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A SIMPLE MATHEMATICAL MODEL FOR AXIALLY SYMMETRIC MOTIONS OF SUBMARINE HULLS DUE TO PROPELLER EXCITATION

by

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TABLE OF CONTENTS

		Page No.
NOMENCLATURE		1
I.	INTRODUCTION	1
II.	THEORY	2
	A. Displacement Components	2
	B. Kinetic Energy and Potential Energy of Shell Deformation	2
	C. Generalized Coordinates and Generalized Forces	3
	D. Equations of Motion	6
	E. Solution of Equations	9
	F. Longitudinal Motions	11
	G. Coupled Longitudinal-Radial Motions	14
DISCUSSION		24
DISTRIBUTION LIST		

$F = \sqrt{\dfrac{1}{1 + 2\dfrac{a^2}{h\ell}\dfrac{\rho_o}{\rho}}}$ [Equation 55]

$\overline{F}$ = Royleigh dissipation function for structural damping

G = modulus of rigidity, $\dfrac{E}{2(1+\nu)}$

h = thickness of shell wall

J_o, J_1 = Bessel functions of real argument

$k = \dfrac{\omega}{c_o}$

$k'_m a = \lambda\sqrt{1 - (\Psi\dfrac{c_r}{c_o})^2}$ [See Equation 55]

$k_m a = \lambda\sqrt{(\Psi\dfrac{c_r}{c_o})^2 - 1}$ [see Equation 56]

K_o, K_1 = Bessel Functions of imaginary argument

$K = \alpha\ 2\ \lambda^2 - A\omega^2$ [Equation 46]

$K' = \alpha(2+2\dfrac{h^2}{12a^2}\ \lambda^4) - A'\omega^2$ [Equations 46 and 58]

$K^* = \sqrt{(\dfrac{c_e}{c_o})^2 F^2 - 1}$ [See Equation 56]

$\overline{K}$ = structural damping coefficient [Equation 25]

ℓ = length of shell

m = number of axial half waves in vibration pattern of shell

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p = natural frequency

P_o = amplitude of propeller force

Q_U = generalized force due to fluid reaction on end caps [Equation 13]

Q_W = generalized force due to fluid reaction on cylindrical surface [Equation 8]

Q_P = generalized propeller force [Equation 15]

s = area of cylindrical surface

S = cross sectional area

t = thickness of end cap

T = kinetic energy of shell cap system

u = longitudinal displacement

U = longitudinal generalized displacement

V = potential energy of deformation of shell-cap system

w = radial displacement of shell

W = radial generalized displacement

x = longitudinal coordinate

X_c = reactive component of cap acoustic impedance [Equation 34]

X_{mo} = reactive component of shell acoustic impedance [Equations 55 - 57]

Y_o, Y_1 = Bessel Functions of real argument

Z_c = acoustic impedance of end caps

α = $\dfrac{\pi E h \ell}{4a(1-\nu^2)}$ [See Equation 43]

$\bar{\beta}$ = $\begin{cases} 2/m\pi & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even} \end{cases}$ [See Equation 61a]

δ = logarithmic decrement for structural damping

γ = Euler's constant

ζ_{mo} = acoustic impedance of cylindrical shell surface in water [Equation 7]

θ_c = resistive component of acoustic cap impedance [Equation 34]

θ'_{mo} = resistive component of acoustic shell impedance [Equations 55-57]

θ = angular coordinate of shell

λ = $\dfrac{m\pi a}{\ell}$

λ_o = wave length of sound wave in water

ν = Poisson's ratio

ρ = mass density of shell material

ρ_o = mass density of water

ϕ_m = phase angle of m the mode

Ψ = $\dfrac{\omega}{\frac{m\pi}{\ell}\sqrt{\frac{G}{\rho}}}$ [See Equation 55]

ω = frequency at resonance forcing frequency due to propeller

NOMENCLATURE

a - mean radius of shell

A = $\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}$ [Equation 42]

A' = $\frac{1}{2}\pi\rho h\ell a + 2\pi\rho_o c_o \frac{X_{mo}}{\omega} \frac{\ell}{2}$ [Equation 42]

A_c - area of end cap

$\overline{A}$ - longitudinal displacement amplitude

$\overline{A}_r$, $\overline{A}_i$ - real and imaginary parts of $\overline{A}$

B = $2\rho_o c_o \pi a^2 \theta_c + \overline{K}$ [Equation 43]

B' = $2\pi a \rho_o c_o \theta_{mo} \frac{\ell}{2} + \overline{K}$ [Equation 43]

c_o - sound velocity in water

c_e = $\sqrt{\frac{E}{\rho}}$

c_r = $\sqrt{\frac{E}{2\rho(1+\nu)}}$

$\overline{C}$ = radial displacement amplitude

$\overline{C}_r$, $\overline{C}_i$ = real and imaginary parts of $\overline{C}$

E - Young's modulus of shell material

A SIMPLE MATHEMATICAL MODEL FOR AXIALLY SYMMETRIC MOTIONS OF SUBMARINE HULLS DUE TO PROPELLER EXCITATION

I. INTRODUCTION

A program is currently being pursued by HYDRONAUTICS, Incorporated to estimate the relative degree of sound radiation from various pulsating sources on a submarine hull. This particular study was initiated to compute the approximate effect that the pulsating propeller force could have on hulls of various dimensions.

Although there may exist more accurate mathematical models for describing this phenomenon than the one employed here, there do not exist, to the writer's knowledge, any closed form formulas. It is the purpose of this study to obtain approximate formulas which can be used to assess the relative importance of the physical parameters without the use of complicated digital programs.

II. THEORY

A. Displacement Components

The hull will be approximated by an elastic cylindrical shell with rigid circular and caps (see Figure 1). The rigid end caps may move longitudinally as a result of the hull shell vibrating in a longitudinal mode of vibration but the caps prevent any radial motion of the shell at the ends. Only the axially symmetric motions of the system will be considered here and it will be assumed that when the shell vibrates in a single mode, the displacements can be approximated as follows:

$$u = U(t) \cos \frac{m\pi x}{\ell} \qquad [1]$$

$$w = W(t) \sin \frac{m\pi x}{\ell}$$

In equation [1] m is the number of axial half waves in the vibration pattern. For the first mode m = 1 and it will be seen later that the hull could vibrate with displacements that are primarily longitudinal with a small radial component or primarily radial with a small longitudinal component. The former will be the fundamental longitudinal mode of the shell and will be of relatively low frequency compared to the latter which is the fundamental radial mode.

B. Kinetic Energy and Potential Energy of Shell Deformation

Under the assumptions of axially symmetric motions the potential energy of deformation for the m th mode of the shell can be written as follows:[1]

$$V = \frac{\pi E h \ell}{4a(1-\nu^2)} \left[U^2\lambda^2 + W^2 + 2\nu\lambda UW + \frac{h^2}{12a^2} (\lambda^4 W^2) \right] \qquad [2]$$

---

[1] R. N. Arnold and G. B. Warburton, Proc. Roy. Soc. London, A 197, 238-254 (1949).

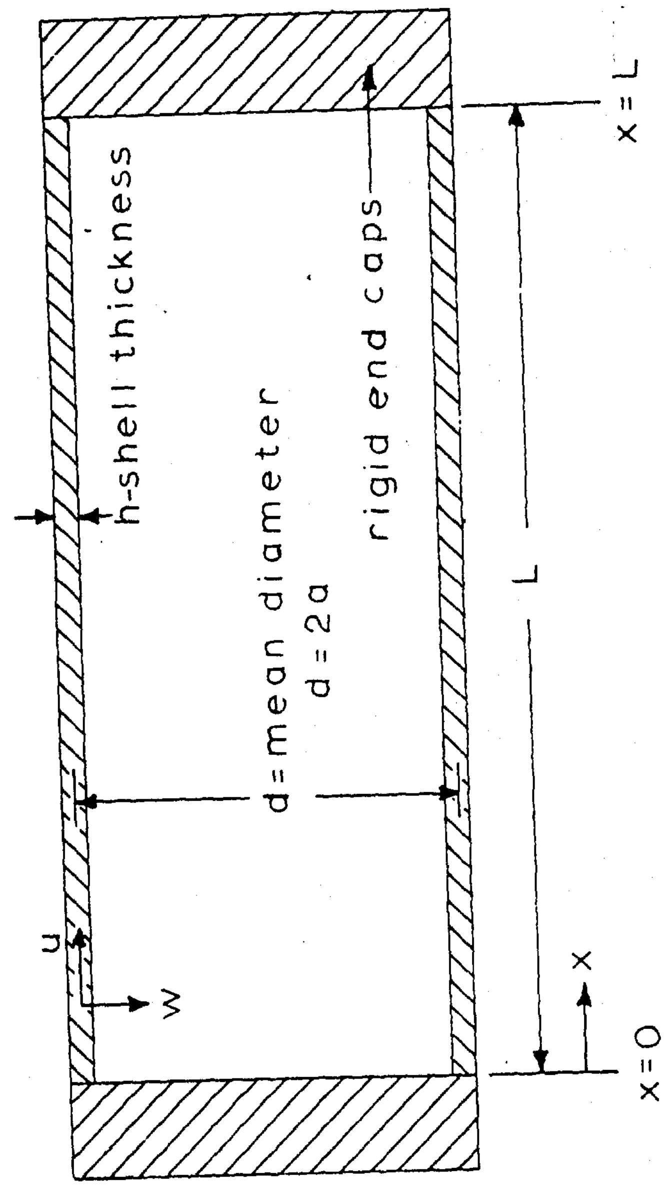

FIGURE I. MATHEMATICAL MODEL OF HULL

and the kinetic energy is

$$T = \frac{\pi\rho h\ell a}{4} [\dot{U}^2 + \dot{W}^2]^* . \qquad [3]$$

In the above equations $\lambda = \frac{m\pi a}{\ell}$.

In deriving the expression for the potential energy the assumptions of thin shell theory have been employed. For a more detailed account of these assumptions the reader is referred to the above reference[1].

C. Generalized Coordinates and Generalized Forces

The generalized coordinates of the system will be chosen as the independent displacements U and W. The generalized forces corresponding to changes in these displacements will be associated with the following:

1. The water pressure on the cylindrical surface arising from radial displacement of the cylindrical surface
2. The water pressure on the end caps arising from longitudinal displacement at the ends
3. The force on the hull due to the pulsating propeller action.

In the analysis presented here the following assumptions will be made with respect to the water pressures:

1. The pressure on the ends due to radial motion of the cylindrical surface will be neglected (this assumption should be valid for primarily longitudinal motions)
2. The pressure on one end due to motion of the other end will be neglected (this assumption should be valid for long shells

*The end cap kinetic energy is neglected in this study. This is discussed at the end of the report.

where the ends are far apart)

3. The pressure on the cylindrical surface due to motion of the ends will be neglected.

Let the generalized force associated with the radial pressure (due to radial displacement w) be Q_W, then the virtual work due to a change in the generalized coordinate W is $Q_W\ \delta W$. If p_r is the pressure on the cylindrical surface due to radial motion then the work done in a displacement δw is

$$\int_s p_r\ \delta w ds \quad (s = \text{area of cylindrical surface})$$

but

$$\delta w = \delta W \sin \frac{m\pi x}{\ell} \ .$$

Therefore

$$Q_W = \int_0^\ell \int_0^{2\pi} p_r \sin \frac{m\pi x}{\ell}\ a d\theta dx \ . \qquad [4]$$

The radial pressure p_r will be taken equal to the value for an infinitely long cylinder the surface of which has the radial displacement proportional to $\sin \frac{m\pi x}{\ell}$. The value for this pressure is given in a previous reference[2] and can be written as follows:

$$p_r = - \rho_o c_o \dot{W} \zeta_{mo} (k_m a) \sin \frac{m\pi x}{\ell} \ . \qquad [5]$$

In formula [5] ρ_o is the density of the medium, c_o is the sound velocity in the medium and ζ_{mo} is the acoustic impedance.

---

[2] M. C. Junger, J. Acoust. Soc. Am., 25, 40-47 (1953)

$$k_m a = \sqrt{\frac{\omega^2 a^2}{c_o^2} - \left(\frac{m\pi a}{\ell}\right)^2} \qquad [6]$$

where ω is the frequency of the vibration. The acoustic impedance can be written in terms of its real and imaginary parts as follows:

$$\zeta_{mo} = \theta_{mo}(k_m a) + i\, X_{mo}(k_m a) \qquad [7]$$

where θ_{mo} is the resistive component (associated with radiation damping) and X_{mo} is the reactive component (associated with virtual mass).

The generalized force given by [4] can then be written as follows:

$$Q_W = - \int_0^{\ell} \int_0^{2\pi} \rho_o\, c_o\, \dot{W}\, \zeta_{mo} \sin^2 \frac{m\pi x}{\ell}\, a\, d\theta\, dx$$

or

$$Q_W = - 2\pi a \rho_o\, c_o\, \dot{W}\, \frac{\ell}{2}\, \zeta_{mo} \qquad [8]$$

Let Q_U be the generalized force due to longitudinal motion of one circular end cap, then

$$Q_U\, \delta U = - \int_{A_c} p_U\, [\delta u]_{x=o,\ell}\, dA_c \quad (A_c = \text{area at end cap}) \qquad [9]$$

In equation [9] p_U is the water pressure due to longitudinal motion of the end cap.

Now

$$[\delta u]_{x=c,\ell} = \delta U .$$

Therefore

$$Q_U = - \int_{A_c} p_U dA_c = F_c \text{ (force on end cap).}$$

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But

$$F_c = -Z_c \dot{U}_c \qquad [10]$$

where Z_c is the acoustic impedance of the circular piston end cap moving against the water.

Thus

$$Q_U = - Z_c \dot{U}_c \qquad [11]$$

The acoustic impedance of the end cap can be written as follows:

$$Z_c = \pi a^2 \rho_o c_o (\theta_c + iX_c) \qquad [12]$$

where θ_c is the resistive component and X_c is the reactive component. The values for θ_c and X_c will be taken as the values for a circular piston radiating from one face into an acoustic medium. The values for these impedances are given in a recent reference.[3] The generalized force due to both end caps will be

$$\overline{Q}_U = 2Q_U = - 2\rho_o c_o \pi a^2 (\theta_c + iX_c) \dot{U} \qquad [13]$$

Lastly, let the generalized force due to the propeller force be Q_p, then

$$Q_p \delta U = P [\delta u]_{x=o,\ell} = P\delta U \qquad [14]$$

Thus

$$Q_p = P = P_o e^{i\omega t} \qquad [15]$$

D. Equations of Motion

Let $\overline{F}$ be the dissipation function associated with structural damping, then the Lagrange's equations of motion for the shell-end cap system will be

[3] S. Hanish, NRL Report 5538, U. S. Naval Research Laboratory, Washington, D.C., Oct. 24, 1960.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{U}}\right) + \frac{\partial \overline{F}}{\partial \dot{U}} + \frac{\partial V}{\partial U} = \overline{Q}_U + Q_p \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{W}}\right) + \frac{\partial \overline{F}}{\partial \dot{W}} + \frac{\partial V}{\partial W} = Q_W \qquad [16]$$

where

$$\overline{F} = \tfrac{1}{2}\ \overline{K}\ (\dot{U}^2 + \dot{W}^2) \qquad [17]$$

in which $\overline{K}$ is proportional to the structural damping force).

Substituting the values for the potential energy, the kinetic energy, the dissipation function, and the generalized forces, the following equations are obtained

$$\tfrac{1}{2}\pi\rho h\ell a\ddot{U} + \frac{\pi E h\ell}{2a(1-\nu^2)}[\ U\lambda^2 + \nu\lambda W] + \overline{K}\ \dot{U} = -2\rho_o c_o \pi a^2(\theta_c + iX_x)\dot{U} + P_o e^{i\omega t}$$

$$\tfrac{1}{2}\pi\rho h\ell a\ddot{W} + \frac{\pi E h\ell}{2a(1-\nu^2)}[W + \nu\lambda U + \frac{h^2}{12a^2}\lambda^4\ W] + \overline{K}\ \dot{W} = -2\pi a\rho_o c_o \dot{W}\frac{\ell}{2}\zeta_{mo} \qquad [18]$$

Assuming that the motions will be harmonic in time the equations can be rewritten as follows

$$\tfrac{1}{2}\pi\rho h\ell a\ddot{U} + 2\rho_o c_o \pi a^2\ \ddot{U}\left(\frac{X_c}{\omega}\right) + 2\rho_o\pi a^2\theta_c\dot{U} + \overline{K}\ \dot{U} + \frac{\pi E h\ell}{2a(1-\nu^2)}[\lambda^2 U + \nu\lambda W] = P_o e^{i\omega t}$$

$$\tfrac{1}{2}\pi\rho h\ell a\ddot{W} + 2\pi a\rho_o c_o \ddot{W}\left(\frac{X_{mo}}{\omega}\right)\frac{\ell}{2} + 2\pi a\rho_o c_o \dot{W}\theta_{mo}\frac{\ell}{2} + \frac{\pi E h\ell}{2a(1-\nu^2)}[W + \nu\lambda U + \frac{h^2}{12a^2}\lambda^4 W]$$

$$+ \overline{K}\ \dot{W} = 0 \qquad [19]$$

or

$$\ddot{U} + \dot{U}\,\frac{2\rho_o c_o \pi a^2\theta_c + \overline{K}}{\tfrac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2\frac{X_c}{\omega}} + \frac{\pi E h\ell}{2a(1-\nu^2)}\ \frac{[\lambda^2 U + \nu\lambda W]}{\tfrac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2\frac{X_c}{\omega}} = \frac{P_o e^{i\omega t}}{\tfrac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2\frac{X_c}{\omega}}$$

$$\ddot{W} + \dot{W}\,\frac{[2\pi a\rho_o c_o \theta_{mo}\frac{\ell}{2} + \overline{K}]}{\tfrac{1}{2}\pi\rho h\ell a + 2\pi a\rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2}} + \frac{\pi E h\ell}{2a(1-\nu^2)}\ \frac{[W + \nu\lambda U + \frac{h^2}{12a^2}\lambda^4\ W]}{\tfrac{1}{2}\pi\rho h\ell a + 2\pi a\rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2}} = 0 \qquad [20]$$

Before going further, it is of interest to note some well known special cases which may be used as a basis with which to compare the more general results to be obtained later. The first case is the vacuum uncoupled longitudinal vibrations of the shell. For this case the equations of motion reduce to

$$\ddot{U} + \frac{\pi E h \ell}{a(1-\nu^2)} \frac{\lambda^2 U}{\pi \rho h \ell a} = \frac{P_o e^{i\omega t}}{\frac{1}{2}\pi \rho h \ell a} \qquad [21]$$

The natural frequency is given approximately by

$$p \approx \frac{m\pi}{\ell} \sqrt{\frac{E}{\rho}} \quad \left(\text{neglecting } \frac{1}{1-\nu^2}\right) \qquad [22]$$

This is the natural frequency of a free-free bar of length ℓ.

The second special case is the in vacuum uncoupled radial vibrations. For this case the equations of motion reduce to

$$\ddot{W} + \frac{\pi E h \ell}{a(1-\nu^2)} \frac{\left[W + \frac{h^2}{12a^2} \lambda^4 W\right]}{\pi \rho h \ell a} = 0 \qquad [23]$$

The natural frequency is given by

$$p = \sqrt{\frac{E}{\rho(1-\nu^2)}} \frac{1}{a} \sqrt{1 + \frac{h^2}{12a^2} \lambda^4} \qquad [24]$$

As $\lambda \rightarrow 0$ this reduces to the well known formula for the radial vibrations of a shell. Formula [24] includes a thickness effect term $\frac{h^2}{12a^2} \lambda^4$ in view of the fact that bending was considered in the expression for the potential energy.

The structural damping constant $\overline{K}$ is as follows:

$$\overline{K} = \frac{p\delta\rho h\ell a}{2} \qquad [25]$$

where p is the natural frequency

δ is the logarithmic decrement for structural damping

$\rho h\ell a$ are as given before.

E. Solution of Equations

Now going back to the general equations and letting

$$U = \overline{A}\, e^{i\omega t}, \quad W = \overline{C}\, e^{i\omega t}$$

the following equations are then obtained for determining the amplitudes $\overline{A}$ and $\overline{C}$

$$-\omega^2 \overline{A} + i\omega\overline{A} \frac{[2\rho_o c_o \pi a^2 \theta_c + \overline{K}]}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} + \frac{\pi E h\ell}{2a(1-\nu^2)} \frac{[\lambda^2 \overline{A} + \nu\lambda \overline{C}]}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}}$$

$$= \frac{P_o}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} \qquad [26a]$$

$$-\omega^2 \overline{C} + i\omega\overline{C} \frac{[2\pi a\rho_o c_o \theta_{mo} \frac{\ell}{2} + \overline{K}]}{\frac{1}{2}\pi\rho h\ell a + 2\pi a\rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2}} + \frac{\pi E h\ell}{2a(1-\nu^2)} \frac{[\overline{C} + \nu\lambda\overline{A} + \frac{h^2}{12a^2}\lambda^4 \overline{C}]}{\frac{1}{2}\pi\rho h\ell a + 2\pi a\rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2}} = 0 \qquad [26b]$$

Collecting terms, we obtain

$$\overline{A}\left[\frac{\pi E h\ell}{2a(1-\nu^2)} \frac{\lambda^2}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} + \frac{i\omega(2\rho_o c_o \pi a^2 \theta_c + \overline{K})}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} - \omega^2\right]$$

$$+ \overline{C}\left[\frac{\frac{\pi E h\ell}{2a(1-\nu^2)}\nu\lambda}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}}\right] = \frac{P_o}{\frac{1}{2}\pi\rho h\ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} \qquad [27a]$$

$$\overline{A}\left[\frac{\pi Eh\ell}{2a(1-\nu^2)}\;\frac{\nu\lambda}{\frac{1}{2}\pi\rho h\ell a+2\pi a\rho_o c_o\frac{X_{mo}}{\omega}\;\frac{\ell}{2}}\right]$$

$$+\;\overline{C}\left[\frac{\frac{\pi Eh\ell}{2a(1-\nu^2)}\left(1+\frac{h^2}{12a^2}\lambda^4\right)}{\frac{1}{2}\pi h\ell\rho a+2\pi a\rho_o c_o\frac{X_{mo}}{\omega}\;\frac{\ell}{2}}+\frac{i\omega\left(2\pi a\rho_o c_o\theta_{mo}\frac{\ell}{2}+\overline{K}\right)}{\frac{1}{2}\pi\rho h\ell a+2\pi a\rho_o c_o\frac{X_{mo}}{\omega}\;\frac{\ell}{2}}-\omega^2\right]=0 \quad (7) \qquad [27b]$$

Equation [27] constitute a set of two complex simultaneous algebraic equations which are to be solved for the complex amplitudes $\overline{A}$ and $\overline{C}$. The solutions will be of the following form

$$\overline{A}=\overline{A}_r+i\,\overline{A}_i$$

$$\overline{C}=\overline{C}_r+i\,\overline{C}_i \qquad [28]$$

where the r subscript refers to the real part and the i subscript refers to the imaginary part. The displacements themselves will then be equal to

$$u=-\sqrt{\overline{A}_r^{\,2}+\overline{A}_i^{\,2}}\;\cos\frac{m\pi x}{\ell}\;\cos\;(\omega t+\phi_m)$$

$$w=\sqrt{\overline{C}_r^{\,2}+\overline{C}_i^{\,2}}\;\sin\frac{m\pi x}{\ell}\;\cos\;(\omega t+\phi_m) \qquad [29]$$

where

$$\phi_m=\tan^{-1}\frac{\overline{A}_i}{\overline{A}_r}$$

ϕ_m is the phase angle between the driving force P_o and the displacements and arises because of the radiation damping and structural damping in the system.

F. Longitudinal Motions

The motions that are of prime importance as a result of propeller excitation are the primarily longitudinal ones. These uncoupled motions in the water will now be studied in more detail. The equation of motion for uncoupled longitudinal motions in the water is

$$\ddot{U} + \frac{\dot{U}\left[2\rho_o c_o \pi a^2 \theta_c + \overline{K}\right]}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} + \frac{\pi E h \ell}{4a(1-\nu^2)} \frac{2\lambda^2 U}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} = \frac{P_o e^{i\omega t}}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} \quad [30]$$

Assuming $U = \overline{A}\, e^{i\omega t}$, we obtain

$$\left[-\omega^2 + \frac{i\omega(2\rho_o c_o \pi a^2 \theta_c + \overline{K})}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} + \frac{\pi E h \ell}{4a(1-\nu^2)} \frac{2\lambda^2}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}}\right]\overline{A} = \frac{P_o}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} \quad [31]$$

So

$$\overline{A} = \frac{\dfrac{P_o}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}}}{\left[\dfrac{\pi E h \ell}{4a(1-\nu^2)} \dfrac{2\lambda^2}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} - \omega^2\right] + \dfrac{i\omega(2\rho_o c_o \pi a^2 \theta_c + \overline{K})}{\frac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega}}} \quad [32]$$

The natural frequency of these longitudinal motions in the water is therefore given by the following frequency equation

$$\frac{\pi E h \ell}{4a(1-\nu^2)} \frac{2\lambda^2}{\frac{1}{2}\pi\rho h\ell a+2\rho_o c_o \pi a^2 \frac{X_c}{\omega}} - \omega^2 = 0 \qquad [33]$$

The values of ω which satisfy [33] are the natural frequencies of the system. In general ω cannot be obtained directly since X_c is a function of ω. For relatively low frequencies, however, the resistive and reactive impedance of an unbaffled circular piston radiating from one side into an infinite medium can be approximated as follows[3]:

$$\theta_c \approx \frac{1}{4}(ka)^2, \; X_c \approx \frac{1}{2} ka \qquad [34]$$

where $k = \frac{2\pi}{\lambda_o} = \frac{\omega}{c_o}$.

Under these circumstances the frequency equation becomes

$$\frac{\pi E h \ell}{4a(1-\nu^2)} \frac{2\lambda^2}{\frac{1}{2}\pi\rho h\ell a+\pi a^3 \rho_o} - \omega^2 = 0$$

The natural frequency can then be obtained in closed form and is as follows:

$$\omega = \sqrt{\frac{\frac{\pi E h \ell}{4a(1-\nu^2)} 2\lambda^2}{\frac{1}{2}\pi\rho h\ell a+\pi a^3 \rho_o}} \qquad [35]$$

or neglecting $\frac{1}{\sqrt{1-\nu^2}}$

$$\omega = \frac{m\pi}{\ell}\sqrt{\frac{E}{\rho}}\sqrt{\frac{1}{1+2\frac{a^2}{h\ell}\frac{\rho_o}{\rho}}} \qquad [36]$$

So the natural frequency of the low frequency longitudinal motions in water is equal to the in vacuum frequency reduced by a factor which depends on the radius of the shell, the thickness of the shell wall and the shell length. It is interesting to note that the factor in the denominator of equation [36] can be written as

$$\frac{2a^2\rho_o}{h\ell\rho} = \frac{4\pi a^3\rho_o}{2\pi a h\ell\rho} = \frac{3M_{sphere}}{M_{shell}} \qquad [37]$$

where M_{sphere} is the mass of a sphere of water which has the same radius as the shell and M_{shell} is the mass of the shell without the end caps. When the shell is oscillating at its natural frequency the amplitude of oscillation is found from equation [32] and is as follows:

$$|\bar{A}| = \frac{P_o}{\omega_n(2\rho_o c_o \pi a^2 \theta_c + \bar{K})} \qquad [38]$$

Using the low frequency approximation for θ_c the following expression is obtained:

$$|\bar{A}| = \frac{P_o}{\omega(2\rho_o c_o \pi a^2 \; \frac{\omega^2 a^2}{4c_o^2} + \frac{\omega\delta\rho h\ell a}{2})} \qquad [39]$$

or

$$|\bar{A}| = \frac{P_o}{\frac{1}{2}\omega_n^2 \rho h\ell a(\frac{\rho_o}{\rho}\,\frac{\pi a^3\omega_n}{c_o h\ell} + \delta)} \qquad [39a]$$

The propeller force is about 10 percent of the mean thrust, the mean thrust is equal to the drag and the drag is given approximately by the following formula:

$$D = \tfrac{1}{2} C_D \rho_o v^2 S$$

where

D = mean drag

C_D = drag coefficient (about .05 for modern submarines)

v = velocity (15-30 knots)

S = cross sectional area

G. Coupled Longitudinal-Radial Motions

Going back to the more general equations [27] the second of the equations gives the following relation between the radial displacement amplitude $\overline{C}$ and the longitudinal amplitude $\overline{A}$ for a given longitudinal driving force P_o

$$\overline{C} = - \frac{\left[\dfrac{\pi E h \ell}{4a(1-\nu^2)} \; \dfrac{2\nu\lambda}{\frac{1}{2}\pi\rho h \ell a + 2\pi a \rho_o c_o \dfrac{X_{mo}}{\omega}\dfrac{\ell}{2}}\right]\overline{A}}{\left[\dfrac{\dfrac{\pi E h \ell}{4a(1-\nu^2)}\left(2 + 2\dfrac{h^2}{12a^2}\lambda^4\right) + i\omega\left(2\pi a \rho_o c_o \theta_{mo}\dfrac{\ell}{2} + \overline{K}\right)}{\frac{1}{2}\pi\rho h \ell a + 2\pi a \rho_o c_o \dfrac{X_{mo}}{\omega}\dfrac{\ell}{2}} - \omega^2\right]} \qquad [40]$$

Therefore substitution into the first of equations [27] gives

$$\overline{A} = \frac{\dfrac{P_o}{A}}{\left[\dfrac{\alpha 2\lambda^2}{A} + i\omega\dfrac{B}{A} - \omega^2\right] - \dfrac{\left[\dfrac{\alpha 2\nu\lambda}{A}\right]\left[\dfrac{\alpha 2\nu\lambda}{A'}\right]}{\left[\dfrac{\alpha\left(2+2\dfrac{h^2}{12a^2}\lambda^4\right)}{A'} + i\omega\dfrac{B'}{A'} - \omega^2\right]}} \qquad [41]$$

where

$$A = \tfrac{1}{2}\pi\rho h \ell a + 2\rho_o c_o \pi a^2 \frac{X_c}{\omega} \qquad A' = \tfrac{1}{2}\pi\rho h \ell a + 2\pi a \rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2} \qquad [42]$$

$$\alpha = \frac{\pi E h \ell}{4a(1-\nu^2)} \qquad B = 2\rho_o c_o \pi a^2 \theta_c + \bar{K} \qquad B' = 2\pi a \rho_o c_o \theta_{mo} \frac{\ell}{2} + \bar{K} \qquad [43]$$

$\bar{K}$ is the structural damping parameter as given before.

Simplifying

$$\bar{A} = \frac{P_o}{[\alpha 2\lambda^2 + i\omega B - A\omega^2] - \dfrac{[\alpha 2\nu\lambda]\ [\alpha 2\nu\lambda]}{[\alpha(2+2\dfrac{h^2}{12a^2}\lambda^4) + i\omega B' - A'\omega^2]}} \qquad [44]$$

Now divide the denominator of the above equation into its real and imaginary parts as follows:

$$\text{Denominator} = \frac{[\alpha 2\lambda^2 + i\omega B - A\omega^2]\ [\alpha(2+2\dfrac{h^2}{12a^2}\lambda^4) + i\omega B' - A'\omega^2] - [\alpha 2\nu\lambda]^2}{[\alpha(2+2\dfrac{h^2}{12a^2}\lambda^4) + i\omega B' - A'\omega^2]} \qquad [45]$$

Let

$$K = \alpha 2\lambda^2 - A\omega^2 \qquad K' = \alpha(2+2\frac{h^2}{12a^2}\lambda^4) - A'\omega^2 \qquad [46]$$

Then

$$\text{Denominator} = \frac{[K+i\omega B]\ [K'+i\omega B'] - [\alpha 2\nu\lambda]^2}{K' + i\omega B'} \qquad [47]$$

$$= \left\{K - \frac{[\alpha 2\nu\lambda]^2 K'}{[K'^2 + \omega^2 B'^2]}\right\} + i\omega\left\{B + \frac{[\alpha 2\nu\lambda]^2 B'}{K'^2 + \omega^2 B'^2}\right\} \qquad [48]$$

For uncoupled longitudinal motions set $\nu = 0$ and the equation reduces to the uncoupled longitudinal case obtained previously.

The natural frequency for coupled motions is given by the solution to the following frequency equation:

$$K - \frac{[\alpha 2\nu\lambda]^2 K'}{[K'^2+\omega^2 B'^2]} = 0 \qquad [49]$$

and the resonant amplitude of the longitudinal component of motion is given by

$$\bar{A} = \frac{P_o}{\omega\left\{B+\frac{[\alpha 2\nu\lambda]^2 B'}{K'^2+\omega^2 B'^2}\right\}} \qquad [50]$$

where ω is the natural frequency. For small values of ω the term $\omega^2 B'^2$ will be considerably smaller than K'^2. Under the assumption that $\omega^2 B'^2 << K'^2$ the frequency equation becomes

$$KK' = [\alpha 2\nu\lambda]^2 \qquad [51]$$

At low frequencies it can be shown (and will be shown later) that A and A' are independent of ω. Therefore the frequency equation reduces to

$$[\alpha 2\lambda^2][\alpha(2+2\frac{h^2}{12a^2}\lambda^4)]-\omega^2\left\{[A][\alpha(2+2\frac{h^2}{12a^2}\lambda^4)]+[A'][\alpha 2\lambda^2]\right\}+AA'\omega^4$$

$$= [\alpha 2\nu\lambda]^2 \qquad [52]$$

If we focus our attention on modes of relatively long wave length where λ is small and if we neglect ν^2 compared to unity in the above equation then the shell end cap system has the following two sets of natural frequencies:

$$\omega_1^2 = \frac{2\alpha}{A'} \qquad \omega_2^2 = \frac{2\alpha\lambda^2}{A} \qquad [53]$$

These frequencies are exactly the uncoupled radial and longitudinal frequencies in water respectively. The lower one, ω_2, is the longitudinal and is the one of greatest practical importance in this study.

In order to simplify the theory further some rather far reaching assumptions have to be made concerning the values of B, B', and K'. Therefore the following section will be devoted to the simplification of these expressions and the justification for this.

If it is assumed that the impedance offered by radial motions can be approximated by the impedance of an infinitely long cylinder in an infinite acoustic medium whose vibration pattern is identical to the finite shell (see recent reference[4]) then

$$B' = 2\pi a \rho_o c_o \frac{\theta_{mo} \ell}{2} \qquad [54]$$

$$K' = \alpha\left(2+2\frac{h^2}{12a^2}\lambda^4\right) - A'\omega^2 \qquad A' = \tfrac{1}{2}\pi\rho h \ell a + 2\pi a \rho_o c_o \frac{X_{mo}}{\omega}\frac{\ell}{2}$$

If $\Psi\, c_r/c_o < 1$ $\qquad \theta_{mo}=0\,;$ $\qquad X_{mo} = \dfrac{\lambda\Psi c_r/c_o\, K_o(k'_m a)}{(k'_m a)K_1(k'_m a)}$

where

$$\Psi = \frac{\omega}{\frac{m\pi}{\ell}\sqrt{\frac{G}{\rho}}}$$

$$\Psi\, c_r/c_o = \frac{c_e}{c_o} F \qquad F = \sqrt{\frac{1}{1+2\frac{a^2}{h\ell}\frac{\rho_o}{\rho}}}$$

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[4] J. Greenspon, Jour. Acoust. Soc. Am., Vol. 32, No. 8, 1017-1025, August 1960.

for resonant longitudinal vibrations $p = \omega = \frac{m\pi}{\ell} c_e F$

$$C_e = \sqrt{\frac{E}{\rho}}$$

$$k_m' a = \lambda\sqrt{1-(\Psi\ c_r/c_o)^2} \qquad [55]$$

If $\Psi\ c_r/c_o > 1$ $$X_{mo} = \frac{\lambda\Psi\ c_r/c_o[J_o(k_m a)J_1(k_m a)+Y_o(k_m a)Y_1(k_m a)]}{(k_m a)\left\{[J_1(k_m a)]^2+[Y_1(k_m a)]^2\right\}}$$

$$\theta_{mo} = \frac{2\ \lambda\ \Psi\ c_r/c_o}{\pi(k_m a)^2\left\{[J_1(k_m a)]^2+[Y_1(k_m a)]^2\right\}}$$

where $$k_m a = \lambda\sqrt{(\Psi\ c_r/c_o)^2-1} = \lambda\sqrt{(c_e^2/c_o^2)F^2 - 1} = \lambda K^*$$

$$K^* = \sqrt{(c_e^2/c_o^2)F^2 - 1}$$

For small $k_m a$ (small λ) and $\Psi\ c_r/c_o > 1$

$$X_{mo} = \frac{\lambda\Psi(c_r/c_o)Y_o(k_m a)}{(k_m a)Y_1(k_m a)} \qquad [56]$$

$$\theta_{mo} = \frac{2\lambda\Psi(c_r/c_o)}{\pi(k_m a)^2\ Y_1\ (k_m a)}$$

where $$Y_o(k_m a) = \frac{2}{\pi}\left(\ell n\ \frac{k_m a}{2} + \gamma\right) \quad \gamma = \text{Euler's Constant}$$

$$Y_1(k_m a) = \frac{2}{\pi k_m a}$$

So

$$X_{mo} = \lambda (c_e/c_o) F \left(\ell n \frac{\lambda K^*}{2} + \gamma\right) \qquad [57]$$

$$\theta_{mo} = \frac{\pi}{2} \lambda (c_e/c_o) F$$

(therefore $\frac{X_{mo}}{\omega} = a \left(\ell n \frac{\lambda K^*}{2} + \gamma\right)$ so that A' is independent of the mode and frequency for small ω).

Thus (neglecting structural damping)

$$B' = 2\pi a \rho_o c_o \frac{\pi}{2} \lambda (c_e/c_o) F \frac{\ell}{2}$$

$$K' = \alpha\left(2+2\frac{h^2}{12a^2}\lambda^4\right) - \omega^2 \left[\tfrac{1}{2}\pi\rho h \ell a + \frac{2\pi a^2 \rho_o c_o \lambda (c_e/c_o) F \left(\ell n \frac{\lambda K^*}{2} + \gamma\right) \frac{\ell}{2}}{\frac{m\pi}{\ell} c_e F a}\right]$$

$$= \frac{Eh\ell}{a}\left[\frac{2\pi}{4(1-\nu^2)} - \lambda^2 F^2 \tfrac{1}{2}\pi - \lambda^2 F^2 \frac{a}{h} 2\pi \frac{\rho_o}{\rho} \left(\ell n \frac{\lambda K^*}{2} + \gamma\right)\right] \qquad [58]$$

For small λ the last two terms will be neglected* leaving

$$K' \approx \frac{Eh\ell}{a} \left[\frac{2\pi}{4(1-\nu^2)}\right] \qquad [58a]$$

Thus inserting the above relations in Equation [50]

$$\overline{\Lambda} = \frac{P_o}{\omega \left\{ 2\rho_o c_o \pi a^2 \theta_c + \frac{(\alpha 2 \nu \lambda)^2 2\pi a \rho_o c_o \frac{\pi}{2} \lambda (c_e/c_o) F \frac{\ell}{2}}{\left[\frac{Eh\ell}{a} \frac{2\pi}{4(1-\nu^2)}\right]^2} \right\}} \qquad [59].$$

---

*Due to the behavior of the logarithm this assumption may be in considerable error for some geometries.

Simplifying and substituting the value for θ_c

$$\overline{A} = \frac{P_o}{\omega\left\{\rho_o c_o \pi a^2 \frac{1}{2} \frac{c_e^2}{c_o^2} F^2\lambda^2 + v^2\lambda^3 2\pi a \rho_o c_o \frac{\pi}{2} \frac{c_e}{c_o} F \frac{\ell}{2}\right\}} \qquad [60]$$

Thus

$$\overline{A} = \frac{P_o}{\omega\lambda^2 \frac{\pi}{2} \rho_o c_o a^2 \frac{c_e}{c_o} F \left\{\frac{c_e}{c_o} F + v^2 m \pi^2\right\}} \qquad [61]$$

Instead of employing the resistive impedance of an infinite cylinder let us use Robey's[5] results for the resistive impedance of a uniformly pulsating ring and average the displacement value for the given mode. The B' can then be written (for small k_a)

$$B' = \overline{\beta} \rho_o c_o \frac{2\pi a \ell}{2} (k_a)(k\ell) \qquad [61a]$$

where ℓ is the length of the cylinder and $k = \frac{\omega}{c_o}$

$$\overline{\beta} = \frac{\int_0^{2\pi} \int_0^{\ell} \sin \frac{m\pi x}{\ell} dx\, a\, d\theta}{\int \int a d\theta dx} = \frac{2\pi a \frac{\ell}{m\pi} \left(-\cos \frac{m\pi x}{\ell}\right)_0^{\ell}}{2\pi a \ell}$$

So

$$\overline{\beta} = \frac{2}{m\pi} \quad \text{if m is odd}$$

$$\overline{\beta} = 0 \quad \text{if m is even.}$$

Substituting this into the equation for $\overline{A}$, we obtain

$$\overline{A} = \frac{P_o}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}\left\{1+\overline{\beta}\, 2 \frac{\ell^2}{a^2} \lambda^2 v^2\right\}} \qquad [62]$$

[5] Robey, D.H., J. Acous.Soc. of Am., V. 27, No. 4, July 1955, pp 706.

or

$$\overline{A} = \frac{P_o}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}\left\{1+2\nu^2 \overline{\beta} m^2\pi^2\right\}} \qquad [63]$$

We will use this expression instead of [61]. The amplitude of radial motion at longitudinal resonance can be written

$$\overline{C} = - \overline{A} \frac{\left[\frac{\alpha 2\nu\lambda}{A'}\right]}{\left[\alpha \frac{(2+2 \frac{h^2}{12a^2} \lambda^4)}{A'} + i\omega \frac{B'}{A'} - \omega^2\right]} \qquad [64]$$

or

$$\overline{C} = - \frac{P_o[\alpha 2\nu\lambda]}{[K+i\omega B][K'+i\omega B'] - [2\alpha\nu\lambda]^2} \qquad [65]$$

neglecting $\omega^2 BB'$ compared to the other terms we obtain

$$\overline{C} = - \frac{P_o\ [\alpha 2\nu\lambda]}{[KK'-(2\nu\alpha\lambda)^2] + i\omega(K'B+B'K)} \qquad [66]$$

At resonance $KK' = (2\nu\alpha\lambda)^2$ (see equation [51]) therefore the radial amplitude at longitudinal resonance is

$$\overline{C} \approx - \frac{P_o[\alpha 2\nu\lambda]}{\omega(K'B + B'K)} \qquad [67]$$

Substituting the value of the longitudinal resonance frequency, the following relation is obtained for $\overline{C}$

$$\overline{C} = - \frac{P_o[2\nu\lambda\alpha]}{\omega(K' 2\pi a^2\rho_o c_o \frac{\omega^2}{4c_o^2} a^2 + K \overline{\beta} \rho_o c_o \pi \ell\lambda^2 \frac{c_e^2}{c_o^2} F^2 \ell)} \qquad [68]$$

Simplifying, we obtain

$$\overline{C} = -\frac{P_o\,[2\nu\alpha\lambda]}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}\left\{K' + \overline{\beta}\,\frac{2\ell^2}{a^2}\,K\right\}} \qquad [69]$$

The volume change of the coupled motions at longitudinal resonance can be written

$$\Delta V(t) = \int_0^{2\pi}\int_0^{\ell} w\,dx\,a\,d\theta + (U_{x=\ell} - U_{x=0})\,A_{cap} \qquad [70]$$

$$= \left\{\frac{\ell}{m\pi}\,2\pi a\overline{C}(-\cos m\pi + \cos 0) + (U_{x=\ell} - U_{x=0})A_{cap}\right\}e^{i(\omega t+\phi)} \qquad [71]$$

where

ϕ = phase angle.

Therefore only for odd m do we obtain a net volume change due to radial motion. For modes with m even there is no volume change at all because the ends are moving in phase with the same displacement. The volume change can be written as follows:

For m odd

$$\Delta V = \left\{\frac{\ell}{m\pi}\,4\pi a\left[-\frac{P_o[2\nu\lambda\alpha]}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}\left\{K' + \overline{\beta}\,\frac{2\ell^2}{a^2}\,K\right\}}\right]\right.$$

$$\left. + \left[\frac{P_o 2\pi a^2}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}\left\{1 + 2\nu^2\,\overline{\beta}\,m^2\pi^2\right\}}\right]\right\} \qquad [72]$$

or since from equations [53] and [46] K = 0 at longitudinal resonance

$$\Delta V = \frac{P_o}{\omega\left\{\rho_o c_o \frac{\pi}{2} a^2\lambda^2 \frac{c_e^2}{c_o^2} F^2\right\}}\left\{\frac{2\pi a^2}{[1+2\nu^2 \bar{\beta} m^2\pi^2]} - \frac{4a \frac{\ell}{m} 2\nu\alpha\lambda}{K'}\right\} e^{i(\omega t+\phi)} \qquad [73]$$

Simplifying,

$$\left|\Delta V\right| = \frac{4 P_o}{\omega\left\{\rho_o c_o \lambda^2 \frac{c_c^2}{c_o^2} F^2\right\}\left\{1+2\nu^2 \bar{\beta} m^2\pi^2\right\}}\left\{\left|2\nu(1+2\nu^2 \bar{\beta} m^2\pi^2)-1\right|\right\} \qquad [74]$$

For m even

$$\left|\Delta V\right| = 0 \qquad [75]$$

DISCUSSION

Approximate closed formula relations have been derived in this report for the coupled longitudinal - radial motions of a cylindrical shell - end cap configuration which is intended to model in a rough manner a submarine hull undergoing axially symmetric motions. A number of rather drastic simplifying assumptions have been made in order to obtain the closed form relations. The rewards of such a simplified analysis are that the relative effects of different parameters can be assessed and problem areas can be defined without resorting to detailed computer analyses.

This simplified approach can be used to point out salient features but it is believed that the assumptions bear some further investigation. For the future it is believed that the following items should be examined:

1. Investigate the accuracy of the simplifications by carrying out computer calculations with the unsimplified equations [26] and [27].
2. Compare the results with available spheroidal shell analyses.
3. Compare the relative magnitude and frequency of sound radiated by the hull with sound radiated by other sources now under investigation.
4. Consider non-resonant response in combinations of modes.
5. Compare the analysis with available experiments.
6. Determine the sound field in more detail.
7. Consider the effect of transverse rings and bulkheads in the shell analysis.
8. Consider the effects of structural damping in more detail.
9. Make parametric studies using existing hulls as examples.
10. Estimate the Q of the various modes.
11. Estimate the effect of mutual impedance between the ends and cylindrical surface.

In this study we have found that volume changes are associated with the odd m modes only, the volume change being given by [74]. The even m modes, on the other hand, are not associated with volume changes and essentially act as a dipole radiator with a source at one end of the shell and a sink at the other.

If stiffeners and rigid bulkheads were added this should tend to decrease the radial motion and allow for more of a net volume change due to combined longitudinal motion and radial motion.

Supposing bulkheads were inserted which were allowed to rotate but would prevent radial motion at their attachment points. This would probably give a radial displacement distribution resembling a continuous sine curve with negative and positive crests. If the number of hills and valleys of radial displacement were equal then the volume change to due radial motion would cancel leaving

$$\left| \Delta V \right| = \frac{4 P_o}{\omega \left\{ \rho_o c_o \lambda^2 \frac{c_e^2}{c_o^2} F^2 \right\}} \quad \text{m odd only}$$

where ω is the natural frequency of longitudinal hull vibrations and can be written as follows:

$$\omega = \frac{m\pi}{\ell} C_e F$$

where

$$C_c = \sqrt{\frac{E}{\rho}}$$

$$F = \sqrt{\frac{1}{1+2 \frac{a^2}{h\ell} \frac{\rho_o}{\rho}}}$$

$$\left(\lambda = \frac{m\pi a}{\ell}\right)$$

In this analysis we neglected the kinetic energy of the end caps. This term would be $\frac{1}{2} M \dot{U}^2$ where $M = \pi a^2 \rho t$ (mass of end cap, t = thickness of cap). The ratio of longitudinal kinetic energy of the shell to the kinetic energy of the two end caps would be

$$\frac{h}{4t} \; \frac{\ell}{a}$$

We limited the analysis to cases where $\ell \gg a$ (h will be of the same order as t).

In order to consider end cap kinetic energy the total kinetic energy would be

$$T = \frac{\pi \rho h \ell a}{4} \; \left[\left(1 + \frac{4t}{h} \; \frac{a}{\ell}\right) \dot{U}^2 + \dot{W}^2 \right]$$

Therefore we merely carry the analysis through with the longitudinal inertia term increased by

$$\left(1 + \frac{4t}{h} \; \frac{a}{\ell}\right)$$

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