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"e^9": Stability Theory and Boundary-Layer Transition

S. A. Berger, J. Aroesty

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Successful low-drag design employs methods of boundary-layer control to delay the transition of unstable laminar boundary layers, but a suitable comprehensive theory is needed to guide prediction and control of boundary-layer transition for low-drag hydrodynamics. This report suggests that such a theory must be substantially different from the current nonlinear stability theories. New and less formal nonlinear theories, which combine the growth rates and frequency dependence of the two-dimensional Tollmien-Schlichting waves (which are the basis of the "e-9" method) and the three-dimensional nonlinear processes of modern stability theory, could ultimately lead to improvements in the understanding and manipulation of the transition process in low-drag hydrodynamics, and to the inclusion of disturbance effects. Until such methods are developed, the "e-9" or similar empirically based methods must be relied on for deisgn and analysis. Refs. (BG)

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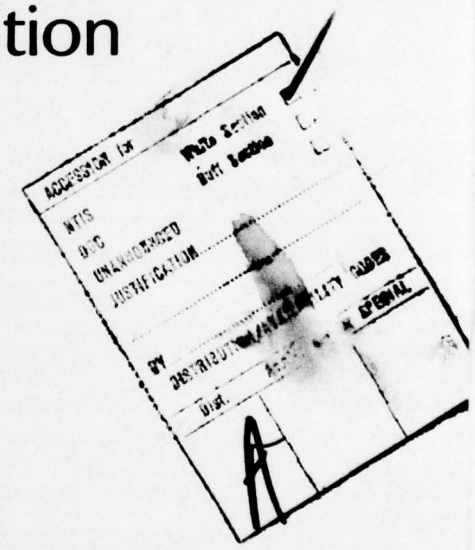

A Report prepared for

DEFENSE ADVANCED RESEARCH PROJECTS AGENCY

Rand
SANTA MONICA, CA. 90406

PREFACE

Rand has been investigating fluid mechanical bases for the design of low-drag submersible vehicles, under the sponsorship of the Tactical Technology Office of the Defense Advanced Research Projects Agency. The impact of fluid mechanics on hydrodynamic design is most critical in the characterization of the location and properties of the transition region between laminar and turbulent flow. A semiempirical method, generally known as the "e^9" method of transition prediction, is widely used as a guide to the effect of body shape, pressure distribution, surface mass transfer, and heating. This report reviews the method and the advances in both linear and nonlinear hydrodynamic stability theory that have occurred in the 20-year interval since first publication of the "e^9" method. Ultimately, it is hoped that these advances could further refine the ability both to predict and to control boundary-layer transition. The report should be useful to hydrodynamicists, designers of submersibles, and others engaged in the application of fluid mechanics to the improvement of underwater vehicle performance. Related Rand reports include:

R-1752-ARPA/ONR -- *Low-Speed Boundary-Layer Transition Workshop*, W. S. King, June 1975.

R-1789-ARPA -- *Controlling the Separation of Laminar Boundary Layers in Water: Heating and Suction*, J. Aroesty and S. A. Berger, September 1975.

R-1863-ARPA -- *The Effect of Wall Temperature and Suction on Laminar Boundary-Layer Stability*, W. S. King, April 1976.

R-1866-ARPA -- *Hydrodynamic Considerations in the Design of Small Submersible Vehicles* (U), C. Gazley, Jr., J. Aroesty, W. S. King, E. R. Van Driest, April 1976 (Confidential).

R-1907-ARPA -- *Buoyancy Cross-Flow Effects on the Boundary Layer of a Heated Horizontal Cylinder*, L. S. Yao and Ivan Catton, April 1976.

R-1966-ARPA -- *The Buoyancy and Variable Viscosity Effects on a Water Laminar Boundary Layer Along a Heated Longitudinal Horizontal Cylinder*, L. S. Yao and I. Catton (in process).

SUMMARY

Successful low-drag design employs methods of boundary-layer control to delay the transition of unstable laminar boundary layers. However, there is not yet a suitable comprehensive theory to guide prediction and control of boundary-layer transition for low-drag hydrodynamics.

Our survey suggests that such a comprehensive theory must be substantially different from the current nonlinear stability theories. This is because nonlinear theory can represent the late stage of the transition process only if disturbance levels are large enough to trigger transition near the first onset of laminar instability. This requirement is completely at odds with the customary low-drag design, where boundary-layer control is used to delay the transition of an unstable boundary layer, where there are great distances between the first laminar instabilities and transition, and where considerable effort is made to reduce disturbance effects.

New and less formal nonlinear theories which combine the growth rates and frequency dependence of the two-dimensional Tollmien-Schlichting waves (which are the basis of the "e^9" method) and the three-dimensional nonlinear processes of current stability theory could ultimately lead to improvements in the understanding and manipulation of the transition process in low-drag hydrodynamics, and to the inclusion of disturbance effects. Until such methods are developed, the "e^9" or similar empirically based methods must be relied on for design and analysis.

Unfortunately, the "e^9" method cannot elucidate the sensitivity of boundary-layer transition to small disturbances such as surface roughness, vibration, or freestream turbulence. This must be dealt with using even more intuitive ad hoc procedures than the original "e^9" method itself, until a more comprehensive theory is developed.

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I. INTRODUCTION

Low-drag submersible design is a creative engineering art form which is becoming a science. The principal area of creativity is the selection of appropriate criteria for locating the transition from laminar to turbulent boundary-layer flow. Ideally, these criteria reflect the impact of body shape, pressure gradient, wall heating, suction, propulsion, freestream disturbances, vibration, surface finish, acceleration, suspended particulates, and noise on boundary-layer transition.

Laminar flow over the forward region of a low-drag body is well understood, and its characteristics can be predicted almost routinely using computerized versions of the equations of motion. These are based on classic, systematic approximations to the Navier-Stokes equations which govern the dynamic behavior of real viscous fluids such as water or air. Turbulent flow characteristics over the aft end of such vehicles can also be predicted, using computer programs which are more empirically based but which are still consistent with similarity laws and considerations of mass, momentum, and energy balance.

The transition region is more problematic: no complete theory analogous to laminar flow is available and the experimental measurements are not so complete as in the turbulent case. For this reason, then, a combination of theory and empiricism is the basis for the best transition prediction.[1]

The "e^9" method currently appears to be the best combination of theory and empiricism for correlating and predicting the effects of body shape, pressure distribution, wall heating, and suction for hydrodynamic flows. Extensions of this method and the linear theory on which it is based include the effects of freestream turbulence, wall roughness, and noise on boundary-layer transition.

Because the "e^9" method is based on stability theory, because its predictions of instability growth rates and "critical" frequencies have been verified experimentally, and because of the importance of the Tollmien-Schlichting instability wave growth in low-drag

applications, the method is taken more seriously by fluid mechanicians than when it was originally suggested by Smith and Van Ingen. While fundamental questions are still open about the process of transition, the "e^9" method and its relationships to both linear and nonlinear stability theory deserve study. This report explores some of these relationships by surveying the relevant assumptions and features of classic hydrodynamic stability theory and the newer work on nonlinear stability. The intention is to review the current status of the nonlinear theory, in order to assess its capacity for improving, elucidating, or supplanting the "e^9" method as the touchstone of current low-drag hydrodynamics.

II. HYDRODYNAMIC STABILITY THEORY

A. LINEAR THEORY

In this section, the mathematical development of the classical linear theory is outlined.

Consider an incompressible, steady, two-dimensional laminar boundary layer on a planar body which is not so strongly curved that centrifugal forces are unimportant. We wish to investigate the response of the boundary layer to an infinitesimal disturbance. To this end we first write down the Navier-Stokes equations for unsteady two-dimensional incompressible flow*

$$\nabla \cdot \vec{q} = 0 \text{ (continuity)}$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\,\vec{q} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{q} \text{ (momentum)} \tag{1}$$

where $\vec{q} = (u, v)$, u and v being the x and y components of velocity, p is the pressure, ρ is the density, ν is the kinematic viscosity ($\nu = \mu/\rho$). The x-axis is chosen to lie in the direction of the free-stream velocity and the y-axis normal to the body.

To analyze the response to an infinitesimal disturbance, we now write each flow quantity as the sum of a mean steady flow quantity and a small fluctuation term

---

*There is no a priori reason to assume that the disturbances are two-dimensional, and in fact they would not generally be so under experimental conditions. One can readily carry out the analysis for three-dimensional disturbances. However, there exists a transformation which reduces the three-dimensional equations for a sinusoidal disturbance to two-dimensional equations. One can then show the flow is more stable to three-dimensional disturbances than to two-dimensional (Squire's Theorem), so the stability analysis can concentrate on the two-dimensional case. (However, see the footnote on p. 7 where it is pointed out that this theorem is valid only for waves that grow in time.) This argument should not be taken to suggest that three-dimensional effects are unimportant for transition to turbulence of two-dimensional boundary layers. In fact, as we shall discuss later, in the latter stages of the transition process, three-dimensional effects play a crucial role.

$$\vec{q}(\vec{r},t) = \vec{U}(\vec{r}) + \vec{q}'(\vec{r},t)$$

$$p(\vec{r},t) = P(\vec{r}) + p'(\vec{r},t) \tag{2}$$

where $|q'| << |\vec{U}|$, $|p'| << |P|$ and $\vec{U} = (U, V)$ and P are solutions of the steady, two-dimensional boundary-layer equations. If we substitute these expressions into the Navier-Stokes equations above and drop all quadratic terms in the disturbance quantities, the resulting equations can be decomposed into one set consisting of zero-order terms only and one set consisting of first-order terms

$$\left.\begin{aligned} \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} &= 0 \\ U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2} \\ 0 &= -\frac{1}{\rho}\frac{\partial P}{\partial y} \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \\ \frac{\partial u'}{\partial t} + u'\frac{\partial U}{\partial x} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} + V\frac{\partial u'}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial x} + \nu\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right) \\ \frac{\partial v'}{\partial t} + u'\frac{\partial V}{\partial x} + U\frac{\partial v'}{\partial x} + v'\frac{\partial V}{\partial y} + V\frac{\partial v'}{\partial y} &= -\frac{1}{\rho}\frac{\partial p'}{\partial y} + \nu\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right) \end{aligned}\right\} \tag{4}$$

Alternatively, these equations could have been obtained by defining an appropriate mean, denoted by an overbar, and assuming $\bar{\vec{q}} = \vec{U}$, $\bar{\vec{u}}' = 0$, $\bar{p} = P$, $\bar{p}' = 0$. Substitution of expressions (2) into the Navier-Stokes equations and then taking the mean yields Eqs. (3); subtraction of Eqs. (3) from the original equations, after neglect of the quadratically small terms, then yields Eqs. (4).

The balance among inertial, pressure, and viscous forces is required for the fluctuation quantities. The dominant convective terms in the

momentum equations are estimated on the basis of the usual boundary-layer ordering of the mean flow terms, $V \ll U$, $\partial/\partial y \gg \partial/\partial x$; the fluctuation velocities, u' and v', are assumed to be equal to each other in magnitude, and their derivatives are also comparable. The length scale for variation in fluctuation velocities is a wavelength, which is roughly of the order of δ, the thickness of the layer. Finally, we assume that the mean flow quantities do not change significantly over a wavelength of the disturbance. We can then consider $U(x,y)$ to be a function of y alone; i.e., $U = U(y)$. Under this set of assumptions, the equations for the fluctuation quantities simplify to

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \nu \nabla^2 u' \qquad (5)$$

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} = -\frac{1}{\rho} \frac{\partial p'}{\partial y} + \nu \nabla^2 v'$$

The implication of the assumption $U = U(y)$ is that stability at a particular x is determined by local conditions at that station independent of all others. This assumption is obviously exact for any truly parallel flow, such as in a channel, but only an approximation for other flows, such as boundary layers which generally grow with streamwise distance. There has been a great deal of recent interest centered on this point. Solutions for nonparallel effects have been obtained through straight-forward perturbations, iteration, and the method of multiple scales by, most prominently, Ling and Reynolds,[2] Bouthier,[3,4] Nayfeh and Saric,[5,6] and Gaster.[7] For boundary layers with pressure gradient, the most extensive calculations of nonparallel effects are those of Wazzan et al.[8] for Falkner-Skan flows. They found these effects to be negligible when β, the Falkner-Skan pressure gradient parameter, was greater than 0.4. Nonparallel terms were most important for negative β, cases for which the minimum critical Reynolds number is small and the parallel flow assumption breaks down. In general, nonparallel effects can be

expected to be relatively unimportant at large Reynolds numbers. At lower Reynolds numbers such effects may have to be taken into account. (For example, see the recent work of Strazisar et al.[9] for a heated flat plate in water, where it was necessary to include these non-parallel effects in comparing experimental results with theory, because of the proximity to the minimum critical Reynolds number.) The remainder of this discussion will be limited to high Reynolds number flows making the parallel flow assumption.

For boundary-layer flow the appropriate boundary conditions are

$$u' = v' = 0 \text{ at } y = 0$$
$$u',\ v' \to 0 \text{ as } y \to \infty \tag{6}$$

Since the coefficients in Eqs. (5) are independent of x and t and the equations are linear, the solution can be obtained most readily by taking a Fourier transform in x and Laplace transform in t. This procedure then says that the behavior of the solution is characterized by the normal mode representation

$$\begin{Bmatrix} u' \\ v' \\ p' \end{Bmatrix} = \begin{Bmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{p}(y) \end{Bmatrix} e^{i(\alpha x - \omega t)} \tag{7}$$

where α is the wave number in the x direction ($= 2\pi/\lambda$, where λ is the wavelength) and ω is the frequency. Either or both α and ω can be complex, and correspondingly so are the amplitude functions $\hat{u}(y)$, $\hat{v}(y)$, and $\hat{p}(y)$. The classical approach has been to assume α is real and ω complex, in which case the imaginary part of ω determines whether disturbances grow or decay in time; hence this is referred to as the temporal approach. In the spatial approach ω is assumed real and α complex, so the imaginary part of α determines whether disturbances grow or decay with increasing x. Under appropriate conditions, the temporal and spatial amplification rates are related by the group

velocity of a packet of waves of the form (7) (see Gaster[10]). Since the spatial theory corresponds more closely to the usual physical situation and seems, at least at present, to lead to the best correlations with transition Reynolds number, the remainder of this discussion will be limited to spatial theory.*

Writing $\alpha = \alpha_r + i\alpha_i$ where the subscripts r and i denote the real and imaginary parts, respectively, and assuming ω is real, a typical disturbance quantity, say u', can be written

$$u' = \hat{u}(y)\, e^{-\alpha_i x}\, e^{i(\alpha_r x - \omega t)} \tag{8}$$

where ω is assumed real. Thus for

$$\begin{aligned} &\alpha_i > 0, \text{ disturbances are damped} \\ &\alpha_i = 0, \text{ disturbances neither grow nor decay ("neutral" disturbances)} \\ &\alpha_i < 0, \text{ disturbances are amplified} \end{aligned} \tag{9}$$

The set of disturbance equations (5) can be reduced to a single equation for v' by cross-differentiating the x and y momentum equations to eliminate p' and then using the continuity equation to eliminate u'. This then leads to the following linear fourth-order equation for $\phi(y)(= v'/\alpha)$

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = \frac{-i}{\alpha Re_\delta}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \tag{10}$$

where $c = \omega/\alpha$ and $Re_\delta = U_e \delta/\nu$, where U_e is the velocity at the edge

*Unfortunately, Squire's theorem that three-dimensional disturbances are more stable than two-dimensional ones only holds for temporally growing disturbances. At present, the question whether three-dimensional spatially growing disturbances are more stable or not than two-dimensional ones cannot generally be answered and must be considered separately for each case.[11]

of the boundary layer and δ is the boundary-layer thickness. (Prior to carrying out this derivation, all variables have been nondimensionalized in terms of U_e and δ.)

Equation (10) is the Orr-Sommerfeld equation. The boundary conditions are

$$\begin{aligned} &\phi(0) = \phi'(0) = 0 \\ &\phi(y),\ \phi'(y) \to 0 \text{ as } y \to \infty \end{aligned} \tag{11}$$

The Orr-Sommerfeld equation plus boundary conditions describe the self-excited or free oscillations within a linear dispersive, dissipative system. It is an eigenvalue problem and will only have solutions if a secular equation of the form

$$F(\alpha,\ Re_\delta,\ c) = 0 \tag{12}$$

is satisfied. Since $c = \omega/\alpha = \omega/(\alpha_r + i\alpha_i)$, Eq. (12) can be written

$$F(\alpha,\ Re_\delta,\ \omega) = 0 \tag{13}$$

Since Re_δ and ω are real and α complex, taking the real and imaginary parts of this equation yields two equations which can be combined to yield

$$F_1(\alpha_r,\ Re_\delta,\ \omega) = 0 \tag{14a}$$

$$F_2(\alpha_i,\ Re_\delta,\ \omega) = 0 \tag{14b}$$

For a given Re_δ and ω, Eq. (14a) yields the wave number of the disturbance, α_r. Setting $\alpha_i = 0$ in Eq. (14b) yields the expression $F_2(0,\ Re_\delta,\ \omega) = 0$, which defines the neutral stability curve in the $Re_\delta - \omega$ plane. Setting α_i = constant yields curves of constant amplification in this plane. Because Eq. (14a) uniquely defines α_r

in terms of Re_{δ_*} and ω, these curves can equally well be plotted in a $Re_{\delta_*} - \alpha_r$ plane.*

Thus far the analysis has been presented for an isothermal, constant-density fluid flow, and modification of the above formulation is required to consider the stability of water boundary layers with heat transfer. Taking temperature to be the only state variable, and assuming viscosity is a function of temperature alone, with all other fluid properties constant (a good assumption for water boundary layers with heat transfer at moderate pressures), Wazzan et al.[12] have shown that the stability of water boundary layers with heat transfer can be adequately treated by the Orr-Sommerfeld equation, Eq. (10), modified to include terms that arise from the variation of viscosity with temperature.

According to Eq. (8), the amplification of a typical disturbance u' between the points x_0 and $x_1 = x_0 + \Delta x$ is given by

$$\left|\frac{u_1'}{u_0'}\right| = e^{-\alpha_i \Delta x} \tag{15}$$

If the flow were truly parallel then α_i = const. and the amount of amplification between two points x_0 and x_1 a finite distance apart would be simply

$$e^{-\alpha_i (x_1 - x_0)}$$

Since our interest is in boundary layers where the flow is nearly parallel, we write instead

$$\left|\frac{u_1'}{u_0'}\right| = e^{-\int_{x_0}^{x_1} \alpha_i(x)dx} \tag{16}$$

---

*The eigenfunctions associated with these eigenvalues correspond to the Tollmien-Schlichting instability waves.

since α_i in such cases will depend on x. This is not exact and is only an ansatz. Confirmation, for the case of the flat plate, is presented by Ross, Barnes, Burns, and Ross.[13] According to Eqs.(7), the amplification is the same for each disturbance quantity, and if denoted by A can be written as

$$A(x_2, x_1) = e^{-\int_{x_0}^{x_1} (\alpha_i^* \, \delta/Re_\delta) \, U_e/\nu \, dx} \tag{17}$$

in nondimensional terms. (We note in passing that $A^{-1} \, dA/dx = -\alpha_i$.)

The traditional approach in laminar stability theory via the Orr-Sommerfeld equation has been to emphasize the calculation of the neutral stability curve, and from this the minimum critical Reynolds number, i.e., the lowest Reynolds number at which <u>any</u> disturbance begins to be amplified. Implicit in this approach is that present in any laminar flow is a continuous spectrum of small disturbances covering the whole frequency or wave-number range, so the Reynolds number for which the first of these disturbances becomes unstable determines the stability characteristics of the flow. Only recently has it become apparent that this minimum critical Reynolds number is not very important in determining the onset of turbulence, particularly when it is desired to delay transition. Since transition occurs, experimentally, only after disturbances have reached a finite magnitude, the total amplification of infinitesimal disturbances and how they vary with disturbance frequency is more important. For truly parallel flows this matter is less delicate, since all of the parameters of the flow are independent of streamwise distance and therefore the curves representing the solution of Eq. (14b), $F_2(\alpha_i, Re_\delta, \omega) = 0$, for various values of α_i = const. immediately give the <u>constant</u> growth rate of a distance of frequency ω. For nearly parallel flows, however, Re_δ changes with x and so does the U(y) which appears in the Orr-Sommerfeld equation. In such cases, the spatial growth of a disturbance of a fixed frequency ω requires that a series of stability plots of Eq. (14b)

be made corresponding to increasing x and that the quantity Eq. (16) or (17) be calculated for increasing x. The significance of the minimum critical Reynolds number becomes more subtle; it could become a surrogate for the amplification characteristics of the boundary layer at the particular station x. (An illustration is given by Van Ingen.[14]) The calculation of Eq. (16) or Eq. (17) must be carried out for all frequencies using the same set of stability plots and beginning at some appropriate initial point x_0. This is done for increasingly larger values of x. At any such x there will generally be one frequency of disturbance that has been most amplified in its passage from x_0 to x; in some sense this frequency may be looked upon as being the most dangerous one. When this procedure was carried out for a number of two-dimensional and axisymmetric boundary-layer flows, it was found (by A. M. O. Smith[15] and Van Ingen[16]) that the experimentally determined position of transition was located near the point where the right-hand side of Eq. (16) or Eq. (17), the total amplification, first attained the value e^9 for <u>any</u> frequency. This correlation led to the formulation of the so-called "e^9" rule for the prediction of transition. According to this rule, transition occurs at that point of the body where any disturbance of arbitrary constant frequency first attains a total amplification factor of e^9. The idea is generalized to the "e^n" method.

Critical Evaluation of the "e^9" Rule

What explanation is there for the widely acknowledged success of the "e^9" rule? In particular, why should a rule based on the total amplification of two-dimensional disturbances work so well in predicting transition when the latter stages of transition are known to be so profoundly affected by three-dimensional effects (e.g., the formation of "spikes" and turbulent "spots" and their growth and coalescence). Further, what justification is there for using linear stability theory up to an amplification factor of 8100 ($\approx e^9$), when any practical free-stream disturbance (of the order of 0.1 percent or larger) multiplied by this factor would, at least formally, violate the assumption of small disturbances, basic to the linear theory.

Logical justification for the use of a maximum amplification criterion may be found in the suggestion of Liepmann[17] that transition occurs at the point where the Reynolds stress first reaches the magnitude of the laminar shear. If one calculates the Reynolds stress from the solutions of the Orr-Sommerfeld equation, then the ratio of these two stresses contains the amplification factor. Unfortunately, it also contains a quantity representing the initial disturbance level. And this deficiency seems to undermine any attempt to develop a rational basis for the "e^9" rule at this time, namely, transition appears to require the attainment of some absolute level of disturbance, and this can be calculated from an amplification factor only if the initial disturbance level and its spectral distribution are known.

As regards the level of ultimate disturbance due to an amplification factor of e^9, we note that although the freestream disturbance level in a modern wind tunnel may be of the order of 0.1 percent, this figure represents the spectrally integrated level. Since much of this disturbance energy is damped out or becomes irrelevant in the filtering Tollmien-Schlichting mechanism, the internalized narrow-band fundamental Tollmien-Schlichting mode may have an initial disturbance maximum at least a hundred times smaller, of the order of 0.001 percent or less at the x corresponding to the location of the minimum critical Reynolds number, R_c (Obremski et al.[18]). This may be even further diminished by the weak coupling that seems to exist between the freestream and the boundary layer as regards the receptivity of the latter to freestream disturbances (Mack, private communication, quotes factors of 10^3 or so between freestream and internal disturbances). According to Klebanoff et al.,[19] the linear theory ceases to be valid (for a flat plate) when the disturbance reaches a level of 1 to 1.5 percent of U_e. Thus, a total amplification factor of the order of 1000, or even more, between the beginning of amplification and the onset of nonlinearity would not be surprising. Since the first appearance of turbulent spots, the beginning of the true transition region, occurs for a flat plate at disturbance levels of the order of 12 to 20 percent, only an additional amplification of

the order of 10 to 15 percent beyond the onset of nonlinearity must be incurred by the three-dimensional, nonlinear, and other mechanisms (Obremski et al.[18]). If this flat-plate data is typical, the streamwise extent of linear amplification covers about 75 to 85 percent of the distance to the beginning of transition, and a total amplification factor of $e^9 (\approx 8100)$ would appear to be of the right order of amplification. Moreover, because of the discrepancy in lengths occupied by these two domains, errors in estimates of the extent of the nonlinear and/or three-dimensional prebreakdown regimes might be expected to cause only small errors in estimates of total length to the beginning of transition. On the other hand, these errors plus the differences due to different levels of initial disturbances in each wind tunnel or in each free flight test should manifest themselves in deviations from the factor e^9. And in fact, as the data of Jaffe et al.[15] demonstrate, the exponent 9 represents an average value, whereas for exact correspondence to experimental transition locations the exponent may be as low as 8 and as high as 12.

With the above objections to the use of linear stability to predict transition, it is worthwhile investigating the extent to which the work of the past nearly two decades on nonlinear effects, both two- and three-dimensional, can shed light on the linear, two-dimensional "e^9" criterion, or perhaps lead to different and more comprehensive predictions of transition. Before doing so, however, let us briefly consider a somewhat different approach to the stability problem, based on global considerations.

B. ENERGY BALANCE

The previous treatment of stability emphasized the behavior of a single spectral component of the disturbance. We can instead look at the growth or decay of the total energy of the disturbance. For incompressible flow, the energy is primarily kinetic. An equation for this quantity can be obtained from the disturbance equations by multiplying each momentum equation by the corresponding velocity disturbance, summing the resulting equations and then integrating

across the layer. Doing so, considering only parallel flow with two-dimensional disturbances periodic in x, and integrating over a wavelength in x, we obtain (in nondimensional terms)

$$\frac{\partial E}{\partial t} = M - \frac{1}{Re} N \tag{18}$$

where

$$E = \iint \frac{1}{2} \rho (u'^2 + v'^2)\, dxdy = \text{total energy of disturbance}$$

$$M = -\rho \iint \overline{u'v'} \frac{d\bar{u}}{dy}\, dxdy = \text{energy transfer between mean flow and disturbance through action of Reynolds stress} \tag{19}$$

$$N = \iint \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 dxdy = \iint \zeta'^2\, dxdy = \text{dissipation integral, rate of dissipation of kinetic energy into heat}$$

Neutral stability occurs when $\partial E/\partial t = 0$, in which case

$$M - \frac{1}{Re_c} N = 0 \text{ or } Re_c = \frac{N}{M} \tag{20}$$

The minimum critical Reynolds number then is given by the minimum of the ratio N/M with respect to all possible disturbances u', v'. If one attempts to make use of this expression, as some early investigators did, by evaluating N and M using simple expressions for u' and v', one finds minimum critical Reynolds numbers that are generally too low when compared to calculations based on the small perturbation equations. Presumably, this is a consequence of using disturbance velocities that are not solutions of the equations of motion. This procedure, however, will yield sufficient conditions for stability. Using this approach, Serrin[20] found that Re = 5.71 is a universal minimum critical Reynolds number for arbitrary fluid motion in a bounded region.

This formulation of the energy method, as embodied in Eq. (18), is based on the temporal growth of disturbances. In accordance with the earlier discussion of the linear theory it would seem more appropriate to formulate the energy method so as to be able to study the *spatial* growth of the total energy of the disturbance. An analysis of this kind seems not yet to have been carried out.

Stuart[21] has successfully employed the energy method for finite disturbances at Reynolds numbers slightly above the minimum critial Reynolds number. Assuming that the modes calculated from the linear theory represent a reasonable approximation to the modes for small but finite amplitudes, he writes

$$u' \cong \alpha A(t)\ \phi'(y)\ e^{i\alpha(x - ct)}$$
$$v' \cong -2i\alpha\ A(t)\ \phi(y)\ e^{i\alpha(x - ct)} \tag{21}$$

where $\phi(y)$ is the fundamental harmonic of the linear disturbance, and c is real since the growth of the amplitude is accounted for explicitly by the amplitude function A(t). Since the disturbance is finite, the distortion of the basic mean flow must be taken into account; this can be represented as

$$\bar{u} = \bar{u}_0(y) + A^2\ \mathrm{Re}\ \bar{u}_1\ (y, t) \tag{22}$$

Substitution into the integrated energy equation, Eq. (18), yields the following nonlinear ordinary differential equation for A(t):

$$\frac{dA}{dt} = c_1\ A + c_2\ |A|^2\ A \tag{23}$$

where c_1 and c_2 are constants that will be defined later. This has been generalized by Stewartson, Hocking, Stuart, and Davey in a number of recent papers. All of this work falls properly under the heading of nonlinear stability and is discussed further in what follows.

Below, the various nonlinear theories are classified according to their principal assumptions or limitations. The major features and predictions of each are very briefly outlined and summarized.

C. NONLINEAR THEORIES

1. Two-Dimensional Disturbances

a. Landau[22]-Stuart[21,23]-Watson[24] Theory. In 1944, Landau[22] heuristically derived an equation governing the evolution of the amplitude of a disturbance starting in the linear regime and extending into the early nonlinear regime when the disturbances are small (in some sense) but not infinitesimal. Stuart[21,23] and then Watson[24] later presented a formal theory. They assumed that for values of the Reynolds number just slightly greater than the minimum critical value Re_c the perturbation solution was of the form

$$A\,\phi(y)\,e^{i\alpha(x - ct)} \tag{24}$$

where $\phi(y)$ is the eigenfunction corresponding to the (real) wave number α and (complex) wave speed c calculated from the classical Orr-Sommerfeld equation at $Re = Re_c$. (This is an important point to which we shall return later.) That there is a single mode of the form (24) is a good approximation in the neighborhood of Re_c, since at this value of Re only this one mode amplifies. In the classical linear stability theory, A is an infinitesimal constant. For $0 < Re - Re_c \ll 1$, it is assumed that A can be replaced by A(t) and is small but not infinitesimal. From the Navier-Stokes equations one can show formally that A(t) satisfies (near $\alpha = \alpha_c$)

$$\frac{dA}{dt} = \alpha_c c_i A + k|A|^2 A \tag{25}$$

where the subscript c denotes values at Re_c, k is a complex constant $(k_r + ik_i)$, and c_i is the imaginary part of c and is proportional to

$(Re - Re_c)$. From Eq. (25) we note that with $Re - Re_c > 0$, and hence $c_i > 0$, if $k_r < 0$, then $|A|$ does not increase indefinitely, but approaches a maximum value, proportional to $c_i^{1/2}$. (This behavior of A(t) leads to the bifurcation theory of Taylor-Couette flow and Bénard convection.) If, on the other hand, $k_r > 0$, then $|A| \to \infty$ at a finite value of t.

Although the theory is strictly not valid for $Re - Re_c < 0$ $(c_i < 0)$, Eq. (25) shows in such cases that if $|A|^2$ is initially sufficiently small then $|A|^2 \to 0$ as $t \to \infty$, whereas if $|A|^2$ is sufficiently large $(> -\alpha_c c_i / k_r)$ then $A \to \infty$ as $t \to \infty$. This latter feature led Stuart to formulate the principle of a threshold of instability, according to which a flow may be stable to infinitesimal disturbances, but unstable to disturbances which are finite.

b. Görtler and Witting.[25] Mean flow streamlines will be distorted by the disturbances and become at some places convex and at others concave. Concave streamlines should lead to the formation of longitudinal vortices in a manner similar to that for a concave surface (Görtler vortices). It was found experimentally, however, that eddies were formed in regions where the streamlines were convex, not concave.[19]

c. Benney and Bergeron.[26] The inviscid solution of the Orr-Sommerfeld equation, originally due to Rayleigh, breaks down in the neighborhood of the critical point, the location at which the disturbance phase velocity is equal to the local flow velocity. The traditional approach to handling this singular behavior is to carry out a local analysis using the full Orr-Sommerfeld equation with the viscous terms included. A possible alternative procedure is to eliminate the singularity by taking into account the nonlinear terms that were not included in the Orr-Sommerfeld equation. More precisely, if one alternatively balances the inviscid terms in the Orr-Sommerfeld equation with the viscous terms in the equation or with the nonlinear terms that were dropped, it is found that there are two possible boundary scales: $y - y_c = 0\ (\nu^{1/3})$ and $y - y_c = 0\ (\varepsilon^{1/2})$, where ν is the kinematic viscosity, and ε is a measure of the magnitude of the nonlinear terms. If $\varepsilon^{1/2} \ll \nu^{1/3}$, the critical layer is dominated by viscous effects, whereas if $\varepsilon^{1/2} \gg \nu^{1/3}$, nonlinear effects are

dominant. In the critical layer, the streamlines are found to have the so-called cat's eye configuration. Unfortunately, it is not clear from the analysis whether the off-neutral solutions are stable or unstable. Unlike the viscous critical layer solution, no phase change occurs across the nonlinear critical layer. Also there is no critical Reynolds number associated with the initiation of waves; it is only necessary that the Reynolds number be large and the disturbance amplitude sufficiently large. A new class of large wave number disturbances is found that do not occur in the classical Orr-Sommerfeld viscous theory. For this new family of disturbances the critical layer is located far from the boundary, with wave speeds close to freestream velocity. The possibility is raised that this is a possible mechanism for freestream disturbances to enter the boundary layer. Stewartson,[27] in a masterful review of stability theory, suggests that the theory is physically unrealizable.

d. Landahl.[28] The nonlinear wave mechanics of Whitham,[29] with extensions and elaborations by Hayes,[30] is extended to include dissipation for laminar shear flows. Under certain conditions, determined by the theory, the solutions exhibit breakdown into high frequency oscillations. It is claimed that this theory explains the main features of the Klebanoff et al.[19] experiments, such as rapid localized onset, formation of hairpin vortices and their downstream evolution. In this theory, three dimensionality plays a secondary role; its principal role is to establish criticality conditions for the two-dimensional flow. Stewartson,[27] again raises serious questions about the assumptions of the theory.

e. Ko, Kubota, and Lees.[31] An integral analysis for a flat plate wake is carried out, which includes nonlinear growth of the disturbances, the effect of the Reynolds stress due to these disturbances on the mean flow, and streamwise variation of mean flow properties. The analysis essentially extends the Stuart-Watson (C.1.a) theory to take account of streamwise variation.

f. Benney and Maslowe.[32] The Benney-Bergeron nonlinear critical Layer theory (see C.1.c above) is extended via the multiple scaling

technique, to allow, for nonneutral modes, the amplitude of the wave to evolve slowly in space and time. The amplitude A is shown to obey the following equation

$$\mu\left[\frac{\partial A}{\partial T} + \omega'(k)\,\frac{\partial A}{\partial X}\right] - \frac{1}{2}\,i\mu^2\omega''(k)\,\frac{\partial^2 A}{\partial X^2} = i\varepsilon^2\,\gamma A^2 A^* \qquad (26)$$

where $\omega(k)$ is the dispersion relation,

ε = amplitude parameter,

γ = "Landau" constant,

μ = distance and time over which the wave is modulated,

$X = \mu x$, $T = \mu t$, "slow" space and time variables,

* denotes complex conjugate.

Note that if $\omega'(k)$ ($= c_g$, group velocity) is real, then new variables $\xi = X - \omega' T$ and $T' = \mu T$ can be introduced, in terms of which the above equation becomes

$$\frac{\partial A}{\partial T'} = i\left(\frac{\omega''}{2}\,\frac{\partial^2 A}{\partial \xi^2} + \gamma A^2 A^*\right), \qquad (27)$$

similar to the equation obtained by Stewartson and Stuart[33] using another approach (see Eq. (29) below). Solutions to Eq. (26) can be obtained by the inverse scattering method.

2. Three-Dimensional Effects

Although three-dimensional effects were observed experimentally much earlier by many investigators, it was the measurements of Klebanoff, Tidstrom, and Sargent[19] which firmly set theoreticians on the path towards developing theories encompassing them, and very decidedly determined the course of nonlinear stability research of the past decade. By demonstrating that beyond the primary stage, governed by linearized stability theory, transition is dominated by nonlinear three-dimensional effects, they firmly established that existing two-dimensional nonlinear theories were incapable of fully explaining the

observed phenomena. Among such two-dimensional theories considered were the generation of higher harmonics, the interaction of the mean flow and the Reynolds stress (C.1.a. and C.1.e), and concave streamline curvature associated with the wave motion (C.1.b). The nonlinear range of boundary-layer instability is associated first with the development of longitudinal or streamwise vortices, followed by the formation of high-frequency oscillations or "spikes," interpreted as "hairpin" eddies, leading ultimately to the birth of turbulent bursts or spots. (Jets and wakes do not exhibit the sudden onset of high-frequency fluctuations nor the intermittent turbulent bursts, and may in general be less subject to three-dimensional effects. Thus the Ko, Kubota, and Lees[31] analysis (C.1.e) may have a greater range of validity in such flows than it has for boundary-layer flows.)

We now very briefly discuss the theories developed in response to the Klebanoff experiments, classified according to the basic approach taken.

a. Modal Analyses. These analyses are all based on Fourier decomposition of the disturbance and analysis of individual modes.

(1) Benney and Lin,[34] Benney,[35,36] Antar and Collins,[37] and Craik.[38] The first paper considered the nonlinear interaction between a single two-dimensional and a single three-dimensional disturbance superimposed on a laminar flow. They found that a mean secondary flow was generated in the form of longitudinal vortices. Benney followed up on this work with Lin by obtaining analytic solutions for shear flow and a linear velocity profile boundary layer. A major weakness of the theory is that the wave number in the spanwise direction is arbitrary, undetermined by the theory. In addition, the wave speeds of both disturbances are assumed to be the same. This latter assumption was challenged by Stuart,[23] who pointed out that for a Blasius boundary layer these phase speeds could differ by as much as 15 percent. Relaxation of the assumption was carried out by Antar and Collins,[37] who found as a result a slow

modulation of the secondary flow for Blasius and Falkner-Skan profiles. At the same time, certain physically unrealistic features are encountered in their solution. Whereas the vortices in all of this work appear in the right places (i.e., corresponding to where they were found by Klebanoff,[19] in the Antar and Collins work their sense of rotation periodically reverses! Craik[38] extended the Benney-Lin[34] analysis in a major way by considering triads of Tollmien-Schlichting waves, consisting of a two-dimensional wave and two oblique waves propagating at equal and opposite angles to the flow direction and such that all three waves have the same phase velocity in the downstream direction. The analysis shows that there can be remarkably powerful resonance interactions, leading to a rapid transfer of energy from the primary shear flow to the disturbance, preferentially to the oblique wave. For a given two-dimensional wave, resonance occurs only for certain oblique waves, suggesting a possible natural selection process for the oblique wave of the Benney-Lin theory.

(2) Greenspan and Benney.[39] The linear stability of time-dependent shear flows, of the type Klebanoff[19] found to exist before the final stages of breakdown, is analyzed. (Although this is a linear theory, it is included in this discussion because it begins with the highly distorted unsteady mean flow profiles found beyond the primary (linear) range and thereby attempts to explain the later (and supposedly nonlinear) stages of transition.) Violent secondary instabilities are found to occur (e.g., in one-half period of the primary oscillation, the energy increases by a factor of 100). The wavelength corresponding to a maximum amplification is one-fifth that of the primary wave--a trend in accord

with the experimental evidence that the scales of motion become smaller in the later stages. Can these secondary instabilities be considered the "spikes" observed?

b. Localized Point (Three-Dimensional) or Line (Two-Dimensional) Disturbances and Their Evolution in Space and Time. It is intrinsic to a modal analysis, based as it is on decomposition into Fourier modes, that the disturbance grows everywhere at the same rate. This uniform growth runs counter to the observed localized intense regions of disturbance in the transition region. The theories in this section do not follow this approach.

Stewartson, Hocking, Stuart, and Davey.[27,33,40-45] The Stuart nonlinear stability analysis (C.1.a) is extended to the space and time modulation of the disturbance amplitude. First, the long-time behavior of an infinitesimal disturbance according to the linearized Navier-Stokes equations is found (by using Fourier transforms in space and a Laplace transform in time):

$$\varepsilon^{\frac{1}{2}} A\phi_1(z) \exp\{i\alpha_c(x - c_c t)\} \text{ (for a 3-D disturbance)} \quad (28)$$

$$\text{where } A \propto \frac{1}{\tau} \exp\left(\frac{-\xi^2}{4a_2\tau} - \frac{\eta^2}{4b_2\tau}\right)$$

$$\text{and } \varepsilon = d_{1r}(Re - Re_c) \ll 1 \quad (d_{1r} \text{ a const., real part of } d_1)$$

where $\phi_1(z)$ is the eigensolution of the Orr-Sommerfeld equation at the minimum critical Reynolds number Re_c, corresponding to wave number α_c and phase speed c_c. (The other quantities are defined below.) Equation (28) represents a modulated wave packet, and suggests scales for a multiple-scale analysis. The amplitude A is found by this approach to satisfy

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} - b_2 \frac{\partial^2 A}{\partial \eta^2} = \frac{d_1}{d_{1r}} A + k|A|^2 A + q\,AB \quad (29)$$

where a_2, b_2, d_1, d_{1r}, k, and q are all constants, B is related to the pressure and

$$\text{"slow" variables} \begin{cases} \xi = \varepsilon^{\frac{1}{2}}(x - c_g t) \ (c_g = \text{group velocity}) \\ \eta = \varepsilon^{\frac{1}{2}}\, y \ (\text{spanwise variable}) \\ \tau = \varepsilon t \end{cases}$$

(B is governed by another differential equation, involving A and therefore coupled to Eq. (29).) This equation governs the slow modulation in space and time of the disturbance wave packet, Eq. (28). The solutions of Eq. (29) are found to have singularities (bursts?, "turbulent spots"?) both for $Re - Re_c > 0$ and $Re - Re_c < 0$. The most recent work by this group is by Hocking[45] on the asymptotic suction profile; thus far this approach has not been extended to flows with slowly varying mean properties. Gaster's experiments (to be published; see also Ref. 46) on centered disturbances in laminar flow confirm the general features of this approach.

D. FREESTREAM DISTURBANCES AND INITIAL DISTURBANCE AMPLITUDE

To the extent that one accepts as plausible the concept that hydrodynamic instability and the beginning of transition are associated with the attainment of some absolute minimum level of disturbance amplitude, it seems to follow that one must assign a key role to the *initial* amplitude and spectrum of the disturbance. For most practical cases of interest the most likely source of this initial disturbance field is the freestream.

Early attempts to incorporate freestream disturbance levels into ad hoc transition criteria were made by Liepmann[17] and Van Driest and Blumer.[47] As noted above, Benney and Bergeron suggested that the additional neutral modes found in their nonlinear critical layer analysis, for which the phase speed is close to the freestream velocity and the critical layer is near the outer edge, may be a possible mechanism for freestream disturbances to enter the boundary layer. It seems that neither they nor other investigators have followed up on this possibility. The forced response of the boundary layer to disturbances externally imposed at the outer edge has been calculated by Criminale[48,49] and Mack.[50] In the same spirit, Rogler and Reshotko[51,52] have studied the interaction between a boundary layer and a low-intensity array of single wavenumber vortices convected at the mean freestream velocity. This latter work has consisted of linear analyses based on the parallel-flow Orr-Sommerfeld equation made inhomogeneous by the external disturbances.

Mack[53] employs an empirical nonlinear coupling relation between the boundary layer and external disturbances in order to account for experimental results in low-speed flow. Mack uses an amplitude rather than amplification rate criteria, and finds that the amplification rate at transition is less than e^9 when the freestream turbulence levels are low, and is greater than e^9 when they are high. More recently, Mack's[54] calculations predict that the amplification rate at transition is e^9 for a low-speed flat plate boundary layer when freestream disturbance levels are .07 percent.

It should be noted that the incorporation of the initial disturbance field into an "e^n" calculation is plausible only if the initial amplitude is sufficiently small.

E. DISCUSSION AND CONCLUSIONS

1. Theoretical Shortcomings

No one has yet been able to patch together a comprehensive theory covering the entire instability-transition regime. It is significant that almost none of the nonlinear stability analyses compare results

with experimental data--they almost exclusively are presented as qualitatively representing the post-primary stages of breakdown. Specifically, here are some of the weak points of these theories that keep us from attaining a comprehensive treatment:

1. The initial amplitude of the disturbance and its spectral distribution within the boundary layer are unknown, in particular as they are influenced by the external disturbances. This is a problem for the linear as well as the nonlinear theories, and as Mack[50] points out will remain a problem even when it becomes possible to solve the full nonlinear, three-dimensional, time-dependent Navier-Stokes equations. Without knowledge of the initial disturbances, transition prediction is always likely to remain a partially ad-hoc or empirical enterprise.

2. Almost all the nonlinear theories of hydrodynamic stability are based on the smallness of an amplitude parameter ε which is proportional to the difference $(Re - Re_c)$, where Re_c is the minimum critical Reynolds number. These theories are valid therefore only for Reynolds numbers near and slightly greater (or less) than Re_c. This limitation is inherent in their representation of the growing disturbance as the mode corresponding to the single unstable wave at Re_c, and the assumption that the amplitude of this mode, although not infinitesimal, is small. They would seem to be invalid then for those wave numbers α which according to linear theory are most amplified (and therefore seemingly most implicated in the transition process). For such α generally first begin to amplify at Re much greater than Re_c, and therefore whereas this newly growing disturbance mode will be infinitesimal when this particular Re, say Re_o, is reached, all the modes which are unstable between Re_c and Re_o will have had time to grow, so that the disturbances at Re_o can be assumed to be neither small nor monochromatic.

2. Impact on Prediction Methodology

Our inquiry into the state of current nonlinear stability theory has been disappointing. The theory involves difficult mathematics, and often obscure physics. It can represent interesting features of late stages of the transition process, provided that disturbance levels are large enough to trigger transition near the minimum critical

Reynolds number. Moreover, the theory is not intended for those important situations in which transition is delayed by shaping, heat transfer, or other means, and where heroic attempts are made to reduce the effects of disturbances.*

Unfortunately, those who require estimates of transition location, for purposes of design and performance optimization, must still make do with more or less ad hoc engineering criteria, or, at best, calculations of the "e^9" type. If disturbance levels are similar to those involved in the original "e^9" data base, then such calculations still appear to be the best and most reliable method for locating transition. In particular, if (1) the initial amplitude of the disturbance is very low, whether ingested from the main stream or internally generated in the boundary layer, and only excites Tollmien-Schlichting waves; and (2) the disturbances are slowly amplifying, as in a boundary layer with a favorable or mildly adverse pressure gradient, so that the disturbance growth over most of the distance to transition is exponential, then the "e^9" method may reasonably be expected to be applicable to predicting transition. However, any real-world departures from the ideality implicit in the "e^9" method must be dealt with on an empirical basis. Thus, to the extent that roughness or vibration are responsible for influencing the flow in ways other than by producing small disturbances that excite or feed Tollmien-Schlichting waves, it is unreasonable to expect the "e^9" method to account for their effect upon transition.

There appears to be little in current nonlinear theory which bears on these problems of design and prediction, or even provides rationalization of the success of the "e^9" approach. The same state of affairs holds at present (see Fasel[55]) for numerical solutions of the Navier-Stokes equations, and the situation with model equations is even more problematic.

There is clearly a need for a new, perhaps less formal, nonlinear theory, which combines the growth rates and frequency dependency of

*These comments, and indeed most of this report, are intended to refer specifically to attached shear layers, such as boundary layers, and not to free shear layers, such as wakes. For the latter class of flows, which are inviscidly unstable, disturbances grow more rapidly, and nonlinear mechanisms come into play much earlier in transition.

the two-dimensional Tollmien-Schlichting waves (which is central to the "e^9" method) and the three-dimensional nonlinear effects described earlier. Until a nonlinear theory can describe transition far downstream of the minimum critical Reynolds number, there is little hope for its relevance to problems of underwater low-drag hydrodynamics. Until that time, the remarkable "e^9" criteria for boundary-layer transition will not be supplanted.

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