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A Dynamic Model for Optimum Bonus Management

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DEFENSE ADVANCED RESEARCH PROJECTS AGENCY

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Formulates an economic framework for managing the
military bonus program, in which bonuses serve
two functions: to create permanent pay differentials
between specialties and to reduce temporary shortages.
The dynamic adjustment model determines the optimum
time path of bonuses for different year groups in a
specialty, subject to the constraint that deviations
from the desired manpower inventory be eliminated
over the assigned period. The optimum structure of
bonuses minimizes the sum of two costs--bonus cost
incurred to reduce shortages and penalty cost assigned
to shortages and overages. Concepts currently used
as rules of thumb by bonus managers, such as criticality,
training costs, and substitutability are parameterized
in the model through demand, supply, and production
functions. The solution methodology and results of
the computer simulation are presented. They demon-
strate that the policy prescribed by the optimization
model results in substantial savings, relative to a
year-group management or a no-bonus policy. (See also
R-1940/1). Bibliog. (Author)

PREFACE

This report was prepared as part of Rand's DoD Training and Man-power Management Program, sponsored by the Cybernetics Technology Office of the Defense Advanced Research Projects Agency (ARPA). With manpower issues assuming ever greater importance in defense planning and budgeting, it is the purpose of this research program to develop broad strategies and specific solutions for dealing with present and future military manpower problems. The goals include the development of new research methods for examining broad classes of manpower prob-lems, as well as specific problem-oriented research. In addition to analysis of current and future manpower issues, it is hoped that this research program will also contribute to a better general understand-ing of the manpower problems confronting the Department of Defense.

Because of its flexibility, the bonus program is potentially one of the most efficient forms of compensation for enlisted personnel. How far this potential efficiency is exploited depends on the criteria that are used to determine when bonuses should be paid. This report formulates bonus management as an economic problem. It provides a framework for a rational determination of optimum bonus levels for a specialty, based on such considerations as training cost, criticality of the specialty, and manning levels in other specialties. These cri-teria are currently considered by specialty managers, but in the absence of an integrating framework in which tradeoffs among the several often incompatible criteria are specified, actual decisions are inevitably based on rules of thumb.

The objective of the model is to minimize the sum of two sources of cost--bonus payments and a penalty cost assigned to shortages--over a period of several years, subject to the constraint that the desired manpower inventory be attained by the end of the period. Results of the computer simulation model show that, taking into account both bonus and shortage costs, the policy prescribed by the model results in sub-stantial savings compared with either a year-group management or a policy

of simply paying no bonuses. The computer simulation model is described
in Roy Danchik, *A Dynamic Model for Optimum Bonus Management: Computer
Program and Mathematical Analysis*, R-1940/1-ARPA, forthcoming.

SUMMARY

Tnis report provides an economic approach to the problem of managing the military bonus program. The bonus program was designed to provide the Department of Defense (DoD) with a flexible form of compensation that could be used to stimulate enlistment and retention of enlisted personnel in particular specialties. The great advantage of bonuses is that they can be targeted at specific specialties or year groups within specialties that are experiencing shortages for variable lengths of time. Because bonuses can be applied selectively, they are potentially a much more cost-effective form of compensation than basic pay, which varies by grade and years of service but is uniform across specialties. If basic pay were the only policy tool available, unnecessarily high wages would have to be paid in attractive specialties in order to eliminate shortages in unattractive specialties.

Administration of the bonus program has suffered from the lack of a framework for determining where bonuses should be paid. The manpower distributions generated by the services' objective force models are adopted as targets, and bonuses are considered where necessary to attain these targets. However, because of budget constraints, not all targets can be reached, and the practical problem is one of trading off between bonus costs and shortages. Bonus levels are determined by taking into consideration such factors as shortages or surpluses in year groups other than the one immediately eligible for a bonus, the cost of training a new recruit relative to the cost of retaining an experienced man, whether the specialty experiences persistent shortages in the career force or is merely understrength in particular cohorts, and the criticality of the specialty in the overall defense mission.

In practice, bonuses are used both to introduce permanent interspecialty pay differentials that are made necessary by underlying differences in supply and demand and to reduce temporary or cohort-specific shortages due to past or present fluctuations in supply or demand. The criteria currently applied reflect this dual role of bonuses, but up to now they have not been integrated into a single framework that yields a

single optimal set of bonuses for a given specialty, after weighing both steady state and short run considerations.

The purpose of this report is to provide such a framework. Bonus management is formulated as an economic problem, with the tradeoffs that have been made implicitly by the practical administration of the bonus program being made explicit. The solution is separated conceptually into two phases. The first phase of the solution determines the optimum normal or steady state set of bonus levels for a particular specialty simultaneously with the optimum or target manpower distribution. First term training cost is a prime determinant of optimum retention, hence of the optimum steady state reenlistment bonus. The second phase determines the optimum temporary deviations from the normal bonus set, given the initial inventory and the constraint that the target inventory be attained within a fixed number of periods. The steady state model developed by Jaquette and Nelson (1974) can be used for the first phase. This report presents a dynamic adjustment model of the second phase. The larger question of allocating a fixed bonus budget among specialties is not addressed.

Overall specialty strength is measured by a production function that is a weighted aggregation function of men in different years of service. It formally structures the notion that the importance of a shortage in one year group depends on manning levels in other year groups and on substitutability between them. Deviations of specialty strength around the target level are evaluated with a demand function. The elasticity of demand is chosen to reflect the criticality of the specialty, which depends on the availability of men in other specialties with similar skills. The supply function embodies the internal supply structure of a military specialty, in which the stock of future senior men is determined by retention of current junior men.

Because the optimum bonus for any one year group depends on present strength within that and other year groups and on the effect of current policy on future strengths, optimum bonus levels for all eligible year groups at all points in the time horizon are solved for simultaneously. The optimum structure of bonuses over time is defined as the structure that maximizes net benefits, subject to the constraint that the arbitrarily chosen starting manpower inventory reach the desired inventory

within the number of periods assigned. Maximization of net benefits is shown to be equivalent to minimizing the sum of the two sources of cost--the bonus cost incurred to reduce shortages and the penalty cost associated with deviating from target strength.

The solution methodology and some illustrative results of the computer simulation model are presented. The total cost of following the optimum bonus policy derived from the model is compared with the cost of both strict year-group management--i.e., paying the bonus neces- sary to achieve target strength in each year group in each year--and with a policy of paying no bonuses but simply setting the accession rate at the steady state level and waiting for these optimum size cohorts to flow through the system. With two alternative sets of parameter values, the cost of year-group management exceeds the cost of the optimum policy by between 30 and 70 percent; and both policies show large savings over a no-bonus policy.

In its present form the computer simulation model accommodates eight year groups and an enlistment and a reenlistment bonus. It could, in principle, be expanded to handle a more realistic representation of a military specialty and modifications necessary to include training as an alternative policy tool to bonuses, which are described in Appen- dix A. The model illustrates the usefulness of this framework for managing bonuses. It demonstrates that a flexible policy that takes into account conditions of substitutability, supply, and criticality permits substantial savings over a rigid policy of year-group management.

ACKNOWLEDGMENTS

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I. INTRODUCTION

INSTITUTIONAL BACKGROUND

The bonus program was designed to provide the Department of Defense (DoD) with a tool to stimulate retention of enlisted personnel in military specialties that are experiencing shortages. The distinguishing characteristic of bonuses as a form of compensation is flexibility. In contrast to basic military pay, which varies by grade and years of service (YOS) but is uniform across specialties and can be revised only by act of Congress, bonuses are paid to particular year groups in designated specialties. These designations can be changed at any time but are normally revised biannually.[1] In the absence of bonuses, rigidity of the pay structure results in shortages and surpluses, because supply and demand conditions vary across specialties at any one time and may vary over time, within any one specialty. If basic pay were the only policy tool available, substantial rents would have to be paid in attractive specialties in order to eliminate shortages in unattractive specialties.

The flexibility of bonuses makes them a potentially more efficient form of compensation than basic pay. How far this potential is exploited depends on the efficiency of the criteria used to determine actual bonus levels. Currently, several not necessarily consistent criteria are used to select specialties qualifying for a bonus. However, there has been no systematic framework for assigning appropriate weights to the various criteria or determining exact bonus levels once the necessary conditions of eligibility are met.

The problem of determining optimum bonus levels is analogous to the problem of setting optimum wages faced by a firm. In the context of the military services, there are several complicating factors.

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[1] Under current authorization (P.L. 93-277), and regulations prescribed by the Secretary of Defense, bonuses can be awarded at three periods of service in a specialty: (1) an enlistment bonus, up to $3000, payable on completion of training; (2) a selective reenlistment bonus (SRB Zone A) at reenlistment between two and six years of service, up to $12,000 ($15,000 maximum for nuclear-trained personnel); and (3) a second selective reenlistment bonus (SRB Zone B), at reenlistment between six and ten years of service, up to $12,000 ($15,000 maximum for nuclear-trained personnel).

First, unlike the firm of standard economic theory, a military specialty does not produce a measurable output, sold at a market price, that can be used as an estimate of the value placed by society on this particular use of resources. If bonuses are to be used efficiently to reduce manpower shortages, efficient target manpower levels must be defined. This requires a measure of the effective strength or output corresponding to various configurations of men of differing levels of skill within the specialty and of the value placed on this output. Once efficient target strength levels have been defined for men in each skill level within a specialty, the bonus problem is solved if these optimum levels are taken as binding constraints and bonuses are the only policy tool available. In practice, targets are not binding because funds have alternative uses and there are alternative policies for obtaining men in shortage specialties, such as retraining from specialties experiencing surpluses. Thus, efficient bonus management requires a measure of the cost or penalty associated with deviating from target levels for any skill group within a specialty and the cost of bonuses relative to the costs of alternative policies.

Second, in addition to the difficulties associated with measuring the level and value of output of a military specialty, the problem of defining optimum bonus awards is complicated by the dynamic characteristics of labor supply. The firm of traditional competitive theory can vary its inputs over time by renting or buying and selling in an external market. The military services, by contrast, recruit senior personnel almost exclusively from the ranks of more junior personnel within the same specialty or group of specialties that constitute a career management field.[1] Individuals cannot normally be discharged within a term of service and have implicit tenure after the second reenlistment until they reach retirement eligibility at 20 years of service.

These supply characteristics of a military specialty imply that optimum bonus policy cannot be determined within the framework of a

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[1]Reliance on in-house supply is probably partly an efficient response to the specific nature of human capital, since specialties are defined in terms of job content, partly an attempt at equity and simplicity in promotion policies.

static model that abstracts from actual inventories. Optimum current bonus levels for junior personnel depend not only on expected relative to target strengths in the year groups eligible for a bonus, but also on shortages or surpluses in year groups not amenable to control by bonuses or other policies and on the implications of current stocks of junior personnel for the future supply of senior personnel. For example, even if the projected number of first term reenlistments is equal to the desired number without a reenlistment bonus, if there is currently a shortage of senior men (for whom second termers are close but not perfect substitutes), it may be optimal to pay a bonus and over-shoot temporarily. Several years of undershooting might follow when the current senior year groups have been replaced by the cohorts that are overstrength.

Optimal bonus management requires recognition of the dual role of bonuses within the present institutional framework of military compensation:

1. Bonuses are to be used to obtain the desired steady state distribution of manpower by year of service (YOS). Used in this context, bonuses would be a permanent component of pay in some specialties, introducing permanent pay differentials across specialties. Such differentials may be optimal be-cause supply conditions may vary even if desired retention rates (demand) are uniform across specialties. For example, in the absence of second term pay differentials, actual re-tention will be negatively related to first term training content if military training has value in the civilian sector. The effects of supply-induced persistent shortages in high training specialties will be magnified if, as dictated by cost effectiveness, desired retention rates are positively related to first term training.[1]

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[1]Use of bonuses for steady state objectives is likely to become more prevalent with the phasing out of Shortage Specialty Proficiency Pay. To the extent bonuses are viewed by enlistees as uncertain, being reviewed biannually, bonuses are an inefficient form of permanent com-pensation, since expected bonus payments will be discounted by the re-cipient, assuming risk aversion.

2. Bonuses are to be used to eliminate temporary shortages. The existing manpower inventory reflects vagaries of supply and demand over the previous 30 years and may be very different from the current desired inventory, which in turn may differ from the expected future desired inventory. Within a context of fluctuating demand and supply, the internal labor supply structure, which severely limits the specialty manager's control on cohort strength beyond the initial entry point, becomes extremely costly. It implies, for example, that a current shortage of senior skilled personnel can be eliminated only by increasing accessions and waiting for the larger cohorts to work their way through the system. Bonuses enable the manager to eliminate the shortage more rapidly by temporarily raising retention rates from existing trained cohorts.

The criteria currently used by the services in formulating their bonus requests to DoD, and by DoD in evaluating these requests, reflect both these steady state and short-run adjustment considerations.[1] The Army, Navy, and Air Force have objective force models that generate desired distributions of men by YOS.[2] Projection models are then used to predict shortages or surpluses relative to desired manning levels. Bonus requests are based on the predicted shortages. The Navy looks at shortages over zones of three YOS groups (YOS five to seven for a SRB Zone A, YOS eight to ten for SRB Zone B). One-step adjustments upward or downward from steady state bonus levels are made for deviations of greater than 10 percent from desired levels. The Air Force adopts a more narrowly focused year-group management policy, looking at projected shortages in the single year group affected by the bonus, and "will normally consider paying a bonus if there is an anticipated shortage of more than 10 percent."[3] Overall manning in the specialty and

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[1] The description of service practices draws on an unpublished report of meetings with representatives of the services in December 1974 (Nelson and Enns, 1975).

[2] The Navy's ADSTAP system of models also solves for steady state bonus award levels by rating. These models are discussed in detail in Jaquette, Nelson, and Smith (forthcoming).

[3] OASD (1974), p. 2.

training costs relative to bonus costs are also considered.[1] The Army as yet has no systematic formula to derive bonus requests from projected shortages. The Marine Corps has no formal objective force or projection models but considers training costs, manning levels, and past bonus levels to determine requests.

As guidelines for evaluating service requests, a recent DoD instruction on the subject lists the following characteristics as qualifying a specialty for an SRB Zone A:

1. Serious undermanning in a substantial number of adjacent career years (three or more) which can be affected by the bonus.
2. The bonus will have a significant effect on decreasing career manning shortages in these problem career years.
3. Chronic and persistent shortages in total career manning.
4. High first term replacement costs, including training.
5. Skill is relatively unattractive compared to other military skills or civilian alternatives.
6. Skill is essential to the accomplishment of Defense missions.
7. Even if the foregoing criteria are not completely satisfied, the SRB level will not be reduced by more than two increments in a given fiscal year.[2]

The precise level of bonus to be awarded is based on a "balanced evaluation" of these criteria.[3] Either implicitly or explicitly, most of the considerations that arise if bonus management is viewed as a strictly economic problem are taken into account in these criteria. What is lacking is a systematic way of reconciling conflicts between the indications of the several criteria and determining precise bonus amounts. This task has been relegated to judgment. When decisions are based on judgment, consistency across specialties, within and between services, is difficult to achieve and even more difficult to prove.

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[1] Ibid.

[2] Department of Defense (1975).

[3] Ibid.

The approach taken in this report is to formulate the bonus management problem as an economic problem, integrating the several relevant considerations into a single framework and explicitly stating the trade-offs that must be made, and have been made implicitly, by decisionmakers. Judgment is not eliminated but is structured as the explicit selection of the values of the parameters of the model.

ECONOMIC FORMULATION OF THE BONUS MANAGEMENT PROBLEM

Assuming that bonuses are to continue to be used to manage both steady state and temporary shortages, the optimal bonus policy for a particular specialty over time may be determined in two stages:[1]

1. The optimal steady state level of bonuses is derived from a static optimization model. In the static model, the problem is to find the cost-minimizing distribution of men by YOS and the corresponding structure of steady state wages necessary to attain the desired manning levels, subject to a constraint on the overall strength of the specialty.[2] The model developed by Jaquette and Nelson (1974) is a prototype of this sort. Once optimum steady state wages are determined, optimum steady state bonuses are the difference between optimum and constrained actual wages. Note that the usefulness of the concept of a steady state bonus does not require the force to attain steady state, which it obviously never does. A steady state bonus is simply the bonus that will normally have to be paid if supply and demand conditions are expected to remain stable for several years.

2. The optimum temporary or transition phase structure of bonuses is then determined from the dynamic adjustment model, which also yields the optimum path of manpower from the starting

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[1]The problem of simultaneous solution of optimum bonus policy in all specialties is ignored.

[2]If steady state wages are considered not variable for some year groups, for example those beyond eligibility for SRB Zone B, the model can be estimated subject to this constraint.

inventory to the steady state. This is the model discussed here. Bonuses in this model are to be interpreted as deviations, positive or negative, from steady state wages. To avoid confusion, the term "wage" will be used to refer to steady state wages. Thus, in the adjustment model, when no bonus is being paid, it is to be understood that the level of compensation is equal to the steady state wage.

Viewed as a problem of finding the optimum long-run equilibrium level of inputs and the optimum rate of adjustment to the new level from a disequilibrium starting point, bonus management is closely analogous to the problem of disequilibrium factor demand addressed by Eisner and Strotz (1963), Lucas (1967), and Nadiri and Rosen (1973). The distinguishing characteristic of these models is that, because the cost of changing factor stocks is positively related to the rate of adjustment, the optimal adjustment path involves a lagged rather than an instantaneous adjustment to the new long-run equilibrium level. The internal supply structure of a military specialty implies similar time-related adjustment costs because the elasticity of supply of men in a particular YOS is positively related to the time allowed for a specified percentage increase in strength. In the short run, an increase in second term pay, for example, increases the supply of second termers only by stimulating retention from the existing cohorts of first termers. In the long run, however, the increased retention rate operates on a larger pool of eligibles, assuming that the first term accession rate responds to the increase in second term pay.[1]

The models developed in the economic literature specify a long-run supply function and a short-run adjustment cost function and solve simultaneously for the long-run equilibrium input and output values and the optimal adjustment path of all variables to these values from their initial levels. These models are not immediately applicable to the military, for two reasons.

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[1] It seems plausible to assume that the decision to enter the military is based on expected earnings over the entire military career.

First, because of the problem of measuring and evaluating output, it seems foolhardy to make the steady state level of output, inputs, and wages depend on a necessarily arbitrary demand function. By separating the steady state and adjustment components, the steady state problem can be formulated as one of cost minimization subject to an output constraint, where the constrained level of output is obtained by applying a production function to the service target input levels. Such a model yields optimal input and wage levels subject to a realistic output constraint, and the Lagrange multiplier gives the marginal cost per unit of output at this desired level of output. The equilibrium input and wage values may then be used as the target or terminal values in the adjustment model and the Lagrange multiplier as the equilibrium price. An arbitrary demand function is then used only for evaluating small deviations of output around this equilibrium level during the adjustment phase.

Second, if inputs in the military specialty are defined as men categorized by YOS who move from one category to another over time, the supply of different inputs is interrelated in a way not found in the models of civilian sector firms. However, the model developed here could be applied to any civilian institution with a rigid promotion pattern.

Section II describes the dynamic adjustment model. An overview of the model precedes more detailed discussion of the production, supply, demand, and overall objective functions for the simplest case in which bonuses are the only policy option. The solution methodology is presented with some illustrative results of the computer simulation model. The cost of following the optimum bonus policy derived from the model is compared with the cost of both strict year-group management (i.e., paying the bonus necessary to reach target strength in each year group) and with a policy of paying no bonuses but simply setting the accession rate at the steady state level and waiting for these optimum size cohorts to flow through the system. With two alternative sets of parameter values, the cost of year-group management exceeds the cost of the optimum policy by between 30 and 70 percent, and both policies show huge savings over a no-bonus policy.

II. A DYNAMIC OPTIMIZATION MODEL OF BONUS MANAGEMENT

OVERVIEW OF THE MODEL

Given an initial inventory of men by YOS and a target inventory
to be attained within a specified time period, the model solves for the
pattern of bonuses over time that maximizes net benefits over the period.
Net benefits are defined as the difference between the (social) value
of output and its (social) cost. The quantity of output is calculated
using a production function that aggregates men in the different pro-
ductivity categories into an overall measure of specialty strength or
output. In the simplest form of the model, productivity categories
correspond to YOS groupings. To convert output to dollar units for
comparison with costs, output is evaluated according to a demand func-
tion that reflects the value placed by society on the output of the
specialty. The value per unit of output at the target or equilibrium
value of output is set equal to marginal cost.[1] The value per unit of
output at other levels of output is then determined using a constant
elasticity demand function. The elasticity of demand parameter is
chosen to reflect the criticality of the specialty, which depends on
its role in the defense mission and the availability of substitutes
from other military specialties or from civilians.

Cost is measured as wages plus bonuses minus inframarginal rents.
On the assumption that military compensation is equal to the supply
price of the marginal recruit, and that this is equal to his potential
civilian wage, which measures his social value in the civilian sector,
this measure of cost corresponds to social opportunity cost. The supply
functions of men in year groups eligible for a bonus incorporate the
steady state reenlistment rate plus a linear response to the bonus.
For a year group not amenable to control by bonuses, the supply func-
tion simply reflects steady state continuation from the previous year
group.

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[1]Marginal cost is equal to the value of the Lagrange multiplier
obtained by solving the steady state problem of finding the cost-
minimizing input mix, subject to an output constraint.

The objective function to be maximized is the sum of net benefits over the transition period. Bonuses are chosen to maximize this function subject to attaining the desired inventory at the terminal time, T. The objective function reduces to a quadratic loss function in deviations of actual from desired input levels, with the penalties assigned to deviations from target being derived from the parameters of the demand, production, and supply functions. Thus, considerations currently used in an *ad hoc* manner by bonus managers--criticality of specialty, substitution possibilities between year groups in the specialty, and the effectiveness of a bonus in reducing a shortage--are systematically related in the model. Outputs of the model include optimum bonuses, distribution of men by years of service, penalty costs, and bonus costs in each year. In addition, the solution methodology generates the shadow value of men in each year group, which indicates where other control policies, such as cross training or early separations, might be used to reduce total costs.

PRODUCTION FUNCTION

The output of the specialty is measured using a nested constant elasticity of substitution (CES) labor aggregation function of the general form:[1]

$$Z = \left[\sum_{i=1}^{n} \delta_i X_i^{-\rho} \right]^{-\frac{\mu}{\rho}},$$

and

$$X_i = \left[\sum_{j \epsilon i} \delta_j^{(i)} L_j^{(i) -\rho_i} \right]^{-\frac{1}{\rho_i}},$$

where Z = specialty output,

X_i = i*th* composite input, and,

$L_j^{(i)}$ = j*th* basic input in i*th* composite.

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[1]Sato (1967); Bowles (1970).

Overall specialty output, Z, is a CES function in [X], and X_i in turn is a CES function in $[L^{(i)}]$. To illustrate the application of this production function to a military specialty, [L] may denote the set of basic inputs within which individuals are perfect substitutes, such as YOS. [X] may denote terms of service. Thus, L_2^1 denotes the number of men in the second year of the first term. X_i denotes number of quality adjusted man-years in the first term.[1] The two-tier formulation permits variation in the elasticity of substitution between pairs of basic inputs. For example, it is possible to specify that men in different year groups in the third term are better substitutes for each other than men in different year groups in the first term, and men in different terms are poorer substitutes than men in different year groups within the same term.[2]

This production specification embodies several simplifying assumptions. If [L] and [X] are interpreted as YOS and term of service respectively, it is implicitly assumed that experience on the job is the sole determinant of productivity. Other productivity-related variables, such as quality of accessions and formal military training, and the associated possibilities of substituting quality for quantity are discussed in Appendix A, together with appropriate modifications of the supply and objective functions.

The overall production function of defense output is assumed separable in labor in each specialty and capital. Suppressing other inputs from the individual specialty production functions presupposes that the relative productivity of different labor categories is independent of other factors. To the extent this is not true, the derived wage and manpower distributions will not be optimal. In principle, however, other factors can be accommodated as separate tiers of the production function. Since they are omitted and assumed fixed in planning bonuses for a single specialty, the returns to scale parameter, μ, is chosen to be less than unity, to reflect diminishing returns to labor. Measuring

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[1]Alternatively, [X] may denote skill levels or grades.

[2]The year-group management approach, which sets individual year-group requirements, implies zero elasticity of substitution between year groups. Rigid application of this approach would be unrealistic and is not employed by DoD in evaluating the services' bonus requests.

all inputs as stocks rather than flows of services precludes the possibility of meeting temporary demand fluctuations by varying rates of utilization of existing stocks.[1] The cost of this option is hard to specify, since the military does not pay for overtime. To the extent it is a useful option, bonus levels derived from the model will exceed the true optimum.

Selection of the parameters of the production function is largely a matter of judgment rather than hard empirical evidence. The techniques used to estimate the substitution parameters of private sector production functions from either time series or cross-sectional response to changes in relative factor prices cannot be applied to military data.[2] In addition to the problem that technology and other factor inputs cannot be assumed constant across specialties or time periods, the basic assumption underlying the procedure, that the input mix is designed to minimize costs, is untenable for the military, at least without allowing for lags of unknown length. The parameters are therefore estimated subjectively, and the model is programmed to facilitate sensitivity analysis of the outcome with respect to all input parameters.

SUPPLY FUNCTION

In the simple model with inputs defined as men categorized by YOS, the supply of men in the ith YOS in year t consists of reenlistments from the previous year group at a constant rate determined by steady state wages, plus the increment induced by the bonus:[3]

$$L_{i,t} = \alpha_{i-1} L_{i-1,t-1} + \beta_i B_{i,t} + K_i , \qquad (1)$$

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[1] See Nadiri and Rosen (1973).

[2] For example, Bowles (1970). For military specialists with close civilian counterparts, estimates of production parameters from civilian data might be used.

[3] Modifications of the model necessary to include cross training as an alternative source of supply are described in Appendix A.

where $L_{i,t}$ = number of men in the ith YOS in the specialty in year t,

α_{i-1} = continuation rate from YOS_{i-1} to YOS_i at steady state wage levels (zero bonus),

$B_{i,t}$ = bonus for reenlistment into YOS_i in year t,

β_i = parameter derived from supply elasticity, and

K_i = steady state lateral entry flow into YOS_i. For $i \neq 1$, K_i may be zero.

This formulation captures the essence of the internal supply structure, although the simplifying assumptions required to obtain a numerical solution to the model do considerable violence to reality. In particular, the parameters α_i and β_i are assumed independent of the size of the cohort eligible for reenlistment. If civilian opportunities or tastes for the military are positively correlated across individuals at all points in their careers, then it is likely that marginal continuation rates will be less than average continuation rates and the average continuation rate, α, will be inversely related to the size of the eligible cohort. Conversely, the absolute response to a given level of bonus award, β_i, is likely to be positively related to the size of the eligible cohort. The specification used here incorporates biases in offsetting directions, with an overestimate of the continuation rate and an underestimate of the bonus effect in the case of an abnormally large cohort, and conversely in the case of a below average size cohort. Bonuses derived from the model should be interpreted as upper and lower bounds on the true optimum for cohorts that substantially exceed or fall short of steady state size.

The assumption that payment of a bonus stimulates supply only in the cohort immediately eligible contrasts with the supply specification of a steady state model, in which (re)enlistment decisions are based on expected earnings over the entire military career, not just the immediate term of service.[1] The dichotomy is appropriate because, by definition, steady state wages correspond to long-run average pay, hence provide a rational basis for calculating expected career earnings.

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[1] See Jaquette and Nelson (1974).

Bonuses, on the other hand, interpreted strictly as transitory devia-
tions from steady state pay, will be perceived to vary from year to
year. It would then be irrational to base expectations of future
bonuses at more senior YOS on current bonuses at those YOS. Given this
uncertainty as to future bonuses, transitory deviations from steady
state supply to a particular term of service are likely to be dominated
by current bonus payments for that term.

The specification in Eq. (1) accommodates both internal and lateral
entry. Within a term of service, the previous year group within the
same specialty is the exclusive source of supply. At the entry point
to a term of service where bonuses are authorized, internal supply is
augmented by bonus-induced reenlistments. The response to the bonus
may but need not be constrained by the number of men in the previous
year group.[1] At the other extreme, if lateral entry is unrestricted and
no more costly than drawing from within the specialty, and if firing
within a term is costless,[2] this can be modeled by omitting the reenlist-
ment term and specifying a positive steady state flow of lateral entrants:

$$L_{i,t} = K_i + \beta_i B_{i,t} \; .$$

Other features that complicate bonus management in practice are
ignored. All reenlistments occur at the beginning of an accounting
period, in response to the bonus set in that period. Early reenlist-
ments and extensions in anticipation of changing bonus levels are not
permitted. The term of commitment is the same for all individuals in

_____

[1] In practice, given realistic assumptions about no-bonus reenlist-
ment rates, such a constraint is unlikely to be binding.

[2] Introducing the possibility of early-out programs would require
modification of the supply functions of both the steady state and ad-
justment models. Risk averse individuals will attach a positive value
(hence, accept lower per period wages) to a commitment of guaranteed
employment for a fixed term from the employer, but attach a negative
value (hence, require higher wages) to committing themselves to serve
for a fixed term. Thus, the net effect of a mutual obligation to a
fixed term of service is uncertain *a priori* and may vary with the length
of the term. However, if the obligation is relaxed on the employer's
side only, as implied by making early-out programs a policy variable,
the supply curve would shift to the left.

a particular YOS[1] and is constant over the time horizon. The estimated cost of an optimum bonus policy under this constraint will be an upper bound on the costs that would result from optimizing simultaneously with respect to bonus policy and *term of commitment.* *Bonuses are paid* in a lump sum at the reenlistment point.[2]

DEMAND FUNCTION

The role of the demand function is to assign a dollar value to deviations from the target level of output in order to weigh the benefits of moving closer to target against the cost, in the form of bonus payments. A constant elasticity functional form is used:

$$P = \alpha \, Z^{-(1/\varepsilon)}$$

where P = price per unit of specialty output,

  α = a scale parameter, and

  ε = elasticity of quantity with respect to price.[3]

The elasticity parameter ε is an own price elasticity—*i.e., it* assumes constant manning levels in related specialties, both substitutes and complements. It is a crude measure of the criticality of the specialty. For example, the elasticity of demand will be low for a combat arms specialty that is crucial to the defense mission and has no close

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[1] The terms of service may differ in length.

[2] The current method of payment in installments over the term can, in principle, be modeled, but it adds complexity because of the dichotomy introduced between cost to DoD and value to the recipient. This problem is handled in the steady state context in the Jaquette-Nelson model. Ignoring it is less serious in the case of a bonus payment, extending typically only over a four year term, than it would be for an entire career earnings stream.

[3] The discussion is in terms of ε, the elasticity of quantity with respect to price, to conform to the conventional definition of the elasticity of demand. The demand specification would be more complex in a complete, general equilibrium formulation of the bonus management problem, in which optimum bonuses for all specialties would be determined simultaneously, subject to an overall budget constraint. The demand function for the output of an individual specialty in that model would include own and cross price elasticities.

substitutes. In principle, the range of possible substitutes extends over different specialties within one service, as well as across services and to civilians. In general, the demand elasticity is likely to be lower if the model is applied to a career manning field (CMF) rather than specialty by specialty within a CMF. The value of the parameter α is found by equating price to marginal cost at the steady state level of output and solving the demand equation for α.[1]

OBJECTIVE FUNCTION AND SOLUTION METHODOLOGY

The objective is to maximize the sum of net benefits--i.e., benefits minus costs--over the transition period. Benefits and costs are defined in terms of social rather than private values. Thus, if DoD is viewed as a producer of defense output, it is assumed to maximize social welfare rather than "private" profit. These differ because both demand and supply functions are assumed to be less than perfectly elastic.[2] Maximization of private net benefits, in the absence of price discrimination, would imply exploitation of monopsony and monopoly power by DoD in purchasing labor services and "selling" output to the public, yielding wage rates less than the value of marginal product of labor and levels of output at which marginal social value exceeds marginal social cost.

An implication of maximizing social rather than private net benefits is that the model generates an optimal labor force mix that does

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[1]
$$P_0(Z_0) = \alpha Z_0^{-(1/\epsilon)} = \lambda(Z_0)$$

where λ = marginal cost.
Marginal cost is given by the value of the Lagrange multiplier obtained by solving the steady state model for the cost-minimizing input mix, subject to producing the output level Z_0.

[2] If perfect price discrimination is exercised in both product and factor markets, social and private benefits and costs converge. Thus, the formulation of the model can alternatively be interpreted as maximizing private benefits with perfect price discrimination. Bonus awards in practice are multiples of individual base pay, which differs by grade, hence across individuals in the same YOS. To the extent this variation in military pay is positively correlated with variation in supply price across individuals, some degree of price discrimination exists in practice.

not minimize DoD budget cost, for a given level of output.[1] This results from treating intramarginal rents as a transfer payment, not a cost of production. As a result, the optimal factor mix contains more of factors in fairly inelastic supply than would the budget cost minimizing factor mix. In any case, maximization of social rather than private benefits is a particular specification, not a necessary feature of the model. The private benefits maximizing formulation is given in Appendix B.

Given certain assumptions, the social benefit (SB) of the output of a military specialty may be measured by the area under the output demand curve, and social opportunity cost (C) by the area under the social factor cost curves.[2] The objective function is then

$$\max \sum_{t=1}^{T} \left\{ \int_{0}^{Z_t} P_t[Z_t(L_t)]dZ_t - \sum_{i=1}^{n} \left[\int_{0}^{L_{i,t}} W_i(L_{i,t})dL_{i,t} \right] \right\}$$

where $P(Z)$ = demand function for specialty output,

  L = n-dimensional vector of labor inputs,

  T = terminal time of planning horizon, and

  $W_i(L_i)$ = cost (inverse supply) function of i*th* labor input.

The expression within the braces represents net social benefits in period t. Taking a second order Taylor expansion of this function with respect to L_t, the vector of labor inputs, yields:

$$SB_t(L_o) + \frac{\partial SB_t}{\partial L_t}\bigg|_{L_o} (L_t - L_o) + \frac{1}{2}(L_t - L_o)' \frac{\partial^2 SB_t}{\partial L_t^2}\bigg|_{L_o} (L_t - L_o)$$

$$- C_t(L_{*t}) - \frac{\partial C}{\partial L_t}\bigg|_{L_{*t}} (L_t - L_{*t})' - \frac{1}{2}(L_t - L_{*t})' \frac{\partial^2 C}{\partial L_t^2}\bigg|_{L_{*t}} (L_t - L_{*t}) \ .$$

---

[1] With perfect price discrimination, social cost minimization coincides with DoD budget minimization.

[2] The rest of the economy is free of distortions; the DoD demand curve is a compensated demand curve and reflects the value placed by society on defense output; the supply price of labor to DoD reflects its civilian opportunity cost. A zero rate of discount is assumed.

Benefits are expanded around L_o, the vector of target input levels. Costs are expanded around L_{*t}, the level of input that would be obtained in period t from steady state reenlistment rates, induced by steady state wages, in the absence of a bonus. Costs are expanded around L_{*t} rather than L_o because the supply curves shift as a function of the size of the cohorts eligible for reenlistment. These variations in beginning-period cohort strength imply movements along a constant output demand function rather than shifts of that function, hence the expansion of benefits around L_o.

After eliminating terms that are either constant or vanish upon differentiation,[1] the objective function reduces to the quadratic terms:

$$\max_{(B_t)} \frac{1}{2} \left[\sum_{t=1}^{T} \overline{L}_t' F \overline{L}_t - B_t' U B_t \right]$$

where $\overline{L}$ = vector of deviations from target input levels,

B = vector of bonuses,

F = matrix of second partial derivatives of benefits function, and

U = matrix of second partial derivatives of cost function.

This is simply a quadratic loss function in the deviations of actual from steady state input and wage levels, $\overline{L}_t$ and B_t. The penalties assigned to these deviations are derived from the parameters of the demand, production, and supply functions. For example, for the case of two inputs, the one-period measure of net benefits is:

$$\frac{1}{2} [F_{11}(L_{1t} - L_{10})^2 + F_{22}(L_{2t} - L_{20})^2 + 2F_{12}(L_{1t} - L_{10})(L_{2t} - L_{20})$$

$$- U_{11} B_1^2 - U_{22} B_2^2]$$

---

[1] The terms involving $L_{*t+1} = \alpha L_t$ vanish upon differentiation as follows:

$$\frac{d}{dL_t} \left[-(C_{t+1}(\alpha L_t) + \frac{\partial SB_{t+1}}{\partial L_t} (\alpha L_t - L_o) \right] = \alpha(w_o - w_o) = 0 .$$

In other words, to a first order approximation, the net value in year t+1 of men added in year t is zero because their wages equal the value of their marginal product.

where $F_{ii} = \dfrac{\partial^2 SB}{\partial L_i^2} = \dfrac{dP}{dZ}\left(\dfrac{\partial Z}{\partial L_i}\right)^2 + P\,\dfrac{\partial^2 Z}{\partial L_i^2}$,

$F_{ij} = \dfrac{\partial^2 SB}{\partial L_i \partial L_j} = \dfrac{dP}{dZ}\,\dfrac{\partial Z}{\partial L_i}\,\dfrac{\partial Z}{\partial L_j} + P\,\dfrac{\partial^2 Z}{\partial L_i \partial L_j}$,

$U_{ii} = \dfrac{\partial^2 C}{\partial L_i^2} = \dfrac{\partial W_i}{\partial L_i} = \dfrac{1}{\beta_i}$.

The first three terms measure the loss in consumer surplus due to non-optimal input levels. The effects of a shortage or surplus both on a factor's own marginal product and on the productivity of other factors are included. The last two terms measure excess of short-run over long-run opportunity cost in the case of a positive bonus, and loss of producer surplus (a negative bonus) in the case where some of the men willing to reenlist without a bonus are rejected. These components of the measure of net benefits, before and after optimization, are illustrated for one factor in Fig. 1.
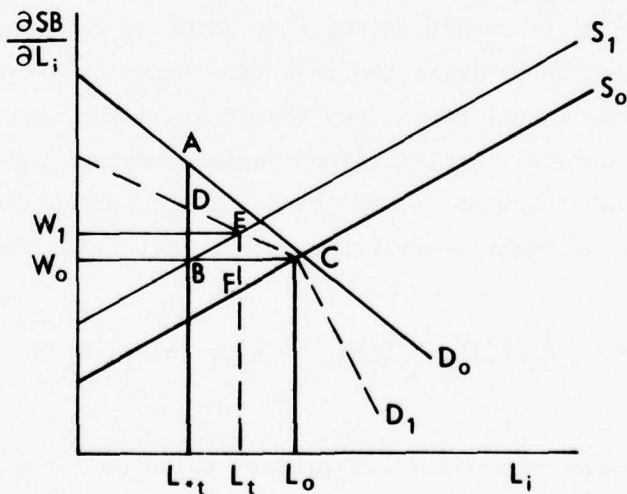

Fig. 1 — Determination of the optimum bonus for L_i

The curve D_o plots the value of the marginal product of L_i on the assumption that L_j is at equilibrium strength.[1] The dotted curve D_1 plots the value of marginal product of L_i for a particular surplus of L_j. If L_i and L_j are substitutes, $(F_{ij} < 0)$, $F_{ij}(L_{jt} - L_{jo}) < 0$. Thus, an overage of L_j reduces both the loss from a shortage of L_i and the incremental value of an overage of L_i by an amount that is proportional to the deviation of L_j. The curve S_o is the supply of L_i when the cohort of potential reenlistees, $L_{i-1,t-1}$, is at steady state strength, such that L_o men reenlist at the steady state wage, w_o. The curve S_1 is the supply curve when the eligible cohort is understrength, such that at w_o only L_{*t} men reenlist.

In the absence of any bonus payments, $L_t = L_{*t}$. The net social benefit associated with L_i, given the overage of L_j, is the loss in consumer surplus or penalty cost of the shortage, ABC - ADC = DBC. Optimization dictates paying a bonus equal to $w_1 - w_o$, which increases manning in this category to L_t. Penalty cost is reduced to CFE, but excess wage costs equal to BEF are incurred, yielding a total cost of BEC.[2]

SOLUTION METHODOLOGY

The problem is to choose the time paths of the control variables--the bonuses--which maximize the objective function subject to the constraints of the supply conditions and of attaining the target vector by the terminal period. Applying Pontryagin's maximum principle, define the Hamiltonian function, H, by joining the supply function, S(L, B), to the objective function with the vector multiplier function, λ':

$$H_t = \frac{1}{2} [\bar{L}'_t F \bar{L}_t - B'_t U B_t] + \lambda'_{t+1} \{S(L_t, B_t)\} .$$

---

[1]The demand curves reflect present value over the expected future career.

[2]This ignores optimization in the L_j market, which would rotate D_1, in order to minimize total costs over the two inputs.

λ is a vector of costate variables that are the dynamic equivalents of the Lagrange multipliers of static problems involving maximization subject to constraints. Each costate variable may be interpreted as the shadow price of the associated state variable.

First order necessary conditions for a maximum are:

$$\frac{\partial H_t}{\partial B_{it}} = 0 \ . \tag{2}$$

At each decision point, the control variables are chosen to maximize the objective function, subject to the supply constraints.

$$L_{i,t} = \alpha_{i-1} \, L_{i-1,t} + \beta_i B_{i,t} + K_i$$

$$\lambda_{i,t} = \frac{\partial H}{\partial L_{it}} \ . \tag{3}$$

The shadow price of each input is equal to its marginal contribution to the objective function, subject to supply constraints.

The first-order conditions indicate the nature of the solution. Expanding Eq. (2) for the first component:

$$\frac{\partial H}{\partial B_1} = \beta_1^2 \, F_{11}B_1 + \beta_1 \sum_{i=2}^{n} F_{1i}\overline{L}_i - \beta_1 B_{1t} + \alpha_1\beta_1\lambda_{2t+1} = 0 \ .$$

$$\hat{B}_{1t} = (1 - \beta_1 F_{11})^{-1} \left[\sum_{i=2}^{n} F_{1i}\overline{L}_{it} + \alpha_1\lambda_{2t+1} \right] \ . \tag{2'}$$

$\hat{B}_{1t}$, the optimal enlistment bonus in year t, is determined by the contribution of an L_1 to output in year t, as measured by the summation of the cross partial derivatives of the gross benefits function, weighted by the manning levels in each labor category,

$$\left(\sum_{i=2}^{n} F_{1i}\overline{L}_{it} \right)$$

plus its shadow value as an L_2 in year $t + 1$, weighted by the probability of continuing from the first to the second YOS $(\alpha_1 \lambda_{2t+1})$.[1]

From Eq. (3),

$$\lambda_{2,t} = \frac{\partial H}{\partial L_{2t}} = \sum_{i=1}^{n} F_{2i}\bar{L}_{it} + \alpha_3 \lambda_{3t+1} \ .$$

$\lambda_{2,t}$, the shadow value of an L_2 at time t, is equal to its contribution to output in year t plus its expected shadow value as an L_3 in year $t + 1$, which in turn incorporates productivity as an L_3 plus expected shadow value the next period, and so on. Thus the expected value of the *ith* input throughout *its future career* is reflected in its shadow price at time t. This is incorporated into the optimum bonus paid to that input category at time t.

The effect on the optimum bonus of substitution possibilities between input categories and of shortages and overages is evident from these first-order conditions. Since $F_{11} < 0$, the denominator in Eq. (2') is positive. The effect on $\hat{B}_{1t}$ of a shortage in the *ith* input category $(\bar{L}_{i,t} < 0)$ is positive if L_i and L_1 are substitutes $(F_{1i} < 0)$, negative if they are complements $(F_{1i} > 0)$. Conversely, a surplus of substitutes decreases $\hat{B}_{1t}$. Similarly, the effect of future shortages and surpluses is embodied in the λ_{2t+1} term. If all inputs are at target levels, all shadow prices and bonuses are equal to zero.

The first-order conditions yield a set of $2n - 2$ difference equations in the n input categories and their corresponding costate variables,[2] and m equations for the m control variables. Particular solutions are defined by the boundary conditions on the state variables, with $L(0)$ corresponding to the initial inventory and $L(T)$ the target inventory.

---

[1] For reenlistment bonuses, the summation includes an own term involving the projected no-bonus shortage.

[2] Equations for L_1 and λ_1 are excluded because the supply equation for L_1 can be simply incorporated into the objective function, dispensing with the need for adding this supply equation as a constraint. This simplification cannot be adopted for the other input categories whose supply includes continuation from previous year groups.

RESULTS

Tables 1 through 5 present results obtained from the computer simulation model. To illustrate the working of the dynamic model, in the absence of an operational steady state model from which to derive the optimum target inventory and wage levels, the target inventory and one wage level, w_j, are set at arbitrary but reasonable levels and are assumed to be optimal. The equilibrium output price, λ, is then derived from the first-order condition of a steady state model:

$$w_j = \lambda \delta_j \left(\frac{Z}{L_j} \right)^{1+\rho} .$$

In this example, average second term wages were set at \$10,000 per year.[1]

In this simple version of the model there are eight basic input categories corresponding to men in YOS 1 through YOS 8, and two composite input categories, corresponding to the first and second term. An enlistment bonus can be paid to YOS 1 and a reenlistment bonus to YOS 5. The elasticity of substitution between men in different year groups within the same term is infinite ($\sigma_1 = \sigma_2 = \infty$).[2] The elasticity of substitution between men in different terms is high in Case I ($\sigma = 10$), low in Case II ($\sigma = .25$). Continuation rates (α) between year groups are .95 within the first term, 1.0 within the second term. The no-bonus reenlistment rate from the first to the second term is .72. The supply elasticity of both first and second termers in response to a bonus is 2.0. The demand elasticity is high (2.0) in Case I, low (.15) in Case II.

Tables 1 and 3 show the manpower inventories for the two cases under the optimum bonus policy derived from the model and under a year-group management policy in which bonuses are set to achieve the target inventories in YOS 1 and 5 in each year, regardless of manning levels in other

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[1]Assuming the typical reenlistee enters the second term (fifth YOS) as an E5 and is promoted to E6 at the end of his seventh YOS, average regular military compensation over the four year term is approximately \$10,000, using October 1975 pay scales.

[2]This assumption effectively reduces the production function to a single tier CES function in two inputs, first and second term men.

Table 1

CASE I[a]--MANPOWER INVENTORIES

Year	Optimal Bonus Policy								Year Group Management							
	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
1	356	307	200	200	206	170	180	220	323	307	200	200	200	170	180	220
2	335	339	292	190	183	206	170	180	323	307	292	190	200	200	170	180
3	330	318	322	277	163	183	206	170	323	307	292	278	200	200	200	170
4	325	313	302	306	207	163	183	206	323	307	292	278	200	200	200	200
5	324	309	298	287	218	163	183	183	323	307	292	278	200	200	200	200
6	324	308	293	282	200	207	163	163	323	307	292	278	200	200	200	200
7	324	308	292	278	200	218	207	207	323	307	292	278	200	200	200	200
8	323	307	293	279	200	200	218	218	323	307	292	278	200	200	200	200
9	323	307	292	278	200	200	200	200	323	307	292	278	200	200	200	200

[a]Parameter Values

$\delta_1 = .25$ $\varepsilon = 2$

$\delta_2 = .75$ $s_1 = 2 =$ supply elasticity to first term

$\sigma = 10$ $s_2 = 2 =$ supply elasticity to second term

$\sigma_{\eta_1} = \infty$ $\beta_1 = .032$

$\sigma_{\eta_2} = \infty$ $\beta_2 = .01$

$\alpha_1 = \alpha_2 = \alpha_3 = .95$ $\alpha_4 L_4(0) = 160$

$\alpha_4 = .72$

$\alpha_5 = \alpha_6 = \alpha_7 = 1.0$

Table 2

CASE I—COSTS ($)

	Optimal Bonus Policy					No Bonus Policy		Year Group Management				
Year	Enlistment Bonus	Reenlistment Bonus	Bonus Cost	Penalty Cost	Total Cost	Penalty Cost	Year	Enlistment Bonus	Reenlistment Bonus	Bonus Cost	Penalty Cost	Total Cost
1	1,040	4,569	121,700	46,856	168,556	125,109	1	0	4,000	80,000	76,185	156,185
2	376	3,911	78,738	53,882	132,621	265,666	2	0	5,600	156,800	35,658	192,458
3	212	2,650	35,828	46,939	82,767	343,023	3	0	6,300	198,450	9,129	207,579
4	54	773	3,031	10,602	13,633	242,857	4	0	0	0	0	0
5	31	-200	216	6,227	6,443	109,838	5	0	0	0	0	0
6	39	-682	2,350	998	3,348	24,028	6	0	0	0	0	0
7	17	-358	645	6,714	7,360	1,969	7	0	0	0	0	0
8	5	-46	11	3,345	3,357	1,941	8	0	0	0	0	0
9	0	0	0	0	0	84	9	0	0	0	0	0
Total			242,520	175,564	418,084	1,114,520				435,250	120,972	556,222

Table 3

CASE II[a]--MANPOWER INVENTORIES

Year	Optimal Bonus Policy								Year Group Management							
	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
1	356	307	200	200	228	170	180	220	323	307	200	200	200	170	180	220
2	327	338	292	190	251	228	170	180	323	307	292	190	200	200	170	180
3	319	310	321	277	184	207	228	170	323	307	292	278	200	200	200	170
4	321	303	295	305	193	184	207	228	323	307	292	278	200	200	200	200
5	324	305	288	280	212	193	184	207	323	307	292	278	200	200	200	200
6	324	308	290	273	200	212	193	184	323	307	292	278	200	200	200	200
7	324	308	292	275	200	200	212	193	323	307	292	278	200	200	200	200
8	323	307	293	278	200	200	200	212	323	307	292	278	200	200	200	200
9	323	307	292	278	200	200	200	200	323	307	292	278	200	200	200	200

[a]Parameter Values

$\delta_1 = .25$ $\epsilon = .15$

$\delta_2 = .75$ $s_1 = 2 = $ supply elasticity to first term

$\sigma = .25$ $s_2 = 2 = $ supply elasticity to second term

$\sigma_1 = \infty$ $\beta_1 = .032$

$\sigma_2 = \infty$ $\beta_2 = .01$

$\alpha_1 = \alpha_2 = \alpha_3 = .95$ $\alpha_4 L_4(0) = 160$

$\alpha_4 = .72$

$\alpha_5 = \alpha_6 = \alpha_7 = 1.0$

Table 4

CASE II--COSTS ($)

	Optimal Bonus Policy					No Bonus Policy		Year Group Management				
Year	Enlistment Bonus	Reenlistment Bonus	Bonus Cost	Penalty Cost	Total Cost	Penalty Cost	Year	Enlistment Bonus	Reenlistment Bonus	Bonus Cost	Penalty Cost	Total Cost
1	1,036	6,769	246,309	93,976	340,285	966,058	1	0	4,000	80,000	388,372	468,372
2	116	6,316	199,682	56,877	256,558	3,533,300	2	0	5,600	156,800	474,559	631,359
3	-127	4,665	109,094	23,401	132,495	5,756,350	3	0	6,300	198,450	145,476	343,924
4	-61	-640	2,109	25,908	28,018	4,099,620	4	0	0	0	0	0
5	30	-757	2,878	2,725	5,603	1,737,970	5	0	0	0	0	0
6	39	-168	165	20,648	20,814	377,119	6	0	0	0	0	0
7	17	313	494	4,657	5,150	33,020	7	0	0	0	0	0
8	49	182	166	24,532	24,699	32,571	8	0	0	0	0	0
9	0	0	0	0	0	1,004	9	0	0	0	0	0
Total			560,897	252,724	813,621	16,537,000				435,250	1,008,405	1,443,655

Table 5

SUMMARY

	Case I			Case II		
	Bonus Cost	Penalty Cost	Total Cost	Bonus Cost	Penalty	Total Cost
Optimal bonus (1)	242,520	175,564	418,084	560,897	252,724	813,621
Year group management (2)	435,250	120,972	556,222	435,250	1,008,405	1,443,655
No bonus (3)	0	1,114,520	1,114,520	0	16,537,000	16,537,000
Net gain (2) - (1)	192,730	-54,592	138,138	-125,647	755,681	630,034
Net gain (3) - (1)	-242,520	938,956	696,436	-560,897	16,284,276	15,723,379

year groups. The model is constrained to reach the target inventory in the ninth year. The starting inventory is given by the first row, except that L_1 and L_5 are determined endogenously by bonus policy. In the absence of a bonus, L_1 would be equal to the steady state number of enlistments (323) and L_5 would be 160, by assumption. Thus the initial condition is one of shortage in all year groups except L_2, which is in equilibrium, and L_8, which is in surplus.

The solutions are consistent with economic theory. In Case I, where demand is elastic, the penalty cost assigned to the initial shortage is low and does not warrant incurring large bonus costs to eliminate it rapidly. In particular, it does not pay to bring L_5 up to strength for the first two years when the L_4 cohorts are understrength. Because of the high elasticity of substitution, this second term shortage can be partially offset by overshooting on accessions for the first two years. In Case II, demand is less elastic, so the shortage implies a greater penalty cost. The elasticity of substitution is lower, so an excess of first termers is a less effective offset to the shortage of second termers. Although L_4 is understrength for the first two years, it is optimal to pay a sufficiently large reenlistment bonus to over-shoot the target for L_5 in those years to compensate for the shortage in more senior cohorts that cannot be augmented. In contrast, the optimum enlistment bonus and resulting accession rate is lower for the first two years than in Case I.

Tables 2 and 4 present the bonus levels and bonus, penalty, and total costs for each year under the alternative policies. In addition, the penalty cost that would be incurred if no bonuses were paid is calculated to provide an alternative benchmark against which to measure the gains from following an optimum bonus policy. The main conclusions to be drawn is that because of its inflexibility, year-group management results in excessive bonuses in Case I, where substitutability is high and the specialty is not critical; and it results in insufficient bonuses in Case II, where low substitutability and inelastic demand make it optimal to overshoot the year-group targets in the second term initially, to com-pensate for shortages in more senior year groups not amenable to control by bonuses. These conclusions are summarized in Table 5. In Case I

the optimal policy, by paying lower bonuses, generates savings that more than offset the higher penalty costs, to give a net gain of $138,138 over year-group management. Alternatively stated, year-group management results in total costs that are roughly 33 percent higher than the optimal policy because it sets requirements that ignore substitution possibilities between specialties and between year groups within a specialty. In Case II, the optimal policy incurs higher bonus costs than year-group management in order to reduce penalty costs. The total excess costs of year-group management are $630,034 or 77 percent higher than the optimal policy. Both cases show huge savings relative to a no-bonus policy.

These results are sensitive to the particular values of the parameters chosen and are intended mainly to illustrate the operation of the model. They demonstrate the general point that year-group management is an excessively costly bonus management policy. A more flexible policy that takes into account conditions of substitutability, supply, and criticality would produce substantial savings.

III. CONCLUSIONS

The criteria currently used by DoD in managing the bonus program
reflect steady state and short run considerations since, given the
rigidity of the basic pay structure, bonuses are the only policy tool
available to introduce either permanent or temporary pay differentials
across specialties to counter uneven conditions of shortage and surplus.
This report shows how the multiple factors that must be considered in
managing bonuses efficiently can be integrated into a unified framework.

The problem is simplified by distinguishing two phases. Optimum
steady state bonuses are determined by permanent features of the spe-
cialty, such as training costs, attractiveness of job content, etc.
Optimum temporary deviations from the steady state are determined by
differences between the actual and the desired steady state inventory.
Since the cost of bringing the actual inventory up to desired strength
depends on the speed of this adjustment, the optimum policy over the
transition phase requires weighing the bonus costs of eliminating a
shortage against the penalty costs of tolerating the shortage temporar-
ily. Bonus costs depend on the predicted no-bonus shortage and the
elasticity of supply in response to a bonus. Penalty costs depend on
the availability of substitutes for the year groups in shortage, from
both within the particular specialty and in other specialties. These
considerations can be quantified by appropriate selection of the param-
eters of the model.

The usefulness of this approach is illustrated by the computer
simulation model. The optimum policy derived from the model achieves
substantial savings relative to a policy of strict year-group manage-
ment. Year-group management pays the bonus necessary to attain target
strength in each year group without regard to conditions of oversupply
or undersupply in other year groups or specialties and without regard
to substitution possibilities between them. The optimum policy pays
lower bonuses than year-group management when the specialty is not
crucial and concentrates the bonus effort on year groups that can be
increased at low cost. These bonus savings more than offset the higher

penalty costs associated with the larger shortage. Conversely, in the case of a highly critical specialty, with shortages in senior year groups that cannot be affected by bonus policy, year-group management tends to pay inadequate bonuses and thus incurs high penalty costs and higher total costs than the optimum policy.

Appendix A

TRAINING OF LATERAL ENTRANTS AS ALTERNATIVE SOURCE OF SUPPLY

If training lateral entrants (E) at a cost $M(E)$ is an alternative to increasing within-specialty reenlistments by bonus payments, the social cost function contains the additional term

$$C_2 = \int_0^E M(E) \ dE \ .$$

Expanding this term around E_o (which may be zero if the optimal steady state use of cross training is zero) yields:

$$C_2 \Big|_{E_o} + M \Big|_{E_o} (E_t - E_o) + (E_t - E_o)' \ \frac{1}{2} \frac{dM}{dE} \Big|_{E_o} (E - E_o) \ .$$

The first term is a constant and can be ignored. The second term will either cancel with a term in the benefits function if $E_o \neq 0$ (since the first-order conditions for a steady state optimum require

$$\frac{\partial SB}{\partial E} \Big|_{E_o} = \frac{\partial C_2}{\partial E} \Big|_{E_o} \ ) \ ,$$

or equal zero, if the optimal steady state level of cross training is zero. dM/dE is the inverse of the slope of the supply function of cross trainees in response to bonus plus training expenditure, which is constant under the linearity assumption. Thus, introducing an external supply source of men in other military specialties who become perfect substitutes for within-specialty reenlistments after an initial training outlay merely adds another quadratic form to the loss function. The training costs may be either budget outlays on formal training or forgone output during an initial period of on-the-job training. The

production function and target input levels are redefined in terms of
the new composite input,

$$L_i' - L_i + E_i \ ,$$

and the composite supply function is

$$L_{i,t}' = \alpha L_{i,t}' + \beta B_{i,t} + \gamma M_{i,t} \ .$$

Maximization simultaneously with respect to bonus and training expenditure yields the optimal policy mix.

Appendix B

MAXIMIZATION OF NET PRIVATE BENEFIT

If the postulated objective is to maximize private rather than social net benefit, the objective function is

$$\sum_{t=1}^{T} \left\{ \int_{0}^{Z_t} \left(P_t + Z_t \frac{dP}{dZ} \right) dZ_t - \sum_{i=1}^{n} \left[\int_{0}^{L_{i,t}} \left(w_i + L_{i,t} \frac{dw_i}{dL_i} \right) dL_{i,t} \right] \right\} .$$

A second-order Taylor expansion of the expression in brackets permits the constant term to be ignored; the first order terms cancel, assuming steady state wages are set so that value of marginal product equals marginal factor cost, and the expression reduces to (omitting time subscripts and using vector notation):

$$\frac{1}{2} \left\{ \left[\left(2 \frac{dP}{dZ} + Z \frac{d^2P}{dZ^2} \right) \left(\frac{\partial Z}{\partial L} \right)^2 + \left(P + Z \frac{dP}{dZ} \right) \frac{\partial^2 Z}{\partial L^2} \right] (L_t - L_o) - 2 \frac{\partial W}{\partial L} (L_t - L_{*t}) \right\} .$$

Appendix C

SOLUTION FOR THE CONTINUOUS CASE

The supply function in continuous time is of the form:

$$S_i = \dot{L}_i = \alpha_{i-1} L_{i-1} + \beta_i B_i - L_i \ .$$

The Hamiltonian is

$$H = \bar{L}'_t \ F \ \bar{L}_t - \bar{L}'_t \ U \ \bar{L}_t + \lambda'_t \ [S(L_t, \ B_t)] \ .$$

First-order necessary conditions for a maximum are:

$$\frac{\partial H}{\partial B_{it}} = - \beta_i \ B_{it} + \lambda_{it} \ \beta_i = 0 \tag{C.1}$$

or

$$B_{it} = \lambda_{it} \ .$$

Thus, the optimal bonus payment to the ith YOS at time t is equal to the shadow price of a man in YOS_i at time t

$$\dot{\lambda}_i = - \sum_{j=1}^{n} F_{ij} (L_j - L_j^o) + \lambda_i - \alpha_i \ \lambda_{i+1} \ . \tag{C.2}$$

Therefore,

$$\dot{\lambda} \gtrless 0 \quad \text{as} \quad \lambda_i \gtrless \sum_{j=1}^{n} F_{ij} (L_i - L_j^o) + \alpha_i \ \lambda_{i+1} \ .$$

Thus, if λ_i, the shadow price of an L_i at time t, exceeds the sum of $\sum_{j=1}^{n} F_{ij} (L_j - L_j^o)$, the current contribution of L_i to the benefits

function plus $\alpha_i \lambda_{i+1}$, the value of an L_{i+1}, discounted by the proba-
bility of continuing from YOS_i to YOS_{i+1}, then $\dot{\lambda}_i$ is positive and the
time path of bonus payments to YOS_i will be increasing, given the con-
dition $B_{it} = \lambda_{it}$. Further, when the net value of an additional man in
all categories is stable over time and equal to zero--i.e., $\lambda_{i=0}$,
$i=1, \ldots, n$--Eq. (C.2) reduces to

$$\dot{\lambda}_i = - \sum_{j=1}^{n} F_{ij}(L_j - L_j^o) = 0 .$$

A sufficient condition for this to obtain is

$$L_j = L_j^o , \qquad j = 1, \ldots, n;$$

i.e., all inputs are at their target levels.[1]

$$\dot{L}_i = \alpha_{i-1} L_{i-1} + \beta_i B_i - L_i . \tag{C.3}$$

The supply conditions are fulfilled at all points over the time
path.

Solving Eq. (C.1) in terms of the state and costate variables and
substituting into Eqs. (C.2) and (C.3) yield a set of 2n linear constant
coefficient differential equations in the 2n state and costate variables.
Initial and terminal conditions on the state variables define a particu-
lar solution to the system.

Define $V(t)$ as the vector of 2n elements, $L(t)$ and $\lambda(t)$, with
initial value $V(0)$ and target value $V(T)$; then the solution is of the
form:

$$V(t) = e^{At} [V(0) - V(T)] + V(T)$$

or

$$V(t) = \hat{V} = e^{At} [V(0) - V(T)] .$$

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[1] If some of the F_{ij} are of oppositive signs, this is not a neces-
sary condition.

Thus, at any point, the deviation between the actual and the desired state vector is an exponential function of the initial discrepancy between actual and desired state vectors.[1]

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[1]This is the vector analog of the Eisner-Strotz solution for the single variable case:

$$S(t) = (S(0) - \hat{S})\ e^{\alpha t} + \hat{S} .$$

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