center
University of Wisconsin-Madison
Madison, Wisconsin 53706

and

Department of Mathematics
The Ohio State University
Columbus, Ohio 43210.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The purpose of the present paper is to improve some results of R. Askey, P. Erdös, G. Freud, L. Ya. Geronimus, U. Grenander, G. Szegö and P. Turan on orthogonal polynomials, Christoffel functions, orthogonal Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. In particular, ~~Turan's~~ the problem will (positively) be answered: is there any weight w with compact support such that for each $p > 2$ the Lagrange interpolating polynomials corresponding to w diverge in L_w^p for some continuous function f? Most of the paper deals with Christoffel functions and their applications. Many limit relations for orthogonal polynomials are found in the assumption that the coefficients in the recursion formula behave nicely.

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