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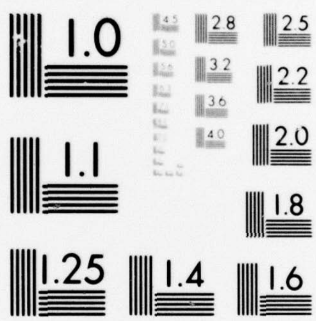

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In this paper, optimal control problems where there is uncertainty in the initial state measurement or where there are uncertain parameters in the state equations are investigated. It is assumed that nature will choose the uncertainty to maximize the cost which the controller is attempting to minimize. Thus a minmax control is sought. Sufficient conditions for a control to be a minmax control are presented. These conditions suggest methods for finding the minmax control and such techniques are described. The application of these methods is illustrated by example problems.

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MINMAX CONTROL OF SYSTEMS WITH UNCERTAINTY IN THE INITIAL STATE AND IN THE STATE EQUATIONS[1]

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Abstract

In this paper, optimal control problems where there is uncertainty in the initial state measurement or where there are uncertain parameters in the state equations are investigated. It is assumed that nature will choose the uncertainty to maximize the cost which the controller is attempting to minimize. Thus a minmax control is sought.

Sufficient conditions for a control to be a minmax control are presented. These conditions suggest methods for finding the minmax control and such techniques are described. The application of these methods is illustrated by example problems.

1. Introduction

In an optimal control problem, only uncertain measurements of the initial state may be available rather than knowledge of the exact initial state. One approach to this problem is to obtain a stochastic description of the uncertainty and choose the control to minimize an expected value. Here the problem is treated in a different fashion. It is assumed that from the measured initial state it is only possible to conclude that the true initial state belongs to some subset of the state space. The objective is to choose a control, based on this measurement, which minimizes the maximum value of the cost over all possible initial states in the subset. Thus it is assumed that nature is perverse and chooses the uncertainty to maximize the cost which the controller is attempting to minimize. For each control there is a guaranteed performance (which is determined by assuming nature maximizes against this control) and the optimal control is the one which achieves the best guaranteed performance.

If Q is the set of possible values for the uncertainty, $\mathfrak{M}$ the set of admissible controls and $J(u(\cdot),q)$ the cost functional, then the problem is to find a $u^*(\cdot) \in \mathfrak{M}$ satisfying for all $u(\cdot) \in \mathfrak{M}$

$$\sup_{q \in Q} J(u^*(\cdot),q) \leq \sup_{q \in Q} J(u(\cdot),q)$$

The first sufficient condition, Theorem 1, is applicable when the cost function has a saddle point. However, examples show that often this is not the case and the second sufficient condition, Theorem 2, can be used when there is no saddle point solution.

In obtaining the second sufficient condition, the initial state uncertainty problem is transformed into a problem with known initial state but having uncertainty in the state equations. Thus the result is also applicable to problems where the mathematical model contains uncertainty in the differential equation describing the evolution of the state. Both sufficient conditions suggest a method for constructing a minmax control and these procedures are described.

Initial state uncertainty problems have been studied in [1] where a general result for the linear quadratic case is obtained. Our results are not limited in application to such problems. The minmax approach to uncertainty has also been investigated in [2]-[8]. The results presented here are different and appear to be applicable to a wider class of problems than those considered in most of these papers since the assumption of the existence of a saddle point solution is not required in Theorem 2 nor are the results limited to linear quadratic problems.

2. Problem Formulation

Consider a system which can be modeled by ordinary differential equations

$$\dot{x}(t) = f(x(t),u(t)), \quad t \in [t_o,t_f] \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control and the time interval $[t_o,t_f]$ is prescribed. A measurement of the initial state, x_{om}, is available and is related to the true initial state, $x(t_o)$, by

$$x(t_o) = x_{om} + q \tag{2}$$

where $q \in Q \subset R^n$ and Q is known. If q were known exactly, then we would have the usual optimal control problem. Here, however, we assume that q is not known exactly but is chosen perversely by nature.

A control $u(\cdot)$ will be called admissible if it is piecewise continuous and $u(t) \in U$ for all $t \in [t_o,t_f]$ where $U \subset R^m$ is a given set. The set of admissible controls will be denoted by $\mathfrak{M}$. We

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shall assume throughout that for every $u(\cdot) \in \mathfrak{m}$ and $q \in Q$ there is a solution of (1) and (2) on $[t_o, t_f]$. The cost or criterion depends on the choice of the control $u(\cdot)$ and the parameter q.

$$J(u(\cdot),q) = \phi(x(t_f)) + \int_{t_o}^{t_f} L(x(t),u(t))dt \qquad (3)$$

The problem is to find the optimal control, $u^*(\cdot)$, based on the measurement x_{om} when the optimality criterion is minmax, i.e., find an admissible control $u^*(\cdot)$ satisfying

$$\sup_{q \in Q} J(u^*(\cdot),q) \le \sup_{q \in Q} J(u(\cdot),q) \qquad (4)$$

for all $u(\cdot) \in \mathfrak{m}$.

One approach to the problem would be to determine $\hat{q}(u(\cdot))$ satisfying for every admissible $u(\cdot)$

$$J(u(\cdot),\hat{q}(u(\cdot))) \ge J(u(\cdot),q) \quad \forall \quad q \in Q$$

and then determine the admissible control which minimizes $J(u(\cdot),\hat{q}(u(\cdot)))$. This approach, however, is not feasible because of the difficulty in determining $\hat{q}(u(\cdot))$.

Alternatively, one can assume q is fixed and find $\hat{u}(\cdot,q)$ satisfying for all $q \in Q$

$$J(\hat{u}(\cdot,q),q) \le J(u(\cdot),q) \quad \forall \quad u(\cdot) \in \mathfrak{m}$$

If $q^o \in Q$ maximizes $J(\hat{u}(\cdot,q),q)$, then $\hat{u}(\cdot,q^o)$ is a candidate for the minmax control. While it may be possible to perform the above steps, the resulting control, $\hat{u}(\cdot,q^o)$, can only be the minmax control if $J(u(\cdot),q)$ has a saddle point solution, i.e., there is a $(u^o(\cdot),q^o)$ satisfying

$$J(u^o(\cdot),q) \le J(u^o(\cdot),q^o) \le J(u(\cdot),q^o)$$

for all $u(\cdot) \in \mathfrak{m}$ and $q \in Q$. This will not always be the case for these initial state uncertainty problems. Judging from the examples, what will usually occur in a problem is that for some values of x_{om} there will be a saddle point solution while for others there will not.

In the next section, we present a sufficient condition for a minmax control when there is a saddle point solution and in Sec. 5 a sufficient condition which applies when a saddle point solution does not exist. In both cases, the sufficient conditions suggest a constructive method for finding the minmax control.

In the following, we assume $f(\cdot,\cdot) : R^n \times R^m \to R^n$, $\phi(\cdot) : R^n \to R$ and $L(\cdot,\cdot) : R^n \times R^m \to R$ are C^1 functions and that $Q = \{q : C(q) \le 0\}$ where $C(\cdot) : R^n \to R^p$ is also C^1.

3. The First Sufficient Condition

<u>Theorem 1</u>. Suppose $\phi(\cdot)$ is convex on R^n. Let $u^*(\cdot)$ be admissible and $h(q) = J(u^*(\cdot),q)$. If there exists a $q^* \in Q$ and a non-positive vector ρ such that

i) $\dfrac{\partial h(q^*)}{\partial q} + \sum_{i \in I} \rho_i \dfrac{\partial C_i(q^*)}{\partial q} = 0$,

$I = \{i : C_i(q^*) = 0\}$

ii) $\dfrac{\partial^2}{\partial q^2}\left(h(q) + \sum_{i \in I} \rho_i C_i(q)\right) \le 0 \quad \forall \quad q \in R^n$

and if there exists an absolutely continuous function $\beta(\cdot) : [t_o,t_f] \to R^n$ with $\beta(t_f) = \dfrac{\partial \phi(x^*(t_f))}{\partial x}$ such that

iii) $\Delta \triangleq L(y,v) + \beta^T(t)f(y,v) - L(x^*(t),u^*(t))$

$- \beta^T(t)f(x^*(t),u^*(t)) + \dot{\beta}^T(t)(y - x^*(t)) \ge 0$

$\forall$ $y \in R^n$, $\forall$ $v \in U$ and for almost all $t \in [t_o,t_f]$ where $x^*(t)$ is the solution of

$$\dot{x}^*(t) = f(x^*(t),u^*(t)), \quad x^*(t_o) = x_{om} + q^*$$

then $u^*(\cdot)$ is a minmax control.

<u>Proof</u>. Conditions (i) and (ii) imply $h(q) \le h(q^*)$ $\forall$ $q \in Q$ or $J(u^*(\cdot),q) \le J(u^*(\cdot),q^*)$ $\forall$ $q \in Q$. Following [9], condition (iii) implies $J(u^*(\cdot),q^*) \le J(u(\cdot),q^*)$. Thus $(u^*(\cdot),q^*)$ is a saddle point solution and, since all pairs $(u(\cdot),q)$ with $u(\cdot) \in \mathfrak{m}$ and $q \in Q$ are playable, $u^*(\cdot)$ is a minmax control. □

Condition (iii) of Theorem 1 is used to show that $J(u^*(\cdot),q^*) \le J(u(\cdot),q^*)$ for all $u(\cdot) \in \mathfrak{m}$. Rather than using a simple sufficiency approach one could use a field theorem such as [10,11] to show that $J(u^*(\cdot),q^*) \le J(u(\cdot),q^*)$. If this is done, the assumption that $\phi(\cdot)$ is convex can be dropped.

This theorem suggests the following procedure for finding a minmax control.

1. Solve the necessary conditions for the optimal control problem (1)-(3) assuming q is known. This yields $\hat{u}(\cdot,q)$.
2. Evaluate $J(\hat{u}(\cdot,q),q)$.
3. Maximize $J(\hat{u}(\cdot,q),q)$ subject to $C(q) \le 0$. Call the maximizing solution q^o.
4. Let $u^*(\cdot) = \hat{u}(\cdot,q^o)$ and evaluate $h(q) = J(u^*(\cdot),q)$.
5. Check the sufficient condition of Theorem 1 with $u^*(\cdot)$ as the candidate. This involves finding a q^* and ρ satisfying (i) and (ii). In (iii), $\beta(\cdot)$ can be taken as the multiplier from the optimal control problem of Step 1 with $q = q^*$.

This procedure often works when there is a saddle point solution for the particular x_{om} under consideration but will not work when there is no saddle point solution. A simple example illustrates this technique.

$$J(u(\cdot),q) = \frac{1}{2} x^2(1) + \frac{1}{2}\int_o^1 u^2(t)dt \qquad (5)$$

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$$\dot{x}(t) = u(t)\ ,\ x(t_o) = x_{om} + q \tag{6}$$

$$Q = \{q : |q| \le 1\} \tag{7}$$

From Step 1 we obtain $\hat{u}(t,q) = -\frac{1}{2}(x_{om} + q)$ and from Step 2 $J(\hat{u}(\cdot,q),q) = \frac{1}{4}(x_{om} + q)^2$. The adjoint variable in Step 1 is $\lambda(t) = \frac{1}{2}(x_{om} + q)$. The maximization problem in Step 3 yields $q^o = 1$ if $x_{om} \ge 0$ and $q^o = -1$ if $x_{om} < 0$. Then $u^*(t) = \hat{u}(t,q^o) = -\frac{1}{2}(x_{om}+1)$ if $x_{om} \ge 0$ while $u^*(t) = -\frac{1}{2}(x_{om}-1)$ if $x_{om} < 0$. Also $h(q) = \frac{1}{4}(x_{om}+1)^2 + \frac{1}{2}(x_{om}+q)^2 - \frac{1}{2}(x_{om}+1)(x_{om}+q)$ if $x_{om} \ge 0$ and $h(q) = \frac{1}{4}(x_{om}-1)^2 + \frac{1}{2}(x_{om}+q)^2 - \frac{1}{2}(x_{om}-1)(x_{om}+q)$ if $x_{om} < 0$.

If $x_{om} \ge 1$, the sufficient condition of Theorem 1 are satisfied with the above $u^*(\cdot)$ and $q^* = +1$, $\rho = -\frac{1}{4}(x_{om}+1)$, $\beta(t) = \frac{1}{2}(x_{om}+1)$ while if $x_{om} \le -1$, they are satisfied with $u^*(\cdot)$ and $q^* = -1$, $\rho = \frac{1}{4}(x_{om}-1)$, $\beta(t) = \frac{1}{2}(x_{om}-1)$. For $-1 < x_{om} < 1$, the sufficient conditions cannot be satisfied and we suspect that there is no saddle point solution when $-1 < x_o < 1$.† In the next sections we present a method for treating such situations.

4. A Transformation

We shall derive sufficient conditions for minmax control when there is no saddle point solution by considering an equivalent problem.

Let $z(t) = x(t) - q$ and

$$k(z(t),u(t),q) = f(z(t) + q, u(t))$$

$$\psi(z(t_f),q) = \phi(z(t_f) + q)$$

$$M(z(t),u(t),q) = L(z(t) + q, u(t))$$

Now consider the optimal control problem

$$\dot{z}(t) = k(z(t),u(t),q) \tag{8}$$

$$z(t_o) = x_{om} \tag{9}$$

$$K(u(\cdot),q) = \psi(z(t_f),q) + \int_{t_o}^{t_f} M(z(t),u(t),q)dt \tag{10}$$

The original problem has been transformed from one with initial state uncertainty to one with initial state known, but with an uncertain parameter in the state equations and cost. For any $u(\cdot) \in \mathfrak{M}$ and $q \in Q$, $K(u(\cdot),q) = J(u(\cdot),q)$. Thus if $u^*(\cdot)$ is a minmax solution to (1)-(3) then $u^*(\cdot)$ is also a minmax solution to (8)-(10).

In the next section, a sufficient condition for a control to be a minmax control for the problem (8)-(10) will be presented. Since any initial state uncertainty problem can be transformed to this form, the condition will also be a sufficient condition for the initial state problem. Unlike Theorem 1, this sufficient condition is applicable when there is no saddle point solution. Of course, the sufficient condition also applies to problems which can be formulated as (8)-(10) and we also have a sufficient condition for optimal control problems where there is parameter uncertainty in the state equations. Results for problems with time varying uncertainty in the state equations are given in [12,13].

† In Sec. 5, we show that this suspicion is confirmed.

5. The Second Sufficient Condition

Consider the problem (8)-(10).

<u>Theorem 2</u>. Suppose $\psi(\cdot,q)$ is a convex function of z on R^n for all q and let $u^*(\cdot) \in \mathfrak{M}$. If there exists

a) a positive integer γ

b) vectors q^i, $i = 1,\cdots,\gamma$

c) absolutely continuous functions

$$\lambda^i(\cdot) : [t_o,t_f] \to R^n,\ i = 1,2,\cdots,\gamma$$

d) scalars $\alpha_i > 0$, $i = 1,\cdots,\gamma$ with $\sum_{i=1}^{\gamma} \alpha_i = 1$

such that

i) $$\lambda^i(t_f) = \alpha_i \frac{\partial\psi(z^{i*}(t_f),q^i)}{\partial z},\ i = 1,\cdots,\gamma$$

where $z^{i*}(\cdot)$ is the trajectory corresponding to $(u^*(\cdot),q^i)$

ii) $q^i \in \mathcal{L}(u^*(\cdot))$, $i = 1,\cdots,\gamma$ where $\mathcal{L}(u^*(\cdot)) = \{q : q \in Q \text{ and } J(u^*(\cdot),q) = \sup_{p \in Q} J(u^*(\cdot),p)\}$

iii) $$\sum_{i=1}^{\gamma} \alpha_i M(y^i,w,q^i) + \sum_{i=1}^{\gamma} \lambda^i(t)k(y^i,w,q^i) - \sum_{i=1}^{\gamma} \alpha_i M(z^{i*}(t),u^*(t),q^i) - \sum_{i=1}^{\gamma} \lambda^i(t)k(z^{i*}(t),u^*(t),q^i) + \sum_{i=1}^{\gamma} \dot{\lambda}^i(t)(y^i - z^{i*}(t)) \ge 0$$

$\forall\ y^i \in R^n$, $i = 1,2,\cdots,\gamma$,

$\forall\ w \in U$ and for almost all $t \in [t_o,t_f]$

then $u^*(\cdot)$ is a minmax control.

<u>Proof</u>. Consider any $u(\cdot)$ and let $z^i(\cdot)$ be the trajectory corresponding to $(u(\cdot),q^i)$ satisfying $z^i(t_o) = x_{om}$. Then

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$$0 \le \sum_{i=1}^{\gamma} \alpha_i M(z^i(t),u(t),q^i) + \sum_{i=1}^{\gamma} \lambda^i(t)k(z^i(t),u(t),q^i)$$

$$- \sum_{i=1}^{\gamma} \alpha_i M(z^{i*}(t),u^*(t),q^i)$$

$$- \sum \lambda^i(t)k(z^{i*}(t),u^*(t),q^i) + \sum_{i=1}^{\gamma} \dot{\lambda}^i(z^i(t)-z^{i*}(t))$$

$$= \sum_{i=1}^{\gamma} \alpha_i M(z^i(t),u(t),q^i) - \sum_{i=1}^{\gamma} \alpha_i M(z^{i*}(t),u^*(t),q^i)$$

$$+ \sum_{i=1}^{\gamma} \frac{d}{dt} \lambda^i(t)(z^i(t)-z^{i*}(t))$$

Integrating the above inequality from t_o to t_f, using (i) and the fact that $z^i(t_o) = z^{i*}(t_o)$, the above inequality becomes

$$0 \le \sum_{i=1}^{\gamma} \int_{t_o}^{t_f} \alpha_i M(z^i(t),u(t),q^i)dt$$

$$- \sum_{i=1}^{\gamma} \int_{t_o}^{t_f} \alpha_i M(z^{i*}(t),u^*(t),q^i)dt$$

$$+ \sum_{i=1}^{\gamma} \frac{\partial\psi(z^{i*}(t_f),q)}{\partial z} (z^i(t_f) - z^{i*}(t_f)) \quad (11)$$

From the convexity assumption and the fact that $\alpha_i > 0$, $i = 1,\cdots,\gamma$

$$\sum_{i=1}^{\gamma} \alpha_i \psi(z^i(t_f),q^i) - \sum_{i=1}^{\gamma} \alpha_i \psi(z^{i*}(t_f)-q^i)$$

$$\ge \sum_{i=1}^{\gamma} \alpha_i \frac{\partial\psi(z^{i*}(t_f),q^i)}{\partial z} (z^i(t_f)-z^{i*}(t_f)) \quad (12)$$

Combining (11) and (12) leads to

$$\sum_{i=1}^{\gamma} \alpha_i \left[\psi(z^i(t_f),q^i) + \int_{t_o}^{t_f} M(z^i(t),u(t),q^i)dt\right]$$

$$\ge \sum_{i=1}^{\gamma} \alpha_i \left[\psi(z^{i*}(t_f),q^i) + \int_{t_o}^{t_f} M(z^{i*}(t),u^*(t),q^i)dt\right]$$

Since $\alpha_i > 0$, $i = 1,\cdots,\gamma$, there exists an $i\epsilon\{1,\cdots,\gamma\}$ such that

$$\psi(z^i(t_f),q^i) + \int_{t_o}^{t_f} M(z^i(t),u(t),q^i)dt \ge$$

$$\psi(z^{i*}(t_f),q^i) + \int_{t_o}^{t_f} M(z^{i*}(t),u^*(t),q^i)dt$$

or,

$$J(u(\cdot),q^i) \ge J(u^*(\cdot),q^i)$$

or,

$$\sup_{q\epsilon Q} J(u(\cdot),q) \ge J(u^*(\cdot),q^i)$$

Since $q^i \epsilon \mathcal{L}(u^*(\cdot))$, $J(u^*(\cdot),q^i) = \sup_{q\epsilon Q} J(u^*(\cdot),q^i)$ and the theorem is proved. □

This theorem suggests the following method for finding a minmax solution when there is no saddle point solution.

1. Transform the initial state uncertainty problem into the form (5)-(7).
2. Choose a number $\gamma \ge 2$ and vectors $q^i \epsilon Q$, $i = 1,\cdots,\gamma$.
3. For this choice of γ and q^i, consider the optimal control problem

$$\hat{K}(u(\cdot)) = \sum_{i=1}^{\gamma} \alpha_i \left[\psi(z^i(t_f),q^i) + \int_{t_o}^{t_f} M(z^i(t),u(t),q^i)dt\right]$$

$$\dot{z}^i(t) = k^i(z^i(t),u(t),q^i),\ z^i(t_o) = x_{om}, i=1,\cdots,\gamma$$

where the α_i are as yet undetermined.

4. Use the necessary conditions for this optimal control problem to determine a candidate $\tilde{u}(t;\alpha_1,\cdots,\alpha_\gamma)$.
5. If possible, choose $(\alpha_1,\cdots,\alpha_\gamma)$ so that for all $i,j\epsilon\{1,\cdots,\gamma\}$

$$\psi(z^i(t_f),q^i) + \int_{t_o}^{t_f} M(z^i(t),u(t),q^i)dt$$

$$= \psi(z^j(t),q^j) + \int_{t_o}^{t_f} M(z^j(t),u(t),q^j)dt$$

6. If such $(\alpha_1,\cdots,\alpha_\gamma)$ exists with $\alpha_i > 0$, $i = 1,\cdots,\gamma$ and $\sum_{i=1}^{\gamma} \alpha_i = 1$, let $u^*(t) = \tilde{u}(t;\alpha_1,\cdots\alpha_\gamma)$ and check the sufficient conditions of Theorem 2. The functions $\lambda^i(\cdot)$ can be taken as the adjoint variables from the problem in Step 3. To verify (ii), one must evaluate $h(q) = K(u^*(\cdot),q)$ and show that q^i, $i = 1,\cdots,\gamma$ maximizes $h(q)$ subject to $C(q) \le 0$.
7. If no α_i, $i = 1,\cdots,\gamma$ can be found with $\alpha_i > 0$, $i = 1,\cdots,\gamma$ and $\sum_{i=1}^{\gamma} \alpha_i = 1$, return to Step 2 and choose a new set (γ,q^i).

What makes this technique difficult to apply is that there is no apparent technique for making a good choice of γ and the vectors q^i in Step 2. It may be necessary to try many combinations before the method will be successful. Nevertheless, the method is a possible one for finding minmax controls and, except for a special linear quadratic case, the only one known to the author.

We next apply this technique to the simple example of Sec. 3. After applying the transformation to the problem (5)-(7), we have

$$K(u(\cdot),q) = \frac{1}{2}(z(1) + q)^2 + \frac{1}{2}\int_0^1 u^2(t)dt$$

$$\dot z(t) = u, \quad z(o) = x_{om}, \quad Q = \{q : |q| \le 1\}$$

In Sec. 3, the minmax control for $x_{om} \ge 1$ and $x_{om} \le -1$ was found. Here we consider $-1 < x_{om} < 1$. Since q is chosen to maximize $K(u(\cdot),q)$, we expect q to be on the boundary of Q. Thus we choose $\gamma = 2$, $q^1 = 1$, $q^2 = -1$. With this choice, the optimal control problem in Step 3 becomes

$$\hat K(u(\cdot),\alpha_1,\alpha_2) = \frac{1}{2}\alpha_1(z^1(1)+1)^2 + \frac{1}{2}\alpha_1\int_0^1 u^2(t)dt$$

$$+ \frac{1}{2}\alpha_2(z^2(1)-1)^2 + \frac{1}{2}\alpha_2\int_0^1 u^2(t)dt$$

$$\dot z_1 = u, \quad \dot z_2 = u, \quad z_1(o) = z_2(o) = x_{om}$$

From the optimal control necessary conditions, we obtain $\tilde u(t;\alpha_1,\alpha_2) = -\frac{1}{2}x_{om} + \frac{\alpha_2-\alpha_1}{2}$ and then from Step 5, $\alpha_1 = \frac{1+x_{om}}{2}$, $\alpha_2 = \frac{1-x_{om}}{2}$. Thus, the minmax candidate is $u^*(t) = -x_{om}$. Note that this can only be a candidate when $|x_{om}| \le 1$ since $u(t)$ is constrained to satisfy $|u(t)| \le 1$.

Using Theorem 2 with

$$u^*(t) = -x_{om}, \quad \gamma = 2, \quad q^1 = 1, \quad q^2 = -1$$

$$\lambda^1(t) = \frac{1+x_{om}}{2}, \quad \lambda^2(t) = \frac{x_{om}-1}{2}, \quad \alpha_1 = \frac{1+x_{om}}{2}, \quad \alpha_2 = \frac{1-x_{om}}{2}$$

conditions (i) and (iii) are readily satisfied. Condition *(ii) is also satisfied since* $h(q) = \frac{1}{2}q^2 + \frac{1}{2}x_{om}^2$ and $q^1 = 1$, $q^2 = -1$ both maximize $h(q)$ subject to $|q| \le 1$.

Combining this result with that of Sec. 3, we can conclude that the minmax control is

$$u^*(t) = \begin{cases} -\frac{1}{2}(x_{om}+1) & \text{if } x_{om} \ge 1 \\ -x_{om} & \text{if } -1 < x_{om} < 1 \\ -\frac{1}{2}(x_{om}-1) & \text{if } x_{om} \le 1 \end{cases}$$

The results presented above are illustrated further with the examples in the next section.

4. Example Problems

<u>Example 1.</u> $J(u(\cdot),q) = \frac{1}{2}(x_1^2(1) + x_2^2(1)) + \int_0^1 u^2(t)dt$

$$\dot x_1 = x_2, \quad x_1(o) = q_1 + x_{10m}, \quad \dot x_2 = u, \quad x_2(o) = q_2 + x_{20m}$$

$$Q = \{q : q_1^2 - 1 \le 0, \ q_2^2 - 1 \le 0\}$$

While this is a linear quadratic problem, it cannot be solved by the methods of [1] since Q is not in the form required there. If q is known, the optimal control is

$$\hat u(t,q) = \frac{24(x_{10m}+q_1) + 18(x_{20m}+q_2)}{29}(t-1) + \frac{6(x_{10m}+q_1) - 10(x_{20m}+q_2)}{29}$$

and

$$J(\hat u(\cdot,q),q) = \frac{12}{29}(x_{10m}+q_1)^2 + \frac{14}{29}(x_{20m}+q_2)^2 + \frac{18}{29}(x_{10m}+q_1)(x_{20m}+q_2) \quad (13)$$

Consider first $x_{10m} = 10$, $x_{20m} = 10$ *and use the* procedure outlined after Theorem 1. The q which maximizes (13) subject to $q \in Q$ is $q^o = \begin{bmatrix}1\\1\end{bmatrix}$ and and minmax candidate is

$$u^*(t) = \hat u(t,q^o) = \frac{462}{29}t - \frac{506}{29}$$

With this control,

$$h(q) = J(u^*(\cdot),q) = \frac{1}{2}\left(\frac{15}{29} + q_2\right)^2 + \frac{1}{2}\left(\frac{404}{29} + q_1 + q_2\right)^2 + 55.536$$

This $u^*(\cdot)$ and $h(q)$ along with $q_1^* = 1$, $q_2^* = 1$, $\rho_1 = -\frac{231}{29}$, $\rho_2 = -\frac{253}{29}$, $\beta_1(t) = \frac{462}{29}$ and $\beta_2(t) = -\frac{462}{29}t + \frac{506}{29}$ satisfy the sufficient conditions of Theorem 1 for $x_{10m} = 10$, $x_{20m} = 10$. Thus $u^*(t) = \frac{462}{29}t - \frac{506}{29}$ is the minmax control.

Next consider the case when the measured initial state values are $x_{10m} = 1$, $x_{20m} = 0$. If one applies the Theorem 1 technique, no information is obtained since the solution obtained this way fails to satisfy the sufficient conditions of Theorem 1. Thus we approach the problem through Theorem 2. The transformed problem is

$$\dot z_1(t) = z_2(t) + q_2, z_1(o) = 1, \quad \dot z_2(t) = u(t), \quad z_2(o) = 0$$

$$K(u(\cdot),q) = \frac{1}{2}\left[(z_1(1)+q_1)^2 + (z_2(1)+q_2)^2\right] + \int_0^1 u^2(t)dt$$

It is expected that the q vector will be on the boundary of Q and thus we try some combination of the four vectors

$$q = \begin{bmatrix}1\\1\end{bmatrix}, \quad q = \begin{bmatrix}1\\-1\end{bmatrix}, \quad q = \begin{bmatrix}-1\\1\end{bmatrix}, \quad q = \begin{bmatrix}-1\\-1\end{bmatrix}$$

We choose $\gamma = 2$ and $q^1 = \begin{bmatrix}1\\1\end{bmatrix}$, $q^2 = \begin{bmatrix}-1\\-1\end{bmatrix}$.

(Other possibilities include $\gamma = 2$, 3 or 4 and a corresponding number of vectors from the above set of four vectors.) For this choice, the optimal control problem of Step 3 in Sec. 5 is

$$\hat{K}(u(\cdot),\alpha_1,\alpha_2) = \frac{1}{2}\alpha_1\left[(z_1(1)+1)^2 + (z_2(1)+1)^2\right] + \frac{1}{2}\alpha_1\int_0^1 u^2(t)dt + \frac{1}{2}\alpha_2\left[(y_1(1)-1)^2 + (y_2(1)-1)^2\right] + \frac{1}{2}\alpha_2\int_0^1 u^2(t)dt$$

$$\dot{z}_1(t) = z_2(t)+1,\ z_1(o) = 1;\ \dot{y}_1(t) = y_2(t)-1,\ y_1(o) = 1$$

$$\dot{z}_2(t) = u(t)\quad,\ z_2(o) = 0;\ \dot{y}_2(t) = u(t)\quad,\ y_2(o) = 0$$

Following Step 4 the solution of this optimal control problem is

$$\tilde{u}(t,\alpha_1,\alpha_2) = \frac{1}{29}(98\alpha_1 - 32\alpha_1^2 - 18\alpha_2 - 32\alpha_1\alpha_2)(t-1) - \frac{1}{29}(48\alpha_1 - 50\alpha_1^2 - 10\alpha_2 - 50\alpha_1\alpha_2)$$

and from Step 5, $\alpha_1 = \frac{99}{114}$, $\alpha_2 = \frac{15}{114}$. Since $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, a minmax candidate is

$$u^*(t) = \tilde{u}\left(t, \frac{99}{114}, \frac{15}{114}\right) = \frac{36}{19}t - \frac{34}{19}$$

We now apply the sufficient condition of Theorem 2 with the above $u^*(\cdot)$ and

$$\alpha_1 = \frac{99}{114},\ \alpha_2 = \frac{15}{114},\ \gamma = 2,\ q^1 = \begin{bmatrix}1\\1\end{bmatrix},\ q^2 = \begin{bmatrix}-1\\-1\end{bmatrix}$$

$$\lambda^1(t) = \frac{1}{(114)^2}\begin{bmatrix}27{,}324\\ -27{,}324(t-1) + 1782\end{bmatrix}$$

$$\lambda^2(t) = \frac{1}{(114)^2}\begin{bmatrix}-2700\\ 2700(t-1) - 3150\end{bmatrix}$$

$$z^{1*}(t) = \begin{bmatrix}\frac{6}{19}t^3 - \frac{17}{19}t^2 + t + 2\\ \frac{18}{19}t^2 - \frac{34}{19}t + 1\end{bmatrix}$$

$$z^{2*}(t) = \begin{bmatrix}\frac{6}{19}t^3 - \frac{17}{19}t^2 - t\\ \frac{18}{19}t^2 - \frac{34}{19}t - 1\end{bmatrix}$$

It is straightforward to verify that conditions (i) and (iii) of Theorem 2 are satisfied. To verify (ii), $J(u^*(\cdot),q)$ is needed.

$$J(u^*(\cdot),q) = \frac{1}{2}\left(\frac{8}{19} + q_1 + q_2\right)^2 + \frac{1}{2}\left(-\frac{16}{19} + q_2\right)^2 + .504$$

To verify (ii) it must be shown that $q^1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $q^2 = \begin{bmatrix}-1\\-1\end{bmatrix}$ maximize $J(u^*(\cdot),q)$ subject to $q \epsilon Q$. This can be done by noting that this nonlinear programming problem satisfies the conditions guaranteeing the existence of a solution, finding candidates from the necessary condition for nonlinear programming problems and evaluating $J(u^*(\cdot),q)$ for these candidates.

Thus all the conditions of Theorem 2 are satisfied and $u^*(t) = \frac{36}{19}t - \frac{34}{19}$ is a minmax solution when $x_{om} = 1$, $x_{20m} = 0$.

This result was obtained by guessing $\gamma = 2$, $q^1 = \begin{bmatrix}1\\1\end{bmatrix}$, $q^2 = \begin{bmatrix}-1\\-1\end{bmatrix}$. If instead we choose $\gamma = 2$, $q^1 = \begin{bmatrix}1\\1\end{bmatrix}$, $q^2 = \begin{bmatrix}1\\-1\end{bmatrix}$ or $q^1 = \begin{bmatrix}1\\1\end{bmatrix}$, $q^2 = \begin{bmatrix}-1\\+1\end{bmatrix}$, we find that α_1 and α_2 do not satisfy $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$. Thus the choice of γ and q^i is crucial and several choices may have to be tried before the minmax solution can be found.

Suppose one assumes the measured value of the initial state is exact and finds the optimal control $u_{opt}(\cdot)$ under this assumption. Then, when $x_{10m} = x_{20m} = 10$, $\sup_{q\epsilon Q} J(u_{opt}(\cdot),q) = 184.55$ while the minmax value is 183.59. When $x_{10m} = 1$ and $x_{20m} = 0$, $\sup_{q\epsilon Q} J(u_{opt}(\cdot),q) = 4.362$ while the minmax value is 3.447. In the latter case a reduction in cost of over 20% may be obtained by using the minmax control as opposed to using the optimal control with the assumption that the measured state is exact.

Example 2. Consider a spring mass system where values of the spring constant lie in a known range but the exact value of the constant is unknown. A force is applied to the mass and the objective is to choose the force to maximize the position of the mass at the final time when the system starts from a known initial state with zero initial velocity. The optimal control problem is

$$J(u(\cdot),k) = -x_1(\pi)$$

$$\dot{x}_1(t) = x_2(t)\ ,\ x_1(o) = \frac{1}{2}$$

$$\dot{x}_2(t) = -k\,x_1(t) + u(t)\ ,\ x_2(o) = 0$$

$$K = \{k : 1 \le k \le 4\}\ ,\ U = \{u(t) : u^2(t) - 1 \le 0\}$$

Here we have assumed the mass is unity, $t_f = \pi$, and k is the spring constant. An optimal control $u^*(\cdot)$ is a control satisfying for all admissible $u(\cdot)$, $\sup_{k\epsilon K} J(u^*(\cdot),k) \le \sup_{k\epsilon K} J(u(\cdot),k)$. If $k = 1$, the optimal control is $u(t) = +1$, $t\epsilon[0,\pi]$. While if $k = 4$, it is

$$u(t) = \begin{cases}-1, & t\epsilon[0,\frac{\pi}{2})\\ +1, & t\epsilon(\frac{\pi}{2},\pi]\end{cases}$$

However since k is unknown, neither of these controls has the minmax property. Since the problem is of the form (8)-(10), Theorem 2 can be used.

Following the procedure outlined after Theorem 2, a minmax candidate is

$$u^*(t) = \begin{cases} -1, & t\epsilon[0,t_1) \\ +1, & t\epsilon(t_1,\pi] \end{cases}$$

where t_1 is defined by $\cos t_1 = \sqrt{3} - 1$, $0 < t_1 < \frac{\pi}{2}$. Using this candidate in Theorem 2 with

$$\gamma = 2,\ k_1 = 1,\ k_2 = 4,\ \alpha_1 = 1 - \frac{1}{\sqrt{3}},\ \alpha_2 = \frac{1}{\sqrt{3}}$$

$$\lambda^1(t) = \begin{cases} \alpha_1 \cos t \\ -\alpha_1 \sin t \end{cases},\quad \lambda^2(t) = \begin{cases} -\alpha_2 \cos 2t \\ \frac{\alpha_2}{2} \sin 2t \end{cases}$$

it can be shown that the above $u^*(\cdot)$ is a minmax control. Condition (ii) is the most difficult to verify since it required showing that $k = 1$ and $k = 4$ are solutions to the nonlinear programming problem of maximizing

$$\frac{2}{k} \cos \sqrt{k}\ (t_1 - \pi) - \left(\frac{1}{2} + \frac{1}{k}\right) \cos \sqrt{k}\ \pi - \frac{1}{k}$$

subject to $1 \le k \le 4$.

7. Concluding Remarks

The optimal control of systems with uncertain initial state measurements or with parameter uncertainty in the state equations has been considered. The optimality criterion was taken to be minmax.

For problems with initial state uncertainty, the initial state measurement space can be divided into two regions. In one region, the problem has a saddle point while in the other there is no saddle point solution. Theorem 1 is applicable when the initial state measurement is in the first region and Theorem 2, while applicable for both regions, is more useful when there is no saddle point solution. At present, there is no simple way to determine a priori if there is a saddle point solution for the measured initial state under consideration.

For situations where there is no saddle point solution, the initial state uncertainty problem was transformed into a problem with initial state known but with uncertain parameters in the state equations. This allowed us to derive Theorem 2, and, in so doing, also obtain results which are applicable to problems that are modeled with uncertain parameters in the state equations.

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