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VARIATIONAL PRINCIPLES OF CONTINUUM MECHANICS (AN ANALYTICAL PO--ETC(U)
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US Army Weapons Command
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JUNE 1973

TECHNICAL REPORT

U.S. ARMY WEAPONS COMMAND
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By

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PREFACE

This monograph is basically the outline of lectures given at the WECOM Research Seminar on Applied Mathematics and Continuum Mechanics held at Watervliet Arsenal, New York, in June 1973.

The purpose of the lectures was to familiarize theoretically oriented engineers with some mathematical techniques which are useful in continuum mechanics. Abstract differentiation and integration is shown to lead directly to the formulation of known or new variational principles. Some applications are also given to specific types of ordinary differential equations which occur in theoretical mechanics. However, the formulation of variational principles for boundary value problems of ordinary differential equations can obviously be extended to a much broader class of problems.

July 1973
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VARIATIONAL PRINCIPLES OF CONTINUUM MECHANICS.

(an analytical point of view.)

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Chapter 1

Abstract Differentiation and Integration.

1.0 Introductory comments.

In this chapter we shall consider primarily problems posed in Hilbert space setting. Most statements offered here are easily translated into Banach space terminology, where we have to keep careful count of which elements belong to the space B and which belong to its dual B*, or even to B**. In the Hilbert space setting we can afford to be sloppy, and the arguments are frequently simplified.

1.1 The concept of a derivative.

To offer generalizations of the concept of a derivative and of the usual necessary condition for the extremum of a function we need to have a look at the concept of a derivative of a function of a single (real) variable, and of two variables. Differentiability of a function f whose domain is D(f), $f(x)\colon\ D(f) \subseteq R \to R$, can be stated as follows: given $x_0 \in D(f)$ and given h, we can express the difference $f(x_0 + h) - f(x_0) = \Delta_h f_{x_0}$ in the form

$$(1) \qquad \Delta_h f_{x_0} = K(x_0;h)\cdot h + \tau(x_0;h),$$

where K is some constant (depending on x_0 and on h) and $\tau(x_0,h)$ has the property

$$\lim_{h\to 0} \left|\frac{1}{h}\right| \, |\tau(x_0;h)| = 0.$$

Moreover (regarding K as a function of h) $\lim_{h\to 0} K(x_0;h)$ exists.

We could of course generalize this by considering $\lim_{h\to 0_+}$, or $\lim_{h\to 0_-}$

or $\limsup_{h\to 0} K(x_0;h)$, etc. obtaining the Dini's right, left, upper and lower derivatives.

We shall deliberately stay with only the simplest concepts. The relation (1) can be rewritten as

$$\Delta_h f_{x_0} = \langle K(x_0;h),h\rangle + \tau(x_0,h).$$

where in one dimension the inner product $\langle , \rangle$ is a simple multiplication.

At this point we have deliberately avoided the form $\frac{\Delta f}{h}$, since generalizations of this form are possible only in spaces where products of this type are defined. In general h will be a vector in an infinite dimensional vector space, f may be some mapping and the product $\frac{\Delta f}{h}$ makes no sense at all. However, the analog of formula (1) has an easy interpretation. It is clear that for the infinite dimensional case one dimensional concepts are inadequate, and we shall consider the concept of a derivative in R^2, and attempt to generalize the basic notions from R^2 to an arbitrary Hilbert space.

We shall consider two basic pointwise maps

$$f: \quad R^2 \to R^2$$

$$\phi: \quad R^2 \to R^1.$$

f will be called an operator, a map, or a transformation, ϕ will be called a functional. Clearly a functional is a map $R^2 \to R^2$ $(R^1 \subset R^2)$. However, the one dimensional range of this map simplifies many arguments, and such maps should be considered separately.

The map f may have the following types of derivatives in R^2

i) partial derivatives

ii) total derivative - or the Jacobian derivative,

iii) directional derivative (in the direction of some two dimensional vector in R^2).

For the functional Φ we introduce the concept of a gradient of Φ. All these concepts have appropriate generalizations in Hilbert space. We can also introduce the appropriate concepts of integration in R^2. Given any simple arc Γ, whose endpoints are p_1, p_2, we can introduce the Riemann integral

$\int_{\Gamma(p_1,p_2)} \langle \underset{\sim}{f}(x,y), d\underset{\sim}{s} \rangle$ If $\underset{\sim}{f}(x,y)$: $R^2 \to R^2$ has the property that for any p_1, $p_2 \in \Omega \subseteq R^2$ such integral is independent of the path, we introduce the idea of a potential functional.

$$\Phi(x_0,y_0) = 0, \quad \Phi(x,y) = \int_{\substack{(x_0,y_0)\\ \Gamma}}^{(x,y)} \langle \underset{\sim}{f}(x,y) \cdot d\underset{\sim}{s} \rangle \quad .$$

We note that the independence of path condition requires no properties apart from Riemann integrability.

The more commonly used conditions curl $\underset{\sim}{f}(x,y) = 0$ in R^3 or

$$\frac{\delta f_1(x,y)}{\delta y} = \frac{\delta f_2(x,y)}{\delta x} \quad \text{in } R^2 \quad , \qquad \underset{\sim}{f} = (f_1, f_2) \quad ,$$

assume differentiability of f, and assume simple connectivity of Ω. Each of these criteria will be shown to have important generalizations in Hilbert space.

1.2 Definitions of derivatives of maps in Hilbert spaces.

By a map in Hilbert space we mean a map (transformation, operator...)

$$f: \ \Omega \subseteq H_1 \rightarrow H_2$$

where Ω is some subset of a Hilbert space H_1, and the range of f is a subset of a Hilbert space H_2.

The simplest case when $H_1 = \mathbb{R}$ is easily disposed, since division by a scalar is defined in H_2. We say that $f: \ \mathbb{R} \rightarrow H_2$ is differentiable at the point $t_0 \in \mathbb{R}$ if

$$\lim_{t \to t_0} \left[\frac{f(t) - f(t_0)}{t - t_0} \right] \quad \text{exists.}$$

Unless otherwise qualified lim. will stand for limit in the norm (strong limit). That is we postulate the existence of a vector $f'(t_0) \in H_2$, such that

$$\lim_{t \to t_0} \left\| \frac{f(t) - f(t_0)}{t - t_0} - f'(t_0) \right\| = 0. \tag{1.1}$$

We can rewrite this in the form (1.1) as an equation in H_2:

$$f(t) - f(t_0) = (t - t_0) \, f'(t_0) + \tau(t;t_0) \tag{1.2}$$

where
$$\lim_{t \to t_0} \left\| \frac{\tau(t;t_0)}{t - t_0} \right\| = 0, \tag{1.3}$$

where we deliberately avoid complications at this point by insisting on this limit being defined instead of possible alternate conditions (one sided limit, lim sup,...)

If we replace convergence in the norm by weak convengence, we have the equivalent definition of a weak derivative:

$f: \mathbb{R} \to H_2$ has a weak derivative at the point $t_0 \in \mathbb{R}$, if there exist a vector $f'(t_0) \in H_2$ such that

$$f'(t_0) = \underset{t \to t_0}{\text{weak limit}} \left(\frac{f(t) - f(t_0)}{t - t_0} \right) , \qquad (1.4)$$

with the formula (1.2) still valid, and condition (1.3) replaced by

$$\left(\frac{\tau(t;t_0)}{t-t_0} \right) \xrightarrow{w} \emptyset, \qquad (1.5)$$

where $\emptyset$ is the zero vector in H_2. These definitions don't make much sense if f is an operator $f: H_1 \to H_2$ (generally H_1 is infinite dimensional).

The definition of directional derivative due to Gateaux generalizes the concept of a directional derivative in $\mathbb{R}^2$. We consider the map $f: \Omega \subseteq H_1 \to H_2$. Take x_0 - an interior point of Ω. Pick a fixed vector $h \in H_1$. For fixed x_0, $h \in H_1$ consider the vector $(f(x_0 + th) - f(x_0)) \in H_2$ where the constant t is picked sufficiently small in absolute value to make sure that $x_0 + th \in \Omega$, which is possible since x_0 was an interior point of Ω. For fixed x_0 and h the difference $\delta f(t) = (f(x_0 + th) - f(x_0))$ is a *function of* the real variable t only. Hence we are back in the previously discussed case $\delta f(t): \mathbb{R} \to H_2$, and we can define the <u>directional derivative</u> of f in the direction of h, computed at x_0, if there exists a vector $f(x_0;h) \in H_2$ such that for all sufficiently small values of t, the formula holds:

$$\delta f(t;x_0,h) = f(x_0+th) - f(x_0) = t \cdot f'(x_0;h) + \tau(t;x_0,h) \qquad (1.6)$$

where $\lim_{t\to 0} \left\| \frac{\tau(t;x_0h)}{t} \right\| = 0$, (see [10]) . (1.7)

(or $\frac{1}{t}\tau(t;x_0h) \overset{w}{\to} \emptyset$ as $t \to 0$ in case of a weak derivative.)

We should point out that the directional derivative $f'(x_0;h)$ does not have to be linear in h. In general $f'(x_0;h_1+h_2) \neq f'(x_0;h_1) + f'(x_0;h_2)$. It is however homogeneous of degree one in h, as can be easily checked from the definition, and $f'(x_0;ch) = cf'(x_0;h)$.

1.3 <u>The Gateaux derivative</u>.

We now assume that $f'(x_0;h)$ exists and is linear in h. Then $f'(x_0;h)$ is called the <u>Gateaux derivative</u> of f in the direction of h, computed at the point x_0. The Gateaux derivative is not necessarily continuous with respect to h either in the strong, or even in the weak topology of H_2. It only implies the existence of a linear operator $L_{(x_0)}$ such that for a fixed $h \in H_2$ and for sufficiently small (in absolute value) $t \in \mathbb{R}$,

$$f(x_0 + th) - f(x_0) = tL_{(x_0)}h + \tau(x_0;th) \quad (1.8)$$

where $\lim_{t\to 0} \left\| \frac{\tau(x_0;th)}{t} \right\| = 0$.

(or $\text{weak}\lim_{t\to 0} \frac{1}{t}\tau(x_0;th) = \emptyset$.) We would like to point out that

$\lim_{t\to 0} \left\| \frac{\tau(x_0;th)}{t} \right\| = 0$ does not imply

$$\lim_{\|h\|\to 0} \left(\frac{\|\tau(x_0;h)\|}{\|h\|} \right) = 0,$$

for arbitrary $h \in H_2$ (of sufficiently small norm). In other words the Gateaux derivative may exist in the direction of a vector h_1, but may fail to exist in the direction of a vector h_2. This is easily checked to be true even in some two dimensional cases) $f: \mathbb{R}^2 \to \mathbb{R}^2$.

1.4 Fréchet differentiation.

Assume that the operator L_{x_0} in (1.8) is linear and bounded, i.e. for fixed $x_0 \in H_2$ we have

i) $L_{x_0}(\alpha h_1 + \beta h_2) = \alpha L_{x_0} h_1 + \beta L_{x_0} h_2$, and there exists $M > 0$ such that

ii) $\|L_{x_0} h\| \leq M(x_0)\,\|h\|$ for all h of sufficiently small morm.

We also assume that for all such $h \in H_2$ it is true that

iii) $f(x_0+h) - f(x_0) = L_{x_0} h + \tau(x_0;h)$ where

$$\lim_{\|h\|\to 0} \frac{\|\tau(x_0;h)\|}{\|h\|} = 0, \qquad \text{(see [7])} . \tag{1.8}$$

Then L_{x_0} is called the Fréchet derivative of the operator f evaluated at x_0. The following condition can replace condition (iii)

iv) $\lim_{t\to 0} \left\| \frac{\tau(x_0,th)}{t} \right\| = 0$ uniformly with respect to h on all bounded subsets of H_2.

The fact that (iii) and (iv) turn out to be equivalent is not trivial. The proof of it can be found in [21].

Theorem. Fréchet derivative exists if and only if the Gateaux derivative exists and satisfies condition (iii). The proof is easy and will not be given here. (See [52].)

1.5 Fréchet differentiation of functionals.

In this case we consider the Fréchet derivative of the map $f: \Omega \subseteq H_1 \to \mathbb{R}$. Let the Fréchet derivative L_{x_0} exist at $x_0 \in \Omega$. Then L_{x_0} is a continuous linear map from $\Omega \subseteq H_1$ into $\mathbb{R}$. Therefore for some $h \in H_1$ we have $L_{x_0} ch = c L_{x_0} h$, $\forall c \in \mathbb{R}$, and by the Hahn Banach theorem L_{x_0} can be extended to all of H_1 without changing its norm.

By the Riesz representation theorem there exists a unique $Z \in H_1$ such that $L_{x_0} h = \langle Z,h\rangle$ for all $h \in H_1$. Hence for all $h \in H_1$ the Fréchet derivative of a functional f evaluated at x_0 is a continuous linear functional L_{x_0}, and $L_{x_0} h = \langle Z_{x_0},h\rangle$ (1.9)

for all $h \in H_1$.

The unique vector Z_{x_0} is called the gradient of f evaluated at x_0. Clearly Z depends on the definition of the inner product in H_1. However the existence of the gradient depends only on the existence of the Fréchet derivative which in turn depends on convergence of our limiting process. Convergence is a topological property, and remains invariant in equivalent topologies. Hence the introduction of an equivalent norm in H_1 will not affect the existence, or non-existence of a gradient (or of the Fréchet derivative). In fact let us introduce a new inner product (the so called energy product, and corresponding energy norm) $[\ ,\ ]$ $[x,y] = \langle Tx,y\rangle$, where T is positive definite, symmetric operator. Denoting by $|||x|||^2 = [x,x] = \langle Tx,x\rangle$, and by $\|y\|^2 = \langle y,y\rangle$, we have $\langle \text{grad } f, h\rangle = [\text{grad}^\# f,h] = \langle T\text{grad}^\# f,h\rangle$ (1.10) where $grad^{\#}$ is the gradient of f in the topology induced by the norm $|||\ |||$. Since relation (1.10) is true for all $h \in H_1$, we have the equality $T \cdot grad^{\#} = grad$. If T^{-1} exists, then $grad^{\#} = T^{-1} \cdot grad$. If the operator T is bounded away from zero (i.e. the spectrum of T does not have zero as a limit point) then there exists a constant $\gamma > 0$, such that for any $x \in H_1 \cap \mathcal{D}(T)$ $\|x\| \leq \frac{1}{\gamma} |||x|||$. ($\mathcal{D}(T)$ denotes the domain of T). Hence convergence (or continuity, or existence of limit points) in the new norm $|||\ |||$ implies convergence (or continuity...) in the old norm $\|\ \|$.

In the final dimensional case all these statements are trivial, and all norms $\langle , \rangle$, $[\ ]$ where $\langle Tx,y \rangle = [x,y]$, and T is positive definite, and symmetric, are equivalent and generate the same topology.

At this point it is appropriate to make a comment about the non-existence of the Fréchet derivative in some Hilbert space H_1. Suppose the Gateaux derivative of Φ (in arbitrary direction h) exists, but fails to be a continuous functional of h in the usual (norm) topology of H_1, i.e. we can find a sequence of vectors $\{h_i\} \in H_1$ such that $\lim_{i\to\infty} \|h - h_i\| = 0$, but $\lim_{i\to\infty} \Phi(h_i) \neq \Phi(h)$. It may be possible to introduce a different, and non-equivalent inner product $\{ , \}$, with a corresponding norm $\|\ \|_{(2)}$ such that $\Phi(h)$ is a continuous functional of h in the new (norm) topology.

If the inner product is fixed, and no attempt will be made to change the topology, then we can use the sloppy notation $\Phi_x(x_0)$, or $\left.\frac{\delta\Phi(\underset{\sim}{x})}{\delta \underset{\sim}{x}}\right|_{x_0}$ to denote the gradient of Φ evaluated at x_0, where Φ is a Fréchet differentiable functional whose values depend on

$\underset{\sim}{x} \in H_1$, i.e. $\Phi: H_1 \to \mathbb{R}$. $L_{x_0}(x) = \langle Z_{x_0}, x\rangle = \langle \frac{\delta\Phi}{\delta \underset{\sim}{x}}\Big|_{x_0}, x\rangle$ is the corresponding Fréchet functional. This notation is both confusing, and sloppy, but it has been consistently in use, and the author of this paper is also guilty of having abused it.

It is partially vindicated by a physical explanation. The Hilbert space setting has obscured our vision to a certain extent, since we identify H with its dual, and we are allowed to neglect the bookkeeping which vector is in H, and which vector is in H* (the dual of H, which is identified with H). This distinction becomes important later on when we try to interpret certain equations of mathematical physics. In the remainder of this paper we shall use the notation grad Φ if Φ is a functional $\Phi: H_1 \to \mathbb{R}$. We shall also find examples where Φ is a functional $\Phi: H_1 \oplus H_2 \oplus \ldots \oplus H_k \to \mathbb{R}$ (or $\mathbb{C}$), that is Φ maps the direct product of Hilbert spaces $H_1, H_2 \ldots H_k$ into the real (or complex) numbers. In this case the notation grad $\Phi_{(H_i)}$ is too cumbersome and we shall write Φ_{x_i} to denote

$$L_{x_i}(h) = \langle Z_{x_i}, h\rangle = \langle \lim_{t\to 0} \left(\frac{f(x_0+th) - f(x_0)}{t}\right), h\rangle$$

with $h \in H_i$, $x_0 \in H_1 \oplus H_2 \oplus \ldots \oplus H_k$. That is we generalize simultaneously the idea of a gradient and of a partial derivative, by allowing the variation of f to take place exclusively in H_i, and regarding the components of $\underset{\sim}{x}$ in H_j, $j \neq i$, as fixed. We shall refer to this derivative as <u>gradient</u> of Φ <u>restricted to</u> the space <u>H_i</u>. This idea plays a crucial part in formulation of multiple variational principles. For example the following equations of mildly nonlinear elasticity corresponds to critical points of a simple functional restricted to respective subspaces of the space $H = \sum_{\oplus} H_i$. The Maxwell and Morrera's stress function relations, the equations of equilibrium, the equations of compatibility, Ricci's equations. (See [53]).All become easily interpreted in this context.

1.6 Some rules of manipulation for gradients of functionals

We shall restrict ourselves first to the simplest case: $\Phi: H \to \mathbb{R}$, $\mathcal{D}(\Phi) \subseteq H$, Φ is Fréchet differentiable in $\mathcal{D} \subseteq H$.

We shall denote by $\Phi_{x(\Omega)}$ the gradient of $\Phi = Z \in \mathcal{D}$, where $\langle Z, \eta\rangle$, $Z \in \mathcal{D}$, $(\eta \in \mathcal{D})$ is the Fréchet derivative of Φ coinciding with the Gateaux derivative in the direction η. We compute the value of the gradient at a point $x_0 \in \mathcal{D}$. $\Phi_x(x_0) = Z(x_0)$. If $Z(x_0) = \emptyset$ (the zero vector) we say that x_0 is a critical point of the functional Φ. A basic theorem of optimization theory (Vaînberg [49]) states a necessary condition for stationary behavior of a functional Φ (particularly for a maximum or minimum) at the point $x_0 \in H$. The point x_0 must be a critical point of Φ, if Φ is Fréchet differentiable at x_0. In particular this is a necessary condition for a local maximum or minimum of a Fréchet differentiable functional.

Example 1. $\Phi = \langle Y_1, Y_2\rangle$

$$Y_1, Y_2 \in H, \qquad Y_1 = C_1 X,$$

$$Y_2 = C_2 X. \quad \text{Then } Z = \Phi_X = (C_1 + C_2) X.$$

Proof of this statement follows from the definition. We compute the Gateaux derivative of Φ in the direction of a vector $\eta \in H$.

$$<\eta, Z> = \lim_{\varepsilon \to 0} \frac{<C_1X+\varepsilon\eta,\ C_2X+\varepsilon\eta>}{\varepsilon} - C_1C_2 <X,X>$$

$$= (C_1 + C_2) <\eta, X> = <\eta, (C_1 + C_2)X>.$$

Hence $\phi_X = (C_1 + C_2)X$.

Example 2. $\phi(X) = \phi_1(X) \cdot \phi_2(X)$ Then the gradient of ϕ:

$$Z = \phi_X = \phi_{1_X} \cdot \phi_2 + \phi_1 \cdot \phi_{2_X}.$$

Similarly

$$\left(\frac{\phi_1(x)}{\phi_2(x)}\right)_X = -\frac{\phi_{1_X}\phi_2 - \phi_{2_X} \cdot \phi_1}{\phi_2^2}$$

Example 3.

$$\phi = -\frac{<Ax,x>}{<X,X>}, \quad A: H \to H, \quad A^* = A, \quad X \neq \emptyset.$$

Then

$$\phi_X = \frac{2}{<X,X>} \cdot (\phi(x)X - AX).$$

1.7 Integration.

Vainberg ([21]) offers the following generalization of the idea of a Riemann integral. Consider f: $H \to H$. We define the integral along a line segment connecting x_0 and $\underset{\sim}{x} \in H$ by

$$\int_0^1 <f(x_0+t(x-x_0)), (x-x_0)> dt = \int_0^1 \phi(t)dt. \qquad (1.11)$$

This formula can be obviously generalized to polygonal paths. The common way to introduce integrals of vector fields along an arc $\Gamma \subset R^3$ is to consider a subdivision of the arc $\Gamma_{[a,b]}$ by points $p_0, p_1, p_2 \ldots, p_n$ and introducing the Riemann sums :

$$\sum_{i=0}^{n} (\vec{p}_{i+1} - \vec{p}_i) \cdot f(\xi_i),$$

where ξ_i is a point contained on the interval $[p_i, p_{i+1}]$. If $f\colon R^3 \to R^3$ is a continuous map and if Γ is "reasonably smooth". (For example we can allow a finite number of corner points without causing any problems). Such Riemann sums generalize to Hilbert (Banach) spaces by simply replacing the three-dimensional inner product, by the appropriate inner product. If f is a continuous map $f\colon H \to H$, and Γ is a smooth arc we can easily show the convergence of the Riemann sums:

$$\lim_{\|p_{i+1}-p_i\|\to 0} \left\{ \sum_{i=0}^{n} \langle (p_{i+1} - p_i, f(\xi_i) \rangle \right\}. \tag{1.12}$$

In the case when Γ is a parametrized arc we can choose the points ξ_i to coincide with p_i, and the integral (1.12) can be reduced to the form

$$\int_0^1 \phi(t)dt.$$

1.8 <u>A basic condition for the existence of a potential.</u>

We generalize (following Vainberg [52]) two well known criteria of R^3. $f\colon H \to H$ is assumed to be a continuous map. Suppose that Ω is a region of H, such that for any $p_1, p_2 \in \Omega$ the integral

$$\int_{\Gamma}^{(p_1,p_2)} \langle \underset{\sim}{f}(\underset{\sim}{x}), d\underset{\sim}{x} \rangle = \int_{t(p_1)=t_1}^{t(p_2)=t_2} \phi(t)dt$$

is independent of Γ, for any parametrized arc Γ, and only depends on the end points p_1, p_2. Then there exists a functional $\Phi(\underset{\sim}{x})$ such that $f(\underset{\sim}{x})$ is the gradient of Φ.

<u>Proof</u>. Choose a (fixed) point $\underset{\sim}{x}_0 \in \Omega \subseteq H$. Define for $p \in \Omega$ the functional

$$\phi(p) = \int_0^1 \langle f(x_0 + t(p - x_0)), \quad (p-x_0) \rangle dt + \phi(x_0),$$

and (without any loss of generality) put $\phi(x_0) = 0$.

If Ω contains the origin, we may choose $x_0 = \emptyset$, and define

$$\phi(x) = \int_0^1 \langle f(t\underset{\sim}{x}), \underset{\sim}{x} \rangle dt.$$

It is easy to check that $f(\underset{\sim}{x})$ is the gradient of Φ.

Note: Observe that we did not make the assumption that $f: H \to H$ is linear.

There is the obvious problem of knowing how to apply this condition. How do you check that this integral is independent of the path for every path in Γ? In most cases this turns out to be an impossible task. At this point we recall another condition for the existence of a potential function in $\mathbb{R}^3$. Assume that Ω is simply connected, that $f: \mathbb{R}^3 \to \mathbb{R}^3$ is smooth and curl (f) exists at all points of Ω, and curl(f) ≡ 0 in Ω. Then f is a gradient. Of course we do not know what curl(f) means in an infinite dimensional space. However in $\mathbb{R}^3$ curl($\underset{\sim}{f}$) ≡ 0 if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad i = 1,2,3.$$

This condition generalizes quite easily to infinite dimensional space. However we shall have to delay its discussion, since we have not defined yet the meaning of second (or higher) order derivatives of a functional. Observe that:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \Rightarrow \frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\partial^2 \Phi}{\partial x_j \partial x_i} .$$

This condition is trivially satisfied in R^3 if $\Phi \in C^2(\Omega)$, (this is known as Tonelli's theorem). In a Hilbert space there is no obvious reason why second derivatives should commute, and this condition assumes a much deeper meaning.

The generalizations of Cauchy's integral condition in higher dimensional complex spaces are too involved to be of practical use in Hilbert spaces. At least this is the opinion of the author, which may of course turn out to be wrong.

Chapter 2

2.1 The Euler-Lagrange equations, and critical points of a functional.

The Euler-Lagrange equations for the problem of extremizing the value of

$$\int_{t_0}^{t_1} \mathcal{L}(x,x',t)\,dt \qquad \text{are}$$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'}\right) = 0 \tag{2.1}$$

The Legendre transformation introduces the generalized momentum

$$p = \frac{\partial \mathcal{L}}{\partial x'} \tag{2.2}$$

Hence (2.1) can be written as

$$\frac{\partial \mathcal{L}}{\partial x} = \dot{p} = -\frac{\partial H}{\partial x} \tag{2.3^a}$$

and $\frac{\partial H}{\partial p} = \dot{x}$ where $H = \dot{x}\,p - \mathcal{L}$. (2.3^b)

We are now ready to formulate the Hilbert space analogs of equations (2.3^a), (2.3^b), and (2.1). The basic ideas of generalization of equations (2.3^a), (2.3^b) go back to Korn who observed dual variational principles in theory of elasticity of materials not necessarily obeying Hooke's law. Friedricks has interpreted Korn's results in terms of modern operator theory [9].

An intuitive approach to the postulation of duality and formulation of dual variational principles originated with Noble [35], and has been successfully applied by a number of authors to problems of mathematical physics (see for example the monograph of A. M. Arthurs [3]).

2.2 Generalized Solutions, energy norms and extremal points of functionals.

The well known case of a close relation between a generalized solution o equation:

$$Au - f = \emptyset \tag{2.4}$$

in a Hilbert space and extreme value of a functional, when A is a positive definite operator, bounded away from zero, and the domain of A is dense in H. In this case we can introduce a new inner product $[u,v] = \langle Au,v\rangle$, and a new norm $|||u|||^2_{(A)} = \langle Au,u\rangle$. Let H_A denote the closure of the new inner product space. We examine the corresponding functional:

$$F(u) = \langle Au,u\rangle - 2\langle u,f\rangle = |||u|||^2_{(A)} - 2\langle u,f\rangle \tag{2.5}$$

Since A is bounded away from zero (meaning $\langle Au,u\rangle \geq C^2 \|u\|^2$ for some $C > 0$, or $|||u||| > C\ \|u\|$) it follows that $|\langle u,f\rangle| \leq \frac{\|f\|}{C} |||u|||$. Hence by Riesz representation theorem there exists $u_0 \in H_A$ such that $\langle u,f\rangle = [u,u_0]$, and $F(u) = |||u - u_0|||^2 - |||u_0|||^2$. Hence F(u) attains a minimum at u_0, and

$$\min_{u \in H_A} F(u) = -|||u_0|||^2 \tag{2.6}$$

As before H_A denotes the Hilbert space obtained by closure of the inner product space with the product $[\cdot,\cdot]$. The point $u_0 \in H_A$ is called the generalized solution of the equation (2.4). If u_0 was in the domain of A, then it is a "genuine"(or classical) solution. Otherwise it is called a generalized solution of (2.4). In fact since A is positive definite, there exists an operator B, such that $A = B^*B$, and for all $u \in \mathcal{D}_A$ & $u \in \mathcal{D}_B$ and $\langle Au,u\rangle = \langle Bu,Bu\rangle$. We have the containment $\mathcal{D}_A \subset \mathcal{D}_B$. However $u_0 \in \mathcal{D}_B$ does not mean that u_0 is necessarily in the domain of A. As an example consider

$$A = -\frac{d^2}{dx^2}, \quad \mathcal{D}_A = C^2[0,1] \subset L^2[0,1], \quad u(0) = u(1) = 0.$$

The generalized solution of $-\frac{d^2}{dx^2} u = f$ corresponds to the minimization problem for the (L_2) functional

$$\langle -\frac{d^2}{dx^2} u, u\rangle - 2\langle f,u\rangle$$

$$= -\int_0^1 (u'' \cdot u)dx - 2\int_0^1 (fu)dx$$

$$= \int_0^1 (u')^2 dx - 2\int_0^1 (fu)dx.$$

The problem is that the existence of the functional

$$F(u) = \int_0^1 (u')^2 dx - 2\int_0^1 (fu)dx$$

requires only $u' \in L_2[0,1]$ and does not require twice differentiability.

$B = i\frac{d}{dx}$ is one of the possible corresponding **square roots** of the operation of the operator $-\frac{d^2}{dx^2}$. **Hence the minimization problem** can be accomplished in the Sobolev space $H^1[0,1]$ instead of $u \in C^2[0,1] \subset L_2[0,1]$. The energy space H_A turns out in this case to be $H^1[0,1]$. Of course in the above case we had a lot of properties of the operator A, (positive definite, bounded away from zero) which made life very easy, and gave us so easily an equivalent extremal problem for the functional F(u). A feature of this functional which we have not mentioned yet was convexity.

Definition. Let $\Phi: \Omega \subseteq H \to R$ be defined on a convex set $\Omega \subseteq H$. Then Φ is called convex if for any $x_i, x_2 \in \Omega$ the following inequality is true:

$$\Phi(\Lambda x_1 + (1-\Lambda)x_2) \leq \Lambda\Phi(x_1) + (1-\Lambda)\Phi(x_2) \qquad 0 < \Lambda < 1.$$

If strict inequality holds, then Φ is called strictly convex. Φ is called concave if $-\Phi$ is convex. We observe that any norm is a convex functional, since it obeys the triangular inequality.

$$\|\Lambda x + (1-\Lambda)y\| \leq \Lambda\|x\| + (1-\Lambda)\|y\|.$$

A study of convex functionals can be reduced to a study of convex sets in H, by introduction of the "epigraph" of the functional Φ, that is of a subset of $R \times H$

$$[\Phi,C] = \{(r,\underset{\sim}{x}) \in R \times H: \ \Phi(x) < r, \ \underset{\sim}{x} \in C \subseteq H\}$$

where C is a convex subset of H. (see Luenberger [55] for an exposition).

2.3 Conjugate convex functionals.

Let Φ be a convex functional Φ: $\Omega \subseteq B \to \mathbb{R}$, defined on a convex set $\Omega \subseteq B$, where B is a Banach space and B^* is its dual. The set Ω^* conjugate to Ω relative to Φ, $\Omega^* \subseteq B^*$, is defined as

$$\Omega^* = \{p \in B^*: \sup_{x \in \Omega} \langle p,x\rangle - \Phi(x) < M \quad \text{for some } M \in \mathbb{R}\}$$

The conjugate functional $\Phi^*(p)$ is defined by the relation

$$\Phi^*(p) = \sup_{x \in \Omega} \{\langle x,p\rangle - \Phi(x)\} .$$

<u>Proposition</u>. If Φ is convex, then so is Φ^*. <u>Proof</u>. For any number Λ, $0 < \Lambda < 1$, check that

$$\sup_{x \in \Omega} \{\langle x, \Lambda p_1 + (1-\Lambda)p_2\rangle - \Phi(x)\}$$

$$\leq \Lambda \sup_{x \in \Omega} \{\langle x, p_1\rangle - \Phi(x)\} + (1-\Lambda) \sup_{x \in \Omega} \{\langle x, p_2\rangle - \Phi(x)\} .$$

A geometric interpretation of the convex functional Φ^* can be intuitively argued as follows. Consider a convex set $\Omega \subseteq \mathbb{R} \times B$, where B is a Banach space. A family of hyperplanes in $\mathbb{R} \times B$ is given by an element of $\mathbb{R} \times B^*$, i.e. an ordered pair (s, p) $s \in \mathbb{R}$, $p \in B^*$. A particular hyperplane is obtained by setting

$\langle (r,x), (s,p) \rangle = c$, $r \in \mathbb{R}$, $x \in B$, or $rs + \langle x,p \rangle = c$, where c is some real number. By rescaling we can always choose $s = -1$, provided $s \neq 0$. Then $C = \langle x,p \rangle - r$ defines a family of hyperplanes of $\mathbb{R} \times B$. $\phi^*(p) = \sup (C)$ is the sup. of values for which $\phi^*(p)$ is a support hyperplane of $[\phi, C]$.

Convexity, and concavity of certain functionals are closely tied with existence theory of maxima and minima. For the time being we shall only concentrate on sufficiency conditions which generalize the Euler-Lagrange equations of classical calculus of variations.

2.4 Critical points of fucntionals and the equation Ax = f.

We shall consider a fairly arbitrary linear operator A mapping $\mathcal{D}_A \subseteq H_1$ into H_1, where H_1 is a Hilbert space. We wish to solve (for $x \in H_1$) the equation $Ax = f$ ($f \in H_1$), where we shall assume that $A = T^*T$ (A is positive definite) and the domain of T is dense in H_1. Therefore T^* (the adjoint of T) is uniquely defined. T, T^* are the linear maps:

$$T: \mathcal{D}_T \subseteq H_1 \to H_2. \quad T^*: \quad H_2 \to H_1$$

Hence the equation $Ax = f$ ($\in H_1$) can be rewritten as a pair of equations

$$T\,x = p \quad (\in H_2)\ , \tag{2.7}$$

$$T^*\,p = f \quad (\in H_1). \tag{2.8}$$

We consider the Hilbert space $H = H_1 \oplus H_2$ whose elements h are ordered pairs $h = (\tilde{x},\tilde{p})$ $(\tilde{x} \in H_1,\ \tilde{p} \in H_2)$, with the product

$$\{h_1, h_2\} = (x_1, x_2) + \langle p_1, p_2\rangle$$

where (,) is the inner product in H_1, and <,> is the inner product in H_2. Consider arbitrary vectors $\tilde{p} \in H_2$, $\tilde{x} \in H_1$ i.e. an arbitrary $\tilde{w} = \{\tilde{x},\tilde{p}\} \in H$ and the corresponding value of the functional $L\colon\ H \to R$

$$L(\tilde{x},\tilde{p}) = \langle T\tilde{x},\tilde{p}\rangle - \frac{1}{2}\langle\tilde{p},\tilde{p}\rangle - (f,\tilde{x}). \qquad (2.9)$$

It is a fairly trivial result that provided the gradient $L_{\tilde{W}}$ is uniquely defined at the point $W_0 = \{x_0, p_0\} \in H$, then the equations (2.7) and (2.8) are necessary and sufficient conditions for vanishing of $L_{\tilde{W}}$. Hence the functional L has a critical point at W_0 if and only if the equations (2.7) and (2.8) are satisfied.

Again the condition $x \in \mathcal{D}_A$ has been replaced by condition $x \in \mathcal{D}_T$, which is in general easier to satisfy since $A = T^*T$. (See the next section for a more rigorous statement.) A more detailed look at equations (2.7) and (2.8) show that if we fix $\tilde{p} \in H_2$ and vary only $x \in H_1$, $L_x = \emptyset$ (in H_1) if and only if (2.8) is satisfied, and vice versa if we fix $\tilde{x} \in H_1$ and vary $p \in H_2$, $L_p = \emptyset$ (in H_2) if and only if (2.7) is satisfied.

These observations have been originally implied by Kato [22]. (We are using the same notation L_x denoting the gradient of the functional $L(x)$ (with p a fixed vector of H_2), always assuming that this gradient exists, (in a fixed topology of the space!) and if $L_x = \emptyset$ at $x = x_0 \in H_1$ then L_x is defined in some neighborhood of the point x_0 in the appropriate topology. Unless otherwise specified we shall use the topology induced by the norm. Again we observe that $\tilde{W}_0 = \{\tilde{x}_0, \tilde{p}_0\}$ which is a critical point of the functional L does not have to be a "genuine solution of the original equation $Ax = f$, since $\tilde{x}_0$ may be in the complement of the domain of A.

The system (2.7), (2.8) has been designated by Noble ([35] and [36]) and Rall ([39]) as a Hamiltonian system. The name is easily explained if we call the functional $\frac{1}{2} \langle p,p \rangle + \langle f,x \rangle = W(p,x)$ the Hamiltonian and observe that (2.7), (2.8) can be rewritten as

$$T x = W_p \tag{2.10}$$

$$T^*p = W_x \tag{2.11}$$

which is the Hamilton's system of canonical equations in the special case $T = \frac{d}{dt}$, $T^* = -\frac{d}{dt}$, with $\underset{\sim}{x}$ being the vector of generalized displacements, and p of generalized momenta.

In anology with the terminology of classical mechanics the fucntional L will be called the action functional or the Lagrangian (integral) functional. We observe the following peculiarity of our discussion: The equation $Ax = f$ is an equation in H_1. The introduction of the "splitting space" H_2 was a result of the decomposition of the operator $A: H_1 \to H_1$ into T^*T: $T: H_1 \to H_2$, $T^*: H_2 \to H_1$

In particular the work of Browder and Gupta indicates that under certain conditions while H_1 is a Banach space, H_2 may be chosen to be a Hilbert space, with resulting theoretical advantages. In this paper H_1 will be assumed to be a Hilbert space. To make rigorous the ideas loosely expressed here we need some additional definitions from functional analysis.

Definition Let T be a (linear) operator mapping some subset of a Banach space B_1 into a Banach space B_2. The graph of T, denoted by G(T) is a subset of $B_1 \times B_2$ consisting of all ordered pairs of the form $\{u, Tu\}$, $u \in \mathcal{D}_T$. The following statement is known as the closed graph theorem. Let $\mathcal{D}_T = B_1$. Then T is bounded if and only if the graph of T is closed.

Definition (Kato [56]) Any linear mainfold C contained in $\mathcal{D}_T \subseteq B_1$ is called the core of T if the set $\{u, Tu\}$, $u \in C$, is a dense subset of G(T). An important theorem due to Von Neumann asserts the following. Let T be a closed operator $T: H_1 \to H_2$; $\mathcal{D}_T$ dense in H_1. Then $A = T^*T: H_1 \to H_1$ is a selfadjoint operator and $\mathcal{D}_A$ is a core of T. Hence $D_T \supseteq \mathcal{D}_A$. If the containment is proper i.e. $D_T \supset D_A$ then the corresponding variational solutions describing the original problem in terms of critical points of some functionals in H_2 will exhibit solutions which are not "genuine" solutions, that is they are not in the domain of A.

2.5 The Legendre transformation

In classical mechanics we encounter the following transformation:

$$p = \frac{\partial L(x,\dot{x},t)}{\partial \dot{x}}, \quad H(x,p,t) = \sum_{i=1}^{n} p_i \dot{x}_i - L(x,\dot{x},t),$$

changing the "Lagrangian formalism" into the "Hamiltonian formalism". To this transformation corresponds the abstract problem of defining what is meant by

$$\frac{\partial L}{\partial (Tx)} \qquad T: \; H \to H_1,$$

and generalizing this concept to an abstract Hilbert space setting. Again perhaps the best starting point is the equation

$$Ax = f \qquad (\in H_1), \qquad A: \; H_1 \to H_1,$$

for which we intend to establish (weak) solutions which correspond to a critical point of some functional L, or to two (or more) critical points of functionals in some spaces possibly other than H_1. The case we have discussed already presumes that A is positive definite and that we can find a space H_2 such that $x \in \mathcal{D}_T$, $Tx = p \in H_2$, $T^*p = f \in H_1$, $\mathcal{D}_T$ dense in H_1, $T^*T = A$.

Of course the choice of H_1 is non-unique, as can be seen by studying even the simpl est examples

$$\left(-\frac{d^2}{dx^2} : H^2(R) \to L_2(R), \quad -\frac{d^2}{dx^2} = T^*T,\right.$$

$$T = i\frac{d}{dx}, \quad T^* = i\frac{d}{dx}, \quad T: \; H^2(\mathbb{R}) \to H^1(\mathbb{C}),$$

$$T^*: \; H^1(\mathbb{C}) \to L_2(\mathbb{R}), \quad \text{or} \quad T = \frac{d}{dx}, \quad T^* = -\frac{d}{dx}.$$

$$T: \; H^2(\mathbb{R}) \to H^1(\mathbb{R}), \quad T^*: \; H^1(\mathbb{R}) \to L_2(\mathbb{R}),$$

showing that A can be factored thru different Hilbert spaces H_1 and H_1'. More involved examples will be offered later of multiple factoring process with corresponding multiple variational principles, in which the choices of intermediate Hilbert spaces are not uniquely determined.

We shall now consider the effect of the vector $Tx \in H_1$ on the value of the Lagrangian functional L. Regarding p as a fixed vector in H_1, we consider L(x,y) as functional, mapping the pair $x \in H_1$, $y \in H_2$ into the real line, where y = Tx.

L_y denotes the gradient of L (whenever it exists) restricted to H_2. ($x \in H_1$ is ignored). Similarly L_x denotes the gradient of L in H_1. It is a straightforward computation that with p regarded as fixed, and with y = Tx, that

$$L_y = p, \tag{2.12}$$

and that

$$\langle Tx, p\rangle - L = \frac{1}{2}\langle p,p\rangle + (f,x) = W \tag{2.13}$$

defines a new functional W(x,p) satisfying

$$\left.\begin{aligned} W_x &= f \\ W_p &= p \end{aligned}\right\} . \tag{2.14}$$

The notation is the same as before. W_x is the gradient of W restricted to the space H_1, and W_p is the gradient of W restricted to the space H_2. We shall describe the relations (2.12), (2.13) as the Legendre transformation. (See [5] for a classical definition.)

Chapter 3

3.1 An example of multiple critical points, and applications to the general theory of solids.

We shall consider the equations of classical elasticity, assuming small strain theory, but not necessarily small rotations. The equations of equilibrium assume the form:

$$\frac{\partial}{\partial x}\left[(1+e_{xx})\ \tau_{xx} + (e_{xy} - \omega_x)\ \tau_{xy} + (e_{xz} + \omega_y)\ \tau_{xz}\right]$$

$$+ \frac{\partial}{\partial y}\left[(1 + e_{xx})\ \tau_{xy} + (e_{xy} - \omega_z)\ \tau_{yy} + (e_{xz} + \omega_y)\ \tau_{yz}\right]$$

$$+ \frac{\partial}{\partial z}\left[(1 + e_{xx})\ \tau_{xz} + (e_{xy} - \omega_z)\ \tau_{yz} + (e_{xz} + \omega_y)\ \tau_{zz}\right] = 0 \tag{3.1}$$

Equations (3.2), (3.3) are obtained by cyclic permutation of letters x, y, z. Here e_{ij} is the linear strain matrix $\varepsilon_{xx} \simeq e_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{xy} \simeq e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\ldots$ ω_i are the rotation components

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{3.2}$$

i.e. $\vec{\omega} = \frac{1}{2}$ curl $(\vec{u})$.

If we denote by α the following stress tensor:

$$\alpha = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} = (I + e) \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}, \tag{3.3}$$

where e is the Jacobian

$$e = \frac{\partial\,(u,v,w)}{\partial\,(x,y,z)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}, \tag{3.4}$$

we can formulate the following sets of equations of equilibrium.

$$B\alpha = \begin{bmatrix} \partial x & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \alpha_{zz} \\ \alpha_{xy} \\ \alpha_{yz} \\ \alpha_{xz} \end{bmatrix} = \emptyset, \tag{3.5}$$

$(\alpha_{ij} = \alpha_{ji})$.

Because of smallness of strains, we identify the Eulerian strain ε_{ij} with the linear strain e_{ij}. We can define the stress components α_{ij} in terms of appropriate Maxwell and Morrera stress functions

$$\begin{bmatrix} X_{xx} & X_{xy} & X_{xz} \\ X_{yx} & X_{yy} & X_{yz} \\ X_{zx} & X_{zy} & X_{zz} \end{bmatrix} = X, \quad X_{ij} = X_{ji}\,, \tag{3.6}$$

$$\underset{\sim}{\alpha} = A\,X, \tag{3.7}$$

where

$$A = \begin{bmatrix} 0 & \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial y^2} & 0 & -\frac{\partial^2}{\partial y \partial z} & 0 \\ \frac{\partial^2}{\partial z^2} & 0 & \frac{\partial^2}{\partial x^2} & 0 & 0 & -\frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x^2} & 0 & -\frac{\partial^2}{\partial x \partial y} & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} \\ -\frac{\partial^2}{\partial y \partial z} & 0 & 0 & \frac{\partial^2}{\partial x \partial z} & -\frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \\ 0 & -\frac{\partial^2}{\partial x \partial z} & 0 & \frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{3.8}$$

The equations of equilibrium become

$$B\,A\,X = \underset{\sim}{y}\,, \tag{3.9}$$

where $\underset{\sim}{y} = (y_1, y_2, y_3)$ is the vector of body forces. The equation (3.9) can be regarded as a vector equation in the Hilbert space $L_2(\Omega)$, where Ω is the region occupied by the elastic body.

We observe that A is symmetric and its conjugate transpose A* is equal to A. If we choose an arbitrary strain distribution $\varepsilon_{ij}(x)$, $\underset{\sim}{x} \in \Omega$, we observe that

$$A^* \varepsilon_{ij} = \mathcal{K} , \tag{3.10}$$

where $\mathcal{K}$ is the incompatibility tensor. The compatibility equations are $\mathcal{K} \equiv \emptyset$, $K = (K_1, K_2, \ldots, K_6)$. Introducing Hilbert space H_1 with the product

$$\langle \underset{\sim}{z}, \underset{\sim}{y} \rangle_{H_1} = \int_{\Omega} \left(\sum_{i=1}^{6} z_i y_i \right) d\underset{\sim}{x},$$

and a space H_2 with an identical definition of an inner product $\langle , \rangle_{H_2}$, we set $A: H_1 \rightarrow H_2$ $A^*: H_2 \rightarrow H_1$.

We have assumed the existence of a functional W, which we shall call the potential energy, such that $W_\alpha = \varepsilon$ (Principle of complementary virtual work). Equivalent statement is that we have assumed the existence of Gibbs' thermodynamic potential.

$$2W = \langle K,X \rangle_{(H_1)} = \langle X, A^* \varepsilon \rangle_{H_1} = \langle AX, \varepsilon \rangle_{(H_2)}$$

$$= \langle \alpha, \varepsilon \rangle_{(H_2)} = \langle B^*U, \alpha \rangle_{(H_2)} = \langle U, B\alpha \rangle_{(H_3)} = \langle U, B\alpha \rangle_{(H_3)}$$

(3.13)

However the pairs $\{U,\phi\}$, $\{\alpha,\varepsilon\}$, $\{K,X\}$ are not vectors (in the appropriate spaces) which can be picked independently of each other. If U denotes the displacement vector $B^*U = \varepsilon$ is the definition of the linear strain.

$$B\,A^*B^*U = \emptyset \tag{3.12}$$

is exactly the set of Ricci's equations. (see Washizu [20] for explanation of their importance in linear elasticity.) The following diagram illustrates the relations (3.5) - (3.12)

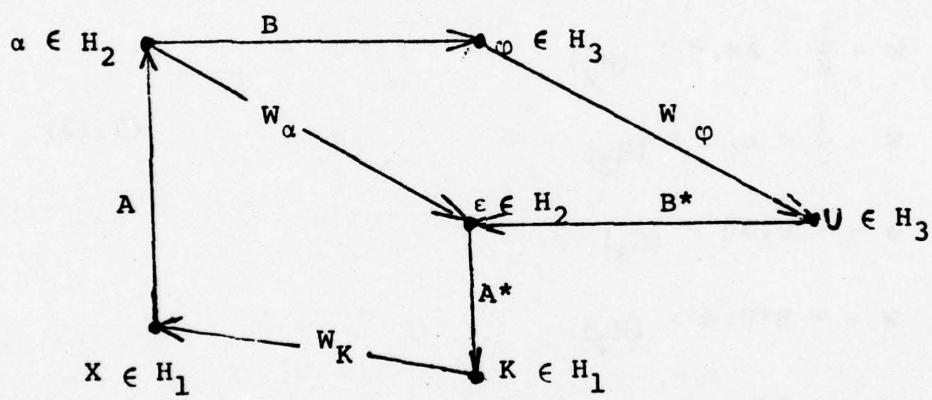

figure 3.1

They are related to each other through the constitutive equations of the solid. For example if Hooke's law is assumed, then we have

$$\alpha_{ij} = C_{ijk\ell}\varepsilon_{k\ell}, \quad (C_{ijk\ell} \text{ non-singular } 9 \times 9 \text{ matrix of anisotropic coefficients}),$$

or

$$\varepsilon_{ij} = \gamma_{ijk\ell}\,\alpha_{k\ell} \quad ,$$

where $\gamma_{ijk\ell}$ is a constant 9 × 9 matrix such that $\gamma_{ijk\ell}\, c_{ijk\ell} = I$ (the 9 × 9 identity matrix)

A well known set of variational principles in classical elasticity can be derived by

a) Assumming a constitutive equation of a solid

b) Restating all basic equations of elasticity as the existence of a critical point of the corresponding functionals:

$$
\begin{aligned}
\Phi_1 &= W - \tfrac{1}{2} < K, x > \quad (H_1) \\
\Phi_2 &= W - \tfrac{1}{2} < x, A^* \varepsilon > \quad (H_1) \\
\Phi_3 &= W - \tfrac{1}{2} < Ax, \varepsilon > \quad (H_2) \\
\Phi_4 &= W - \tfrac{1}{2} < \alpha, \varepsilon > \quad (H_2) \\
\Phi_5 &= W - < U, B\alpha > \quad (H_3) \\
\Phi_6 &= W - < B^* U, \alpha > \quad (H_2) \\
\Phi_7 &= W - < U, \varphi > \quad (H_3)
\end{aligned}
\qquad (3.13)
$$

Of course we have deliberately ignored the basic problem of solving the basic boundary value problems of Elasticity (using Muskhelishvili's terminology) and concentrated on the formulation of fundamental principles with natural boundary conditions. (See definition and discussion in section 6 & 7)

In fact should another loop be added to our mapping diagram

(3.1) we can immediately formulate the corresponding critical point statement, hence a variational principle for the corresponding functional.

3.2 Possible (future) applications to the basic theories of solid materials.

The basic thermodynamics laws applicable to solid materials can be summarized in the equations (3.13), (3.14) and inequalities (3.15), (3.16) below. The solid occupies a region $\Omega \subseteq E^3$, in which the following relations are valid.

$$(3.14) \qquad E = \frac{1}{2} (\rho_0/\rho)\tau_{ij} \dot{C}^{ij} - h^i{}_{,i} + \dot{Q} \quad ,$$

(first law of thermodynamics)

$$\begin{cases} (3.14^a) \quad \psi = \psi(C^{ij}, \theta, q^\alpha) \quad , \\ (3.14^b) \quad \tau_{ij} = \tau_{ij}\ (C_{mn}, \theta, q^\alpha). \end{cases}$$

(the principle of material indifference)

$$\begin{cases} \theta \geq 0 \quad , & \text{(Absolute temperature is non-negative).} \\ \theta\dot{S} \geq 0 & \text{(Positive rate of dissipation).} \end{cases} \qquad (3.15)$$

$$h^i\theta_{,i} \leq 0 \qquad (3.16)$$

(Heat conduction in the direction of negative temperature gradient).

The symbols used here have the following meaning. E is the internal energy, C_{ij} is the Cauchy-Green strain tensor, ρ is the mass density $\rho(0) = \rho_0$, θ the temperature, S entropy, ψ the free energy, h^i the heat flow, Q the heat supply per unit of mass.

Dots denote differentiation with respect to time, commas - covariant derivatives. Summation notation is used unless otherwise stated. q^α are additional independent variables, called internal variables, which are not necessarily observable, i.e. the implicit variable theorem may be not applicable to relations (3.14).

It has been shown by Velanis [48] and others that the stress distribution consistent with the first and second law of themodynamics satisfies the equation

$$\frac{1}{2} \tau_{ij} - \left(\frac{\rho}{\rho_0}\right) \frac{\partial \psi}{\partial C^{ij}} = 0, \qquad (3.15)$$

where $\psi = \psi(C^{ij}, \theta, q^\alpha)$, (is specified in equation (3.14[a]).) That is, assuming no constitutive equations, and choosing τ_{ij}, C^{ij} independently of each other, the correct choice will produce a critical point of the functional :

$$\left(\frac{1}{2} W - \frac{\rho}{\rho_0} \psi\right) = \Phi(C^{ij}, \ldots),$$

where $W(C^{ij})$ satisfies the relation

$$W_{C^{ij}} = 2 \tau_{ij}, \qquad (3.16)$$

i.e. τ_{ij} is the gradient of $\frac{1}{2}$ W in the appropriate product of Sobolëv spaces, which are subsets of $L_2(\Omega) \supset L_2(\bar{\Omega})$. Recalling that the entropy S satisfies the relation

$$- S = \frac{\partial\psi}{\partial\theta} , \tag{3.17}$$

and that the temparture θ is always positive, the first law of thermodynamics can be written as:

$$-\dot{S} = \frac{d}{dt} \left(\frac{\partial\psi}{\partial\theta} \right) = \frac{1}{\theta} \dot{Q} - h^i_{,i} - \frac{\partial\psi}{\partial q^\alpha} \dot{q}^\alpha , \tag{3.18}$$

(which makes no reference to constitutive equations for the material.) We shall attempt to follow at this point a fascinating idea of Ilyushin [19], who suggested that the material has no concept of our idea of time, and that processes within a solid should be parametrized with respect to a "material time", which was later defined by Rivlin by the relation $d\tau = (dC^{ij}\, dC^{ij})^{1/2}$ ([40]). Also see [47], [48]. In particular the internal variables $\underset{\sim}{q}$ should only depend on τ, and preferably in a simple manner. We assume

$$\tau = \tau(C^{ij}(t), q^\alpha(t), \psi(t), t) , \tag{3.19}$$

and

$$\frac{d\tau}{dt} = \frac{\partial\tau}{\partial C^{ij}} \dot{C}^{ij} + \frac{\partial\tau}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial\tau}{\partial\psi} \dot{\psi} + \frac{\partial\tau}{\partial t} \tag{3.20}$$

We hypothesize that

$$\frac{d\tau}{dt} > 0 \ , \tag{3.21}$$

which is a sensible physical assumption, since otherwise the material would react to future physical condition, or could, in a manner of speaking predict the future. The other assumption frequently made here is the existence of a steady state condition. The assumption that $\underset{\sim}{C} = 0$ is possible independently of q^α, S, θ, or t, or $\dot{q}^\alpha = 0$, etc. implies that each term could be zero. Hence that $\frac{d\tau}{dt} = 0$ is possible, and also that each term of the sum (3.20) is non-negative. This is a very basic and a non-trivial assumption. For the time being we shall try to avoid it, assuming $\frac{d\tau}{dt} > 0$ for all $t \in (-\infty, +\infty)$.

Since $\frac{d\tau}{dt} > 0$, we can use implicit function theorem and express t as a function of τ:

$$t = \Phi(\tau), \quad C^{ij}(t) = \hat{C}^{ij}(\tau), \quad q^\alpha(t) = \hat{q}^\alpha(\tau), \quad S(t) = \hat{S}(\tau), \text{ etc.}$$

It also follows from $\frac{d\tau}{dt} > 0$ that

$$-\frac{\partial\psi}{\partial q^\alpha} \cdot \frac{dq^\alpha}{d\tau} \geq 0 \qquad \alpha = 1,2,\ldots,m. \tag{3.22}$$

It is a standard argument (see Velanis [47]) that there is a set of constitutive equations of the form

$$\frac{dq^{\alpha}}{d\tau} = f_{\alpha}(C^{ij}, q^{\beta}, \psi),$$

otherwise the inequality (3.22) can be violated.

Following this discussion the following relations could be hypothesized:

$$\frac{1}{2}\sigma_{k\ell} = \frac{\partial \psi}{\partial C^{k\ell}} = \alpha_{ijk\ell}(\tau) * L_1 C^{ij}(\tau) \quad (3.23)$$

$$-S = \frac{\partial \psi}{\partial \theta} = \beta_{ij}(\tau) * L_2 C^{ij} + \gamma(\tau) * L_3 \theta \quad (3.24)$$

$\sigma_{k\ell}$ is one of the internal variables $(k,\ell = 1,2,3)$, * is the convolution operation:

$$f * g = \int_{\tau_0}^{\tau} f(\tau-\tau')\, g(\tau')\, d\tau',$$

L_1, L_2, L_3 are linear differential operators, which are convolutions (see Mikusiński [33]). These relations reduce to usual assumptions of materials with memory (in the material time parametrization), if the form of the operators L_1, L_2, L_3 are specifically given as in [45], or [46].

For purposes of variational formulation we shall leave the constitutive relations in the form (3.23) and (3.24).

It is apparent that in the "material time" parametrization (3.23) is Hooke's law if $\alpha_{ijk\ell}(\tau)$ is independent of τ, while L_1

is the identity operator. (i.e. $(* L_1(x)) = x$).

Of course the equations (3.23), (3.24) have to be consistent with the first and second laws of thermodynamics.

$$- S \geq 0 \Rightarrow L_3 \theta \frac{\partial \gamma}{\partial \tau} + L_2 C^{ij} \frac{\partial \beta_{ij}}{\partial \tau} \geq 0 \qquad (3.25)$$

$$h_{i,i}(\tau) = \theta L_2 C^{ij} * \frac{d\beta}{dt} + L_3 \theta * \frac{d\gamma}{dt}$$

$$- \frac{\partial \psi}{\partial \sigma_{ij}} L_1 C^{ij} * \frac{d\alpha}{dt} + \frac{dQ}{d\tau}, \qquad (3.26)$$

$$- \frac{\partial \psi}{\partial \sigma_{ij}} L_1 C^{ij} * \frac{d\alpha}{dt} \geq 0; \qquad (3.27)$$

(i,j not summed in 3.27)

If we put $\theta = \text{const} = \theta_0$, $Q \equiv \text{const.}$, $h_{i,i} \equiv 0$, i.e. ignore the thermal effects, we obtain mechanical laws

$$\theta_0 \cdot \bar{L}_2 (C^{ij} * \beta) - \frac{\partial \psi}{\partial \sigma_{ij}} \cdot \bar{L}_1 (C^{ij} * \alpha) = 0 \qquad (3.28)$$

$$\theta_0 \cdot \bar{L}_2 (C^{ij} * \beta) \leq 0. \qquad (3.29)$$

We can only hypothesize at this point that the constitutive equations must be of such form that the equations (3.23), (3.24) or (3.26), (3.27) represent a critical point of some functionals Φ_1, Φ_2, and that

$$A^* q^\alpha = f_\alpha(C^{ij}, q^\beta, \psi) = \Phi_{3_{p_\alpha}} \cdot$$

$$A\, p_\alpha = \varphi_\alpha(C^{ij}, q^\beta, \psi) = \Phi_{3_{q^\alpha}} \cdot$$

$$A^* = -\frac{d}{d\tau} \qquad A^*:\ H_2 \to H_1,$$

$$A = \frac{d}{d\tau} \quad , \qquad A:\ H_1 \to H_2 \ ,$$

where we expect f_α, φ_α to be convolutions. (3.15) is of course of the required form

$$\left(\frac{\rho_0}{\rho}\right) \tau_{ij} = \Phi_{4_{C^{ij}}} \text{ , where we identify } \Phi_4 = \psi,$$

corresponding to the variational principle

$$B(\sigma^{ij}) = \Phi_{4_{C^{ij}}} \text{ , where B is a linear operator, which is}$$

mapping σ^{ij} space into τ_{ij} space, with properties of these spaces still undetermined.

A simplified versions of such equations of state have been also suggested in Russian literature

$$\tau_{ij} = \omega_{ij\alpha\beta}\, \varepsilon^{\alpha\beta}, \quad \text{where } \varepsilon^{\alpha\beta} \text{ is the linear strain component}$$

$$\text{and where } \omega_{ij} = \frac{1}{4}\left(\frac{\partial}{\partial C^{\alpha\beta}} + \frac{\partial}{\partial C^{\alpha\beta}}\right)\left(\frac{\partial}{\partial C^{ij}} + \frac{\partial}{\partial C^{ji}}\right) \Phi(\underset{\sim}{C}) + y^{\alpha i}\tau^{\alpha\beta},$$

and $\tau^{j\beta} = \frac{1}{2}\left(\frac{\partial}{\partial C^{i\beta}} + \frac{\partial}{\partial C^{\beta i}}\right)\Phi(\underset{\sim}{C})$,

leading to variational principle of Hu, corresponding to a critical point of the functional:

$$I_1(\tau,\varepsilon,u) = \int_\Omega [\frac{1}{2}\omega^{ij\alpha\beta}\varepsilon_{\alpha\beta}\varepsilon_{ij} - \tau^{ij}(\varepsilon_{ji} - \nabla_i u_j)]d\underset{\sim}{v}$$

$$- \int_{2\Omega} (p^j u_j\, ds - v \cdot \tau^{ij} u_j) d\underset{\sim}{s}.$$

(see O. Guz [17]).

For computational purposes it is not good enough to know that certain equations of solid mechanics represent a critical point of a functional. Iterative techniques which have been eminently successful in such computations always had the additional information that this functional attains a maximum or a minimum. To make certain that this occurs we need the generalization of the concept of second and higher order derivatives, This will be done in the next chapter. It is not clear at this point how to formulate "a universal variational principle" which would incorporate the equations of state, (if it exists?), and we shall stop conjecturing at this point and return to an area where variational principles are easily established.

Chapter 4.

The second derivative.

4.1. Heuristic comments.

Suppose $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $\underset{\sim}{y} = (y_1, y_2, \ldots, y_m)$ are vectors in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively, and that we have the functional relation $x_1 = x_1(\underset{\sim}{y})$, $x_2 = x_2(\underset{\sim}{y}) \ldots x_n = x_n(\underset{\sim}{y})$. i.e. a map $\mathbb{R}^m \to \mathbb{R}^n$. What meaning do we assign to $\dfrac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}}$? In this case we have suggested in Chapter 1 that the $n \times m$ (Jacobian) matrix of partial derivatives

$$\left(\frac{\partial x_i}{\partial y_j}\right) \begin{cases} i = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{cases}$$

plays the part of the Fréchet derivative. If $X \in H_1$, $Y \in H_2$, where H_1, H_2 are infinite dimensional Hilbert (or Banach) spaces, then the meaning of the Jacobian matrix has to be redefined. However we have to recognize that even in the finite dimensional case $\underset{\sim}{X} \in \mathbb{R}^n$, $\underset{\sim}{Y} \in \mathbb{R}^m$ the Jacobian matrix is not an element of either $\mathbb{R}^n$ or $\mathbb{R}^m$, but a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^m$. Its conjugate transpose $\left\{\dfrac{\partial y_i}{\partial x_j}\right\}^*$ is a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$. To even attempt to define what is meant by $\dfrac{\partial^2 X}{\partial Y^2}$ we need a few basic concepts of functional analysis.

4.2. Tensor products.

For purposes of clarity we shall carefully distinguish between elements of a Banach or Hilbert space H, and its algebraic dual. In the definition of the algebraic dual we use the assumption of linearity only. No boundedness (or continuity) properties are assumed for the functionals in H*.

Let E be a Banach space, and E* its algebraic dual. We consider a Cartesian product of $E^* \times E^*$, i.e. all posible sets of ordered pairs $\{\varphi, \psi\}$ $\varphi \in E^*$, $\psi \in E^*$ (denoted by $(\varphi \otimes \psi)$), and define the map $E \to R$ by the operation $\psi(X) = \langle \psi, X \rangle$ which is defined for any $X \in E$. We shall use the following notation. Vectors subscripted by indices i, j, k, ℓ ... will denote elements of E*, while superscripts will denote elements of E. Our algebraic operation then assumes the formal representation:

$$(\varphi_i \otimes \psi_j)X^j = \langle \psi_j, X^j \rangle \varphi_i \quad , \tag{4.1}$$

where $\langle \psi_j, X^j \rangle$ is a real number obtained by applying the linear map $\psi_j \in E^*$ to the vector $X^j \in E$. Hence $(\varphi_i \otimes \psi_j)$ is a linear map whose domain is E and whose range is contained in E, and is of dimension one, or zero. In the same manner we define the tensor products $(\varphi^i \otimes \psi^j)$: $E^* \to E^*$

$$(\varphi^i \otimes \psi_j): \ E \to E^*$$

$(\varphi_i \otimes \psi^j)$: $E^* \to E^*$ by the algebraic rules:

$$(\varphi^i \otimes \psi_j)X^j = \langle \psi_j, X^j \rangle \varphi^i \tag{4.2}$$

$$(\varphi_i \otimes \psi^j) X_j = \langle X, \psi^j \rangle \varphi_i \, . \tag{4.3}$$

Again we see that the range of this map is of dimension ≤ 1. Higher order tensor products are defined in the same manner. For example $(\varphi_i \otimes \psi_j \otimes \eta_k) \in E^* \times E^* \times E^*$ is defined as a map from $E \times E$ into E, or as a map from E into $E \times E$.

$$(\varphi_i \otimes \psi_j \otimes \eta_k) \; (X^k \otimes Y^j) = \langle \eta_k, X^k \rangle \cdot \langle \psi_j, Y^j \rangle \varphi_i \tag{4.4}$$

We assume associative property of this algebraic operation in the following sense:

$$((\varphi_i \otimes \psi_j \otimes \eta_k) X^k) \; Y^j$$

$$= \langle \eta_k, X^k \rangle \; (\varphi_i \otimes \psi_j) \; Y^j$$

$$= \langle \eta_k, X^k \rangle \; \langle \psi_j, Y^j \rangle \varphi_i$$

$$= (\varphi_i \otimes \psi_j \otimes \eta_k) \; (X^k \otimes Y^j) \tag{4.5}$$

We shall not assume that such compositions of mappings are commutative, and indeed such assumption can not be consistently supported in the general case. Moreover we try to pattern our discussion to agree with the usual engineering ideas of what a "tensor" is.

We find no reason why the notation $\mathfrak{J}^{ij}$ could not be used to denote a specific tensor product $(\varphi^i \otimes \psi^j)$. At this point we are going to make further assumption (which generally may exclude some important considerations in mechanics, but seems justified in solid mechanics). We are going to restrict the spaces E, E*, (E**) to be topological duals of each other. Hence all linear maps defined above are now assumed to be continuous. This could be labelled "finite energy hypothesis". Hence if our discussion concerns Hilbert spaces, we can use Riesz representation theorem. This will considerably simplify the definitions of higher order derivatives which are given below.

Let $f: E \to E$ be a continuous function. We shall use the notation $Y = f(X)$, or $Y = Y(X)$, $X \in \mathcal{D}_f$. We define the Gateaux derivative of f, (which also may be called the directional derivative of Y with respect to X in the direction of h) as

$$\frac{\partial Y}{\partial X_h} = \lim_{t\to 0} \frac{Y(X+th) - Y(X)}{t} = \psi(X,h) \in E. \tag{4.6}$$

provided this limit exists. We claim that for any $Z \in E^*$, $\langle Z, \psi \rangle$ is a linear functional of h, i.e. $\langle Z, \psi(X,ch) \rangle = c\langle Z, \psi(X,h) \rangle$ and $\langle Z, \psi(X,h_1+h_2) \rangle = \langle Z, \psi(X,h_1) \rangle + \langle Z, \psi(X,h_2) \rangle$. Each follow easily from assumption of the existence of the limit (4.6) in some open region of E. If we assume continuous dependence of ψ on h then by Milgram - Lax theorem (the bilinear version of Riesz representation theorem) $c\langle Z,\psi\rangle$ can be written as $\langle AZ,h\rangle$, but in this case the operator A is the operator of multiplication by a constant and

$$\langle Z, \frac{\partial Y}{\partial X_h}\rangle = c\langle Z,h\rangle.$$

Again if we assume that c depends continuously on Y, (linearity is obvious) then using Riesz representation theorem we conclude that there exists a vector $\varphi \in E^*$ such that $c = \langle\varphi,Y\rangle$. Hence

$$\langle\frac{\partial Y}{\partial X_h}, Z\rangle = \langle\psi,Z\rangle = \langle\varphi,Y\rangle\ \langle h,Z\rangle \tag{4.7}$$

Again using the Lax-Milgram theorem we can rewrite this product

$$\langle\psi,Z\rangle = \langle A\varphi,Z\rangle\ \langle Y,Bh\rangle \tag{4.8}$$

where A,B are linear operators

$B: \ E \rightarrow E^*$

$A: \ E^* \rightarrow E.$

Since this is true for abritrary $Z \in E^*$, we have equality

$$\frac{\partial Y}{\partial X_h} = \langle Y,Bh\rangle\ A\varphi = \langle Y,\xi\rangle\ A\varphi = \langle Y,\xi\rangle\mu, \quad \text{where } \mu = A\varphi.$$

Hence the operator $\frac{\partial}{\partial X_h}$ can be represented as a tensor product

$$\frac{\partial}{\partial X_h} = (\mu \otimes \xi), \ \mu \in E, \ \xi \in E^*$$

$$\frac{\partial}{\partial X_h} Y = \langle Y, \xi \rangle \mu. \tag{4.9}$$

Similar argument gives us a representation

$$\frac{\partial Y}{\partial X_h} = \langle h, \xi \rangle \mu \tag{4.10}$$

where ξ is of the form $\xi = CY$, and C is a linear operator, $C\colon E \to E^*$. In this entire argument the domain of the operators A,B,C, is assumed dense in a sufficiently small neighborhood of a region considered for the respective vectors φ, h, Y (in E, E* respectively), and the existence of adjoint operators allows the necessary manupulations of the products <,>. We can generalize the concept of a gradient by observing that our assumption of continuity with respect to h, allows to define grad $Y = Y_X = (\mu \otimes \xi)$, since

$$\frac{\partial Y}{\partial X_h} = \langle \xi , h \rangle \mu.$$

(Recall the definition of a gradient of a functional.) In particular if there exists a functional $V(X,x)$, $x \in E$, $X \in E$, such that $Y = \text{grad } V(X,x) \in E^*$ (or $Y = V_x$), then we can denote by

$$V_{xX} = (\mu \otimes \xi) = Y_X.$$

If we assume that $V = V(X,x)$, $X,x \in E$, then a theorem of Vaînberg states that

$$V_{Xx} = V_{xX}. \qquad (4.11)$$

(This does not imply $(\mu \otimes \xi) = (\xi \otimes \mu)$!) (see [49], chapter 2). In fact commutativity of the "second derivatives" is a necessary condition for the existence of the "potential functional" V.

The idea of second derivative is easily generalized to the case when $V: E \times E^* \to \mathbb{R}$ is a functional depending continuously on $x_1, x_2, \ldots, x_k \in E$ $X_{k+1}, X_{k+2}, \ldots, X_n \in E^*$,

$$E = E_1 \otimes E_2 \otimes \ldots \otimes E_k, \quad E^* = E^*_{k+1} \otimes E^*_{k+2} \otimes \ldots \otimes E^*_n$$

and $X_1 \in E_1 \ldots X_n \in E^*_n$, $E_1, E_2, \ldots, E_k$ being subspaces of E $E^*_{k+1} \ldots E^*_n$ of E^*, where $V_{X_1} \ldots V_{X_n}$ can be regarded as generalications of the idea of partial differentials. Consider a functional $V(X^i, Y_j)$ (not necessarily linear) whose values in $\mathbb{R}$ depend on $X^i \in E$, $Y_j \in E^*$. Then provided all the derivatives shown exist, we can construct the following diagram :

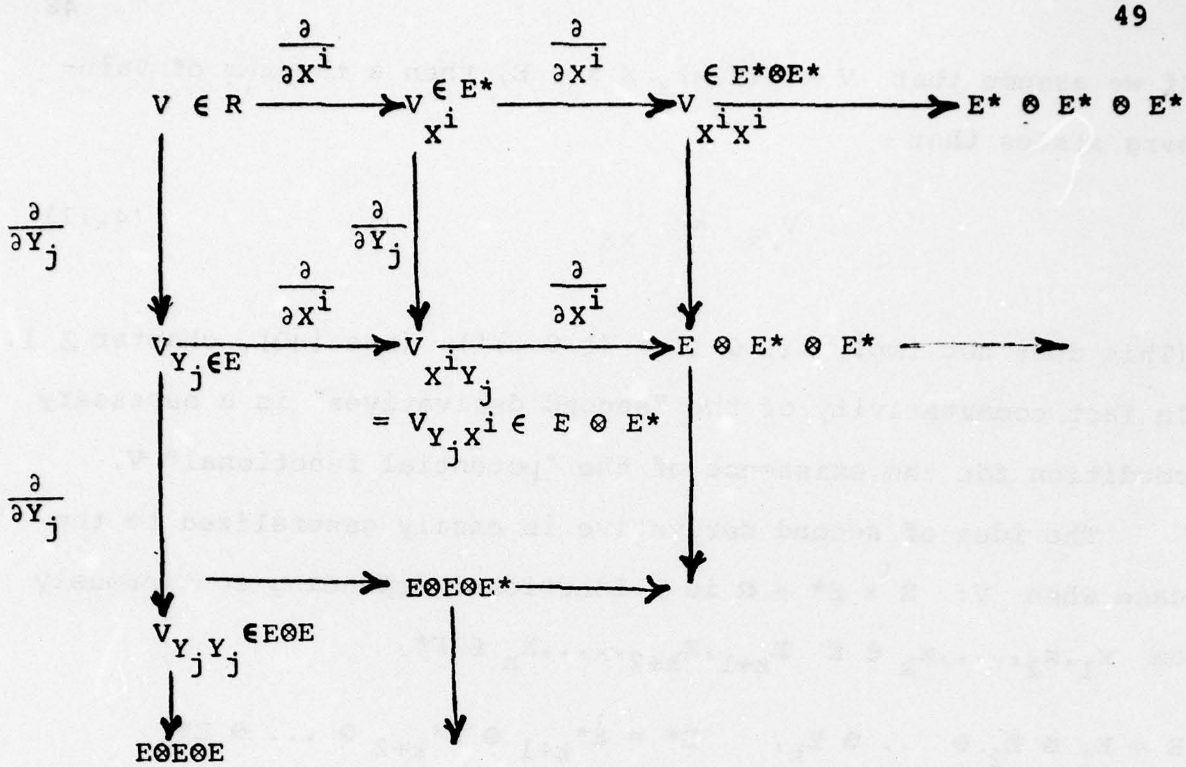

Figure 4.1

It shows the increasingly higher order tensor products obtained by performing repeatedly the gradient operation.

<u>Positive (positive definite) second order tensor product</u>.

Since $(\xi \otimes \eta)$, $\xi \in E$, $\eta \in E$, (or $(\mu \otimes \nu)$ $\mu \in E^*$, $\nu \in E^*$) can be regarded as maps from E* into E* (or from E to E) we can define the concept of positive property of such map by the usual definition: $(\xi \otimes \eta)$ $\xi, \eta \in E$ is <u>positive</u> if

$$\langle(\xi \otimes \eta)\, Y, Y\rangle \geq 0 \quad \text{for all } Y \in E^*. \tag{4.12}$$

Similarly $(\mu \otimes \nu)$, $\mu \in E^*$, $\nu \in E^*$ is positive if $\langle(\mu \otimes \nu) X, X\rangle \geq 0$ for all $X \in E$. $(\xi \otimes \eta)$ is called <u>positive definite</u> if $\langle(\xi \otimes \eta) Y, Y\rangle \geq 0$ for all $Y \in E^*$, and $\langle(\xi \otimes \eta)Y,Y\rangle = 0$ implies $Y = \emptyset \in E^*$. Similar definition is given for $(\mu \otimes \nu)$, $\mu \in E^*$, $\nu \in E^*$.

<u>Vainberg's lemma and some of its consequences</u>.

Let E_1 be a subspace of a Banach space E, and V a functional $V\colon E \to \mathbb{R}$, the value of V depending on a vector $X \in E_1$, and on $Y \in E_2$, where $E_1 \oplus E_2 \subseteq E$, $E_1 \cap E_2 = \emptyset$. Suppose that V is defined in some neighborhood of $X_0 \in E_1$. Then a necessary condition for X_0 to be an extremal point of V, when V is restricted to E_1, is $V_X = \emptyset$, $V|_{E^1}$. (| means restricted to). A sufficient condition for a min. (max) of $V|_{E^1}$ is that $V_X = \emptyset$, V_{XX} is positive definite (negative definite).

<u>Examples of application</u>.

Consider the behavior of the (non-linear) functional

$V(X,P) = \langle AX,X\rangle \cdot \langle X,X\rangle^{-1} + \psi(P)$ $X \in H_1$, $P \in H_2$ where H_1, H_2 are Hilbert spaces, $A\colon H_1 \to H_1$.

Let P be fixed (in H_2). Find sufficient conditions for a local minimum of V (in H_1). We compute the gradient of V in H_1:

$$V_X = 2\langle X,X\rangle^{-1} \cdot (AX - V(X)X)$$

Hence a necessary condition for an extremum of V to occur at

$X_0 \in H_1$ is: $AX - V(X)X = \emptyset$ when $X = X_0$, that is $AX_0 = \Lambda X_0$, where $\Lambda = V(X_0)$, which means that X_0 is an eigenvector, and $V(X_0)$ the corresponding eigenvalue of A.

In what follows let us assume that the multiplicity of Λ_0 is one, and Λ_0 is the lowest eigenvalue of A. The second derivative is computed as follows:

$$(\varphi \otimes \psi)h = V_{XX}h = \lim_{t \to 0} \frac{V_X(X+th) - V_X(X)}{t}$$

$$= 2t^{-1} \{\langle X+th, X+th\rangle^{-1} (AX + tAh - V(X+th)[X+th])$$

$$- \langle X,X\rangle^{-1} (AX - V(X)X)\}$$

$$= \lim_{t \to 0} 2t^{-1} \left\{ \frac{AX + tAh - V(X+th)(X+th)}{\langle X,X\rangle + t\langle h,X\rangle + t^2\langle h,h\rangle} - \frac{AX-V(X)X}{\langle X,X\rangle}\right\} .$$

Having assumed the continuity of V, we obtain as $t \to 0$

$$V_{XX}h = \frac{2(Ah - V(X)h)}{\langle X,X\rangle} .$$

Hence

$$V_{XX} = 2\left(\frac{A - V(X)I}{\langle X,X\rangle}\right) .$$

Hence at the point X_0 where $V_X(X_0) = \emptyset$, we have

$$V_{XX}\ (X=X_0) = 2\ \frac{A - V(X_0)I}{<X_0,X_0>}$$

where I is the identity map ,

or

$$V_{XX}(X=X_0) = 2(A - \Lambda_0 I) \cdot <X_0,X_0>^{-1}$$

Hence the extremum of V at X_0 occurs on a small neighborhood of $X_0 : N_{X_0} \in H$, *only* if for every vector $\xi \in N_{X_0}$ it is true that

$$<A\xi,\xi> - \Lambda_0<\xi,\xi> - <V_X(\xi),\xi> > 0 \quad \begin{cases} \xi \neq X_0 \\ \xi \neq \emptyset \end{cases}$$

or $<A\xi,\xi> - \Lambda_0<\xi,\xi> - <V_X\xi,\xi> < 0$

In particular if A is positive definite, completely continuous, and Λ_0 is the lowest eigenvalue, then the Rayleigh quotient V attains its minimum value at X_0, such that

$$V(X_0) = \Lambda_0 \text{ (since } <V_X(X_0), X_\theta> = 0).$$

It is clear that in a sufficiently small neighborhood of X_0, V_{XX} is positive definite and a local minimum takes place for V(X) at X_0. A global theory is much harder, and it is unreasonable to expect that the signs of first and second abstract derivative in some neighborhood should imply anything about the global behavior of a given functional.

An Example: Thin plate theory.

The plate satisfies the linear equation

$$\nabla^2 (D(x,y)\, \nabla^2 w(x,y)) - (1-\nu)\, \Diamond^4(D,w) = g(x,y) \tag{4.13}$$

in a region $\Omega \subseteq R^2$.

(Here $\Diamond^4(A,B) = \frac{\partial^2 A}{\partial x^2}\frac{\partial^2 B}{\partial y^2} - 2\frac{\partial^2 A}{\partial x\,\partial y}\frac{\partial^2 B}{\partial x\,\partial y} + \frac{\partial^2 B}{\partial x^2}\frac{\partial^2 A}{\partial y^2}$.)

with boundary conditions $w \equiv \frac{\partial w}{\partial n} \equiv 0$ on $\partial\Omega$. Where the boundary $\partial\Omega$ of Ω is assume smooth, except for a finite number of external corners. (i.e. corners like the one shown below are not permitted.)

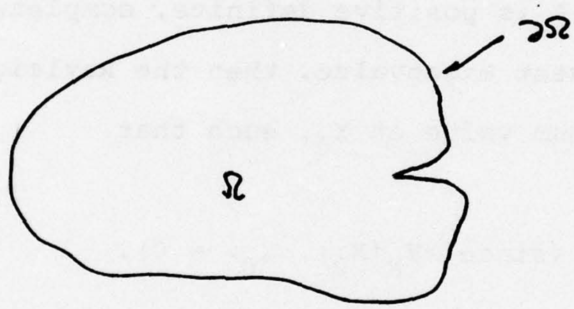

figure 4.2.

This can be recognized as the Euler-Lagrange equation for the functional

$$\Phi = \int_{\Omega} \{D(\nabla^2 w)^2 - D(1-\nu)\,(\Diamond^4(w,w))\}\, dxdy \tag{4.14}$$

It is not very convenient (or very practical in most engineering applications)

to assume the existence of all derivatives appearing in the formal Euler-Lagrange equations (4.14) for the functional $\phi(w)$. Physically we only require twice differentiability of w, and L_2 property of the derivatives ($w \in H^2(\Omega)$). We introduce the following maps

$$N: \quad H^2(\Omega) \rightarrow H^2(\Omega)$$

where N is the following positive definite matrix

$$N = \begin{bmatrix} D(x,y) & \nu D(x,y) & 0 & 0 \\ \nu D(x,y) & D(x,y) & 0 & 0 \\ 0 & 0 & (1-\nu)D(x,y) & 0 \\ 0 & 0 & 0 & (1-\nu)D(x,y) \end{bmatrix} .$$

$D(x,y) > 0$ in Ω, and $0 < \nu < \frac{1}{2}$ for physical reasons.

T is a 2×4 matrix for first order differential operators, T^* is the transpose of T :

$$T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

Then

$$\underset{\sim}{M} = -NT \text{ grad } w = (-NT \text{ grad})w = Aw \tag{4.15}$$

is the well known relation between the moments

$$\underset{\sim}{M} = (M_{xx}, M_{xy}, -M_{yx}, M_{yy})$$

and the displacement w(x,y).

$$A: \quad H^2(\Omega) \longrightarrow L_2(\Omega)$$

At this point of our discussion it is more convenient to introduce the "modified" moments defined by the formula:

$$\mathcal{M} = (-(N^{1/2}).T.\text{grad})w \tag{4.16}$$

where we take the positive square root $N^{1/2}$ of the operator N, as follows

$$N^{1/2} = \begin{bmatrix} \sqrt{1-\beta^2}\ \sqrt{D} & \beta\sqrt{D} & 0 & 0 \\ \beta\sqrt{D} & \sqrt{1-\beta^2}\ \sqrt{D} & 0 & 0 \\ 0 & 0 & \sqrt{(1-\nu)D} & 0 \\ 0 & 0 & 0 & \sqrt{(1-\nu)D} \end{bmatrix},$$

where $\beta = \left(\frac{1}{2} + \frac{\sqrt{1-\nu^2}}{2} \right)^{\frac{1}{2}}$.

It can be checked that $N^{1/2}\ N^{1/2} = N$, and $N^{1/2}$ is again a positive definite (and invertible) matrix, and fortunately $N^{1/2}$ is symmetric. The mapping

$$A:\ \ H^2(\Omega) \to L_2(\Omega)$$

has a formal adjoint $A^*\ L_2(\Omega) \to \mathcal{D}^*(\Omega)$, and $Aw = \mathcal{M}$, $A^*\mathcal{M} = q$ (4.17) where the derivatives now become distributional derivatives, $L_2(\Omega)$ being regarded now as a subset of $\mathcal{D}^*(\Omega)$, where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$.

$$A^* = \text{div}\ (T^*N^{1/2}), \qquad (4.18)$$

i.e.

$$A^*\mathcal{M} = \text{div}\ (T^*(N^{1/2}\mathcal{M})) = \text{div}\ (T^*\underset{\sim}{M}) = q(x,y) \qquad (4.19)$$

The distributional character of q(x,y) being supported by the usual interpretation of admitting point loads, or point moments in engineering practice. Hence we have the following mapping diagram

$$w \xrightarrow{\mathcal{A}} \mathcal{M} \xrightarrow{\mathcal{A}^*} q$$

where $\mathcal{A}^*\mathcal{A}\, w = q$ is the original differential equation of static deflection of the plate :

$$\nabla^2(D\nabla^2 w) - (1-\nu)\,\diamondsuit^4(D,w) = q. \qquad (4.13)$$

We can observe that following our theory in chapter 2, the set of equations:

$$\left.\begin{aligned} \mathcal{A}w &= \mathcal{M} \qquad (4.20^a) \\ \mathcal{A}^*\mathcal{M} &= q \qquad (4.20^b) \end{aligned}\right\}$$

represents two variational principles. We introduce the Hamiltonian

$$W = \langle q,w\rangle + \frac{1}{2}\{\mathcal{M},\mathcal{M}\}$$

where

$$\{A,B\} = \int_\Omega (A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4)\,dx\,dy$$

while

$$\langle a,b\rangle = \int_\Omega (a \cdot b)dxdy.$$

The dual variational principles given for the static plate problem have been given in [25] by the author, and may be summarized in the observation

$$W_{\mathfrak{M}} = \mathfrak{M} = \mathfrak{A}w \tag{4.21^a}$$

$$W_w = q = \mathfrak{A}^*\mathfrak{M} \tag{4.21^b}$$

where $W_{\mathfrak{M}}$ denotes the gradient of W restricted to the Hilbert space $H_{\mathfrak{M}}$ (with the product $\{\cdot,\cdot\}$, while W_w is the gradient of W restricted to H_w (with the product $\langle,\rangle$. The examination of second Fréchet derivatives i.e. of the tensor products $(W_{\mathfrak{M}})_{\mathfrak{M}}$, $(W_w)_w$ reveals that the corresponding Lagrangian functional

$$L = \{\mathfrak{A}w,\mathfrak{M}\} - W \tag{4.22}$$

attains a minimum over the admissible $w \in H^2(\Omega)$ with $\mathfrak{M}$ regarded as fixed, and a minimum over the admissible $\mathfrak{M} \in L_2(\Omega)$ with w regar - ed as fixed at the critical points corresponding to the solution of a system of equations (4.20^a), (4.20^b). Combining this system into the form $AA^*w = q$ gives a single equation (4.13) which is the basic static deflection equation of thin plate theory.

The dual variational principles given here were originally discovered by the author in [25]. At this point we wish to make a more detailed examination of the equation (4.13) and find alternate variational formulation for this equation.

We have the following mapping diagram:

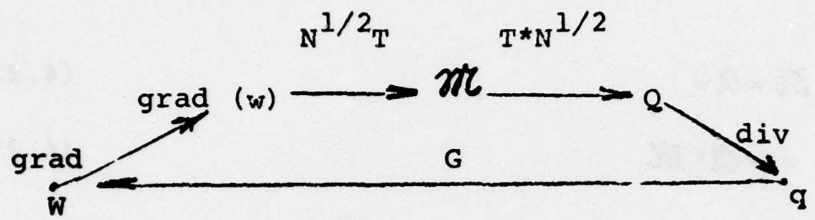

figure 4.2

Let us concentrate on the following portion of the diagram (4.2)

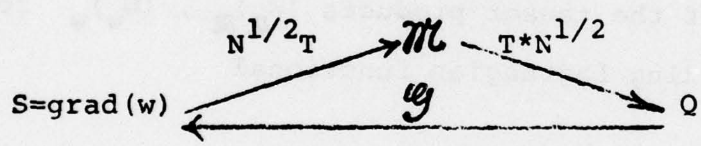

figure 4.3

G is the Green's function, or influence function in engineering terminology: $w = G * q$, where * is the convolution integral. $\mathcal{G}$ is the corresponding map $\mathcal{G} * \underset{\sim}{Q} = \nabla w = \underset{\sim}{S}$, which is to be determined, but whose existence is not hard to prove, if we assume the existence of the Green's function $G(x,y)$. (We do not write $G(x,\xi,y,\eta)$ since the convolution product takes care of introducing the "translated" variables ξ,η.)

The dual equations corresponding to the diagram (4.3) are

$$N^{1/2}TS = \mathcal{M} = W_{\mathcal{M}} , \qquad (4.23^a)$$

$$T^*N^{1/2}\mathcal{M} = Q = W_S , \qquad (4.23^b)$$

except that W is a functional defined in different pair of Hilbert spaces: $H_{\mathcal{M}}$ which is the same as before, and $H_S = L_2^{(2)}(\Omega)$ if point moments are excluded, and only point loads allowed as loads which are not represented by functions. Note: For a classification of admissible (distributional) loads of thin plate theory see author's paper [31].

If point moments are allowed H_S can not be embedded in $L_2(\Omega)$ and is not a Hilbert space, but only an inner product space which can be at best regarded as a Rigged Hilbert space in the terminology of Gelfand and Shilov (see [14] volume 4, chapter 1, section 4). For the purpose of this discussion we identify H_S with $L_2^{(2)}(\Omega)$ with the inner product

$$(A,B)_{(2)} = \int_\Omega [A_1(x,y)B_1(x,y) + A_2(x,y)B_2(x,y)]dxdy.$$

The components of the vector Q are recognized as the shears

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y}$$

$$Q_y = \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y}$$

The generalized Hamilton's canonical equations are

$$N^{1/2}TS = \mathfrak{M} = W_{\mathfrak{M}} \, , \tag{4.23^a}$$

$$T^*N^{1/2} = Q = W_S \, , \tag{4.23^b}$$

$$L = \frac{1}{2}\{\mathfrak{M},\mathfrak{M}\} + (S,Q) = \{N^{1/2}TS,\mathfrak{M}\} - W \, , \tag{4.24}$$

$$\left.\begin{array}{l} L_{\mathfrak{M}}|_{H_{\mathfrak{M}}} = \emptyset \\ \\ L_S|_{H_S} = \emptyset \end{array}\right\} \, ,$$

are exactly the equations (4.23^a), (4.23^b) expressed as restrictions of appropriate Fréchet derivatives of L to the spaces $H_{\mathfrak{M}}$ and H_S respectively.

The tensor products $W_{\mathfrak{M}\mathfrak{M}}$, $W_{\mathfrak{M}S}$, $W_{S\mathfrak{M}}$, W_{SS} can be arranged in a 2 × 2 matrix form of operators:

$$\begin{bmatrix} I & N^{1/2}T \\ T^*N^{1/2} & T^*NT \end{bmatrix} = \begin{bmatrix} W_{\mathfrak{M}\mathfrak{M}} & W_{\mathfrak{M}S} \\ W_{S\mathfrak{M}} & W_{SS} \end{bmatrix} .$$

(I is the identity operator.) $W_{\mathfrak{M}S} = (W_{S\mathfrak{M}})^*$ is a necessary condition for integrability of the differential system (4.23^a), (4.23^b), while positive definite nature of I: $H_{\mathfrak{M}} \to H_{\mathfrak{M}}$ and T*NT: $H_S \to H_S$ assure the existence of a double minimum, i.e. L attains a minimum in $H_{\mathfrak{m}}$ (for a fixed S) if the equation (4.23^a) is satisfied, and a minimum in H_S (for a fixed $\mathfrak{m}$) if the equation (4.23^b) is satisfied. These variational principles suggest numerical techniques of the type introduced by Greenspan in [6] for computation of approximate solutions, and these principles may be simpler than the known principles suggested by equations (4.21^a), (4.21^b). The advantage of obtaining a symmetric form of corresponding operators, and of not having the matrix N appearing in only one set of equations are considerable in actual computation. The positive definite nature of the matrix N allowed us to restate our problem in terms of $\underset{\sim}{\mathfrak{m}}$ rather than $\underset{\sim}{M}$. Physical limitations on such representation are clear: $D(x,y) > 0$ for all $x,y \in \bar{\Omega}$, (we do not have holes in Ω. If we do, let us relabel what the region Ω really is!), and $^1/_2 > \nu > 0$ which is readily recognized by any engineer as the expected behavior of a physical solid, namely that the solid does not expand volumetrically under pressure.

Chapter 5.

Boundary Value Problems

5.1 Integration by parts formulas.

The geometric theory associated with differential forms is due to Grassman. The expository texts include H. Flanders [6], P.K. Rashevskii "Geometric theory of partial differential equations (in Russian). The basic axiom for exterior product of two differentials is

$$dx_i \wedge dx_j = -dx_j \wedge dx_i,$$

where the symbol $\wedge$ is frequently omitted. Denoting by $\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}} = D\binom{x}{y}$ the Jacobian tensor product corresponding to a coordinate transformation, we can derive the general formula

$$\sum_{i_1 < i_2 \cdots < i_k} a_{i_1 i_2 \cdots i_k} \, dx_1 \, dx_2 \cdots dx_k$$

$$= \sum_{j_1 < j_2 \cdots < j_k} a'_{j_1 \cdots j_k} \, dy_1 \, dy_2 \cdots dy_k$$

where $a'_{j_i \cdots j_k} = D\begin{pmatrix} x_{i_1} & \cdots & x_{i_k} \\ y_{j_1} & \cdots & y_{j_k} \end{pmatrix} a_{i_1 \cdots i_k}$.

The generalized Stokes theorem can now be stated for an arbitrary orientable manifold Ω, whose boundary $\partial\Omega$ is smooth.

$$\int_{\Omega} d\alpha = \int_{\partial\Omega} \alpha \qquad (5.1)$$

(See for example a monograph by de Rham). The (exterior) differential forms provide a natural setting for certain class of boundary value problems. The formula (5.1) embraces all classical results of changing integration over Ω to a corresponding boundary integral. We shall not pursue the subject, and try to offer another exposition of algebras of differential forms on manifolds. We shall only assume that appropriate formulas for integration by parts exist, and can be deduced in the special cases by the use of formula (5.1).

We first offer an analogous definition of a Fréchet derivative of a functional $\Phi(X)$ such that the domain of the functional Φ is the union of two Hilbert spaces of functions. i.e. $X \in H_1 \cup H_2$, where H_1 is a Hilbert space of functions whose domain is a set $\Omega \subseteq R^n$, with a product $(\ , \ )_{(\Omega)}$, and H_2 is a space of functions whose domain is $\partial\Omega$, (the boundary of Ω) the inner product in H_2 deing designated by $(\ , \ )_{\partial\Omega}$. Again the Fréchet derivative in each space H_1 and H_2 is defined by the formulation identical to section 1 of chapter 1. The formulation of a gradient of Φ in H_1 and H_2 presents no problems, provided the class of functions which we have in mind restricted to Ω and to $\partial\Omega$ respectively do form appropriate Hilbert (or at least Banach) spaces. A very serious problems of formulation arises when boundary values are not functions (and possibly not even distributions over some "reasonable" test space). For the time being we shall assume that this is not the case and all possible boundary values are functions which belong to a Hilbert space $H_2(\partial\Omega)$. Whenever we use a symbol $f \in H_1 \cup H_2$, we mean that the same function $f: \Omega \cup \partial\Omega \to \mathbb{R}$ is considered simultaneously as an element of H_1 if $x \in \mathcal{D}_f$ is a point of Ω, and of H_2 if $x \in \mathcal{D}_f$ is a point on $\partial\Omega$.

If f is not sufficiently smooth in the neighborhood of $\partial\Omega$, then the integration by parts formulas which form the special case of (5.1) are not applicable, and the behavior of products $(f,\psi)_\Omega$ and $(f,\varphi)_{\partial\Omega}$ is completely unrelated. Hence certain continuity (and smoothness) properties must be specified for functions in H_1 near $\partial\Omega$ if the formulation of boundary value problems is to have a unique solution. To offer a trivial example we consider solutions for the Dirichlet's problem in the unit disc of the complex plane, looking for $L_2(\Omega) \cup L_2(\partial\Omega)$ solutions with no continuity requirements. Suppose we wish to solve the problem $\nabla^2 u \equiv 0$ in Ω (the open unit disc) $u \equiv 0$ on $\partial\Omega$ (the unit circle). In the class of differentiable solutions continuously approaching the boundary value there is only one solution $u \equiv 0$ in Ω. If no continuity is required on approaching $\partial\Omega$, then $u \equiv 5\frac{1}{2}$ in Ω, $u \equiv 0$ on $\partial\Omega$ is another possible solution among infinitely many candidates for the solution of this problem. In fact it is analytic in Ω. However the possibility of discontinuity on approaching the boundary made a specification of the boundary condition completely worthless. This fairly trivial example interprets the meaning of our previous remark that in general the behavior on Ω, and the behavior on $\partial\Omega$ of functions in the class $L_2(\Omega) \cup L_2(\partial\Omega)$ is completely unrelated. We have to digress into discussion of the following basic question. Let B_1, B_2 be some Banach spaces of functions whose domain is $\Omega \subseteq \mathbb{R}^n$. Let A be a given operator A: $B_1 \rightarrow B_2$. Let $f \in B_2$ be given. Let B_3 be some other Banach space of functions whose domain is $\partial\Omega$. Let $\varphi \in B_3$ be given. Does there exist a vector $x \in B_1$, such that x satisfies the equation $Ax = f \in B_2$ and that the values of function $x: \Omega \rightarrow \mathbb{R}$ converge (in some sense) to $\varphi: \partial\Omega \rightarrow R$, in some neighborhood of $\partial\Omega$ in Ω. That is can f be continuously extended to $\Omega \cup \partial\Omega$ so that (in some sense) it coincides with φ on $\partial\Omega$? In the problems of mathematical physics "in some sense" usually is understood to be pointwise convergence almost everywhere. Usually the physical interpretation requires that a solution of such problem should exist (since some physical process is going on, representing a solution), and frequently we would like to assert that such a solution should be unique. If a unique solution does exist to this problem it is called <u>well posed</u>, or properly posed boundary value problem. An entirely different question is the continuous dependence (stability) of such solution on the boundary data (function) $\varphi: \partial\Omega \rightarrow \mathbb{R}$. The common question is will "small" variations in φ result in "small" variations in the solution? Of course mathematically the above sentence makes little sense, but it can be easily translated into a rigorous statement (given $\varepsilon > 0$, there exists $\delta > 0$ such that δ - neighborhood of φ in B_3 is mapped into ε - neighborhood of x in B_1 under the inverse map $(f,\varphi) \to x$, with the existence of such inverse map assured by well posedeness of the problem.) The well posedeness of the boundary value problem implied the existence of an inverse operator $\mathcal{B}^{-1}$ of the operator $\mathcal{B}: x \to (f,\varphi) \in B_2 \cup B_3$, i.e. of $\mathcal{B}^{-1}: (f,\varphi) \to x$. The stability in the sense given above implies that $\mathcal{B}^{-1}$ is a bounded operator. Boundary value problem which is well posed and stable is called in the literature <u>properly posed in the sense of Hadamard</u>. For a classical example of a problem which is unstable and therefore improperly posed in the sense of Hadamard consider the Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u \equiv 0$$

in half plane $y \geq 0$ ($\subset R^2$), with boundary conditions $u\big|_{y=0} = \varphi(x)$ $\frac{\partial u}{\partial y}\Big|_{y=0} = \psi(x)$, $-\infty < x < +\infty$, where $\varphi, \psi \in C(-\infty,+\infty)$. (See Mikhlin [32], chapter 9, § 5 for a discussion of this problem, and chapter 25 for a general discussion of the well posedness of boundary value problems for partial differential operators of mathematical physics.) The main thrust of this discussion is the following conclusion: the boundary value problem is not well posed or ill posed "by itself". The well posedeness depends on the topologies of the spaces B_1, B_2, B_3, and the problem may be well posed for some choices of B_1, B_2, B_3, and ill posed for other choices. In particular we must answer the problem of existence and boundedness of the operator $\mathcal{B}^{-1}$: $B_2 \cup B_3 \to B_1$. From this point of view it is easy to interpret the following statement which physically sounds ridiculous, but is mathematically correct. For certain class of boundary functions (analytic of certain slow growth rate) the backward heat conduction problem is well posed. Here the boundary is the physical space boundary of the solid body and the time hypersurface $t = t_1$ in R^{n+1}. Hence for certain choices of spaces B_1, B_2, B_3 and for certain classes of operators A: $B_1 \to B_2 \oplus B_3$ the formulas of the type (5.1) make sense. Assuming that B_1, B_2, B_3 are Hilbert spaces and the domain of A is dense in B_1, and therefore A^*: $B_2 \to B_1$ is uniquely defined, we may have a unique representation of the form $(u,A^*v)_{(\Omega)} - \langle Au,v\rangle_{(\Omega)} = (Bu,v)_{(\partial\Omega)}$, for $u \in S \subseteq B_1$, where S is some class of functions. We have defined the inner products $(\ ,\ )_{(\Omega)}$ in B_1 $\langle\ ,\ \rangle_{(\Omega)}$ in B_2, and $(\ ,\ )_{\partial\Omega}$ in B_3. In such cases A restricted to S is called the formal adjoint of A^*. In the context of this definition for example the operators $\frac{d}{dt}$ and $-\frac{d}{dt}$ are formal adjoints of each other on $C^1[0,1] \subset R$, where $C^1[0,1]$ is imbedded in the Sobolëv space $H^1[0,1]$. We observe that in that case $(Bu,v)_{\partial\Omega} \equiv uv|_{x=1} - uv|_{x=0}$. There is no way of defining uniquely $(Bu,v)_{\partial\Omega}$ if we attempt to embed the problem in $L_2[0,1]$. This remark illustrates the fact that a setting for many problems of mathematical physics is provided by Sobolëv spaces. See [18] for a comprehensive study of their properties, or see Mikhlin [32] for an expository account.

5.2 Critical points of functionals associated with boundary value problems.

Suppose A is a linear map A: $H(\Omega) \to H(\Omega)$, where $H(\Omega)$ is a Hilbert space of functions whose domain is Ω, $H_2(\partial\Omega)$ a Hilbert space of functions whose domain is $\partial\Omega$, the domain of A is $\mathcal{D}_A (\subseteq H(\Omega))$, which is dense in $H(\Omega)$, and f is some function f: $H(\Omega) \to H(\Omega)$. We formulate the following class of functionals on $H(\Omega) \oplus H_2(\partial\Omega)$

$$\Phi(u,\varphi): \quad H(\Omega) \oplus H_2(\partial\Omega) \to \mathbb{R},$$

$$\Phi(u,v) = \langle Au,v\rangle_{(\Omega)} - \langle f(u),v\rangle_{(\Omega)} + \Gamma(u,v).$$

(We have assumed for convenience that $H(\Omega)$ $H_2(\partial\Omega)$ are Hilbert spaces but this can be generalized.) $\Gamma(u,v)$ is a continuous functional $H_2 \times H_2 \to \mathbb{R}$, i.e. (u,v): $\Gamma(u|_{\partial\Omega}, v|_{\partial\Omega}) \to \mathbb{R}$, is a continuous functional whose domain are ordered pairs of functions u,v restricted to $\partial\Omega$, and regarded as elements of $H_2(\partial\Omega)$. As before the inner products $\langle Au,v\rangle_\Omega$ and $\langle f,v\rangle_{(\Omega)}$ are inner products in $H(\Omega)$. f is regarded as a fixed element of $H(\Omega)$, u, v are regarded as independent of each other. In a more general case f is a map $H(\Omega) \to H(\Omega)$ dependent of u. Computing the gradient of Φ in $H(\Omega) \oplus H_2(\partial\Omega)$ we have

$$\Phi_v = (Au - f(u))_{H(\Omega)} \oplus \Gamma_v(u,v)_{H_2(\partial\Omega)} \qquad \Phi_v \in (H_2 \oplus H)^*.$$

Here $Au - f(u)$ is a vector in $H(\Omega)$ and $\Gamma_v(u,v)$ is a vector in $H_2(\partial\Omega)$. (Since we have assumed a Hilbert space structure for $H(\Omega)$ and $H_2(\partial\Omega)$ there is no mathematical necessity for keeping track of duality. Physically it is of course important to keep the gradient in the dual space to $H(\Omega) \oplus H_2(\partial\Omega)$, distinguishing which are the generalized displacements, and which are the generalized forces, and keeping the physical dimensions correct. But vanishing of the gradient implies that each vector in the direct sum of spaces $H(\Omega) \oplus H_2(\partial\Omega)$ must vanish separately. Hence $\Phi_v = \emptyset$ is a simplified notation for saying that

$$Au - f(u) = \emptyset \quad \text{in } H(\Omega). \tag{5.2}$$

$$g(u) = \emptyset \quad \text{in } H_2(\partial\Omega). \tag{5.3}$$

Therefore a function $u \in H(\Omega) \oplus H_2(\partial\Omega)$ which satisfies both conditions (5.2) and (5.3) corresponds to a critical point of the functional $\Phi(u,v)$. We observe that no assumption was made concerning self adjoint (or positive definite) or other specific properties of the linear map A, and up to this point even linearity was not used. We need to identify a given boundary value problem

$$Au = f(u) \quad \text{in } \Omega \tag{5.4}$$

$$g(u) = 0 \quad \text{on } \partial\Omega \tag{5.5}$$

with a critical point of the functional

$$\langle Au,v\rangle_{\Omega} - \langle f(u),v\rangle_{\Omega} + (g(u),v)_{\partial\Omega} = \phi(u,v).$$

Until now v is independent of u, but quite arbitrary. We can now postulate that v is a solution of the adjoint system

$$A^*v - \frac{\partial f(u)}{\partial u} v = 0 \quad \text{in } \Omega, \tag{5.6}$$

$$\frac{\partial g(u)}{\partial u} \cdot v = 0 \quad \text{on } \partial\Omega. \tag{5.7}$$

That is v is chosen such that $\phi_u = 0$, and that the given boundary value problem corresponds to a dual critical points of ϕ, provided A^* is a (true) adjoint of A. This means that g(u) is so chosen that for all u, v $\in$ S, $\langle g(u),v\rangle_{(H_2(\partial\Omega))} = 0$, and $\langle Au,v\rangle_{(H(\Omega))} - \langle u,A^*v\rangle_{(H(\Omega))} = 0$.

The problem which frequently arises at this point is that A^* (the adjoint of A) may not be uniquely defined in $H(\Omega) \oplus H_2(\partial\Omega)$, and additional information may be needed concerning properties of the (physically) admissible elements of $H(\Omega)$ in the neighborhood of $\partial\Omega$. Our previous remarks concerning formally adjoint operators A, A^*. $(H(\Omega) \to H(\Omega))$ are applicable here. Presume that A, and A^* are formal adjoints mapping a subset S of $H(\Omega)$ into $H(\Omega)$, such that for any $u \in S \subseteq H(\Omega)$ we have an appropriate formula (in $H \oplus H_2$)

$$\langle Au,v\rangle_{(\Omega)} - \langle u,A^*v\rangle_{(\Omega)} = B(u,v)_{\partial\Omega}. \quad (5.8)$$

We see that the existence of the appropiate functional

$$\phi(u,v): \quad (H(\Omega) \oplus H_2(\partial\Omega)) \times (H(\Omega) \oplus H_2(\partial\Omega)) \to \mathbb{R}$$

whose critical points are represented by (5.4), (5.5) and by some equivalent conditions of the form (5.6), (5.7) is now harder to determine. (When A* was a true adjoint of A we wrote down the appropriate functional $\phi(u,v)$ without any trouble!)

We have the relationships:

$$Au = f(u) \qquad \text{in } \Omega,$$

$$g(u) = 0 \qquad \text{on } \partial\Omega,$$

or more generally $g(u,v) = 0$ on $\partial\Omega$. And for all $v \in S \subseteq H$

$$\langle Au,v\rangle_{(\Omega)} - \langle u,A^*v\rangle_{(\Omega)} + B(u,v)_{(\partial\Omega)} = 0 \quad (5.9)$$

where subscripts (Ω), $(\partial\Omega)$ denote the inner products in $H(\Omega)$ and $H_2(\partial\Omega)$ respectively. Denoting as the Hamiltonian the functional $W(u,v)$:

$$\langle f(u),v\rangle_{\Omega} = W(u,v)$$

we have the relation

$$W_v = f(u) \qquad \text{in } \mathbb{R} \text{ ,}$$

while

$$W_u = \frac{\partial f(u)}{\partial u} v \text{ , where the meaning of } \frac{\partial}{\partial u} \text{ has to be defined analogously to section 4.}$$

The corresponding Lagrangian functional is denoted by $\tilde{L}$ and is defined by the relation

$$\tilde{L}(u,v) = \langle Au,v\rangle_{(\Omega)} - W$$

$$= \langle u,A^*v\rangle_{(\Omega)} - W + B(u,v)_{(\partial\Omega)} \text{ .}$$

Here the same symbol v is used to denote v as an element of $H_{(\Omega)}$ and of $H_{2(\partial\Omega)}$.

We seek, however, a "true" Lagrangian $L(u,v) : H \oplus H_2 \rightarrow \mathbb{R}$, given by a relation:

$$L = \tilde{L}(u,v) + C(u,v)_{(\partial\Omega)}$$

$$C(u,v)_{\partial\Omega} : H_2 \times H_2 \xrightarrow{\text{cont.}} \mathbb{R},$$

i.e. $C(u,v)_{\partial\Omega}$ is a continuous map: $H_2(\partial\Omega) \times H_2(\partial\Omega) \to \mathbb{R}$.
The "true" Lagragian $L(u,v)$ should correspond to a multiple variational principle. In this case since only two independent (vector) variables are present we should have the dual critical points:

$$L_v = \phi, \quad L_u = \phi.$$

$Au = f(u)$, $g(u,v) = 0$ can be replaced by the following equation in H and H_2 respectively.

$$Au - W_v = L_v = \phi \in H \quad (u,v|_{\Omega}) \tag{5.10a}$$

$$g(u,v) = L_v = \phi \in H_2 \quad (u,v|_{\partial\Omega}), \tag{5.10b}$$

This suggests that $L(u,v)$ should have the following form.

$$L = \langle Au,v\rangle_{\Omega} - W(u,v)_{(\Omega)} + C(u,v)_{(\partial\Omega)},$$
$$\text{with } C_{v|\partial\Omega} = g(u,v)_{(\partial\Omega)} \tag{5.11}$$

$$L_u = A^*v_{(\Omega)} + B(u,v)_{u(\partial\Omega)} - W_{u(\Omega)} + C_{u(\partial\Omega)} \tag{5.12}$$

Hence $L_v = \phi$ (in $H \oplus H_2$) expresses the initially given equation $Au = f(u)$ and the boundary condition

The "symmetry" formula (5.14) is written below in full:

$$< u,A^*v > \; - < u,W_{uv}v >_{\Omega} + < u,C_{uv}v >_{\partial\Omega}$$

$$= <Au,v>_{\Omega} - <W_{vu}u,v>_{\Omega} + <B_{vu}u,v>_{\partial\Omega} + <C_{vu}u,v>_{\partial\Omega} \quad . \tag{5.14a}$$

The formula (5.14) was suggested by A. M. Arthurs in [2], who also has given the following set of sufficiency conditions for the existence of the Lagrangian functional:

$$<u,W_{uv}v>_{(\Omega)} = <W_{vu}u,v>_{(\Omega)} \tag{5.15a}$$

$$<u,A^*v>_{(\Omega)} = <Au,v>_{(\Omega)} + <B_{vu}u,v>_{(\partial\Omega)} \tag{1.15b}$$

$$<u,C_{uv}v>_{(\partial\Omega)} = <C_{vu}u,v>_{(\partial\Omega)} \tag{1.15c}$$

The condition (5.15a) is a necessary condition for the existence of a Hamiltonian function W(u,v) whose domain is H(Ω). (Not $H \oplus H_2$!).

The condition (1.15b) implies the following equality:

$$<B_{vu}u,v>_{\partial\Omega} = B(u,v) .$$

$$C_v = g(u,v) = \emptyset \quad \text{on } \partial\Omega. \tag{5.10c}$$

On the other hand the requirement

$$L_u = \emptyset \quad (\text{in } H \oplus H_2)$$

requires the vanishing of vectors

$$A^*v - W_u = \emptyset, \quad \text{in } H(\Omega), \tag{5.13a}$$

and of

$$B(u,v)_u + C_u = \emptyset \quad \text{in } H_2(\partial\Omega). \tag{5.13b}$$

The "independence of path" condition of Vaînberg for the existence of L requires that the tensor products L_{uv} and L_{vu} are adjoints of each other, that is

$$\langle L_{uv}\hat{v},\hat{u}\rangle_{H\oplus H_2} = \langle \hat{v}, L_{vu}\hat{u}\rangle_{H\oplus H_2} \tag{5.14}$$

for any admissible pair $(\hat{u},\hat{v})$ $\hat{u},\hat{v} \in S \subseteq H \oplus H_2$, implying that $\hat{u},\hat{v}$ satisfy the required continuity conditions on $\partial\Omega$, such that the formula (5.9) is valid.

(see [4] equation 20).

However the "symmetry" condition $L_{uv} = L_{vu}^*$ is satisfied if for admissible $u,v \in S \subseteq H$ (with suitable continuity conditions in some neighborhood of $\partial\Omega$).

$$<u,A^*v>_{(\Omega)} + <u,C_{uv}v>_{(\partial\Omega)}$$

$$= <Au,v>_{(\Omega)} + <B_{vu}u,v>_{(\partial\Omega)} + <C_{vu}u,v>_{(\partial\Omega)}$$

This implies that $C(u,v)_{(\partial\Omega)}$ must satisfy the relations:

$$B(\hat{u},\hat{v}) - <B_{vu}\hat{u},\hat{v}>_{(\partial\Omega)}$$

$$= <C_{vu}\hat{u},\hat{v}>_{(\partial\Omega)} - <\hat{u},C_{uv}\hat{v}>_{(\partial\Omega)}$$

However

$$\left.\begin{aligned} C_v &= g(\hat{u},\hat{v}), \\ C_u &= -B(u,v)_u \end{aligned}\right] \quad \text{for all admissible } \hat{u},\hat{v}\,\big|_{(\partial\Omega)}.$$

by our previous arguments. (see 5.10c, 5.10b).

The existence of the "boundary potential" $C(u,v)$ such that $C_{vu} = C_{uv}$ implies $(g(u,v))_u = -B_{uv}$.

Since $H_2(\partial\Omega)$ is a Hilbert space C_{vu}, C_{uv}, $g(u,v)_u$, B_{uv}, B_{vu} are all linear maps $H_2 \to H_2$, and the above equalities make sense.

Hence under these assumptions the secessary conditions for the existence of the Lagrangian $L\colon\ H \oplus H_2 \to \mathbb{R}$ are:

(1.16) $W_{uv}\ (\Omega) = W_{vu}\ (\Omega)$

(1.17) $B(\hat{u},\hat{v})\,|\partial\Omega = \langle B_{vu}\hat{u},\hat{v}\rangle_{\partial\Omega} = \langle\hat{u},B_{uv}\hat{v}\rangle_{\partial\Omega}$,

(1.18) $g(u,v) = C_v\ \ (\partial\Omega)$

(1.19) $-B_u = C_u$

If $C(u,v)$, $(\partial\Omega)$ can be chosen to satisfy all conditions (1.16) -(1.19) then of course we have nothing to worry about.

In general however this is not a convenient set of conditions, and it can be readily checked that choosing for example $C = -B$, we end with the original condition suggested by the author in 1966 [24]: $g = B_v(H_2)$, which it is possible to satisfy only in special cases. To restate realistically, the problem of finding a functional $L\colon\ (H \oplus H_2) \to \mathbb{R}$ for quite arbitrary boundary condition $g(u,v) = 0$, we have to sacrifice some conditions on the list (1.16) - (1.19).

The problem can be restated as follows:

Does there exist $v \in H \oplus H_2$, and $C\colon\ H_2 \to \mathbb{R}$ such that

$$A^*v - W_u = \phi \quad \text{in } H(\Omega)$$

$$\left.\begin{aligned} C_u &= -B(u,v)_u \\ C_v &= g(u,v) \end{aligned}\right] \quad \text{in } H_2(\partial\Omega)$$

where $W\colon\ H \to \mathbb{R}$ is given by $W = \langle f(u),v\rangle_{(\Omega)}$?

The answer to this question depends on the nature of the boundary conditions and on the operator A.

If such v, C(u,v) can be found, then the problem of finding a solution to the given boundary value problem is replaced by a pair of variational principles for the corresponding Lagrangian functional $L(u,v)\colon\ H \oplus H_2 \to \mathbb{R}$.

We observe at this point that if L(u,v) can be found, then the tensor products L_{uu}, L_{vv}, L_{uv}, L_{vu} satisfy the equalities

$$L_{uv} = (-W_{uv} + A^*)_\Omega + (B_{uv} + C_{uv})_{\partial\Omega}$$

$$L_{vu} = (-W_{vu} + A)_\Omega + (C_{vu})_{\partial\Omega}$$

$$L_{uu} = (-W_{uu})_{\Omega} + ((C + B)_{uu})_{\partial\Omega}$$

$$L_{vv} = (-W_{vv})_{\Omega} + (C_{vv})_{\partial\Omega}$$

The remarks in the previous chapter explain the following necessary conditions for an extremal behavior of $L(u,v)$ at the point $u = u_o$, $v = v_o$.

a) L_{uv} is the adjoint of L_{vu}

b) The signs of the operators L_{uu}, L_{vv} are defined and remain constant in some neighborhood of the point v_o, $v_o \in (H \oplus H_2) \times (H \oplus H_2)$.

Examples of applications of multiple variational principles, and of boundary value problems posed in this manner will be given in the next chapter. Many examples of dual variational principles can be found in the monograph of Arthurs [3] and in the papers of the British applied mathematicians who have pursued this topic using a largely heuristic approach. See for example Arthurs [3], or Robinson [41] for additional bibliography. Some of the most difficult problems concerning classification of the operators or of the boundary conditions are still largely unsolved.

Chapter 6.

Some applications of necessary conditions for the existence of a critical point of a functional to the theory of ordinary differential equations. Many problems of physics are mathematically modelled by differential equations, which under some assumptions may reduce to a system of ordinary differential equations of the form $\mathcal{L}(W(x)) + f(x\ t) = \varphi(t)$ (6.1)

where $\mathcal{L}$ is an ordinary differential operator which need not be linear, w and f are vectors in a Hilbert space (of functions) H. The domain of functions in H is n-dimensional Euclidean space R^n.

As an example we shall consider a one-dimensional case studied by Poincaré.

$$\left.\begin{array}{l} (a(t)\cdot\tilde{p}(t))' = c(t)\cdot f(x) - \varphi(t) \\ \psi(x)\cdot x' = -\tilde{p}(t) \end{array}\right\} \quad (6.2)$$

$$\left({}' = \frac{d}{dt}\right)$$

Physically the system of two equations (6.2) interprets the motion of a mass, which may decay with time in a force field which is also time dependent.

Interpreting p as the angular momentum $p = p_\theta = a(t)\cdot r^2\cdot\theta$, and introducing a constraint $r = \Psi(\theta)$, we obtain

$$\left.\begin{array}{l} p_\theta = a(t)\,[\Psi(\theta)]^2\cdot\theta' \\ (p_\theta)' = -c(t)\,\hat{f}(r,\theta) = -c(t)\,f(\theta)\quad, \end{array}\right\} \quad (6.3)$$

which is identical with the system (6.2).

Combining equations (6.3) into a single equation, we obtain $\left\{a(t)\,[\Psi(\theta)]^2\,\theta'\right\}' + c(t)\,f(\theta) = \varphi(t),\ t \geq t_o$, (6.4) which is an equation of the form (6.1).

For physical reason it is usually assumed that $a(t) > 0 \quad \forall$ $t \in [t_0, \infty)$. $c(t)$ may assume negative values. We also assume that $\psi(\theta) \geq 0$.

We recall that in the nonlinear motion studied by Poincaré the physical meaning of $\psi(\theta)$ was the distance from the origin . Moreover in this case, we also observe that $p(\theta)$ is undefined at $\psi(\theta) = 0$, and the motion must not pass through the origin. Hence we assume that $\psi(\theta) > 0$ and along the entire trajectory $[\psi(\theta)]^{-1}$ is a continuous function of θ. We can now redefine $p = \sqrt{a}\,\tilde{p}$, i.e. $p(t) = -\sqrt{a(t)}\,\psi(x(t))\,x'(t)$, and rewrite the system (6.3) as

$$\left.\begin{aligned} x'(t) &= [a(t)]^{-1/2}[\psi(x(t))]^{-1}\,p(t) \\ \left(\sqrt{a(t)}\,p(t)\right)' &= c(t)\,f(x(t)) - \varphi(t) \end{aligned}\right\} \quad (6.4)$$

Here $\sqrt{\ }$ denotes the positive square root!

We introduce a potential function

$$K(x) = \int_{x_0}^{x} \psi(\xi)\,d\xi \qquad (6.5)$$

where x_0 can be taken as the origin if $\int_0^x \psi(\xi)\,d\xi$ exists, or we can put $x_0 = +\infty$ if $\int_{-\infty}^{x} \psi(\xi)\,d\xi$ exists. Clearly $dK/dx \neq 0$ and K^{-1} is defined.

In most cases the choice of "the zero potential level" x_0 is immaterial, as long as the equations of motion (6.2) can be rewritten as a generalized Hamiltonian system

$$\mathcal{A}K(x) = p, \qquad (6.6\underline{a})$$

$$\mathcal{A}^* p = c(t)\ f(x) - \varphi(t), \qquad (6.6\underline{b})$$

$$\text{where } \mathcal{A}\,\cdot = (\sqrt{a}\ \frac{d}{dt}\,\cdot), \qquad (6.7\underline{a})$$

$$\mathcal{A}^{*}\cdot = -\frac{d}{dt}\ (\sqrt{a}\ .). \qquad (6.7\underline{b})$$

The operators $\mathcal{A}$ and $\mathcal{A}^*$ are formal adjoints of each other. Hence, we have a system of generalized Hamilton's canonical equations already discussed in chapter 2.

Suppose that we seek weak solutions of equations (6.6$\underline{a}$), (6.6$\underline{b}$) in the Sobolev space $W_2^1\ [\alpha,\beta)$ where $[\alpha,\beta)$ is some interval of time (See [44], Chapter 2). We allow $\beta = +\infty$ in some considerations. For obvious physical reasons we cannot stipulate twice continuous differentiability of solutions, restricting ourselves to piecewise continuous applied force $\varphi(t)$, since in many instances we may consider applied (outside) forces which are only piecewise continuous. In this physical setting, we stipulate

$$p(t)\ \varepsilon\ L_2\ [\alpha,\beta),\ \varphi(t) \text{ and } c(t)\ f(x(t))$$

piecewise continuous in $[\alpha,\beta)$, and we seek solutions

$$x(t)\ \varepsilon\ W_2^1\ [\alpha,\beta).$$

The finite work condition is

$$\int_\alpha^\beta \left| c(t)\ f(x(t))\ \cdot x(t) \right| dt < \infty .$$

Because of piecewise continuity of $f(x)$ (and of $x(t)$) it suffices if $c(t) \in L_2\,[\alpha,\beta]$ - for a finite interval $[\alpha,\beta]$, for the finite work condition to be fulfilled.

We also assume for a finite interval $[\alpha,\beta]$

$$a\,(t) > 0 \text{ for all } t \in [\alpha,\beta], \tag{6.9}$$

$$\left.\begin{aligned} &\psi(x) \in C^1\,(-\infty,+\infty), \\ &\psi(x) > 0 \text{ for all } x \in \mathbb{R}, \\ &f(x) \in \text{piecewise } C\,(-\infty,+\infty\,). \end{aligned}\right\} \tag{6.10}$$

Then we seek $W_2^1\,[\alpha,\beta)$ solution for the problem posed by the equations (6.2) with boundary conditions of the form:

$$\left.\begin{aligned} a_1\,x(\alpha) + b_1\,p(\alpha) &= 0 \\ a_2\,x(\beta) + b_2\,p(\beta) &= 0 \end{aligned}\right\} \tag{6.11}$$

For the sake of simplicity we shall consider first the boundary value problem posed by the simpler conditions

$$\left.\begin{aligned} W\,(\alpha) &= W_\alpha \\ W\,(\beta) &= W_\beta \end{aligned}\right\} \tag{6.12$\underline{a}$}$$

or by

$$\left.\begin{aligned} p\,(\alpha) &= p_\alpha \\ p\,(\beta) &= p_\beta \end{aligned}\right\} \tag{6.12$\underline{b}$}$$

Since it is easy to see how to generalize such results to the more general case of (6.11), Here w_α, w_β, p_α, p_β are real numbers. We remark that the results obtained below may be also derived under somewhat weaker hypothesis.

Since $\psi(x) = \frac{dK(x)}{dx} \neq 0$, we can introduce an invertible transformation $\hat{K}$:

$$x(t) \xrightarrow{\hat{K}} K\ (x(t)) = \hat{R}\ (t).$$
$$x(t) = \hat{K}^{-1}\ (\hat{R}(t)) \tag{6.13}$$

The problem can now be posed in the $L_2[\alpha,\beta]$ setting with the usual product $<u,v> \triangleq \int_\alpha^\beta u(t)\ v(t)dt$. We also introduce the inner product called energy product:

$$[u,v] \triangleq \int_\alpha^\beta a(t)\ u(t)\ v(t)\ dt \tag{6.14^{a}}$$

The Hilbert space obtained by closing this inner product space with respect to the energy norm will be called the energy space. $||\cdot||$ will denote the $L_2[\alpha,\beta]$ norm: $||u||^2 \triangleq <u,u>$, and $|||\cdot|||$the energy norm: $|||u|||^2 \triangleq [u,u]$.

Ignoring the boundary conditions,the Hamiltonian for our problem can be given by

$$W(x,p) = \frac{1}{2} <p,p> + c(t)\ \Phi(K) - \varphi(t)K \tag{6.15}$$

where $$\Phi(K) = \int_0^K f\,\hat{K}^{-1}(\xi)\,d\xi, \tag{6.16}$$

while $$K(x) = \int_{x_\alpha}^{x} \psi(\xi)\,d\xi, \quad fK^{-1}(K) \equiv f(x).$$

(Note that defining $K(x) = \int_0^x \psi(\xi)\,d\xi$, or $\int_\infty^x \psi(\xi)d\xi$ would introduce no real difficulties into all subsequent argrments).

The corresponding Lagrangian functional or the action functional is given by

$$L = W - \langle \mathcal{O}K, p\rangle. \tag{6.17}$$

With the introduction of this functional we have the canonical system:

$$\mathcal{O}K = W_p \tag{6.18a}$$

$$\mathcal{O}^*p = W_K \quad . \tag{6.18b}$$

$\mathcal{O}$ and $\mathcal{O}^*$ are only formal adjoints of each other. To make certain that variational principles of chapter 4 and 5 can be directly applied to the system (6.18a), (6.18b), we need to either include the appropriate boundary terms (see conditions (6.12a) - (6.12b)), or to check that the boundary conditions are natural. In this case the check is performed by integration by parts. By definition:

$$\langle \mathcal{O}K,p\rangle = \int_\alpha^\beta \left(\sqrt{a(t)} \cdot \left(\frac{d}{dt}\ K(x)\right) \cdot \sqrt{a}\ \psi(x) \cdot x'(t)\right)dt$$

$$= \int_\alpha^\beta [a\ (t)\ \psi(x)\ x'\ \frac{d}{dt}\ (K(x))]dt$$

$$= a(t)\ \psi(x)\ x'\ K(x)\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \{K(x)\ \frac{d}{dt}\ (a(t)\ \psi(x)\ x')\}\ dt$$

$$= a(t)\ \psi(x)\ K(x)\ x'\Big|_{\alpha}^{\beta} + \langle K, \mathcal{A}^{*}p\rangle$$

Imposing the boundary conditions (6.12a) we see that $\mathcal{A}$ and $\mathcal{A}^*$ are true adjoints if

$$a(t)\ \psi(x)\ x'(t)\ K(x(t))\Big|_{\alpha}^{\beta} = 0$$

Since $\psi(x(t))$, and $a(t) > 0$ and $t \geq t_0$, vanishing on the boundary of $x'(t)$, therefore of $p(t)$, or of $K(x(t))$ implies that $\mathcal{A}^*$ is a true adjoint of $\mathcal{A}$ and the corresponding boundary conditions at $t = \alpha$ and $t = \beta$ will be referred to as the natural boundary conditions.

We check the sign of the appropriate tensor products:

$$L_{KK} = W_{KK} = C(t)\ \hat{f}'(\hat{K}) \tag{6.19a}$$

where $\hat{f}'(\hat{K}) = \frac{\partial}{\partial \hat{K}}\ f(x(\hat{K}))$.

$$L_{pp} = W_{pp} = I \tag{6.19b}$$

(I is the identity operator.)

$$L_{Kp} = -\mathcal{A} \tag{6.19c}$$

$$L_{pK} = (-\mathcal{A}^*)^* = -\mathcal{A} \tag{6.19d}$$

Hence, if $c(t)$ does not change its sign on $[\alpha, \beta]$ and provided that $\hat{f}'(\hat{K}) \neq 0$ in some neighborhood, a critical point K_0, we are assured of the existence of a variational principle. The extremal behavior of $\mathcal{L}$ corresponds to a solution of the differential equation

(6.2). Depending on the sign of $f'(K)$ an algorithm can be constructed in the spirit of Greenspan's techniques in [16]. Morever, the existence of such a critical point of L implies the correctness of equations ($6.18\underline{a}$), ($6.18\underline{b}$) which are equivalent to the original system of equations (6.2). However, the sufficiency condition for existence of a corresponding variational solution of the problem posed by (5.2) is: c(t), a(t) are of constant sign on (α,β), while $\hat{f}(K)$ is Fréchet differentiable in some region containing the possible critical point K_0.

$K(x(t)) \in W_2^1 [\alpha,\beta]$, hence $p \in W_2^1 [\alpha,\beta]$. A corresponding function x(t) which results in a stationary behavior of L is defined as the <u>weak</u> <u>solution</u> of the differential equation (6.2).

<u>COMMENT</u>.

We note that the L_2 setting of the variational problem was not only mathematically convenient, but also reflected some very strong physical assumptions. (It was a sufficiency condition for finite work!)

6.2 The essential boundary conditions

We shall consider the system (6.2) with boundary conditions

$$W(\alpha) = W_\alpha, \qquad (6.20^{\underline{a}})$$

$$W(\beta) = W_\beta, \qquad (6.20^{\underline{b}})$$

or

$$p(\alpha) = p_\alpha, \qquad (6.21^{\underline{a}})$$

$$p(\beta) = p_\beta. \qquad (6.21^{\underline{b}})$$

where we assume that the boundary conditions are not natural, and that $\mathcal{A}$ and $\mathcal{A}^*$ are only formal adjoints of each other.

As before we formulate the problems in (real) spaces $W_2^1 [\alpha,\beta]$ and $L_2 [\alpha,\beta]$. We modify the definition of the inner product by adjoining a discrete product: $[u,v] = \sqrt{a(\alpha)}\, u(\alpha)\, v(\alpha)$.

We extend the domain of the operator $\mathcal{A}$ in the following manner. For every $f \in W_2^1 [\alpha,\beta]$, we define

$$\hat{\mathcal{A}} : f(t) \longrightarrow \sqrt{a}\, \frac{d}{dt} f(t) \qquad \forall\, t \in (\alpha,\beta)$$

$$: \begin{bmatrix} f(\alpha) \\ f(\beta) \end{bmatrix} \longrightarrow \begin{bmatrix} \lambda\, f(\alpha) + (\lambda-1)\, f_\beta \\ (\lambda-1)\, f(\beta) + \lambda\, f_\alpha \end{bmatrix} \qquad (6.22)$$

$$\text{at } \{t = \alpha\} \cup \{t = \beta\}$$

$$\hat{\mathcal{A}}^* : g(t) \longrightarrow -\frac{d}{dt} (\sqrt{a} \cdot g(t)) \qquad t \in (\alpha,\beta)$$

$$: \begin{bmatrix} g(\alpha) \\ g(\beta) \end{bmatrix} \longrightarrow \begin{bmatrix} (\lambda - 1)\ g(\alpha) + \lambda\ g_\beta \\ \lambda\ \ g(\beta) + (\lambda - 1)\ g_\alpha \end{bmatrix} \qquad (6.23)$$

at $\{t = \alpha\} \cup \{t = \beta\}$.

f_α, g_α, f_β, g_β are some a priori given real numbers.

We check the properties of the product

$\{\hat{\mathcal{A}}f, g\}$ using integration by parts:

$$\{\hat{\mathcal{A}}f, g\} = \langle \mathcal{A}f, g\rangle + \lambda[f(\alpha), g(\alpha)] +$$

$$(\lambda - 1)\ [f_\beta, g(\alpha)] + (\lambda - 1)\ [f(\beta), g(\beta)] + \lambda[f_\alpha, g(\beta)] \qquad (6.24)$$

$$\{f, \hat{\mathcal{A}}^*g\} = \langle f, \mathcal{A}^*g\rangle + (\lambda - 1)\ [f(\alpha), g(\alpha)\ ] +$$

$$\lambda[f(\alpha), g_\beta] + \lambda[f(\beta), g(\beta)] + (\lambda - 1)\ [f(\beta), g_\alpha] \qquad (6.25)$$

This is manipulated in to the form

$$\{\hat{\mathcal{A}}f, g\} = \{f, \mathcal{A}^*g\} + \lambda\Big([f(\alpha), g_\beta] - [f_\alpha, g(\beta)]\Big) +$$

$$(\lambda - 1)\ \Big([f(\beta), g_\alpha] - [f_\beta, g(\alpha)]\Big) \qquad (6.26)$$

We can eliminate the term with $(\lambda - 1)$ coefficient by putting $\lambda = 1$, or the term with λ by putting $\lambda = 0$.

Suppose the boundary conditions are of the form:

$$f(\alpha) = f_\alpha$$

$$g(\beta) = g_\beta \qquad (*)$$

Then substitution of $\lambda = 1$ gives us

$$\{\hat{\mathcal{Q}}f, g\} = \{f, \hat{\mathcal{Q}}^*g\} + \Big([f(\alpha), g_\beta] - [f_\alpha, g(\beta)]\Big)$$

and we have the equality

$$\{\hat{\mathcal{Q}}f, g\} = \{f, \mathcal{Q}^*g\}$$

if the boundary conditions (*) are fulfilled. Similar arg ment works for boundary conditions

$$\left.\begin{array}{l} g(\alpha) = g_\alpha \\ f(\beta) = f_\beta \end{array}\right\} (**)$$

where we need to put $\lambda = 0$ to arrive at the same conclusion.

In the case of *boundary conditions*

$$\left.\begin{array}{l} f(\alpha) = f_\alpha \\ f(\beta) = f_\beta \end{array}\right\} (***)$$

we need to use some value of λ, such that $\lambda \neq 0$, $\lambda \neq 1$.

Since $g(\alpha)$ and $g(\beta)$ are <u>not</u> given, the constants g_α, g_β are also not given a priori and we have to regard them as unknown functions. Hence, identifying g_α with the variable $g(\alpha)$ and g_β with $g(\beta)$, we have

$$\{f, \hat{\mathcal{Q}}^*g\} = \{\hat{\mathcal{Q}}f, g\} \pm \{\lambda[(f(\alpha) - f_\alpha), g(\beta)] +$$

$$(1 - \lambda)\ [(f(\beta) - f_\beta), g(\alpha)]\}, \quad \lambda \neq 0, \lambda \neq 1.$$

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Hence $\{\hat{\mathcal{A}}f, g\} = \{f, \mathcal{A}^*g\}$ if the values assumed by $f(t)$ at $t = \alpha$, and $t = \beta$ are exactly the values specified f_α and f_β respectively.

An identical argument works for the case of boundary conditions

$$\left.\begin{aligned} g(\alpha) &= g_\alpha \\ g(\beta) &= g_\beta \end{aligned}\right\} \quad (****)$$

We are ready to introduce the Hamiltonian and Lagrangian functionals for the equations (6.2) with boundary conditions of either one of the types (*), (**), (***), or (****).

The Hamiltonian is given by

$$W(x, p) = \frac{1}{2} \langle p, p \rangle + c(t)\, \Phi(K) - \varphi(t)\, K$$

which is identical with (6.15), while the Lagrangian is:

$$W(x, p) - \{K, \mathcal{A}^*p\} = L(x, p) \qquad (6.27)$$

If boundary conditions are of the form $x(\alpha) = x_\alpha$, $x(\beta) = x_\beta$ we can replace them by $K(x(\alpha)) = K_\alpha$, $K(x(\beta)) = K_\beta$ since K is a known function of x.

The vanishing of Fréchet derivatives of the Lagrangian functional corresponds to the following sets of equations.

$$W_p - \mathcal{A}K = 0 \text{ in } (\alpha, \beta) \qquad (6.28)$$

$$\left.\begin{aligned}(K(\alpha) - K_\alpha)\ p(\beta) &= 0\\ (K(\beta) - K_\beta)\ p(\alpha) &= 0\end{aligned}\right\}\ \text{on}\ \{\alpha\} \cup \{\beta\} \qquad (6.28^{a})$$

$$W_K - \mathcal{J}^* p = 0 \quad \text{in } (\alpha, \beta) \qquad (6.29)$$

As before we check the signs of the tensor products

$$L_{KK} = W_{KK} = c(t) - \hat{f}'(\hat{K}) \quad \text{in } (\alpha, \beta) \qquad (6.30)$$

$$L_{pp} = W_{pp} = I \quad \text{in } (\alpha, \beta)$$

$$= \begin{bmatrix} K(\beta) - K_\beta, & 0 \\ 0 & K(\alpha) - K_\alpha \end{bmatrix} \text{ on } \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad (6.31)$$

and we check the relation:

$$W_{pK} = (W_{Kp})^*$$

We can now state the following variational principles:

Choosing (arbitrarily) x(t) such that K(x(t)) satisfies the given boundary conditions $K(x(\alpha)) = K_\alpha$, $K(x(\beta)) = K_\beta$, then computing p(t) to satisfy the relation $W_p = \mathcal{J} K$, the actual solution of (6.2) and consequently of (6.29) will minimize the Lagrangian L given by relation (6.27) if $\left(c(t) - \hat{f}'(\hat{K}(t))\right)$ is positive on (α, β) and maximize it if $\left(c(t) - f'(\hat{K}(t))\right)$ is negative on (α, β). No variational principle exists if $\left(c(t) - f'(\hat{K}(t))\right)$ changes signs. In that case the best possible statement is that the solution of (6.2) will correspond to a local stationary behavior of the Lagrangian.

Vice-versa if we choose arbitrarily p(t) and compute K(x(t)) using the relation $W_K = \mathcal{A}^* p$, then the choice of p which will minimize the Lagrangian corresponds to the solution of the system (6.2). Since only once differentiable functions were considered in our variational arguments, the solutions we talk about are weak solutions in Sobolëv space W_2^1 (or $H^1(\alpha,\beta)$ in a commonly used symbolism.)

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Systems Research Division
Research, Development & Engineering Directorate

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

(6)

3. REPORT TITLE

VARIATIONAL PRINCIPLES OF CONTINUUM MECHANICS (AN ANALYTICAL POINT OF VIEW).

(9)

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report. June 1972 — June 1973

5. AUTHOR(S) (First name, middle initial, last name)

(10) Vadim Komkov

6. REPORT DATE

(11) July 1973

(12) 1Ø3p.

7a. TOTAL NO. OF PAGES

100

7b. NO. OF REFS

54

8a. CONTRACT OR GRANT NO.

b. PROJECT NO.

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

(14) AMSWE-RE-HT-TR-73ØØ1

10. DISTRIBUTION STATEMENT

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Headquarters, US Army Weapons Command
Rock Island, Illinois 61201

13. ABSTRACT

Methods of functional analysis are used to clasify known variational principles of continuum mechanics of solids, and to derive new principles. Boundary value problems of continuum mechanics are shown to be representable in this formalism. Applications are given to the general theory of non-linear equations occurring in classical mechanics.

042 425

1B

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT

Security Classification