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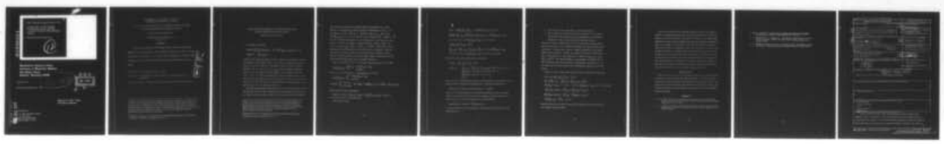

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ON THE DUAL OF THE LINEARLY CONSTRAINED MULTI-FACILITY LOCATION PROBLEM WITH ARBITRARY NORMS

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ON THE DUAL OF THE LINEARLY CONSTRAINED MULTI-FACILITY LOCATION PROBLEM WITH ARBITRARY NORMS†

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ABSTRACT

In this article the dual of the multi-facility location problem with arbitrary norms is developed. The formulation allows any number of linear constraints in the primal. It is shown that the multipliers associated with the linear equations of the dual are the optimal facility locations of the primal.

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*Visiting Professor in the Department of Management Sciences, University of Waterloo, Waterloo, Ontario in 1976-77.

ON THE DUAL OF THE LINEARLY CONSTRAINED MULTI-FACILITY LOCATION PROBLEM WITH ARBITRARY NORMS†

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We consider the problem

$$\text{minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} w_{1ij} K_{1ij}(x_j - a_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{2ij} K_{2ij}(x_i - x_j)$$

$$\text{subject to } \sum_{j=1}^{n} A_j x_j \leq b.$$

m and n are the number of existing and new facilities, respectively, and d is the dimension of the facility space. w_{1ij} is the non-negative weight and $K_{1ij}(\cdot)$ is the norm on R^d, to be used between existing facility i and new facility j for $1 \leq i \leq m$, $1 \leq j \leq n$. w_{2ij} is the non-negative weight and $K_{2ij}(\cdot)$ is the norm on R^d, to be used between new facilities i and j for $1 \leq i < j \leq n$. $a_i \in R^d$ is the location of existing facility i for $1 \leq i \leq m$, and $x_j \in R^d$ is the unknown location of new facility j for $1 \leq j \leq n$. ℓ linear constraints on the locations of the new facilities are expressed using n $\ell \times d$ matrices A_j for $1 \leq j \leq n$ and the ℓ-vector b.

This article generalizes and synthesizes some results obtained by Love [1] and Planchart and Hurter [4]. Using elements from conjugate function theory as exposited by Witzgall [5], we shall find the dual of the location problem.

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Let superscript c denote the conjugate function corresponding to a convex function, and superscript o denote the polar corresponding to a norm. It is convenient to let $x^T = (x_1^T, x_2^T, \ldots, x_n^T)$ and $A = (A_1, A_2, \ldots, A_n)$. Then $Ax = \sum_{j=1}^{n} A_j x_j$. For $1 \le i \le m$, and $1 \le j \le n$, let $B_{1ij} = (\overset{1}{0},0,\ldots 0,\overset{j}{I},0,0,\ldots,\overset{n}{0})$ be a d x nd matrix consisting of n d x d submatrices, each of which is a null-matrix, except the <u>jth</u>, which is an identity matrix. Then $x_j = B_{1ij}x$. For $1 \le i < j \le n$, let $B_{2ij} = (\overset{1}{0},0,\ldots,0,\overset{i}{I},0,0,\ldots,0,\overset{j}{-I},0,0\ldots,\overset{n}{0})$ be a d x nd matrix consisting of n d x d submatrices, each of which is a null-matrix, except the <u>ith</u>, which is an identity matrix and the <u>jth</u>, which is a negative identity matrix. With this notation, we can write down a number of functions occurring in the location problem and find the conjugate functions.

For $1 \le i \le m$, and $1 \le j \le n$, let $f_{1ij}(x) = w_{1ij}K_{1ij}(B_{1ij}x - a_i)$, so we have

$$f_{1ij}^c(B_{1ij}^T y_{1ij}) = \begin{cases} a_i^T y_{1ij} & \text{if } K_{1ij}^o(y_{1ij}) \le w_{1ij} \\ \infty & \text{otherwise} \end{cases}$$

For $1 \le i < j \le n$ let $f_{2ij}(x) = w_{2ij}K_{2ij}(B_{2ij}x)$, so we have

$$f_{2ij}^c(B_{2ij}^T y_{2ij}) = \begin{cases} 0 & \text{if } K_{2ij}^o(y_{2ij}) \le w_{2ij} \\ \infty & \text{otherwise} \end{cases}$$

Let $f_3(x) = \begin{cases} 0 & \text{if } Ax \le b \\ \infty & \text{otherwise} \end{cases}$. Then $f_3^c(z_3) = \sup\{z_3^T x : Ax \le b\} = \inf\{b^T y_3 : A^T y_3 = z_3, y_3 \ge 0\}$

by the theory of linear programming.

Finally, let $f(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} f_{1ij}(x) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{2ij}(x) + f_3(x)$, so the location problem is: minimize $f(x)$.

For the dual we get:

$$f^c(0) = \inf\{\textstyle\sum_{i=1}^m \sum_{j=1}^n a_i^T y_{1ij} + \inf\{b^T y_3 : A^T y_3 = z_3,\ y_3 \geq 0\}:$$

$$\textstyle\sum_{i=1}^m \sum_{j=1}^n B_{1ij}^T\ y_{1ij} + \sum_{i=1}^{n-1} \sum_{j=i+1}^n B_{2ij}^T\ y_{2ij} + z_3 = 0;\ K_{1ij}^o(y_{1ij}) \leq w_{1ij},$$

$$1 \leq i \leq m,\ 1 \leq j \leq n;\ K_{2ij}^o(y_{2ij}) \leq w_{2ij},\ 1 \leq i < j \leq n\} =$$

$$\inf\{\textstyle\sum_{i=1}^m \sum_{j=1}^n a_i^T\ y_{1ij} + b^T y_3:$$

$$\textstyle\sum_{i=1}^m y_{1ij} - \sum_{i=1}^{j-1} y_{2ij} + \sum_{i=j+1}^n y_{2ij} + A_j^T y_3 = 0,\ 1 \leq j \leq n;\ K_{1ij}^0(y_{1ij}) \leq w_{1ij},$$

$$1 \leq i \leq m,\ 1 \leq j \leq n;\ K_{2ij}^o(y_{2ij}) \leq w_{2ij},\ 1 \leq i < j \leq n,\ y_3 \leq 0\}.$$

The dual of the location problem may be written as

$$\text{maximize } -\textstyle\sum_{i=1}^m \sum_{j=1}^n a_i^T\ y_{1ij} - b^T y_3$$

$$\begin{aligned} \text{subject to} \quad & \textstyle\sum_{i=1}^m y_{1ij} - \sum_{i=1}^{j-1} y_{2ij} + \sum_{i=j+1}^n y_{2ij} + A_j^T y_3 = 0 \text{ for } 1 \leq j \leq n, \\ & K_{1ij}^o(y_{1ij}) \leq w_{1ij} \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n, \\ & K_{2ij}^o(y_{2ij}) \leq w_{2ij} \text{ for } 1 \leq i < j \leq n, \\ & y_3 \geq 0. \end{aligned}$$

This dual may be transformed back into the location problem, using differentiable duality theory as exposited by Luenberger [3]. Consider the problem (P):

$$\text{maximize } p(y) \quad \text{subject to } g(y) \leq 0,\ h(y) = 0 \quad (y \in R^N)$$

where $p(\cdot)$ is a differentiable concave real-valued function, $g(\cdot)$ is a differentiable vector-valued function with convex component functions and $h(\cdot)$ is an affine function. A dual of problem (P) is this problem (D):

$$\text{minimize } d(x) = \max\{p(y) + x^T h(y) : g(y) \leq 0\}.$$

In a similar manner to the development by Luenberger [3], we may arrive at the following facts:

1. $p(y) \leq d(x)$ for all y with $g(y) \leq 0$, $h(y) = 0$ and all x.

2. Let a constraint qualification hold for problem (P). If y_o solves problem (P) with multiplier vector x_o corresponding to the equality constraints, then x_o solves problem (D) and $p(y_o) = d(x_o)$.

These facts cannot be applied directly to the dual of the location problem, since the functions in the nonlinear constraints are norms and thus not differentiable. For many norms, however, a simple transformation will yield a differentiable function. Consider, for instance, the norm K on R^d given by $K(x) = \ell_p(C_x)$ where C is a non-singular d x d matrix and $\ell_p(z_1, z_2, \ldots, z_d) = (\sum_{i=1}^{d} |z_i|^p)^{1/p}$. The inequality $K(x) \leq w$ is equivalent to $(K(x))^p \leq w^p$, in which inequality the left-hand side is a differentiable function for $p > 1$. For simplicity, we do not exhibit such transformations explicitly in the following.

Taking the dual of the location problem as (P), we obtain for problem (D):

$$d(x) = \max\{-\sum_{i=1}^{m} \sum_{j=1}^{n} a_i^T y_{1ij} - b^T y_3 +$$

$$\sum_{j=1}^{n} x_j^T(\sum_{i=i}^{m} y_{1ij} - \sum_{i=1}^{j-1} y_{2ij} + \sum_{i=j+1}^{n} y_{2ji} + A_j^T y_3):$$

$$K_{1ij}^o(y_{1ij}) \leq w_{1ij},\ 1 \leq i \leq m,\ i \leq j \leq n;\ K_{2ij}^o(y_{2ij}) \leq w_{2ij},\ 1 \leq i < j \leq n,\ y_3 \geq 0\}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \max\{(x_j - a_i)^T y_{1ij} : K_{1ij}^o(y_{1ij}) \leq w_{1ij}\} +$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \max\{(x_i - x_j)^T y_{2ij} : K_{2ij}^o(y_{2ij}) \leq w_{2ij}\} +$$

$$\max\{(\sum_{j=1}^{n} A_j x_j - b)^T y_3 : y_3 \geq 0\}.$$

Using the properties of polars of norms [5], problem (D) is thus identical to the original location problem.

The location problem can be solved by solving its dual by some standard nonlinear programming algorithm, the optimal locations being the optimal multiplier vectors corresponding to the equality constraints. Our computational experience, however, seems to indicate that, from the viewpoint of computation time, it is somewhat more efficient to solve the location problem directly, using differentiable approximations to the norms involved [2]. (Some may find the dual form easier to program, however, due to the simple nature of the derivatives of the objective function and constraints.) The special structure of the dual can be exploited to solve it using decomposition as suggested by Love [1] and Planchart and Hurter [4]. Even this technique for solving the location problem seems less efficient than either solving the primal form or solving the dual directly with a nonlinear programming routine.

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