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by
JACK SCHECHTMAN

OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA • BERKELEY

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by

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ABSTRACT

A "grain producer" produces and stores a grain, say wheat or corn, and after each crop must decide how much to sell and how much to store. He faces a T period planning horizon, a downwards sloping demand and a storage cost, an increasing function of the amount stored. On the supply side we will assume that the area planted, the inputs such as fertilizers, labor etc. are fixed. The weather with all the unfavorable damage by frost, bugs, and burning sun will determine the final yield of the crop, i.e. we are assuming that the crop of each year is a random variable. The objective of the producer is to maximize the total *expected accumulated profit. The concept of* "liquid stock" is introduced. A sensitivity analysis is carried studying the effect of changes in the parameters of the problem on the optimal policy.

A GRAIN STORAGE PROBLEM, WITH RANDOM PRODUCTION

by

Jack Schechtman

0. INTRODUCTION

Ever since Joseph laid up corn in the land of Egypt for storage against the seven years of famine to come, men have kept on hand a carry-over, large or small, of food. Traders have tried to make a gain by a change in the supply or demand. Storage always has been an important factor in smoothing the supply. In this paper we will consider a simple model in which a "grain producer" produces and stores a grain, say wheat or corn, and after each crop must decide how much to sell and how much to store. He faces a T period planning horizon, a downwards sloping demand, and a storage cost, an increasing function of the amount stored. On the supply side we will assume that the area planted, the inputs such as fertilizers, labor, etc. are fixed. The weather with all the unforseeable damage by frost, bugs, and burning sun will determine the final yield of the crop, i.e. we are assuming that the crop of each year is a random variable. The objective of the producer is to maximize the total expected accumulated profit. This model is similar to the one considered by Samuelson [4] except that we consider the yield of a crop a random variable as opposed to a yield known with certainty.

In this paper we will use the basic concepts of competitive prices and policies developed in [5].

Section 1 contains the basic facts and theorems that will be used in this paper and it is included for sake, of completeness. The deterministic case is considered in Section 2. Section 3 considers the stochastic case, and it is shown that when we follow the optimal policy the stock level will stay in an interval $[a,\bar{y}]$ forming a renewal process. The level $\bar{y}$ will be important in the definition of "liquid stock," an analogue to "liquid excess capital" considered by Keynes in "A Treatise on Money, Vol. II." It's basic property is that if the stock level is above $\bar{y}$ the producer will sell an amount of grain such that the stock available in the next period will be smaller than the previous one no matter what happens with the crop yield. A discussion of this concept and related ones is given in Section 5. The question of how a "liquid stock" can appear after the process has reached its normal course is discussed in Section 6, where the effect of changes in the demand schedule, crop yield, storage cost and deterioration factor are considered.

1. THE MODEL

In this section we will introduce the model and state the basic definitions and results. For proofs of these results the reader could refer to [5] and for a discussion of the concepts for deterministic case to [1].

A grain producer manages the production and storage of a grain. The area planted, labor and physical inputs are constant, with a fixed cost K . The storage cost is an increasing function of the amount stored. The demand is a downwards sloping function and the supply at each period is a random variable. The producer problem is to decide how much to sell, how much to store at each period in order to maximize the total expected profit accumulated during T periods. A period is defined as the time between crops and we will simply call it "year." It is assumed that the yield is available at the beginning of the year and that the decision of how much to store and how much to sell are done at the same time.

1.1 Finite Time Horizon Problem

The problem is divided in T periods that are numbered as $1,2, \ldots, T-1,T$ in that order. We denote by:

y_t : amount of grain available at the beginning of period t .

c_t : amount that is decided to be sold at the beginning of period t .

x_t : amount that is decided to be stored at the beginning of period t .

W_t : the yield that is produced during period t but is available only at the beginning of period $t+1$.

λ : deterioration factor, i.e. if x is the amount stored for the next period, then the amount available at the beginning of it is λx , $0 \leq \lambda \leq 1$.

δ : discount factor that actualizes the profits in different periods.

$P(c)$: demand function, a decreasing and continuous function of the amount sold c .

$\rho(x)$: storage cost function, an increasing and continuous function of the amount stored x .

$u(c)$: the revenue function, i.e., $u(c) = P(c)c$.

The problem can now be formulated as

$$\max E \sum_{t=1}^{T} (u(c_t) - \rho(x_t))$$

$$\text{subject to } c_t + x_t = y_t$$

$$c_t \geq 0 \ , \ x_t \geq 0$$

$$\text{and } y_{t+1} = \lambda x_t + W_t$$

where E is the expectation operator. The random variable W_t $t = 1, \ldots, T$ will be assumed to be i.i.d.*

1.2 Dynamic Programming Formulation of the Model

If $V^t(y)$ is the maximum present value of the expected profit that can be achieved by using an initial amount of grain y optimally through t periods, then the usual Dynamic Programming formulation leads us to the following functional equations:

*The constant cost k , is not included since it will not affect the optimal policies.

$$V^t(y) = \max \left\{u(c) - \rho(x) + \delta EV^{t-1}(\lambda x + W_t)\right\}$$

$$(1.2.1) \qquad \text{s.t.} \quad c + x = y$$

$$c \geqq 0 \ , \ x \geqq 0 \ .$$

Definition:

A policy for a T period problem is a sequence of pair of functions $\{c^t(y), x^t(y)\}_{t=1,\ldots,T}$ such that $x^t(y) + c^t(y) = y$, $x^t(y) \geqq 0$, $c^t(y) \geqq 0$.

Definition:

A T period policy is optimal if $(c^t(y), x^t(y))$ is a solution of (1.2.1) for $t \geqq 1$.

Definition:

A T period policy is competitive if there exist a sequence of nonnegative continuous functions $\{p^t(y)\}_{t=1,\ldots,T}$ such that

i) $c^t(y)$ maximize $u(c) - p^t(y)c$

$$\text{s.t.} \quad c \geqq 0$$

ii) $x^t(y)$ maximize $-\rho(x) + \delta Ep^{t-1}(\lambda x_t(y) + W)(\lambda x + W) - p^t(y)x$

$$\text{s.t.} \quad x \geqq 0$$

iii) $p^o(y) = 0$.

Conditions i) and ii) have very natural economic interpretations, they decentralize the producer activity in two parts, the first tells the producer how to choose the correct amount to sell, $c^t(y)$, by maximizing the profit when he buys the grain at a price $p^t(y)$,*

---

*The reader should note the difference in notation between the competitive prices $p^t(y)$ denoted in general by lower case letter and the actual market price $P(c)$ denoted by a capital letter.

the second tells him how to choose the correct amount to be stored, $x_t(y)$, by maximizing the expected intertemporal profit obtained by the storage activity.

The next theorem will be stated without a proof, the interested reader might refer to [5] for this result and others that will be used here.

Theorem 1.1:

If a policy $\{c^t(y),x^t(y)\}_{t=1,\dots,T}$ is competitive then it is optimal. Furthermore if we assume that the competitive prices $\{p^t(y)\}_{t=1,\dots,T}$ are decreasing functions then $\{c^t(y),x^t(y)\}$ are nondecreasing and continuous functions of y .

The last result is true without any assumption about $u(c)$ and $\rho(x)$ except those about continuity. In order to obtain some qualitative results and the converse of Theorem 1 it will be necessary to assume that $u(c)$ is strictly concave, increasing and differentiable, $\rho(x)$ differentiable, nondecreasing and convex.

The following facts will be stated without a proof:

Theorem 1.2:

Whenever $u(c)$ is strictly concave, increasing, differentiable and $\rho(x)$ nondecreasing, differentiable and convex, the following statements hold.

A. The function $V^t(y)$ is strictly concave, nondecreasing for all t and furthermore $V^t(y)$ is differentiable for all $y > 0$.*

---

*In general the differentiability at $y = 0$ will depend on the specific problem.

B. If a policy $\{c^t(y), x^t(y)\}_{t=T,\ldots,1}$ is optimal then it is competitive and furthermore $(c^t(y), x^t(y))$ satisfy the following conditions:

(1) $p^t(y) \geqq u'(c^t(y))$ with equality if $c^t(y) \geqq 0$.

(2) $p^t(y) \geqq -\rho'(x^t(y)) + \delta\lambda E p^{t-1}(\lambda x^t(y) + W)$ with equality if $x^t(y) > 0$ where $p^t(y) = V^{t'}(y)$ the derivative of $V^t(y)$.

C. The prices $p^t(y)$ are continuous decreasing functions. The continuity is defined on $(0,\infty)$ or $[0,\infty)$ depending on whether $V^t(y)$ is differentiable or not at $y = 0$.

D. The competitive price functions are increasing with t i.e. $p^t(y) \geqq p^{t-1}(y)$ for all $t > 1$ and consequently $c^t(y) \leqq c^{t-1}(y)$ and $x^t(y) \geqq x^{t-1}(y)$ for all $t > 1$.

E. The limiting policy defined by $c(y) = \lim_{t\to\infty} c^t(y)$ and $x(y) = \lim_{t\to\infty} x^t(y)$ are continuous and nondecreasing functions.

From now on we will assume that the conditions of Theorem 2 hold.

In general it will be interesting to compare the stochastic problem, i.e. $W \varepsilon [a,A]$ is a nondegenerate random variable with those in which $W \equiv a$, $W \equiv A$, or $W \equiv E\bar{W}$. If $p_a^t(y)$, $\left(p_A^t(y), p_{\bar{W}}^t(y)\right)$ is the price for a problem in which $W \equiv a$, $(W = A$, $W = \bar{W} = EW)$ then the following results are true.

<u>Theorem 1.3</u>:

$p^t(y) \leqq p_a^t(y)$, $\left(p_A^t(y) \leqq p^t(y)\right)$ and consequently $c_a^t(y) \leqq c^t(y)$, $\left(c^t(y) \leqq c_A^t(y)\right)$; $x_a^t(y) \geqq x^t(y)$, $\left(x_A^t(y) \leqq x^t(y)\right)$ furthermore if $u'(c)$ and $-\rho'(x)$ are convex functions then $p_{\bar{W}}^t \leqq p^t(y)$ and consequently $c_{\bar{W}}^t(y) \geqq c^t(y)$ and $x_{\bar{W}}^t(y) \leqq x^t(y)$.

The last theorem tells us that if we assume the pessimistic (optimistic, average) point of view of getting always the smallest yield (largest yield, the average yield), for the same amount of grain available we will store a greater (smaller, greater) amount than if we assume that $W \varepsilon [a,A]$ is a nondegenerated random variable.

1.3 Infinite Time Horizon Problem

In this section infinite time horizon problems are considered. The producer problem is now:

$$\max E \sum_{t=1}^{\infty} \delta^{t-1}(u(c_t) - \rho(x_t))$$

$$\text{s.t.} \quad c_t + x_t = \lambda x_{t-1} + W_{t-1} \qquad t \geqq 2$$

$$x_1 + c_1 = y_1$$

$$\text{and} \quad c_t \geqq 0 \ , \ x_t \geqq 0 \qquad t \geqq 1 \ .$$

The stationary characteristics of the model leads us to consider only stationary policies, i.e. the decision of how much to sell and how much to store are only functions of the amount at hand and not of the period considered.

Definition:

A policy is a pair of functions $(c(y),x(y))$ such that $c(y) \geqq 0$, $x(y) \geqq 0$ and $c(y) + x(y) = y$. That is if y is the stock at hand $c(y)$ define how much to sell, and $x(y)$ how much to store for the next period.

In many cases the series $E \sum_{t=1}^{\infty} \delta^{t-1}(u(c_t) - \rho(x_t))^*$ might diverge. In order to compare policies it will be necessary to introduce the following definitions:

Definition:

A policy $(c(y),x(y))$ overtakes another policy $(c(y),x(y))$, starting with the same initial stock y_1, if there exist a T_0 such that

$$E \sum_{t=1}^{t=T} \delta^{t-1}(u(\hat{c}_t) - \rho(\hat{x}_t)) - (u(c_t) - \rho(x_t)) > 0$$

for all $T > T_0$.

Definition:

A policy is strongly optimal if it overtakes all the stationary policies.

Definition:

A stationary policy $(c(y),x(y))$ is optimal if

$$\liminf_{T \to \infty} E \sum_{t=1}^{T} u(\hat{c}_t) - \rho(\hat{x}_t) - (u(c_t) - \rho(x_t)) \geqq 0$$

for any stationary policy $(c(y),x(y))$.

The next definition and the following theorem will play a fundamental role in the rest of the paper.

---

*In general given a policy we will denote by y_t the corresponding stock available at the beginning of period t, $c_t = c(y_t)$, $x_t = x(y_t)$ the corresponding amount that will be sold at that period and stored for the next period, respectively.

Definition:

A policy $(c(y),x(y))$ is competitive if there exist a nonnegative, nonincreasing function $p(y)$ such that

i) $c(y)$ maximize $u(c) - p(y)c$
$\quad c \geqq 0$

ii) $x(y)$ maximize $-\rho(x) + \delta E p(\lambda x(y) + W)(\lambda x + W) - p(y)x$
$\quad x \geqq 0$

An immediate consequence of the definition of competitive policy is the following theorem:

Theorem:

If a policy is competitive then the following conditions hold.

(1.3.1) $\quad p(y) \geqq u'(c(y))$ with equality if $c(y) > 0$

(1.3.2) $\quad p(y) \geqq -\rho'(x(y)) + \delta\lambda E p(\lambda x(y) + w)$ with equality if $x(y) > 0$.

For finite time horizon problem, we stated a theorem that tells us that if a policy is competitive then it is also optimal, for infinite time horizon problems this result is not necessarily true. The next theorem give us a sufficient condition for a competitive policy to be optimal.

Theorem:

If a policy $(c(y),x(y))$ is competitive and

(1.3.3) $$\lim_{t\to\infty} E\left\{\delta^t p(y_t)y_t\right\} = 0 \text{ then}$$

the policy is also optimal.

The question of finding optimal solutions for infinite time horizon problems is in general very difficult. Good candidates, are the policies obtained by taking the limit as $t \to \infty$ of $(c_t(y), x_t(y))$, which are well defined functions. Conditions i) and ii) will be immediately satisfied whenever we can show that $p_t(y)$ converges as $t \to \infty$ to a function $p(y)$, and for this it will suffices to show that there exist a bound for $p_t(y)$, since $p_t(y) \geqq p_{t-1}(y)$ for all t.

In general a bound will be obtained by comparing the stochastic case with the deterministic case in which we assume $W \equiv a$. For condition iii) it will be necessary to consider each case by itself.

2. DETERMINISTIC CASE

In this section we will consider the case $w \equiv a$. We will obtain bounds for the prices $p_a^t(y)$, study the behavior of the stock level when we follow the limiting policy and finally it will be shown that the limiting policy is optimal.

Lemma 2.1:

If $0 < \delta\lambda < 1$ and $\rho(x) \geqq 0$, or $\delta\lambda = 1$ and $\rho(x) > 0$ then $c_a^t(y) = y$ for $0 \leq y \leq a$ and $c_a^t(y) \geq a$ for $y \geq a$.

Proof:

To show that $c_a^t(y) = y$ for $0 \leq y \leq a$ it suffices to show that $x_a^t(a) = 0$ since $x_a^t(y)$ is a nondecreasing function. Suppose by contradiction that $x_a^t(a) > 0$. From (B.2) we have that

$$\begin{aligned} p_a^t(a) &= -\rho'\left(x_a^t(a)\right) + \delta\lambda p_a^{t-1}\left(\lambda x_a^t(y) + a\right) \\ &< p_a^{t-1}\left(\lambda x_a^t(a) + a\right) \\ &\leqq p_a^t\left(\lambda x_a^t(a) + a\right) \quad \text{by (D)} \end{aligned}$$

and consequently $a > \lambda x_a^t(a) + a$, since $p_a^t(y)$ is a decreasing function, which is a contradiction. ■

Corollary 2.2:

For all t, $p_a^t(y) \leqq u'(a)$ if $y \geqq a$, and $p_a^t(y) = u'(y)$ if $0 < y \leqq a$, and the limiting policy satisfy the following optimality conditions if either $a > 0$ or $u'(0) < \infty$, (2.1) $$p_a(y) = u'(c_a(y))$$

(2.2) $$p_a(y) \geqq -\rho'(x_a(y)) + \delta\lambda p_a(\lambda x_a(y) + a)$$

where $p_a(y) = \lim_{t\to\infty} p_a^t(y)$.

Proof:

From Lemma (2.1) we get that

$$p_a^t(y) = u'\left(c_a^t(y)\right) \leqq u'(a) \quad \text{for all} \quad t \ , \ y \geqq a$$

and

$$p_a^t(y) = u'(y) \quad \text{for} \quad 0 < y \leqq a \ .$$

Now taking the limit in (B.1) and (B.2) we will get (2.1) and (2.2). ■

From now on whenever we consider infinite time horizon problems we will assume that either $a > 0$ or $u'(0) < \infty$.*

The next theorem tells us how the stock level changes when we follow the limiting policy.

Theorem 2.3:

If $0 < \delta\lambda < 1$ and $\rho(x) \geqq 0$, or $\delta\lambda = 1$ and $\rho'(0) > 0$, then, for the limiting policy, $y^{t+1} \leqq y^t$, $t \geqq 1$; and if y^1 is the initial stock, then there exist an integer $T(y^1)$ such that $y^t = a$ for $t \geqq T(y^1)$.

Proof:

First we will show that $y^{t+1} \leqq y^t$ for all $t \geqq 1$. Of course $y^t \geqq a$, for $t \geqq 1$ since $w = a$, if $y^t > a$ and $x^a(y^t) = 0$.

---

*Cases in which $u'(0) = +\infty$ can also be handled, but a discussion will depend on the specific function $u(c)$.

Then $y^{t+1} = a$ and hence $y^{t+1} < y^t$. If $y^t > a$ and $x^a(y^t) > 0$, then

$$p_a(y^t) = -\rho'(x^a(y^t)) + \delta\lambda p^a(y^{t+1})$$
$$< p_a(y^{t+1})$$

hence, $y^{t+1} < y^t$ since $p_a(\cdot)$ is nondecreasing. Now, if $y^t = a$ using the fact that $x_a(a) = 0$ we will have that $y^{t+1} = a$ and hence $y^{t+1} \leqq y^t$.

The existence of $T(y^1)$ is immediate when $y^1 \leqq a$ since $c_a(y^1) = y^1$ and so $T(y^1) = 1$. For $y^1 > 0$ suppose by contradiction that for all t, $x_a(y^t) > 0$. From the optimality condition (2.1) we will have

$$p_a(y^1) = -\rho'\left(x_a(y^1)\right) + \delta\lambda p_a(y^2)$$
$$p_a(y^2) = -\rho'\left(x_a(y^2)\right) + \delta\lambda p_a(y^3)$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$p_a(y^t) = -\rho'\left(x_a(y^t)\right) + \delta\lambda p_a(y^{t+1})$$

Adding up we get

$$(2.3)\quad p_a(y^1) = \sum_{i=1}^{t} -\rho'\left(x_a(y^i)\right) + (\delta\lambda - 1)\sum_{i=2}^{t} p_a(y^t) + \delta\lambda p_a(y^{t+1}) .$$

Since $y^t > a$ for all t then

$$p_a(y^t) \leqq p_a(y^{t+1}) \leqq p_a(a) = u'(a) .$$

Now, since $x^t > 0$, from (2.3) and using the fact that $p(y^t) \geqq p(y^1)$ for all t we get

$$p_a(y^1) < t(\lambda\delta - 1)p_a(y^1) + \delta\lambda u'(a) - \sum_{i=1}^{t} \rho'\left(x_a(y^1)\right)$$

or, if $\rho'(0) > 0$,

$$p_a(y^1) < -t(1 - \delta\lambda)p_a(y^1) + \delta\lambda u'(a) - t\varepsilon$$

which is a contradiction since the last inequality cannot hold for all t . ∎

Theorem 2.4:

If $0 < \delta < 1$ the limiting policy is optimal.

Proof:

It suffices to show that $\lim_{t\to\infty} \delta^t p_a(y^t)y^t = 0$. From previous results we have that

$$\delta^t p_a(y^t)y^t \leqq \delta^t u'(a)y^1 \text{ , for all } t \text{ .}$$

The theorem now follows from the fact that $0 < \delta < 1$. ∎

Figures (2.1), (2.2), (2.3) and (2.4) represents the behaviour of y^t , c^t , x^t and $P^t = P(c^t)$, where the P^t are the market prices.

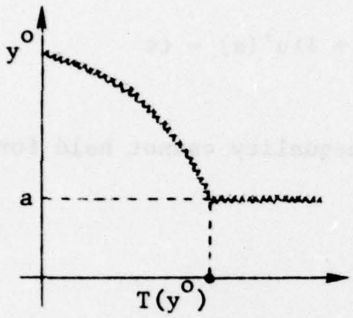


FIGURE 2.1

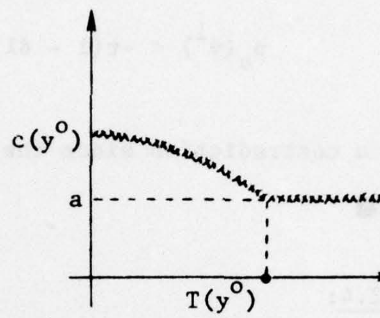


FIGURE 2.2

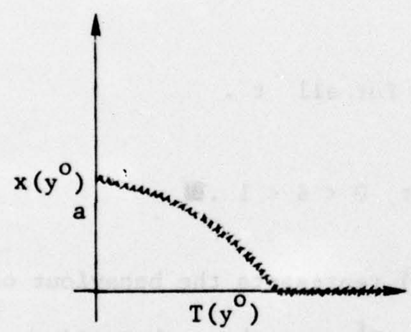


FIGURE 2.3

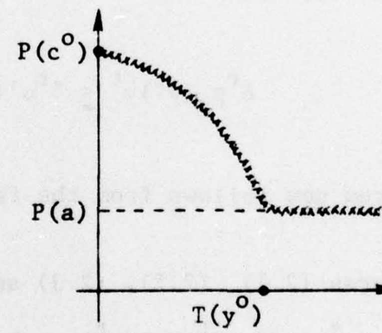


FIGURE 2.4

3. STOCHASTIC CASE

In this section we will study the stochastic case, i.e. $w \varepsilon [0,A]$ is a nondegenerated random variable and Prob $[w = a] = \alpha > 0$.

First we will introduce the competitive condition for the limiting policy.

Lemma 3.1:

The limiting policy satisfies the following relations:

$$p(y) = u'(c(y)) \quad \text{if} \quad y > 0 \tag{3.1}$$

$$p(y) \geqq -\rho'(x(y)) + \delta\lambda Ep(\lambda x(y) + w) \quad \text{with equality if} \quad x(y) > 0 . \tag{3.2}$$

Proof:

Using Theorem 1.3 and Lemma 2.1 we get

$$p^t(y) \leqq p_a^t(y) \leqq u'(a) \text{ , for all } \; y \geqq a$$

$$c^t(y) \geqq c_a^t(y) \geqq a \text{ , for all } \; y \geqq a$$

$$c_t(y) > 0 \text{ , for all } \; y > 0 .$$

Now taking the limit in (B.1) and (B.2) we get (3.1) and (3.2). ■

Theorem 3.2:

If $0 < \delta < 1$, the limiting policy is optimal.

Proof:

As we did in Theorem (2.4) it suffices to show that

$$\lim_{t \to \infty} \delta^t p(y^t) y^t = 0 .$$

When $0 < \lambda < 1$ we have that

$$y^t \leqq \lambda^{t-1} y^1 + A(1 + \lambda + \cdots + \lambda^{t-2})$$
$$\leqq \lambda^{t-1} y^1 + A \frac{\lambda}{1 - \lambda} .$$

So, $\delta^t p(y^t) y^t \leqq \left(\delta^t \lambda^{t-1} y^1 + \delta^t A \frac{\lambda}{1 - \lambda}\right) u'(a)$ now taking the limit as $t \to \infty$ and using the fact that $0 < \lambda < 1$ we get that $\lim_{t\to\infty} \delta^t p(y^t) y^t = 0$.* ■

For the deterministic case we have shown that there exist a $T(y^1) < \infty$, such that for all $t \geqq T(y^1)$, $y^t \equiv a$. That is to say, whatsoever is the initial stock y^1 , after a finite number of periods the stock level will always be equal to $\underline{a}$.

For the stochastic case this result is not necessarily true, but we will try to get an analogous result for this case, i.e., we will show that the stock level will vary randomly, but will return to the level $\underline{a}$ with probability one, and after it reaches this level it will start all over again going out and coming back to this same level, and will constitute a Renewal Process.

The next theorem tells us that there exist a stock level $\bar{y}$ such that for all $t \geqq 1$ $y^t \in [a,\bar{y}]$.

Theorem 3.3:

If $0 < \delta\lambda < 1$ or $\delta\lambda = 1$ and $\rho'(x) > 0$ then there exist a $\bar{y}$ such that for all $y \geqq \bar{y}$, $x(\bar{y}) + A \leq \bar{y}$.

---

*For $\lambda = 1$ we have $y^t \leqq y^1 + tA$, so, $\delta^t p(y^t) y^t \leqq \delta^t t u'(a) y^1$ and hence $\lim_{t\to\infty} \delta^t t u'(a) y^1 = 0$ since $0 < \delta < 1$.

Proof:

First we will show that there exist a $\bar{y}$ such that $x(\bar{y}) + A < \bar{y}$. Suppose by contradiction that not, i.e., for all y, $x(y) + A \geqq y$ or $y - x(y) = c(y) \leqq A$. Now, since $c(y)$ is a continuous and nondecreasing function we will have that $\lim_{y\to\infty} c(y) = M < \infty$. Taking the limit in $p(y) = u'(c(y))$ as $y \to \infty$ we get that $\lim_{y\to\infty} p(y) = u'(M)$. From (3.2) we have

$$\begin{aligned} p(y) &= -\rho'(x(y)) + \delta\lambda Ep(\lambda x(y) + w) \\ &< \delta\lambda Ep(\lambda x(y) + w) - \rho'(x(y)) . \end{aligned}$$

Now, taking the limit and using the fact that $\lim_{y\to\infty} x(y) = \infty$ we get

$$u'(M) < \lambda\delta u'(M) - \lim_{y\to\infty} \rho'(x(y))$$

which is a contradiction, since either $0 < \lambda\delta < 1$ or $\lambda\delta = 1$ and $\rho'(x) > 0$. Consequently there exist a $\bar{y}$ such that $x(\bar{y}) + A < \bar{y}$. Finally as $c(y)$ is a nondecreasing function it follows that $c(y) \geqq c(\bar{y}) > A$ for all $y \geqq \bar{y}$ and consequently $x(y) + A \leqq y$ for all $y \geqq \bar{y}$. ■

Corollary 3.4:

There exist a unique $\bar{y}$ such that $\lambda x(\bar{y}) + A = \bar{y}$.

Proof:

There exist one since for $y = a$, $x(a) = 0$ and $\lambda x(a) + A > a$, and for $y > \bar{y}$, $\lambda x(y) + A < y$, and $x(y)$ is a continuous function.

To show that it is unique suppose that there exist two $\bar{y}_1$ and $\bar{y}_2$ say $\bar{y}_1 < \bar{y}_2$. For $\bar{y}_1$ and $\bar{y}_2$ we have $\lambda x(\bar{y}_1) + A = \bar{y}_1$, $\lambda x(\bar{y}_2) + A = \bar{y}_2$ and

$$c(\bar{y}_1) = y_1\left(\frac{\lambda - 1}{\lambda}\right) + \frac{A}{\lambda} \tag{3.3}$$

and

$$c(\bar{y}_2) = y_2\left(\frac{\lambda - 1}{\lambda}\right) + \frac{A}{\lambda} . \tag{3.4}$$

Subtracting from (3.3) the relation (3.4) we get

$$c(y_2) - c(y_1) = \left(\frac{\lambda - 1}{\lambda}\right)\left(\bar{y}_2 - \bar{y}_1\right) < 0$$

or

$$c(y_2) < c(y_1)$$

which is a contradiction since $c(y)$ is nondecreasing. ■

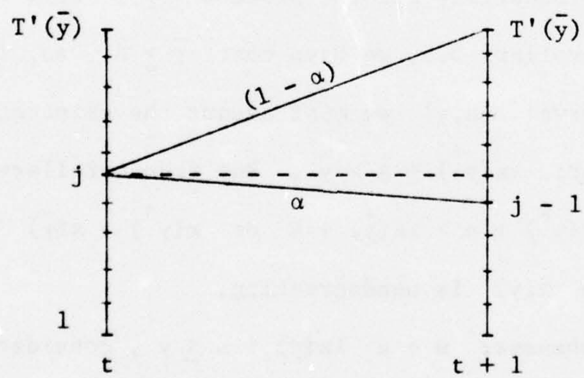
It is an immediate consequence the following corollary

<u>Corollary 3.5</u>:

The level $\bar{y}$ such that $\lambda x(\bar{y}) + \bar{y} = \bar{y}$ satisfies the following inequalities $A \leqq \bar{y} \leqq \frac{A}{1 - \lambda}$.

We should mention that the result of Theorem 3.3 is not necessarily true when $\delta\lambda = 1$, $\rho'(x) = 0$. In (6) we consider the case $\delta = 1$, $\lambda = 1$ and $\rho'(x) = 0$ and obtain the result that $0 \leqq c(y) \leqq \bar{w}$ for all y , $y^t \xrightarrow[\text{s.t.}]{} \infty$ (converges almost sure) and $\lim_{t\to\infty} c(y) = \bar{w}$.

Now, we will argue that the process $\{y^t\}$ is a Delayed Renewal Process. Let x_0 be the number of periods that the process takes to reach for the first time the state $\underline{a}$ and let x_j be the number of periods between the $(j + 1)^{th}$ and the j^{th} time the process reaches the state $\underline{a}$. The random variables $x_0, x_1, \ldots, x_j, \ldots$, are independents and $x_1, \ldots, x_j$ identically distributed, so, the process $\{x_j\}$ is a Delayed Renewal Process. If $y^1 = a$ we will have a Renewal Process. To show that the random variables $\{x_j\}$ have finite expected values, we will consider the following associated process: consider the process in which we have $T'(\bar{y}) = (T(\bar{y}) + 1)$ states, where $\bar{y}$ is defined as in Theorem 3.3 and $T(\bar{y})$ is the number of periods that it would take to reach the level $\underline{a}$ if we make the hypothesis that w assumes always its smallest value $\underline{a}$. Now define two mutually exclusive events, $w = a$ or $w \neq a$. If $w = a$ and we are at state j we move to state $j - 1$, otherwise we go to state $T'(\bar{y})$. The following figure shows how the new process behaves, where the number in the arcs are the transition probabilities.



If z is the random variable that represents the number of periods that the process takes to reach the state 1 starting from the state $T'(\bar{y})$, it can be shown that

$$Ez = \frac{1 + \alpha + \alpha^2 + \cdots + \alpha^{T'(\bar{y})}}{\alpha^{T'(\bar{y})-1}} .$$

Now, observing that $x_j \leqq z$ we will have that $Ex_j \leqq Ez$.

From the property that the process $\{y^t\}$ constitutes a Delayed Renewal Process and the uniqueness of $\bar{y}$ we can state the following theorem.

Theorem 3.6:

With probability one the process $\{y^t\}$ will lay in the interval $[0,\bar{y}]$ and if $w = a$ we will have $\lambda x(y) + a \leqq y$ and if $y \in [a,\bar{y}]$ and $w = A$ $\lambda x(y) + A \geqq y$.

Proof:

From the fact that $\{y^t\}$ constitutes a Delayed Renewal Process we have that with probability one the process $\{y^t\}$ will reach the state $\underline{a}$. From Corollary 3.5, we have that $\bar{y} \geqq A$, so, in order to get out of the interval $[a,\bar{y}]$ we must assume the existence of a $y^* < \bar{y}$ such as that: $\lambda x(y^*) + A > \bar{y}$. But from corollary it follows that $\lambda x(y^*) + A > \lambda x(\bar{y}) + A$ or $x(y^*) > x(\bar{y})$ which is a contradiction since $x(y)$ is nondecreasing.

To show that whenever $w = a$ $\lambda x(y) + a \leq y$, consider the optimality condition (3.2) and assume that $x(y) > 0$.

$$\begin{aligned} p(y) &= -\rho'(x(y)) + \delta\lambda Ep(\lambda x(y) + w) \\ &< p(\lambda x(y) + a) \end{aligned}$$

which implies from the monotonicity of $p(y)$ that $y > \lambda x(y) + a$.
When $x(y) = 0$ and $y \geq a$ it is immediate that $\lambda x(y) + a \leqq y$.

Finally to show that with probability one $\lambda x(y) + A \geqq y$. Suppose that for some $y^* \in [a,\bar{y}]$ $\lambda x(y^*) + y < y$ but, this would imply the existence of $\bar{y}_1 < \bar{y}$ such that $\lambda x(\bar{y}_1) + A = \bar{y}_1$ which contradicts Corollary 3.4.■

From Theorem 3.6 it will follow that a possible sample for the process $\{y^t\}$ is the one represented as in Figure 3.1.

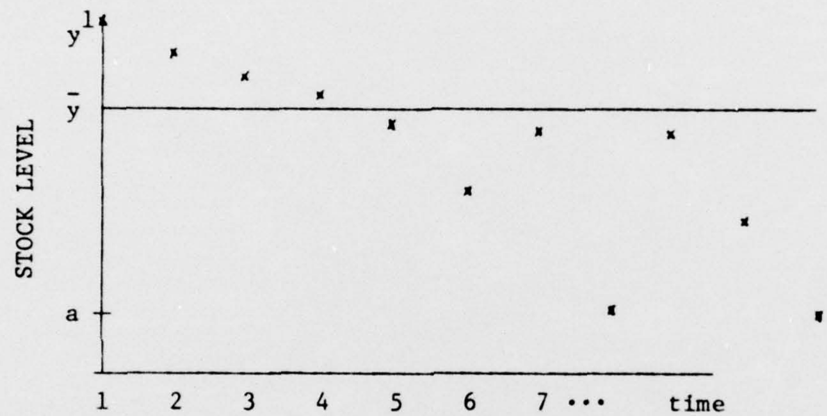


FIGURE 3.1

If we agree to call $w = a$ a bad crop and $w = A$ a bump crop* an interesting question is about the information that the change in the market prices between two crops gives us about the occurrence of a bad or a good crop, namely does an increase (decrease) in prices mean a bad (good) crop? From the previous result it will follow that this is not necessarily true for instance as $y^t > \bar{y}$ we will have that $y^{t+1} < y^t$ independent of the result of the crop and consequently $p^{t+1} > p^t$.

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*Assume for simplicity that w is *binary*.

The study of the sensitivity of the level $\bar{y}$ with respect to changes in the parameters of the problem will be done in the next section.

4. SENSITIVITY ANALYSIS

In this section we will study the effect of changes in the profit schedule, crop yield, deterioration factor in the prices, policies, etc.

Whenever we compare the prices, policies of two problems one obtained from the other by some change we will denote, in the following results, the prices, policies, function for the new problem by $p_*^t(y)$, $\left(c_*^t(y), x_*^t(y)\right)$, $u_*(c)$ etc.

Theorem 4.1:

If $\rho'(x) > \rho_*'(x)$ $(\delta_* > \delta)$ then $p_*^t(y) \geqq p^t(y)$ $\left(p_*^t(y) > p^t(y)\right.$ if $\left.x_*^t(y) > 0\right)$ and consequently $c_*^t(y) \leqq c^t(y)$ $\left(c_*^t(y) < c_*^t(y)\right.$ if $\left.x_*^t(y) > 0\right)$.

Proof:

The proof will be done by induction. First for $t = 1$ it is true since $p_*^1(y) = p^1(y) = u'(y)$ and $x_*^1(y) = 0$ for all $y \geqq 0$. Now suppose that it is true for $t - 1$, and by contradiction that it is not for $\underline{t}$ i.e. for some y , $x_*^t(y) > 0$ and $p_*^t(y) \leq p^t(y)$, so, $c_*^t(y) \geqq c^t(y)$ and consequently $x_*^t(y) \leqq x^t(y)$. Now, from the optimality condition (B.2) we get

$$p^t(y) = -\rho'(x^t(y)) + \lambda\delta E p^{t-1}(\lambda x^t(y) + w))$$

$$\leqq -\rho'\left(x_*^t(y)\right) + \lambda\delta E p^{t-1}\left(\lambda x_*^t(y) + w\right)$$

by the monotonicity of $p_*^{t-1}(y)$ and $-\rho_*'(x)$. Now using the induction hypothesis and the fact that $-\rho'\left(x_*^t(y)\right) < -\rho'(x^t(y))$ we get

$$p^t(y) < -\rho'_*\left(x^t_*(y)\right) + \lambda\delta E p^{t-1}_*\left(\lambda x^t_*(y) + w\right)$$
$$\leqq p^t_*(y)$$

which is a contradiction. Similar arguments will hold if we consider $\delta < \delta^*$. ■

The next theorem considers the case in which there exist a uniform change in the revenue schedule, namely $u_*(c) = ku(c)$ where k is a real number.

Theorem 4.2:

If $u_*(c) = ku(c)$ $0 < k < 1$ $(k > 1)$ $\rho'(x)$ increasing then $p^t_*(y) \leq kp^t(y)$ $\left(p^t_*(y) \geq kp^t(y)\right)$.

Proof:

The proof will be done by induction. It is true for $t = 1$ since $p^1_*(y) = ku'(y)$ and $p^1(y) = u'(y)$. Now suppose it is true for $t - 1$ and by contradiction that is not for t i.e. $p^t_*(y) > kp^t(y)$. Hence, $ku'\left(c^t_*(y)\right) > ku'(c^t(y))$ or $u'\left(c^t_*(y)\right) > u'(c^t(y))$ and hence $c^t_*(y) \leq c^t(y)$ and $x^t_*(y) > x^t(y)$. Now from the optimality condition (B.1) we get

$$p^t_*(y) = -\rho'\left(x^t_*(y)\right) + \delta\lambda E p^{t-1}_*\left(\lambda x^t_*(y) + w\right)$$
$$< -\rho'(x^t(y)) + \delta\lambda E p^{t-1}_*(\lambda x^t(y) + w)$$
$$< -\rho'(x^t(y)) + \delta\lambda k E p^{t-1}(\lambda x^t(y) + w) \quad \text{by induction hypothesis}$$
$$\leqq -\rho'(x^t(y))k + \delta\lambda k E p^{t-1}(\lambda x^t(y) + w) \quad \text{since } 0 < k < 1$$
$$\leqq kp^t(y)$$

which is a contradiction. ■

The last theorem tell us that storage cost is an important factor in explaining changes in the producer's behavior when he foresees a change in the revenue schedule. In the absence of it no changes will occur at least in our simplified version of the reality.

An important question is what happens with the optimality policy, when the random variable w changes. We will be looking to effects related in some sense to technological changes in the yield related either to a net improvement in the crop yield, i.e., $w^* = w + w'$ where w' is non-negative random variable, or to a reduction in the probability of getting "smaller values" for the crop yield. The mathematical concept that captures these ideas is the one related to stochastic order; the next definition introduces it.

Definition:

A random variable x with distribution F is stochastically larger (smaller) than y with distribution G if $F_x(x) \leqq G_y(x)$. This relation will be denoted by $x \succsim y$. The following facts follow immediately from the definition.*

Theorem 4.3:

i) If $x = y + z$ where z is a nonnegative random variable then $x \succsim y$.

ii) If $h(x)$ is a nonincreasing function and $x \succsim y$ then $Eh(x) \leq Eh(y)$.

Theorem 4.4:

If $w^* \succsim w$ then $p^t_*(y) \leqq p^t(y)$.

*For a proof see for instance [2].

<u>Proof</u>:

The proof will be done by induction. First it is true for $t = 1$ since $p_*'(y) = p'(y) = u'(y)$. Now let us assume that it is true for $t - 1$, and by contradiction that is is not for t , i.e., $p_*^t(y) > p^t(y)$ and consequently that $x_*^t(y) > x^t(y)$ and $c_*^t(y) < c^t(y)$. Now using the optimality condition (B.1) we get

$$\begin{aligned} p_*^t(y) &= -\rho'\left(x_*^t(y)\right) + \delta\lambda E p_*^{t-1}\left(\lambda x_*^t(y) + w^*\right) \\ &\leq -\rho'\left(x_*^t(y)\right) + \delta\lambda E p_*^{t-1}\left(\lambda x_*^t(y) + w\right) \end{aligned} \tag{4.1}$$

since w^* w and $p_*^{t-1}(y)$ is a nonincreasing function. Now from (4.1) using the induction hypothesis and the fact that $x_*^t(y) > u^t(y)$ we get that

$$\begin{aligned} p_*^t(y) &\leqq -\rho'(x^t(y)) + E\delta\lambda p^{t-1}(\lambda x^t(y) + w) \\ &\leqq p^t(y) \end{aligned}$$

which is a contradiction. ■

In order to study the effect to a change in the deterioration factor we first need a lemma.

<u>Lemma 4.5</u>:

If $u'(y)y$ is increasing and $\rho'(x)x$ decreasing then $p_t(y)y$ is also increasing for all t .

<u>Proof</u>:

The proof will be done by induction. First it is true for $t = 1$ since $u'(y)y$ is increasing. Now assume that it is true for $t - 1$. From the optimality conditions (B.1) and (B.2) we get

$$p^t(y)x^t(y) = -\rho'(x^t(y))x^t(y) + \delta\lambda E p^{t-1}(\lambda x^t(y) + w)x^t(y)$$

$$(4.2) \qquad = -\rho'(x^t(y))x^t(y) + \delta E(\lambda x^t(y) + w)p^{t-1}(\lambda x^t(y) + w) - Ewp^{t-1}(\lambda x^t(y) + w)$$

and

$$(4.3) \qquad p^t(y)c^t(y) = u'(c^t(y)c^t(y) \; .$$

Now adding up (4.2) and (4.3) we get

$$(4.4) \qquad p^t(y)y = u'(c^t(y))c^t(y) - \rho'(x^t(y))x^t(y) + \delta E(\lambda x^t(y) + w)p^{t-1}(\lambda x^t(y) + w) - Ewp^{t-1}(\lambda x^t(y) + w) \; .$$

Let $y' > y$, hence, from the monotonicity of $x^t(y)$ and $c^t(y)$ we have that $x^t(y') \geq x^t(y)$ and $c^t(y') \geq c^t(y)$ with at least one strict inequality. So from (4.4) using the monotonicity of $u'(y)y$ the induction hypothesis and the fact that $p^{t-1}(y)$ is nonincreasing we get

$$\begin{aligned} p^t(y)y &\leqq u'(c^t(y'))c^t(y') - \rho'(x^t(y')x^t(y') \\ &\quad + \delta E(\lambda x^t(y') + w))p^{t-1}(\lambda x^t(y') + w) - Ewp^{t-1}(\lambda x^t(y') + w) \\ &= u'(c^t(y')c^t(y') - \rho'(x^t(y'))x^t(y') + \delta E\lambda x^t(y')p^{t-1}(\lambda x^t(y') + w) \\ &< p^t(y')y' \; . \blacksquare \end{aligned}$$

<u>Corollary 4.6</u>:

If $u'(y)y$ is increasing, $\rho'(x)x$ decreasing and $\lambda^* > \lambda$ then $p^t_*(y) \geq p^t(y)$.

Proof:

The proof will be done by induction. First it is true for $t = 1$ since $p_*^{\prime}(y) = p'(y) = u'(y)$. Now let us assume that $p_*^{t-1}(y) \geqq p^{t-1}(y)$ and by contradiction that $p_*^t(y) < p^t(y)$, so, $x_*^t(y) < x^t(y)$. Now from the optimality condition (B.1) we get

$$
\begin{aligned}
p^t(y)x^t(y) &= -\rho'(x^t(y))x^t(y) + E\delta(\lambda x^t(y) + w)p^{t-1}(\lambda x^t(y) + w) \\
&\qquad - Ewp^{t-1}(\lambda x^t(y) + w) \\
&\leqq -\rho'(x^t(y)x^t(y) + \delta E(\lambda^* x^t(y) + w))p^{t-1}(\lambda^* x^t(y) + w) \\
&\qquad - Ewp^{t-1}(\lambda^* x^t(y) + w) \\
&= (-\rho'(x^t(y)) + \delta\lambda^t E\lambda^* p^{t-1}(\lambda^* x^t(y) + w))x^t(y) \\
&< -\rho' \, x_*^t(y) \; + \delta\lambda^t E\lambda^* p^{t-1} \, \lambda^t x_*^t(y) + w \, x^t(y) \\
&\leqq p_*^t(y)x^t(y) \quad \text{which is a contradiction.} \blacksquare
\end{aligned}
$$

An interesting question is the one related to the other direction of the previous theorem, namely would $u'(y)y$ decreasing and $\lambda^* > \lambda$, implies that $p_*^t(y) \leq p^t(y)$; the answer is not and suffices to consider a deterministic problem $w = a > 0$ and $u(c) = -\frac{1}{c}$ and $\rho(x) = 0$ and compute it for a two period problem.

Now we will study the effect of changes in the profit schedule, marginal cost of storage, deterioration factor and the crop yield in the stock level $\bar{y}$ defined in Theorem (3.3).

First we need a lemma.

Lemma 4.7:

If $\rho'(x)$ is increasing then $p(y)$ is strictly decreasing.

Proof:

Suppose by contradiction that if not, i.e., there exists y' and y'', $y'' > y'$ such that $p(y') = p(y'')$. If $x(y'') = 0$, the theorem immediately follow since $p(y') = u(y')$, $p(y'') = u(y'')$ and $u'(y') > u'(y'')$. Now suppose that $x(y'') > 0$, from the optimality condition (3.2) we get

$$p(y'') = -\rho'(x(y'')) + \delta\lambda E p(\lambda x(y'') + w) . \tag{4.5}$$

But, $p(y') = p(y'')$ implies that $c(y') = c(y'')$ and $x(y'') = x(y') + y'' - y'$. So

$$\begin{aligned} p(y'') &= -\rho'(x(y'')) + \lambda\delta E p(\lambda x(y'') + w) \\ &\leqq -\rho'(x(y'')) + \lambda\delta E\rho(\lambda x(y') + w) \quad \text{by monotonicity of } p(y) \\ &< -\rho'(x(y')) + \lambda\delta E p(\lambda(x(y)) + w) \\ &\leqq p(y') \end{aligned}$$

which is a contradiction. ■

Theorem 4.8:

If $\rho_*'(x) < \rho(x)$ $(\delta_* > \delta)$ then $\bar{y}_* > \bar{y}$.

Proof:

Suppose by contradiction that if not, i.e., $\bar{y}_* < \bar{y}$. Now from the definition of $\bar{y}_*$ and $\bar{y}$ we have that

$$\lambda x_*(\bar{y}) + A < \bar{y} = \lambda x(\bar{y}) + A$$

or

$$x_*(\bar{y}) < x(\bar{y})$$

which is a contradiction. Now suppose that $\bar{y}_* = \bar{y}$, hence $x_*(\bar{y}_*) = x(\bar{y})$, $c_*(\bar{y}_*) = c(\bar{y})$ and $p_*(\bar{y}_*) = p(\bar{y})$. From the optimality condition (3.2) we get

$$\begin{aligned} p_*(\bar{y}_*) &= -\rho_*'(x_*(y)) + \delta_*\lambda Ep_*(\lambda x_*(y) + w) \\ &= -\rho_*'(x_*(y)) + \delta_*\lambda Ep(\lambda x_*(y) + w) \quad \text{from Theorem 4.1}^* \\ &= -\rho_*'(x(y)) + \delta_*\lambda Ep(\lambda x(y) + w) \\ &> -\rho'(x(y)) + \delta\lambda Ep(\lambda x(y) + w) \\ &= p(\bar{y}) \end{aligned}$$

which contradicts Theorem 4.1 if we take the limit as $t \to \infty$ there.

<u>Theorem 4.9</u>:

If $\rho'(x)$ is increasing, $w^* \succ w$ ($w^* \prec w$) then $\bar{y}_* < \bar{y}$ ($\bar{y}_* > \bar{y}$) .

<u>Proof</u>:

Suppose by contradiction that $w^* \succ w$ and $\bar{y}_* > \bar{y}$, so, from the definition of $\bar{y}_*$ and $\bar{y}$ we get

$$\lambda x_*(\bar{y}) + A > \bar{y} = \lambda x(\bar{y}) + A$$

or $x_*(\bar{y}) > x(\bar{y})$ which is a contradiction. Now suppose that $\bar{y}_* = \bar{y}$, hence $x_*(\bar{y}_*) = x(\bar{y})$, $c_*(\bar{y}_*) = c(\bar{y})$ and $p_*(\bar{y}_*) = p(\bar{y})$. Now, from the optimality condition (3.2) we get

---

*Taking the limit as $t \to \infty$.

$$(4.5)\quad \begin{aligned} p_*(\bar{y}_*) &= -\rho'(x_*(\bar{y}_*)) + \delta\lambda E p_*(\lambda x_*(\bar{y}_*) + w^*) \\ &< -\rho'(x_*(\bar{y}_*)) + \delta\lambda E p_*(\lambda x_*(\bar{y}_*) + w) \end{aligned}$$

since $w^* > w$ and $p_*(y)$ is a decreasing function. From (4.5) and Theorem 4.4 we get

$$\begin{aligned} p_*(\bar{y}_*) &< -\rho'(x(\bar{y})) + \delta\lambda E p(\lambda x(\bar{y}) + w) \\ &= p(\bar{y}) \end{aligned}$$

which contradicts Theorem 4.4 if we take the limit as $t \to \infty$ there.■

<u>Theorem 4.10</u>:

If $U_*(c) = ku(c)$, $0 < k < 1$, $(1 < k)$ then $\bar{y}_* < \bar{y}$ $(\bar{y}_* > y)$, $\rho'(x) > 0$.

<u>Proof</u>:

From Theorem 4.2 taking the limit as $t \to \infty$ we get $p_*(y) \le kp(y)$, $c_*(y) \ge c(y)$ and $x_*(y) \leqq x(y)$. First suppose by contradiction that $\bar{y}_* > \bar{y}$, so, $\lambda x_*(\bar{y}) + A > \bar{y} = \lambda x(\bar{y}) + A$ or $x_*(\bar{y}) > x(\bar{y})$ which is a contradiction. Now assume that $\bar{y}_* = \bar{y}$ so, $x_*(\bar{y}_*) = x(\bar{y})$ and $c_*(\bar{y}_*) = c(\bar{y})$ and consequently $p_*(\bar{y}_*) = kp(\bar{y})$. From the optimality condition (3.2) multiplied by k we get

$$\begin{aligned} p_*(\bar{y}_*) = kp(\bar{y}) &= -k\rho_*'(x(\bar{y})) + k\delta\lambda E p(\lambda x(\bar{y}) + w) \\ &= -k\rho'(x_*(y_*) + k\delta\lambda E p(\lambda x_*(\bar{y}_*) + w) \\ &\geqq -k\rho'(x_*(\bar{y}_*) + \delta\lambda E p_*(\lambda x_*(\bar{y}_*) + w) \\ &> -\rho'(x_*(\bar{y}_*) + \delta\lambda E p_*(\lambda x_*(\bar{y}_*) = w) \\ &= p_*(\bar{y}_*) \text{ a contradiction.}■ \end{aligned}$$

Theorem 4.11:

If $u'(y)y$ is increasing, $\rho'(x)x$ nondecreasing $\lambda^* > \lambda$ then $\bar{y}_* > \bar{y}$.

Proof:

From Theorem 4.6 after taking the limit as $t \to \infty$ we get $p_*(y) \geqq p(y)$ and $c_*(y) \leqq c(y)$, $x_*(y) \geqq x(y)$.

Suppose by contradiction $\bar{y}_* \leq \bar{y}$. So,

$$\lambda^* x_*(\bar{y}) + A \leq \bar{y} = \lambda x(\bar{y}) + A$$

or

$$\lambda^* x_*(\bar{y}) \leq \lambda x(\bar{y}) < \lambda^* x(\bar{y})$$

or $x_*(\bar{y}) < x(\bar{y})$ which is a contradiction.■

Observe that the last result is only true whenever $\rho'(x) > 0$, for the case in which $\rho'(x) = 0$ we will have that $\bar{y}_* = \bar{y}$, and $c_*(y) = c(y)$, $x_*(y) = x(y)$.

5. LIQUID STOCK AND RELATED CONCEPTS

When we look at the data of the stock of some product for which the production is affected by seasonal variations that are not controllable, say a grain, one question that arises is how much of this stock is held in order to overcome the fluctuations and how much might be in "excess." The answer of this question is closely related to the concept of working capital and liquid capital considered by Keynes [3]:

> "Working capital has to provide for carrying stocks between harvests (for such carrying is a form of "process") and also fluctuations in the "carry-over" from one harvest to another, in so far as such carry-over is required by the unavoidable variations of individual harvests around the mean harvest. On the other hand a net prospective surplus, taking one season with another, due to a mistake involving a relative over-production, belongs to liquid capital."

In Section 3 we proved the existence of a stock level $\bar{y}$ such that if the initial stock level is above $\bar{y}$ the behavior of the stock level as a function of time we look like Figure 5.1. That is, it decreases steadily, and after a finite number of periods it will remain below $\bar{y}$ changing in a random fashion.

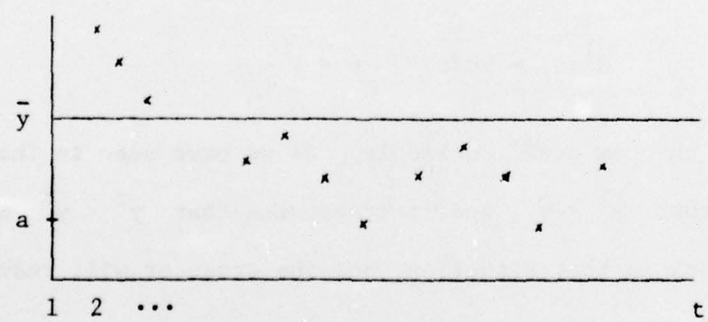

FIGURE 5.1

Motivated by that result we could define "liquid stock" at time t, for a given economy characterized by its demand, storage cost, and deterioration factor and the yield by $L_t = \max \{y_t - y, 0\}$.

As we have already seen, the "liquid stock" will disappear after a finite number of periods. So, when we look at the data of some product we must be careful in considering it as a "liquid stock." In general a considerable part of it must be kept in order to overcome the variation in the production, and should not be taken in advance as a kind of "surplus" that could be traded for other products without any harm to the consumption, as usually we are led to think when we have a sequence of "good crops."

A question that naturally arises in our simple model, is how a "liquid stock" can appear when the process is in its normal course that is, the stock level lies in the interval $[a,\bar{y}]$.

Several ways can be considered in order to try to answer that question. First let us consider when the producer foresee a slump in the market. He expects his profits to be a fraction of the previous profit schedule, in mathematical terms

$$U^I(c) = ku(c) \qquad 0 < k < 1$$

where $U^I(c)$ is the new profit schedule. As we have seen in Theorem 4.10 this will imply that $\bar{y}^I < \bar{y}$, and if it happens that $y^t < y^I$ we will have a liquid stock in that situation, and the producer will reduce its stock steadily.

We should mention again, that the appearance of a "liquid stock" due to a slump in the market can only be explained in the presence of a positive marginal storage cost.

Another important case to consider is the one related to technological changes in the crop yield, either by a net increase in the yield or by a reduction in the probability of smaller crops, say for instance by new irrigation techniques. As have been seen from Theorem 4.9, again in the presence of a positive marginal storage cost, a "liquid stock" may appear and the society will consume more for the same level of stock available.

Another interesting question related to price behavior is the one related to "backwardation of prices" considered by Keynes and Houtacker: 'We say that we have a "backwardation in prices" when the spot price may exceed the forward price.' In spite of the fact that we do not have an explicit market for future products in our simple model, at least we could try to answer the question of what happens in two extreme situations, the first is when we have a "liquid capital" and the second a shortage of the product, say the stock level is such that $x(y) = 0$. In the first we will have that no backwardation can occur since whatsoever is the result of the yield a reduction in the stock held will occur; and as a consequence an increase in the "future price." On the other hand when we have that $x(y) = 0$ we always have a decrease (or at least it does not increase) of the "future price" no matter what is the result of the crop yield. Still left is the important question of "normal backwardation," that we hope might be explained as soon as we introduce in our model an explicit place for future market.

BIBLIOGRAPHY

[1] Gale, D., "Nonlinear Duality and Qualitative Properties of Optimal Growth," in INTEGER AND NONLINEAR PROGRAMMING, J. Abadie, ed., North-Holland, Amsterdam, (1970).

[2] Karlin, S., "Dynamic Inventory Policy with Varying Demands," in MATHEMATICS IN MANAGEMENT SCIENCE, A. Veinott, ed., The MacMillan Co., New York, (1965).

[3] Keynes, J. M., A TREATISE ON MONEY, VOL. II: THE APPLIED THEORY OF MONEY, MacMillan and Co., London, (1930).

[4] Samuelson, P. A., "Intertemporal Price Equilibrium: A Prologue to the Theory of Speculation," in the Selected Scientific Paper of P. A. Samuelson, edited by Stiglitz, Vol. 2, pp. 946 sgg.

[5] Schechtman, J., "Competitive Prices, Dynamic Programming Under Uncertainty, A Nonstationary Case," ORC 76-19, Operations Research Center, University of California, Berkeley, (June 1976).

[6] Schechtman, J., "An Income Fluctuation Problem," Journal of Economic Theory, Vol. 12, pp. 218-241, (1976).

[7] Schechtman, J. and A. Cortines Peixoto Filho, "Utilizacao de Politicas Competitivas na Determinacao de Solvcoes Otimas para Problemas de Programacao Dinamica com Incertexa," IV, Simposio Brasileiro de Pesquisa Operacional, (In Portuguese), Rio de Janeiro, Brazil, (June 1975).

[8] Williams, John B., "Speculation and the Carryover," The Quarterly Journal of Economics, Vol. L, pp. 436 sgg., Cambridge, (1935-36).

[9] Working, H., "The Theory of Price of Storage," The American Economic Review, Vol. XXXIX, pp. 1254 sgg., (1949).