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REGRESSION WITH GIVEN MARGINALS. (U)

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REGRESSION WITH GIVEN MARGINALS

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ABSTRACT

We consider the class of regression functions $\mathcal{M}(F, G) = \{m(x) = E[Y|X = x], (X, Y) \in \Pi(F, G)\}$ where $\Pi(F, G)$ denotes the set of random vectors with marginal distributions F and G. A characterization of $\mathcal{M}(F, G)$ is given together with a representation for the projection operator it induces in an appropriate Hilbert space. Applications are indicated.

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REGRESSION WITH GIVEN MARGINALS

Richard A. Vitale

1. Introduction

Let $\Pi(F, G)$ denote the class of random vectors (X, Y) with marginal distributions F and G $(X \sim F,\ Y \sim G)$. We will consider the associated class of regression functions

$$\mathcal{M}(F, G) = \{m(x) = E[Y | X = x],\ (X, Y) \in \Pi(F, G)\} .$$

The motivation for looking at this class is similar in spirit to that of isotonic regression (from which we will in fact borrow a result): the extent to which auxiliary information be incorporated into the regression process. Knowledge of marginal distributions, in particular, is natural in certain types of problems. We may consider a census in which bivariate observations are collected, the marginal distributions are assumed given (as from a previous survey), and regression is desired. Alternatively, there is the problem of optimal, non-linear prediction in a time series $\{X_i\}$. If F is the equilibrium distribution of the X_i, then the optimal one-step predictor (squared error loss) is $E[X_{i+1} | X_i = x] \in \mathcal{M}(F, F)$ (see [3], [5], [6] for related discussions of this problem).

In section 2, we present a characterization of $\mathcal{M}(F, G)$ for a large class of F and G. The proof follows directly

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from methods in [10]. Characterizations of the type indicated have been investigated from a variety of points of view and we refer the reader to [7], [9] for other discussions and references. It can be fairly stated that the common ancestor of all such approaches is the fertile theorem of Hardy, Littlewood and Pólya [4, p. 49] on the averaging properties of doubly stochastic matrices. In section 3, we investigate further the structure of $\mathcal{M}(F, G)$ by considering it as a convex subset of an appropriate Hilbert space and examining the induced projection operator. The discussion is motivated by a statistical estimation problem.

2. Characterization of $\mathcal{M}(F, G)$

In what follows we shall regard F and G as fixed and satisfying

(A1) F and G are each supported on all of R^1 and are invertible.

(A2) $EY^2 = \int_{-\infty}^{+\infty} y^2 G(dy) < \infty$.

The first assumption can be weakened considerably, but we present it to avoid side-issues. The second insures that $\mathcal{M}(F, G)$ is a subset of $L_2[(-\infty, +\infty); F]$, the Hilbert space of real-valued functions on R^1 square integrable with respect to the measure determined by F (this can be seen directly by noting $EY^2 = E_X E[Y^2|X] \geq E_X(E[Y|X])^2$).

Turning to the characterization of $\mathcal{M}(F, G)$, we note that if $m(x) = E[Y|X = x] \in \mathcal{M}(F, G)$, *then with the application* of marginal probability transformations $U = F(X)$, $V = G(Y)$, we have $m(x) = E[G^{-1}(V)|U = F(x)]$, where U and V are each uniformly distributed on $[0, 1]$. This is essentially the object of study of [10] and with only minor modifications, the methods employed there yield the following result.

Theorem 1. The following statements are equivalent.

(i) $m \in \mathcal{M}(F, G)$.

(ii) m lies in the closed convex hull $(L_2[(-\infty, +\infty); F])$ of functions of the form $G^{-1} \circ T \circ F$.

(iii) $\int_0^x m(F^{-1}(T(u)))du \geq \int_0^x G^{-1}(u)du$

for all $x \in [0, 1]$ (with equality at $x = 1$) and all $T \in \mathcal{T}$.

Here $\mathcal{T} = \{T : [0,1] \to [0,1]$ one-one, Borel-measurable, measure-preserving$\}$. We note that if $m \circ F^{-1}$ is non-decreasing, then the strongest inequality in (iii) occurs upon taking $T(u) = u$, i.e.,

$$\int_0^x m(F^{-1}(u))du \geq \int_0^x G^{-1}(u)du \ .$$

The equality condition in (iii) amounts to $\int_{-\infty}^{+\infty} m(x)F(dx) = \int_{-\infty}^{+\infty} yG(dy)$ or $Em(X) = EY$. Finally, for the projection problem it will be useful to note that the mapping $h \in L_2[(-\infty,+\infty);F] \to h \circ F^{-1} \in L_2[[0,1]; \mu =$ Lebesgue measure$]$ induces an isomorphism between the two spaces. The image $\mathcal{M}_0$ of $\mathcal{M}(F,G)$ under the mapping can be described as follows.

Corollary. The following are equivalent.

(i) $m_0 \in \mathcal{M}_0$.

(ii) m_0 lies in the closed convex hull $(L_2[[0,1];\mu])$ of functions of the form $G^{-1} \circ T$.

(iii) $\int_0^x m_0(T(u))du \geq \int_0^x G^{-1}(u)du$

for all $x \in [0,1]$ (with equality at $x = 1$) and all $T \in \mathcal{T}$.

Proof. Change of variables.

Remark. From (ii), it is evident that for each $T \in \mathcal{T}$, $m_0 \in \mathcal{M}_0 \iff m_0 \circ T \in \mathcal{M}_0$.

3. Projection

Under the assumption $(X, Y) \in \Pi(F, G)$, a natural criterion for judging an estimate $\hat{m}(x)$ of the unknown regression function $m(x)$ is the squared error loss

$$E[m(x) - \hat{m}(x)]^2 = \int_{-\infty}^{+\infty} [m(x) - \hat{m}(x)]^2 F(dx) .$$

It is evident that this loss can be reduced (or at least made no larger) by constructing a new estimate $\tilde{\hat{m}}(x)$ which is the projection of $\hat{m}$ onto the convex $\mathcal{m}(F, G)$. For this reason, it is of interest to investigate the projection operator associated with $\mathcal{m}(F, G)$ in $L_2[(-\infty, +\infty);F]$: that is, for $h \in L_2[(-\infty, +\infty);F]$, we seek the (unique) element $\tilde{h} \in \mathcal{m}(F, G)$ which yields

$$\int_{-\infty}^{+\infty} [h(x) - \tilde{h}(x)]^2 F(dx) = \inf_{m \in \mathcal{m}(F, G)} \int_{-\infty}^{+\infty} [h(x) - m(x)]^2 F(dx)$$

($\sim$ throughout will denote projection in the appropriate space). A feature of this projection is that if a constant is added to h, then $\tilde{h}$ remains the same: this can be seen by expanding

$$\int_{-\infty}^{+\infty} [h(x) + c - m(x)]^2 F(dx) = \int_{-\infty}^{+\infty} [h(x) - m(x)]^2 F(dx)$$

$$+ c^2 + 2c \int_{-\infty}^{+\infty} h(x)F(dx)$$

$$- 2c \int_{-\infty}^{+\infty} m(x)F(dx)$$

and noting that the first term alone depends on m since, as we have noted, $\int_{-\infty}^{+\infty} m(x)F(dx) \equiv \int_{-\infty}^{+\infty} yG(dy)$ for $m \in \mathcal{M}(F, G)$. This being the case, we shall have occasion to invoke the normalization

$$\text{(A3)} \quad \int_{-\infty}^{+\infty} h(x)F(dx) = \int_{-\infty}^{+\infty} yG(dy)$$

and, equivalently, for $\ell = h \circ F^{-1}$

$$\text{(A3)}' \quad \int_0^1 \ell(u)du = \int_0^1 G^{-1}(u)du \ .$$

We now investigate the projection operator, isolating the main aspects of the argument in two lemmas. Some notation will prove to be convenient: let $I(x) = \int_0^x G^{-1}(u)du$ and let capitalization generally indicate integration, e.g. $L(x) = \int_0^x \ell(u)du$. If $A(x) \in C[0,1]$, then denote by $A^*(x)$ the convex minorant of A (i.e. the greatest convex function less than or equal to A).

<u>Lemma</u>. Let $\ell \in L_2[[0,1];\mu]$ be non-decreasing (a.e.) and satisfy (A3)'. The projection $\tilde{\ell}$ of ℓ onto $\mathcal{M}_0$ satisfies

$$\tilde{L}(x) = \int_0^x \tilde{\ell}(u)du = L(x) - (L - I)^*(x) \ .$$

<u>Proof</u>. The proof will be given first for step functions and then extended.

(I) For a fixed integer $N \geq 1$, suppose that ℓ is of the form

$$\ell(u) = \sum_{j=0}^{N-1} \ell_j I_{[x_j, x_{j+1}]}(u), \quad x_j = \frac{j}{N}, \ \ell_j \leq \ell_{j+1} \ .$$

We argue first that it is enough to restrict attention to candidates for projection which are similarly non-decreasing step functions: given $n \in \mathcal{M}_0$, we apply the Cauchy-Schwarz inequality to get

$$\int_0^1 [\ell(u) - n(u)]^2 du = \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [\ell_j - n(u)]^2 du \geq \sum_{j=0}^{N-1} \frac{1}{N} (\ell_j - n_j)^2$$

where $n_j = N \int_{x_j}^{x_{j+1}} n(u)du$. The lower bound is attained for $n(u)$ identically constant on sub-intervals. Moreover, it can further be reduced by rearranging the n_j to be non-decreasing ([4, theorem 378]). If $n_j^{(T)}$ are the rearranged values, then we have

$$\int_0^1 [\ell(u) - n(u)]^2 du \geq \int_0^1 [\ell(u) - n^{(T)}(u)]^2 du$$

where $n^{(T)}(u) = \sum_{j=0}^{N-1} n_j^{(T)} I_{[x_j, x_{j+1}]}(u)$. We now show that $n^{(T)}(u) \in \mathcal{M}_0$. Since $n^{(T)}(u)$ is non-decreasing (a.e.), by the remark after theorem 1, it is enough to show that $N^{(T)}(x) = \int_0^x n^{(T)}(u)du \geq I(x)$ with equality at $x = 1$. The latter condition follows from the normalization (A3)'. Since $I(x)$ is convex and $N^{(T)}(x)$ is piece-wise linear, it is enough to verify the inequality constraints at the nodes $\{x_j\}$. We have

$N^{(T)}(x_k) = \int_0^{x_k} n^{(T)}(u)du = \frac{1}{N} \sum_{j=0}^{k-1} n_j^{(T)}$, which is the integral of $n(u)$ over k of the sub-intervals. Equivalently, it is equal to $\int_0^{x_k} n(T(u))du$

for some T which appropriately permutes the sub-intervals. By (ii) of the corollary, this is bounded from below by $I(x_k)$.

We now have a discrete problem to solve:

$$\text{minimize} \quad \sum_{j=0}^{N-1}{}' (\ell_j - n_j)^2$$

subject to (a) the n_j are non-decreasing,

and (b) $\sum_{j=0}^{k-1}{}' n_j \geq I(x_{k-1})$, $k = 1, \ldots, N-1$ with equality at $k = N$.

Imposing only constraint (b), the problem is treated in [1, pp. 46-51] as a generalized isotonic regression. Letting L and $\tilde{L}$ denote the partial sum vectors of ℓ and the solution vector $\tilde{\ell}$ respectively and setting $I = (I(x_1), I(x_2), \ldots, I(x_N))$, we have

$$\tilde{L} = L - (L - I)^*$$

where * here denotes the convex minorant of a vector. A straightforward argument shows that $\Delta_k^2(L - I)^* \leq \Delta_k^2(L - I)$ (Δ_k^2 denoting a second difference). Hence

$$\Delta_k^2 \tilde{L} = \Delta_k^2[L - (L - I)^*] = \Delta_k^2 L - \Delta_k^2(L - I)^* \geq \Delta_k^2 I \geq 0 .$$

It follows that $\tilde{L}$ is convex and that $\tilde{\ell}$ is non-decreasing. Thus (a) is satisfied automatically.

Translating the solution of the discrete problem into step function terms, we get $\tilde{L}(x) = L(x) - (L - I)^*(x)$.

(II) If $\ell(u)$ is not a step function, then for each $N \geq 1$, approximate $\ell(u)$ with

$$\ell_N(u) = \sum_{j=0}^{N-1} [N \int_{x_j}^{x_{j+1}} \ell(u)du] \, I_{[x_j, x_{j+1}]}(u) .$$

By (I), we have

$$\tilde{L}_N(x) = L_N(x) - (L_N - I)^*(x) . \tag{1}$$

Now as $N \to \infty$, $\ell_N \to \ell$ and $\tilde{\ell}_N \to \tilde{\ell}$ in $L_2[[0,1];\mu]$. Since $[\int_0^x \ell_N(u)du]^2 \leq \int_0^x \ell_N^2(u)du \to \int_0^x \ell^2(u)du$, the dominated convergence theorem yields $L_N(x) \to L(x)$. Similarly, $\tilde{L}_N(x) \to \tilde{L}(x)$. Further, since $L_N \to L$ uniformly and $*$ operates continuously in the uniform norm, $(L_N - I)^* \to (L - I)^*$. Taking limits $(N \to \infty)$ in (1) yields the lemma.

If ℓ is not monotone, then some additional preparation is required to obtain its projection on $\mathcal{M}_0$. For $\ell \in L_2[[0,1];\mu]$, define $\ell_\uparrow \in L_2[[0,1];\mu]$ as the increasing rearrangement of ℓ. There exists a measure-preserving transformation $S_\ell : [0,1] \to [0,1]$, not necessarily one-one, such that $\ell = \ell_\uparrow \circ S_\ell$ ([8]).

Lemma. Let $\ell \in L_2[[0,1];\mu]$ and satisfy (A3)'. Then if $\tilde{\ell}$ and $\tilde{\ell_\uparrow}$ are the projections of ℓ and $\ell_\uparrow$ respectively onto $\mathcal{M}_0$,

$$\tilde{\ell} = \tilde{\ell_\uparrow} \circ S_\ell .$$

Remark. The construction for $\tilde{\ell_\uparrow}$ has been given in the previous lemma.

Proof. If $\ell \in L_2[[0,1];\mu]$, then $\ell_\uparrow \in L_2[[0,1];\mu]$. Using a change of variables, we have

$$\int_0^1 [\ell_\dagger(u) - g(u)]^2 du = \int_0^1 [\ell(u) - (g \circ S_\ell)(u)]^2 du$$

and taking infima over $g \in \mathcal{M}_0$

$$\int_0^1 [\ell_\dagger(u) - \ell_\dagger(u)]^2 du = \inf_{g \in \mathcal{M}_0} \int_0^1 [\ell(u) - (g \circ S_\ell)(u)]^2 du$$

$$= \int_0^1 [\ell(u) - (\widetilde{\ell_\dagger} \circ S_\ell)(u)]^2 du .$$

The lemma will follow if we can show

(i) $$\inf_{g \in \mathcal{M}_0} \int_0^1 [\ell(u) - (g \circ S_\ell)(u)]^2 du = \inf_{g \in \mathcal{M}_0} \int_0^1 [\ell(u) - g(u)]^2 du$$

and

(ii) $\widetilde{\ell_\dagger} \circ S_\ell \in \mathcal{M}_0$.

Each is a consequence of the identity $\mathcal{M}_0 \circ S_\ell = \mathcal{M}_0$, that is, $g \circ S_\ell \in \mathcal{M}_0 \iff g \in \mathcal{M}_0$. The point of interest is that S_ℓ may not be one-one. However, Brown [2, theorem 3] has shown that there exists a sequence $\{T_n\} \subseteq \mathfrak{J}$ such that $g \circ T_n \to g \circ S_\ell$. Accordingly, if $g \in \mathcal{M}_0$, then $g \circ T_n \in \mathcal{M}_0$ (see the remark after the corollary of section 1) and since $\mathcal{M}_0$ is closed $\lim_{n \to \infty} g \circ T_n = g \circ S_\ell \in \mathcal{M}_0$. Conversely, if $g \circ S_\ell \in \mathcal{M}_0$, then using an approximating sequence $\{T_n\}$

$$\|g \circ S_\ell - g \circ T_n\|_{L_2[[0,1];\mu]} = \|g \circ S_\ell \circ T_n^{-1} - g\|_{L_2[[0,1];\mu]} \to 0 .$$

Since $g \circ S_\ell \circ T_n^{-1}$ for each n and $\mathcal{M}_0$ is closed, we have $g \in \mathcal{M}_0$.

We can now state our main result.

Theorem 2. Let $h \in L_2[(-\infty, +\infty); F]$ and satisfy (A3). Let $(h \circ F^{-1})_\uparrow$ be the increasing rearrangement of $h \circ F^{-1}$ with $h \circ F^{-1} = (h \circ F^{-1})_\uparrow \circ S$. Then the projection $\tilde{h}$ of h onto $\mathcal{M}(F, G)$ is given by

$$\tilde{h} = \widetilde{(h \circ F^{-1})_\uparrow} \circ S \circ F$$

where $\widetilde{(h \circ F^{-1})_\uparrow}$ satisfies

$$\int_0^x \widetilde{(h \circ F^{-1})_\uparrow}(u)du = J_1(x) - J_2^*(x)$$

and $J_1(x) = \int_0^x (h \circ F^{-1})_\uparrow (u)du$, $J_2(x) = J_1(x) - \int_0^x G^{-1}(u)du$.

Proof. Together with the indicated isomorphism between $L_2[[0,1];\mu]$ and $L_2[(-\infty, +\infty); F]$, the statement combines the two lemmas.

4. Concluding Remarks

We have investigated the structure of $\mathcal{M}(F, G)$ through a characterization result and an examination of the induced projection operator. Despite the rather formidable description of the latter, computational versions have proved to be accessible. In particular, the operations $*$ and $\dagger$ together with the extraction of the measure-preserving transformation S are reasonably straightforward (a discussion of some relevant algorithms can be found in [1]).

As in isotonic regression, the fact that analytical resources are available to attack the problem investigated here suggests that other nonlinear regression problems may be amenable to similar treatment.

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