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ADAPTIVE ARRAY PERFORMANCE WITH SPATIALLY DISPERSED INTERFERENCE

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TECHNICAL REPORT 3832-1

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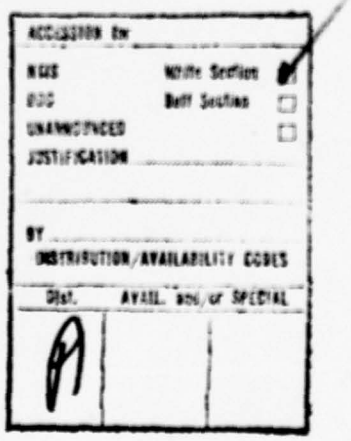

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In this report we discuss adaptive array performance when a continuous sector of interference is incident upon the array. A model is developed for a two-element array with an arbitrary number of incident signals. From this model the array weights are derived and formulas for the array performance are developed. It is then shown how this model is readily extended to the case of a continuous interference sector.

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From these analytical results, the array performance is then examined for specific cases of spatially dispersed interference. It is seen that the array performance is not always degraded as the interference angular sector becomes larger.

ABSTRACT

In this report we discuss adaptive array performance when a continuous sector of interference is incident upon the array. A model is developed for a two-element array with an arbitrary number of incident signals. From this model the array weights are derived and formulas for the array performance are developed. It is then shown how this model is readily extended to the case of a continuous interference sector.

From these analytical results, the array performance is then examined for specific cases of spatially dispersed interference. It is seen that the array performance is not always degraded as the interference angular sector becomes larger.

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The material contained in this report is also used as a thesis submitted to the Department of Electrical Engineering, The Ohio State University as partial fulfillment for the degree Master of Science.

CONTENTS

I. INTRODUCTION

Signal processing antenna arrays have been studied for many years. For receiving arrays used in communication and radar systems, it is important that the output signal to noise ratio (SNR) be optimized. It is highly desirable that this optimization be automatic. That is, for each change in the antenna or signal environment (e.g., change in direction of arrival of the signals, change in electronic components, etc.), the system should modify itself until the SNR is again optimum. One method of achieving this modification is through internal feedback. Antenna systems of this type are called adaptive arrays.

Shor [1] was one of the first to suggest an adaptive process which maximized the SNR of an array of hydraphones. Widrow [2] et al suggested a feedback technique which minimized the mean-square error between the array output and a reference signal. Applebaum [3] and Griffiths [4] discussed similar concepts. An early experimental adaptive array was built by Riegler and Compton [5].

The behavior of an adaptive array with spread spectrum communication signals has been studied by Reinhard, Huff, Compton and others [7,8,10,11,13,14]. Berni [9] has suggested a method of angle of arrival estimation using an adaptive array. A four-element array capable of arrival angle estimation and sensor communications has been implemented and experimental results are described by Swarner and Berni [15].

In this report we study the performance of adaptive arrays based on the LMS algorithm [2,3,4] when interference is incident on the array from an angular sector of finite width. Previous studies have concentrated on the case of interference signals that arrive from a single direction is space. We begin by assuming a large number of interference signals to be incident on the array (many more than degrees of freedom of the array). The signals are assumed to be uncorrelated and to arrive from different directions within a certain sector. The array performance is studied as a function of the size of the sector. The case of discrete interfering signals is generalized to the case of radiation arriving from a continuous sector.

In section II certain mathematical preliminaries are investigated. By analyzing a one-element array, the effect of time varying coefficients in the differential equations for the weights is studied. It is found that under certain conditions these time varying components may be neglected without greatly affecting the weight solutions. In section III the general solution for the weights of a two-element array with an arbitrary number of incident CW signals is developed. In sections IV and V this solution is used to study the system performance. In section VI the case of many discrete signals is extended to a continuous column of inpinging radiation. The weight solutions are derived. Finally, in section VII numerical examples of array performance in a many-signal environment are presented.

II. THE FEEDBACK ALGORITHM AND THE ARRAY STRUCTURE

An N-element antenna array with quadrature weighting of the input signals is shown in Fig. 1. In this figure the quadrature signals from the first antenna are denoted $x_1(t)$ and $x_2(t)$, those from the second are $x_3(t)$ and $x_4(t)$, and so forth. The weights associated with these 2N signals (N antenna elements) are similarly indexed. The array output may be written as

$$(1) \qquad s(t) = \sum_{i=1}^{2N} w_i x_i(t) \ .$$

We define $y_i(t)$ to be the signal incident on the i-th antenna. $x_i(t)$ will be referred to as the element signal. The error signal is

$$(2) \qquad \epsilon(t) = R(t) - \sum_{i=1}^{2N} w_i x_i(t)$$

where R(t) is the reference signal. Realistically, it must be assumed that the desired signal contains modulation components that are unknown at the receiver. Hence, the reference signal cannot be made exactly equal to the desired part of the incoming signal, but can only approximate it in some sense. For correct operation in an interference rejection system, it is necessary to generate a reference signal which resembles the desired signal and correlates poorly with interference [14,25]. The array will then act to drive the error signal to zero in the mean-square sense. Such interference rejection is achieved by placing spatial nulls in the direction of interfering

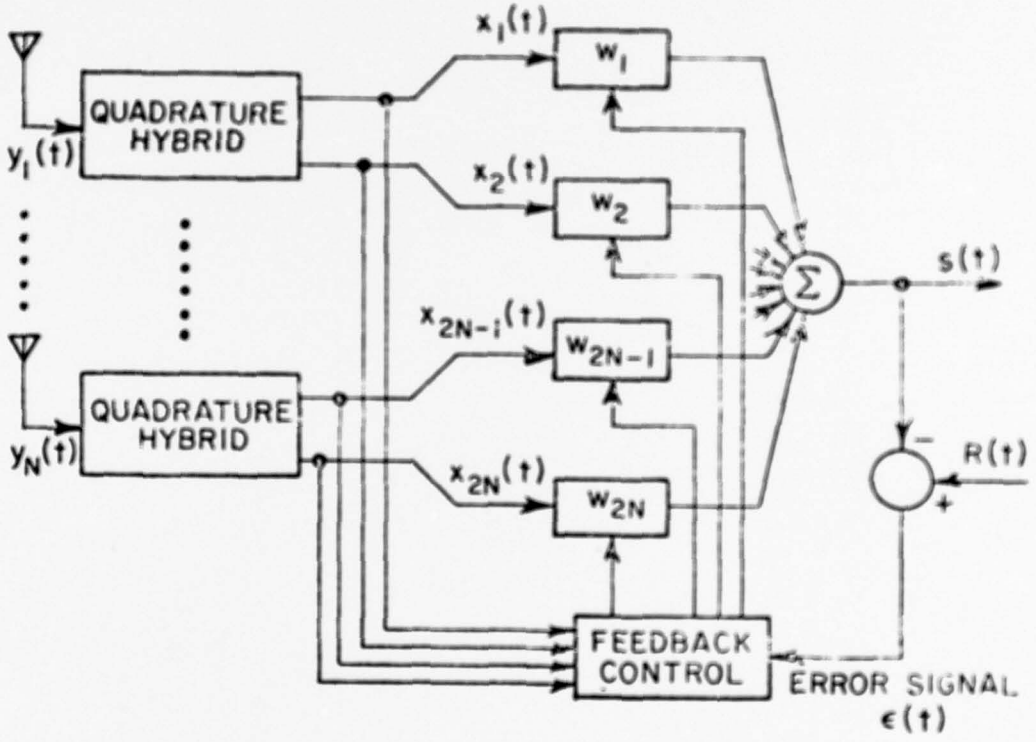


Fig. 1.--Adaptive array processing.

transmissions. One way to obtain a reference signal is to derive it from the array output. Certain difficulties may arise, however, in attempting to derive the reference signal from the array output. Some of these are discussed in detail in [8,14,25].

From Eq. (2) the squared error becomes

$$(3) \qquad \varepsilon^2(t) = R^2(t) - 2R(t) \sum_{i=1}^{2N} w_i x_i(t) + \sum_{i=1}^{2N} \sum_{j=1}^{2N} w_i w_j x_i(t)\, x_j(t).$$

The mean-square error is thus:

$$(4) \qquad \varepsilon^2(t) = R^2(t) - 2 \sum_{i=1}^{2N} w_i \overline{R(t)x_i(t)} + \sum_{i=1}^{2N} \sum_{j=1}^{2N} w_i w_j \overline{x_i(t)x_j(t)}$$

where the overbar represents the action of a low-pass filter as will be discussed later in this section.

Differentiating Eq. (4) with respect to w_i yields

$$(5) \qquad \nabla_{w_i} [\overline{\varepsilon^2(t)}] = \overline{\frac{\partial \varepsilon^2(t)}{\partial w_i}} = -2\overline{x_i(t)\varepsilon(t)} .$$

The feedback system is based on the so called LMS algorithm [2,3,4]. Each weight is controlled by the relation

$$(6) \qquad \frac{dw_i}{dt} = - k\nabla_{w_i} [\overline{\varepsilon^2(t)}] .$$

Then from Eq. (5), the feedback equation becomes

$$(7) \qquad \frac{dw_i}{dt} = 2k\,\overline{\epsilon(t)x_i(t)}$$

which leads to the feedback structure shown in Figs. 2a and 2b. When Eq. (2) is used to substitute for $\epsilon(t)$ in Eq. (7), and all the terms involving w_i are collected on the left, it is found that the weights satisfy the system of differential equations given by

$$(8) \qquad \frac{dw_i}{dt} + 2k \sum_{j=1}^{2N} \overline{[x_i(t)x_j(t)]}w_j = 2k\overline{[R(t)x_i(t)]} \; .$$

The system of Eq. (8) may be solved (under certain simplifying conditions) to yield the time response of the weighting coefficients $w_i(t)$. From the weight solutions, the performance of the system under various conditions may be studied.

Before examining the method of solution we discuss the meaning of the overbar. We have stated previously that the overbar represents the action of an ideal low-pass filter. For example, if $x_i(t)$ and $R(t)$ in Eq. (4) each contain two CW signals of different frequency, their product will contain d.c. terms along with components at the sum and difference frequencies. Typically, multipliers used in an adaptive array will not pass the sum frequencies. For example, in one array implemented at Ohio State [15], the array processing was done at 70 M-Hz. Trans-conductance multipliers were used, which did not pass frequencies higher than 100 M-Hz. Therefore, sum frequency terms (at 140 M-Hz) were not passed.

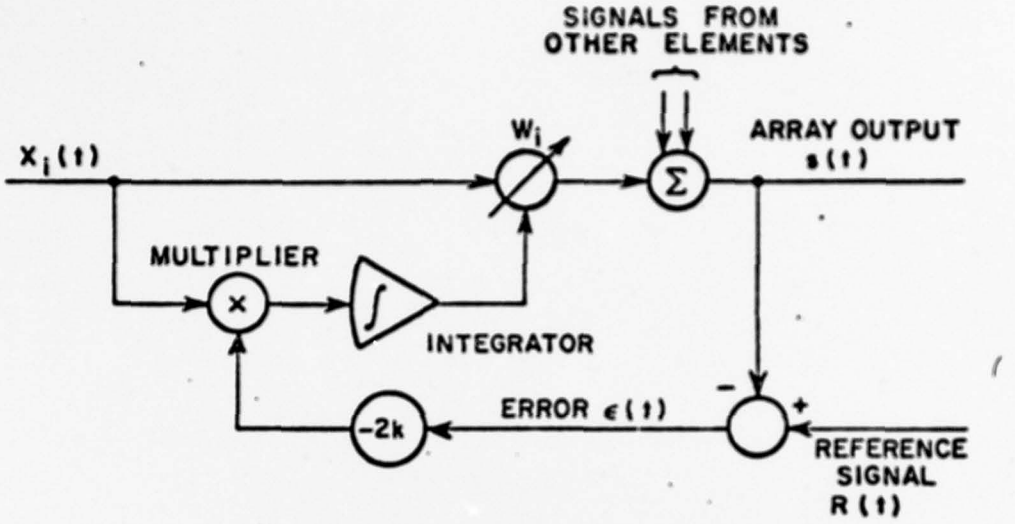


Fig. 2a--Basic feedback loop.

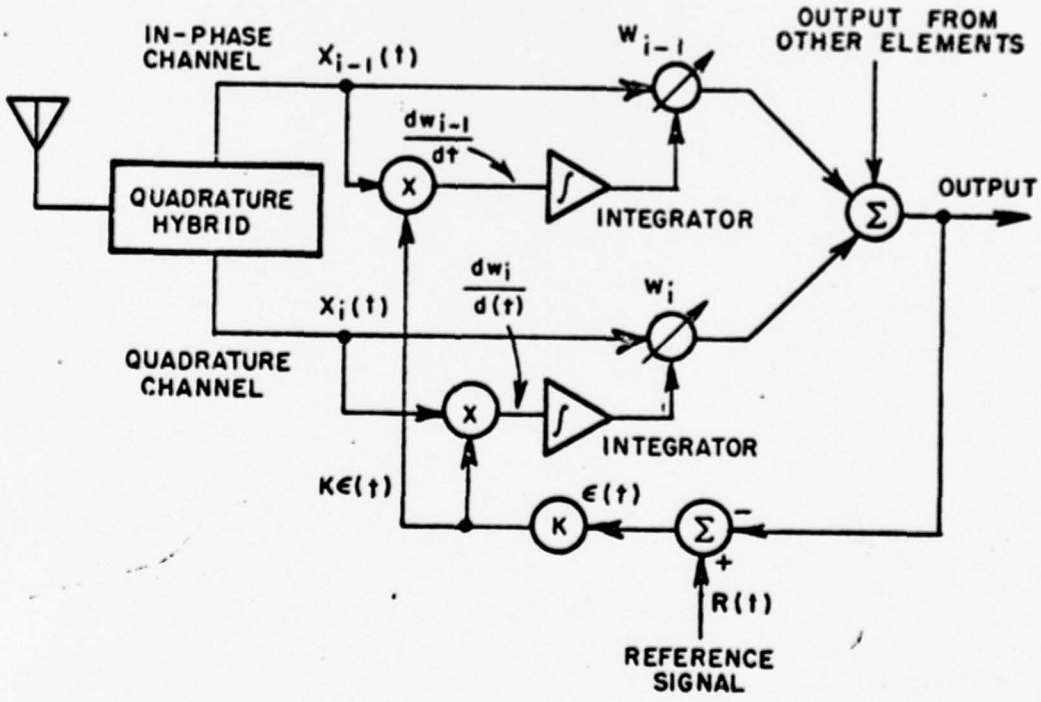


Fig. 2b--Feedback loop for each element.

The difference frequency terms, however, were within the pass-band. In previous studies of adaptive arrays with CW signals, the effects of these difference frequency terms have been neglected[5-11,13,15,16,21-25]. In this section, we discuss the conditions under which these terms may be neglected without affecting the solutions of the weights. It will be shown that there exists a cutoff frequency beyond which the effect of these difference terms may be neglected. It will also be demonstrated that the power contained in frequency components below this cutoff frequency will be small compared to the power in the d.c. component. These conditions will allow us to neglect all but the d.c. terms in Eq. (8).

Let us suppose that a one-element array has two CW signals incident upon it. The signal is

$$y(t) = \sqrt{2}\, a_1 \cos(\omega_1 t) + \sqrt{2}\, a_2 \cos(\omega_2 t). \tag{9}$$

We assume ideal reference signal processing is available (i.e., the reference signal is perfectly coherent with the desired signals and is unaffected by the weights), and the reference signal R(t) is given by

$$R(t) = \cos(\omega_1 t) + \cos(\omega_2 t). \tag{10}$$

In other words, R(t) is coherent with both signals (i.e., we have chosen both signals to be desired). Upon quadrature splitting we

obtain for the element signals

$$x_1(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t) \tag{11}$$

and

$$x_2(t) = a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t). \tag{12}$$

Since the overbar represents an ideal low-pass filter with a cutoff frequency higher than $\omega_1-\omega_2$ but lower than $2\omega_1$ and $2\omega_2$, the terms $\overline{x_i(t)x_j(t)}$ are

$$\overline{x_1(t)x_1(t)} = \frac{a_1^2 + a_2^2}{2} + a_1 a_2 \cos[(\omega_1-\omega_2)t] \tag{13}$$

$$\overline{x_2(t)x_2(t)} = \frac{a_1^2 + a_2^2}{2} + a_1 a_2 \cos[(\omega_1-\omega_2)t] \tag{14}$$

and

$$\overline{x_1(t)x_2(t)} = \overline{x_2(t)x_1(t)} = 0. \tag{15}$$

We also find

$$\overline{R(t)x_1(t)} = \frac{a_1 + a_2}{2} + \frac{a_1 + a_2}{2} \cos[(\omega_1-\omega_2)t]. \tag{16}$$

and

$$(17) \qquad \overline{R(t)x_2(t)} = -\frac{a_1 + a_2}{2}\sin[(\omega_1-\omega_2)t] \ .$$

When Eqs. (13) - (17) are substituted into Eq. (8), the differential equations describing the weights can be written in matrix form as

$$(18) \quad \frac{d}{dt}\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} + 2k \begin{pmatrix} \frac{a_1^2+a_2^2}{2} + a_1a_2\cos(\Delta\omega t) & 0 \\ 0 & \frac{a_1^2+a_2^2}{2} + a_1a_2\cos(\Delta\omega t) \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} =$$

$$k \begin{pmatrix} a_1+a_2 + (a_1+a_2)\cos(\Delta\omega t) \\ -(a_1+a_2)\sin(\Delta\omega t) \end{pmatrix},$$

where

$$(19) \qquad \Delta\omega = \omega_1-\omega_2$$

Let us examine the equation for weight $w_1(t)$. We notice that the system of Eqs. (18) is uncoupled, so we can solve for the weights independently of one another. The equation describing the reponse of the in-phase weight ($w_1(t)$) is then

$$(20) \qquad \frac{dw_1(t)}{dt} + [A + B\cdot\cos(\Delta\omega t)]w_1(t) = [C + D\cdot\cos(\Delta\omega t)]\ u(t)$$

where

(21) $$A = k \cdot (a_1^2 + a_2^2)$$

(22) $$B = 2 \cdot k \cdot a_1 \cdot a_2$$

(23) $$C = k \cdot (a_1 + a_2)$$

(24) $$D = k \cdot (a_1 + a_2)$$

and u(t) is the unit step function. We arbitrarily choose the initial value of $w_1(t)$ to be zero at t = 0-. The Fourier transform of $w_1(t)$ exists and Eq. (20) may be written in the frequency domain as

(25) $$(j\omega) \cdot W_1(\omega) + A \cdot W_1(\omega) + \frac{B}{2} \cdot [W_1(\omega+\Delta\omega) + W_1(\omega-\Delta\omega)] =$$
$$C \cdot [\pi \cdot \delta(\omega) + \frac{1}{j\omega}] + \frac{D\pi}{2} \cdot [\delta(\omega-\Delta\omega) + \delta(\omega+\Delta\omega)] + \frac{j\omega}{(\Delta\omega)^2-\omega^2}$$

where $W_1(\omega)$ is the Fourier transform of $w_1(t)$, $\delta(\omega)$ is the Dirac delta function and

(26) $$j = \sqrt{-1}.$$

Solving Eq. (25) for $W_1(\omega)$ and taking the inverse Fourier transform shows

(27) $$W_1(t) = [e^{-At} \cdot u(t)] * [C \cdot u(t)] + [e^{-At} \cdot u(t)] * [D \cdot \cos(\Delta\omega t) \cdot u(t)]$$

$$- [B \cdot w_1(t) \cdot \cos(\Delta\omega t)] * [e^{-At} \cdot u(t)]$$

where "*" represents convolution in the time domain, defined by

(28) $$f(t) * g(t) = \int_{-\infty}^{\infty} f(t-\tau) \cdot g(\tau)d\tau = \int_{-\infty}^{\infty} f(\tau) \cdot g(t-\tau)d\tau = F(\omega) \cdot G(\omega).$$

The first two convolutions of Eq. (27) are easily performed. The third may be written in integral form yielding as an expression for $w_1(t)$:

(29) $$w_1(t) = \frac{C}{A} - \frac{C}{A} e^{-At} + \frac{A \cdot D \cdot e^{-At}}{A^2+(\Delta\omega)^2} + \frac{D \cdot [A \cdot \cos(\Delta\omega t)+(\Delta\omega) \cdot \sin(\Delta\omega t)]}{A^2 + (\Delta\omega)^2}$$

$$-B \cdot \int_{-\infty}^{t} e^{-A(t-\tau)} \cdot \cos(\Delta\omega\tau) \cdot w_1(\tau)d\tau \ .$$

We can generate an asymptotic expansion for $w_1(t)$ valid for large $(\Delta\omega)$. Continually integrating the final term of Eq. (29) by parts will yield a solution of the form

$$(30)\quad B \cdot \int_{-\infty}^{t} e^{-A(t-\tau)} \cdot \cos(\Delta\omega\tau) \cdot w_1(\tau) \cdot d\tau = \frac{a_1(t,\Delta\omega)}{A^2 + (\Delta\omega)^2} \cdot w_1(t) + \frac{a_2(t,\Delta\omega)}{(A^2+(\Delta\omega)^2)^2} \cdot w_1(t)$$

$$+ \frac{a_3(t,\Delta\omega)}{(A^2+(\Delta\omega)^2)^3} \cdot w_1(t) + \cdots \ .$$

When this result is used in Eq. (29), the solution for $w_1(t)$ may be approximated by the first term of the series if $\Delta\omega$ is high enough.

Integrating Eq. (30) by parts we find

$$(31)\ B \cdot \int_{-\infty}^{t} e^{-A(t-\tau)} \cdot \cos(\Delta\omega\tau) \cdot w_1(\tau)d\tau = \frac{[A \cdot B \cdot \cos(\Delta\omega t) + B \cdot (\Delta\omega) \cdot \sin(\Delta\omega t)]}{A^2 + (\Delta\omega)^2} \cdot w_1(t)$$

$$- \frac{B \cdot e^{-At}}{A^2+(\Delta\omega)^2} \cdot \int_{-\infty}^{t} e^{A\tau}(A \cdot \cos(\Delta\omega\tau) + (\Delta\omega) \cdot \sin(\Delta\omega\tau))\dot{w}_1(\tau)d\tau \ .$$

Recall from Eq. (20) that

(32) $\dot{w}_1(\tau) = [C+D\cdot\cos(\Delta\omega\tau)]u(\tau) - [A+B\cdot\cos(\Delta\omega\tau)] \cdot w_1(\tau).$

Substituting for $\dot{w}(\tau)$ into the last integral of Eq. (31) only those terms of Eq. (32) which contribute to $a_1(t,\Delta\omega)$ (i.e., $\dot{w}_1(\tau) = D\cos(\Delta\omega\tau)\,u(t) - B\cos(\Delta\omega\tau)\,w_1(\tau)$) yields

(33) $$\frac{Be^{-At}}{A^2+(\Delta\omega)^2}\int_{-\infty}^{t} e^{A\tau}(A\cdot\cos(\Delta\omega\tau)+(\Delta\omega)\cdot\sin(\Delta\omega\tau))\cdot\dot{\omega}_1(\tau)d\tau =$$

$$\frac{B\cdot e^{-At}}{A^2+(\Delta\omega)^2}\int_0^t \frac{e^{A\tau}}{2} \cdot (AD-A\cdot B\cdot w_1(\tau))d\tau \ .$$

Continued integration by parts of Eq. (34) again results in integrals containing $\dot{w}_1(\tau)$. Substituting in each case for $\dot{w}_1(\tau)$ only those terms of Eq. (32) which contribute to $a_1(t,\Delta\omega)$ enables us to write Eq. (33) as

(34) $$\frac{B\cdot e^{-At}}{A^2+(\Delta\omega)^2}\int_0^t \frac{e^{A}}{2} \ (A\cdot D-A\cdot B\cdot w_2(\tau))d\tau = \frac{B\cdot D}{A^2+(\Delta\omega)^2}\ (1-e^{-At})$$

$$\frac{-B\cdot w_1(t)}{A^2+(\Delta\omega)^2} + \frac{B\cdot C\cdot t\cdot e^{-At}}{A^2+(\Delta\omega)^2} \ .$$

We can then substitute the result of Eq. (34) for the integral expression in Eq. (31). Upon substituting Eq. (31) into Eq. (29), we find

$$(35)\quad w_1(t) = \frac{C}{A} - \frac{C}{A}e^{-At} + \frac{A\cdot D\cdot e^{-At}}{A^2+(\Delta\omega)^2} + \frac{D\cdot(A\cdot\cos(\Delta\omega t)+(\Delta\omega)\cdot\sin(\Delta\omega t))}{A^2+(\Delta\omega)^2}$$

$$- \frac{(A\cdot B\cdot\cos(\Delta\omega t)+B\cdot(\Delta\omega)\cdot\sin(\Delta\omega t))\cdot w(t)}{A^2+(\Delta\omega)^2} - \frac{B\cdot D\cdot(1-e^{-At})}{A^2+(\Delta\omega)^2}$$

$$+ \frac{B^2\cdot w_1(t)}{A^2+(\Delta\omega)^2} - \frac{B\cdot C\cdot t\cdot e^{-At}}{A^2+(\Delta\omega)^2} .$$

We can now solve Eq. (35) for $w_1(t)$. After a long period of time (i.e., after the transients have died out), we obtain for $w_1(t)$:

(36)

$$w_1(t) \underset{t\to\infty}{\sim} \frac{C}{A} + \frac{A\cdot D\cdot\cos(\Delta\omega t)+D\cdot(\Delta\omega)\cdot\sin(\Delta\omega t)-C\cdot B\cdot\cos(\Delta\omega t) - \frac{B\cdot(\Delta\omega)\cdot C}{A}\cdot\sin(\Delta\omega t)}{A^2+(\Delta\omega)^2+A\cdot B\cdot\cos(\Delta\omega t)+B\cdot(\Delta\omega)\cdot\sin(\Delta\omega t)+B^2}$$

The first term on the right side of Eq. (36) is the steady-state term which would occur if $B\cdot\cos(\Delta\omega t)$ and $D\cdot\cos(\Delta\omega t)$ were neglected in Eq. (20). Eq. (36) is bounded by

$$(37)\quad \lim_{t\to\infty} w_1(t) < \frac{C}{A} + \frac{A\cdot D + D\cdot(\Delta\omega) - C\cdot B - B\cdot(\Delta\omega)\cdot C/A}{A + (\Delta\omega) + B - AB - B\cdot(\Delta\omega)} \quad .$$

In other words we have replaced the sin $(\Delta\omega t)$ and $\cos(\Delta\omega t)$ terms with unity in the numerator and minus one in the denominator. From Eq. (37) we notice that the second term will be small compared to C/A when

$$(38)\quad \Delta\omega >> \frac{A\cdot D}{C} = k\cdot(a_1^2 + a_2^2) = \omega_o .$$

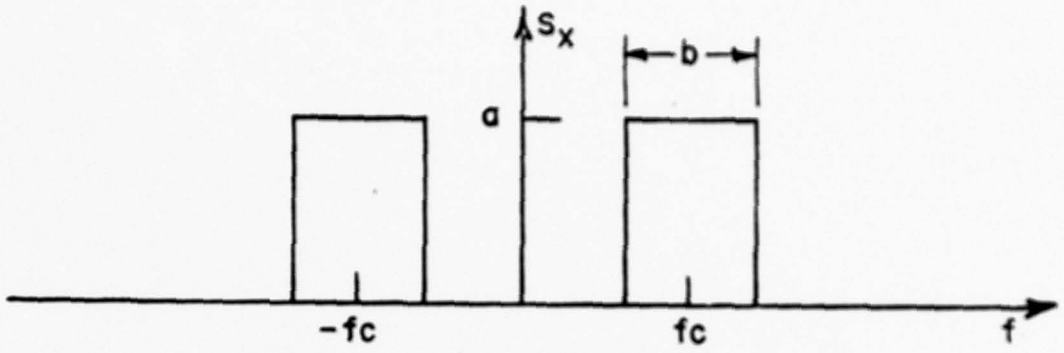
In other words, when the difference frequency $\Delta\omega$ is much larger than the product of the loop gain constant (k) and the sum of the signal powers, the steady-state solution for Eq. (20) will be approximately C/A. We see then, that when the inequality given in Eq. (38) is satisfied, the difference frequency terms ($B\cdot\cos(\Delta\omega t)$ and $D\cdot\cos(\Delta\omega t)$) in Eq. (20) may be neglected. For example, in one adaptive array built at Ohio State [15] , the value of ω_o was typically between 100 Hz and 1 k-Hz. In this array, difference frequencies greater than 1k-Hz did not affect the weight solution.

In an adaptive array for communications, the signal $x_i(t)$ will be a bandlimited process centered at some non-zero carrier frequency. (For example, the Ohio State adaptive array [15] operated at 70 M-Hz with a 10 M-Hz bandwidth.) It will be adequate for the present discussion to assume that $x_i(t)$ has a flat power spectral density of

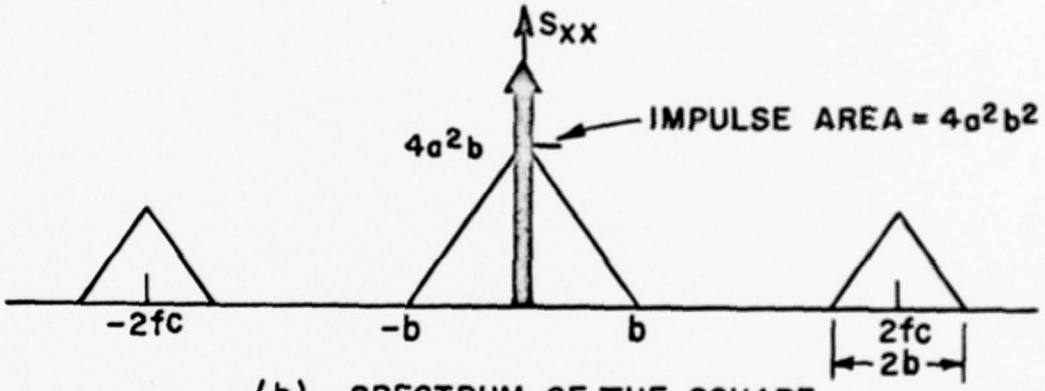
height a and bandwidth b, as shown in Fig. 3a. The power spectral density of $x_i^2(t)$ will then be as shown in Fig. 3b* [26]. The multipliers remove the frequencies centered about $2f_c$. Hence the signal $\overline{x_i^2(t)}$ has spectral density shown in Fig. 3c.

Thus, each term $\overline{x_i(t)\ x_j(t)}$ in Eq. (8) contains a d.c. component (the impulse at $\omega = 0$ in Fig. 3c) and a time-varying component (the power contained in the continuous spectrum from 0 to b in Fig. 3c). From the asymptotic solution to Eq. (8) given in Eq. (37), however, it is clear that only those frequency components of $\overline{x_i(t)\ x_j(t)}$ that lie below the cutoff frequency ω_o will have an effect on the solution to Eq. (8). We may safely ignore all frequency components of $\overline{x_i(t)\ x_j(t)}$ above ω_o. Returning to the Ohio State adaptive array as an example (with 10 M-Hz bandwidth), we see that the time-varying portion of $\overline{x_i(t)\ x_j(t)}$ will have baseband frequency components from 0 to 10 M-Hz. The feedback loop bandwidth (ω_o) is, however, only 1 k-Hz. Thus, the total power in the time-varying part of $\overline{x_i(t)\ x_j(t)}$ is only about 10^{-4} of the d.c. power. The part from 0 to 1 k-Hz need not be included in the differential equations (8). Referring to the one-dimensional differential equation of Eq. (20), we see that the situation is equivalent to having $B \ll A$. In this case, we may safely neglect the B term in constructing the solution to Eq. (20).

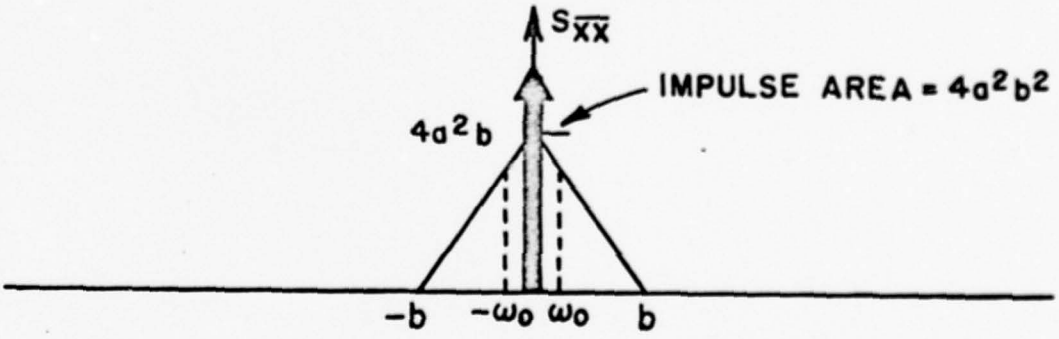
*In general $\overline{x_j(t)} = x_i(t-\tau_j)$, where τ_j is some time delay. Thus, $x_i(t)\ x_j(t) = x_i(t)\ x_i(t-\tau_j)$. For this discussion, we assume $\tau_j = 0$, so $x_i(t)\ x_j(t) = x_i^2(t)$. The value of τ_j is not important for the arguments advanced here, which are only qualitative.



(a) INPUT SIGNAL SPECTRUM



(b) SPECTRUM OF THE SQUARE



(c) MULTIPLIER OUTPUT SPECTRUM

Fig. 3.

Thus we will solve Eq. (8) by including only the constant (d.c.) terms of the products $\overline{x_i(t)\ x_j(t)}$. To reiterate the arguments above, the validity of this procedure rests on two observations. First, the solutions to Eq. (8) exhibit a "cutoff frequency" effect. Time-varying terms in the products $\overline{x_i(t)\ x_j(t)}$ whose frequencies are higher than this cutoff do not affect the solution. Second, in practical array designs for communication systems, the value of this cutoff frequency is very small compared to the RF bandwidth. As a result, the power contained in the time-varying part of $\overline{x_i(t)\ x_j(t)}$ below the cutoff is extremely small compared to the power in the constant term. For this reason, we may neglect the time-varying portion of $\overline{x_i(t)} \cdot \overline{x_j(t)}$ entirely in solving Eq. (8).

When the element signals are random, the multiplier output is the d.c. portion of the product of two random processes. We note that this quantity is the same as the infinite time average of the product of the two processes.

Having shown that only the constant part of $x_i(t)\ x_j(t)$ needs to be included in Eq. (8), we further note that the constant part may be obtained from the infinite time average of $x_i(t)\ x_j(t)$. Moreover, when all $x_i(t)$ are assumed to be ergodic processes, the time average may be replaced by an ensemble average. We now return to the discussion of Eq. (8). If we define the matrices

$$(39)\quad \Phi = \begin{pmatrix} \overline{x_1(t)x_1(t)} & \overline{x_1(t)x_2(t)} & \cdots \\ \overline{x_2(t)x_1(t)} & & \\ \vdots & & \\ \vdots & & \\ \vdots & & \end{pmatrix},$$

$$(40)\quad S = \begin{pmatrix} \overline{R(t) \cdot x_1(t)} \\ \overline{R(t) \cdot x_2(t)} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

and

$$(41)\quad W = \begin{pmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ \vdots \\ \vdots \end{pmatrix},$$

then Eq. (8) can be written in matrix form as

$$(42)\quad \frac{dW}{dt} + 2k\Phi\, W = 2kS.$$

Let us consider the response of Eq. (42). When there is more than one antenna element, the system of equations will, in general, be coupled in Φ. In order to solve this system, we first make a

rotation of coordinates into the principal axis of Φ. Let

$$W = Rn \tag{43}$$

where R is a 2N x 2N orthogonal coordinate rotation matrix,

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots \\ r_{21} & & \\ \vdots & & \\ \vdots & & \end{pmatrix} \tag{44}$$

and

$$n = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \vdots \end{pmatrix} \tag{45}$$

represent a new system of coordinates for the weights. By substituting Eq. (43) into Eq. (42) and multiplying on the left by R^{-1}, Eq. (42) becomes

$$\frac{d\eta}{dt} + 2k[R^{-1}\Phi R]\eta = 2kR^{-1}S \; . \tag{46}$$

If R is chosen so that $R^{-1}\Phi R$ is diagonal,

$$(47)\quad R^{-1}\Phi R = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \\ 0 & 0 & \lambda_3 & \\ \cdot & & \ddots & \\ \cdot & & & \ddots \\ \cdot & & & \ddots \end{pmatrix}$$

then the components of η lie along the principle axes of Φ and the system of equations is uncoupled. We define

$$(48)\quad P = R^{-1}S = \begin{pmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

and Eq. (41) becomes simply

$$(49)\quad \frac{d\eta}{dt} + 2k\Lambda\eta = 2kP \ .$$

We refer to the components of η as the "normal weights" of the array.

The form of the general solution to Eq. (49) depends on the matrix Φ. Since Φ is real and symmetric, its eigenvalues are necessarily real. Furthermore, Φ is non-negative definite [21]. Since none of the eigenvalues of Φ can be negative, the solutions to Eq. (49) will not contain any exponentially growing terms. Furthermore, none of the eigenvalues can be zero when there is element noise in the array [21]. By "element noise" we mean random noise due to RF components behind each element of the array. This type of noise is incoherent from one channel to the next. Element noise

should not be confused with a directional noise signal received by the array; such noise would be highly correlated between elements.

When element noise is present, the element signals (following the quadrature hybrids) are of the form

$$x_i(t) = n_i(t) + s_i(t) \tag{50}$$

where $n_i(t)$ is the noise component and $s_i(t)$ is the received signal component of $x_i(t)$ (see Figs. 2). When this $x_i(t)$ is substituted into Eq. (39), Φ is found to be

$$\Phi = \Phi_{s+n} = \sigma_n^2 I + \Phi_s \tag{51}$$

where we use Φ_{s+n} to denote Φ when both signal and noise are present and Φ_s when only signal is present. σ_n^2 denotes the mean-square value of $n_i(t)$:

$$\overline{n_i^2(t)} = \sigma_n^2 \tag{52}$$

(we assume all $n_i(t)$ have the same mean-square value), and I denotes the identity matrix. To derive Eq. (52), we have made use of the assumption that

$$\overline{n_i(t)n_j(t)} = 0 \qquad \text{for } i \neq j, \tag{53}$$

and

$$(54)\quad \overline{n_i(t)s_j(t)} = 0 \qquad \text{for all } i,j.$$

Since the matrix $\sigma_n^2 I$ is unaffected by a transformation of the type $R^{-1}(\sigma_n^2 I)R = \sigma_n^2 I$, the same orthonormal matrix which diagonalizes Φ_S will also diagonalize Φ_{S+n}. Hence, each eigenvalue of Φ_{S+n} must be equal to σ_n^2 plus the corresponding eigenvalue of Φ_S.

The form of Φ, the eigenvalues and the rotation matrix will also be dependent upon the number of signals incident on the array. If we assume the incoming signals to be uncorrelated with one another and the element signals to be of the form

$$(55)\quad x_i(t) = n_i(t) + \sum_{j=1}^{n} s_{ij}(t)$$

then Φ_{S+n} will be

$$(56)\quad \Phi_{S+n} = \sigma_n^2 I + \sum_{j=1}^{n} \Phi_{Sj}$$

where Φ_{S_j} is the Φ_S when only signal $s_j(t)$ is incident on the array. Since the rotation matrix R will be made up of the eigenvectors of Φ_{S+n}, its determination will become increasingly difficult as the number of signals increases. A method of determining R and Λ for an arbitrary number of signals is presented in Appendix I. We will,

however, in the next section present a transformation which will allow solutions for the weights to be determined and the order of the system reduced by a factor of 2 with an arbitrary number of signals present.

III. THE GENERAL SOLUTION

The two-element adaptive array behaves in a manner similar to higher order systems in that it has pattern flexibility. It is therefore capable of suppressing directional interference signals and its anti-jam performance may be studied. In this section we develop the general weight solutions for a two-element adaptive array assuming CW signals and element noise are present at the array input.

The system of equations describing the response of a two-element adaptive array is from Eq. (42),

$$(57)\quad \frac{d}{dt}\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{pmatrix} + 2k\,\Phi\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{pmatrix} = 2k\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} .$$

When the signals incident on the array are uncorrelated, it can be shown that Φ will be of the form

$$(58)\quad \Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ -\phi_{12} & \phi_{11} & -\phi_{14} & \phi_{13} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ -\phi_{32} & \phi_{31} & -\phi_{34} & \phi_{33} \end{pmatrix} .$$

It has been demonstrated by Compton [21] that when Φ exhibits the type of symmetry of Eq. (58), the system of Eq. (57) can be reduced to

the form

$$(59)\quad \frac{d}{dt}\begin{pmatrix} w_1-jw_2 \\ w_3-jw_4 \end{pmatrix}+2k\begin{pmatrix} \phi_{11}+j\phi_{12} & \phi_{13}+j\phi_{14} \\ \phi_{31}+j\phi_{32} & \phi_{33}+j\phi_{34} \end{pmatrix}\begin{pmatrix} w_1-jw_2 \\ w_3-jw_4 \end{pmatrix} = 2k\begin{pmatrix} s_1-js_2 \\ s_3-js_4 \end{pmatrix}$$

(we will use $w_i(t)$ and w_i interchangably throughout). We have then formed a set of differential equations in terms of complex weighting coefficients. Recall that w_1 and w_3 are the in-phase weights and w_2 and w_4 the quadrature weights of the array. The quadrature channel represents a 90^o phase delay over all frequencies. This quarter cycle delay is symbolized mathematically by the operator -j.

The general configuration of a two-element adaptive array is shown in Fig. 4. Let us suppose these are n CW signals incident upon this array from angles θ_i off braodside. We will assume m of these signals to be desired and (n-m) to be directional CW jammers. This situation is illustrated for n=2, m=1 in Fig. 5.

We define

$$(60)\quad \tilde{w}_1 = w_1-jw_2$$

and

$$(61)\quad \tilde{w}_2 = w_3-jw_4$$

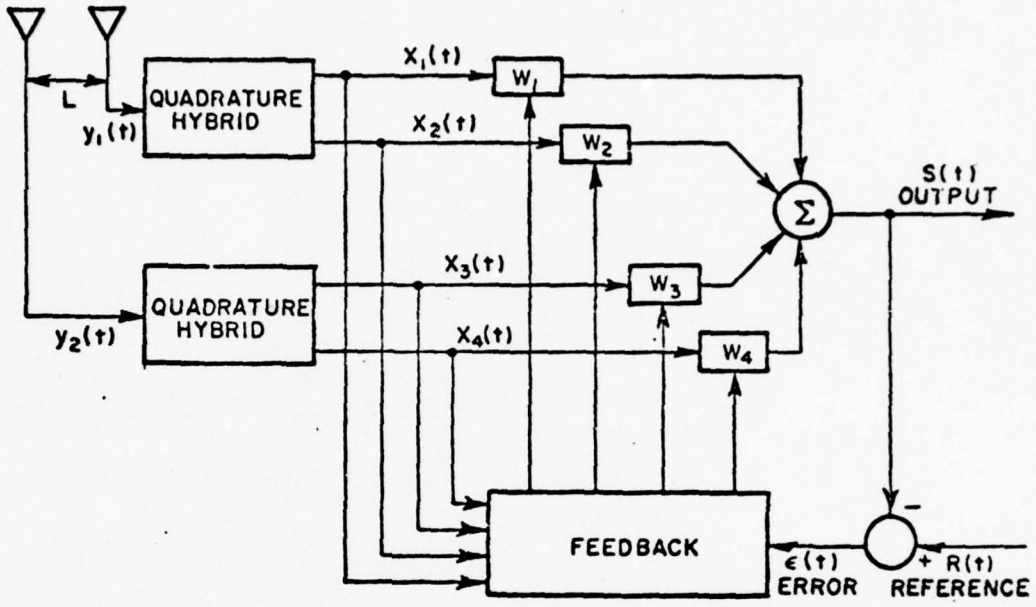


Fig. 4--Two element array.

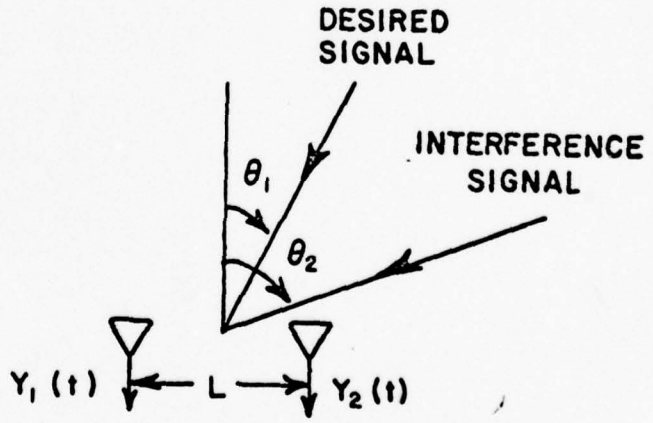


Fig. 5--Signal angles in space.

where "~" indicates a complex variable. We also denote the column vector

$$(62)\quad \tilde{W} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} .$$

The complex correlation matrix $\tilde{Q}$, will be defined as

$$(63)\quad \tilde{Q} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

where

$$(64)\quad \tilde{A} = \phi_{11} + j\phi_{12}$$

$$(65)\quad \tilde{B} = \phi_{13} + j\phi_{14}$$

$$(66)\quad \tilde{C} = \phi_{31} + j\phi_{32}$$

$$(67)\quad \tilde{D} = \phi_{33} + j\phi_{34} \quad .$$

Finally, if we define the vector $\tilde{S}$ to be

$$(68)\quad \tilde{S} = \begin{pmatrix} \tilde{s}_1 \\ \tilde{s}_2 \end{pmatrix} = \begin{pmatrix} s_1 - js_2 \\ s_3 - js_4 \end{pmatrix} ,$$

then Eq. (57) can be written in matrix form as

$$(69)\quad \frac{d}{dt}\tilde{W} + 2k\tilde{Q}\tilde{W} = 2k\tilde{S} .$$

Assuming CW signals are present, the received signals will be of the form

$$(70)\quad s_i(t) = \sqrt{2} \cdot \sigma_{si} \cdot \cos(\omega_i t)$$

where

σ_{s_i} = the signal amplitude

and

ω_i = the signal radian frequency.

Specifically, at the second antenna (see Fig. 5), the signal is given by

$$(71)\quad y_2(t) = \sum_{i=1}^{n} \sqrt{2} \cdot \sigma_{s_i} \cdot \cos(\omega_i t)$$

and at the first antenna,

$$(72)\quad y_1(t) = \sum_{i=1}^{n} \sqrt{2} \cdot \sigma_{s_i} \cdot \cos(\omega_i t - \alpha_i)$$

where α_i is the phase shift between antennas due to the propagation delay,

$$(73) \quad \alpha_i = \frac{2\pi L}{\lambda_{\omega i}} \sin \theta_i$$

(L is the element spacing and $\lambda_{\omega i}$ is the free-space wavelength at frequency ω_i.)

With element noise present, the in-phase and quadrature element signals are

$$(74) \quad x_1(t) = s_1(t) + n_1(t) = \sum_{i=1}^{n} \sigma_{s_i} \cdot \cos(\omega_i t - \alpha_i) + n_1(t)$$

$$(75) \quad x_2(t) = s_2(t) + n_2(t) = \sum_{i=1}^{n} \sigma_{s_i} \cdot \sin(\omega_i t - \alpha_i) + n_2(t)$$

$$(76) \quad x_3(t) = s_3(t) + n_3(t) = \sum_{i=1}^{n} \sigma_{s_i} \cdot \cos(\omega_i t) + n_3(t)$$

$$(77) \quad x_4(t) = s_4(t) + n_4(t) = \sum_{i=1}^{n} \sigma_{s_i} \cdot \sin(\omega_i t) + n_4(t).$$

The factor $\sqrt{2}$ was included in Eqs. (71) and (72) to make the in-phase and quadrature signals in Eqs. (74) - (77) have unit amplitude.

Substituting Eqs. (74) - (77) into Eq. (39) yields for Φ:

(78)

$$\Phi = \begin{pmatrix} \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n & 0 & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cdot\cos\alpha_i & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\sin\alpha_i \\ 0 & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 & -\frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cdot\sin\alpha_i & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cos\alpha_i \\ \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cdot\cos\alpha_i & -\frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cdot\sin\alpha_i & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 & 0 \\ \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cdot\sin\alpha_i & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2\cos\alpha_i & 0 & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i} + \sigma_n^2 \end{pmatrix}$$

We observe Φ exhibits the symmetry of Eq. (58).

Suppose furthermore that the reference signal is given by

$$(79) \quad R(t) = A \cdot \sum_{i=1}^{m} \cos(\omega_i t)$$

where A is some non-zero constant. In a practical system where the reference signal is derived from the desired part of the array output, A will be chosen to have a value compatible with the equipment. This fixed operating level is generally achieved by inserting a limiter somewhere in the reference signal generating network [15,25]. In Eq. (79), the reference signal is coherent with the first m signals. If Eqs. (74) - (77) and Eq. (79) are substituted into Eq. (40), we find

$$(80)\quad S = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = A \begin{pmatrix} \overline{\sum_{i=1}^{m} \cos(\omega_i t) \sum_{i=1}^{n} \sigma_{s_i} \cos(\omega_i t - \alpha_i) + n_1(t)} \\ \sum_{i=1}^{m} \cos(\omega_i t) \sum_{i=1}^{n} \sigma_{s_i} \sin(\omega_i t - \alpha_i) + n_1(t) \\ \vdots \end{pmatrix} .$$

Since the reference signal is uncorrelated with the directional jammers and element noise, we obtain

$$(81)\quad S = \frac{A}{2} \begin{pmatrix} \sum_{i=1}^{m} \sigma_{s_i} \cos \alpha_i \\ - \sum_{i=1}^{m} \sigma_{s_i} \sin \alpha_i \\ \sum_{i=1}^{m} \sigma_{s_i} \\ 0 \end{pmatrix} .$$

Substituting Eq. (81) into Eq. (68) yields for $\tilde{S}$:

$$(82)\quad \tilde{S} = \begin{pmatrix} s_1 - j s_2 \\ s_3 - j s_4 \end{pmatrix} = \frac{A}{2} \begin{pmatrix} \sum_{i=1}^{m} \sigma_{s_i} \cdot e^{j\alpha_i} \\ \sum_{i=1}^{m} \sigma_{s_i} \end{pmatrix} .$$

We can also obtain the reduced form of Φ. From Eqs. (64) - (67) and Eq. (78) we obtain

$$(83)\quad \tilde{Q} = \begin{pmatrix} \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{j\alpha_i} \\ \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{-j\alpha_i} & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 \end{pmatrix} .$$

The system of equations describing the response of the array may now be determined. Making the appropriate substitutions into Eq. (69) yields

$$(84)\quad \frac{d}{dt}\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} + 2k\begin{pmatrix} \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{j\alpha_i} \\ \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{-j\alpha_i} & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 \end{pmatrix}\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = k\begin{pmatrix} \sum_{i=1}^{m}\sigma_{s_i} e^{j\alpha_i} \\ \sum_{i=1}^{m}\sigma_{s_i} \end{pmatrix}$$

We notice that the equations are coupled in $\tilde{Q}$. Solving the system will require a rotation into the principal axes of $\tilde{Q}$. Had the order of the system not been reduced, we would have a system of four coupled differential equations to solve. The coordinate rotation to uncouple the system would be extremely tedious. Diagonalizing the system of Eq. (84) will be obviously much simpler.

We begin by determining the eigenvalues of $\tilde{Q}$. Since $\tilde{Q}$ is Hermitian, its characteristic values are necessarily real. Let

$$(85)\quad |\tilde{Q}-\lambda I| = \begin{vmatrix} \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 - \lambda & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{j\alpha_i} \\ \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 e^{-j\alpha_i} & \frac{1}{2}\sum_{i=1}^{n}\sigma_{s_i}^2 + \sigma_n^2 - \lambda \end{vmatrix} = 0 .$$

Hereafter $\sum_{i=1}^{n}$ will be represented by $\sum_{i}$. Expanding Eq. (85) yields

$$(86)\quad \left(\frac{1}{2}\sum_{j}\sigma_{s_i}^2 + \sigma_n^2 - \lambda\right)^2 - \frac{1}{4}\left(\sum_{i}\sigma_{s_i}^2 e^{-j\alpha_i} \sum_{i}\sigma_{s_i}^2 e^{j\alpha_i}\right) = 0.$$

Solving the above quadratic equation for λ gives for the eigenvalues of $\tilde{Q}$:

$$(87)\quad \lambda_1 = \sigma_n^2 + \frac{1}{2}\sum_{i}\sigma_{s_i}^2 + \frac{1}{2}\left(\sum_{i}\sigma_{s_i}^4 + 2\sum_{\substack{i}}\sum_{\substack{j \\ i\neq j}}\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{1/2}$$

and

$$(88)\quad \lambda_2 = \sigma_n^2 + \frac{1}{2}\sum_{i}\sigma_{s_i}^2 - \frac{1}{2}\left(\sum_{i}\sigma_{s_i}^4 + 2\sum_{\substack{i}}\sum_{\substack{j \\ i\neq j}}\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{1/2}$$

From Eqs. (87) and (88) we can now determine the eigenvectors of $\tilde{Q}$. Let $\tilde{E}_i$ denote the eigenvector associated with λ_i. We require

$$(89)\quad (\tilde{Q} - \lambda_i I)(\tilde{E}_i) = 0.$$

Substituting Eqs. (83) and (87) into Eq. (89) yields

(90)

$$\begin{pmatrix} -\frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{\substack{i\\ i\neq j}}\sum_j \sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{1/2} & \frac{1}{2}\sum_i \sigma_{s_i}^2 e^{j\alpha_i} \\ \frac{1}{2}\sum_i \sigma_{s_i}^2 e^{-j\alpha_i} & -\frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{\substack{i\\ i\neq j}}\sum_j \sigma_{s_i}^2\sigma_{s_i}^2\cos(\alpha_i-\alpha_j)\right)^{1/2} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\varepsilon}_{11} \\ \tilde{\varepsilon}_{21} \end{pmatrix} = 0.$$

We then obtain two equations of the form

(91) $$\tilde{F}\tilde{\varepsilon}_{11} + \tilde{G}\tilde{\varepsilon}_{21} = 0$$

and

(92) $$\tilde{G}^*\tilde{\varepsilon}_{11} + \tilde{F}\varepsilon_{21} = 0$$

where

(93) $$\tilde{F} = -\frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{\substack{i\\ i\neq j}}\sum_j \sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{1/2},$$

(94) $$\tilde{G} = \frac{1}{2}\sum_i \sigma_{s_i}^2 e^{j\alpha_i}$$

and superscript "*" denotes the complex conjugate. However, if we notice

(95) $\tilde{F}^2 = \tilde{B}\tilde{B}^*$

then the eigenvectors can be chosen to be

$$(96)\quad \tilde{E}_1 = \begin{pmatrix} -\tilde{F} \\ \tilde{G}^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2 \sum_{\substack{i \\ i\neq j}} \sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j) \right)^{1/2} \\ \frac{1}{2} \sum_i \sigma_{s_i}^2 \, e^{-j\alpha_i} \end{pmatrix}$$

We hereafter denote $\sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n}$ by $\sum_{i\neq j} \sum$.

Similar calculations show the second eigenvector to be

$$(97)\quad \tilde{E}_2 = \begin{pmatrix} \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2 \sum_{i\neq j} \sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j) \right)^{1/2} \\ -\frac{1}{2} \sum_i \sigma_{s_i}^2 \, e^{-j\alpha_i} \end{pmatrix}.$$

If from the eigenvectors we form the matrix

$$(98)\quad \tilde{Z} = (\tilde{E}_1, \tilde{E}_2)$$

then

$$(99)\quad \tilde{Z}^{-1} \tilde{Q} \tilde{Z} = \Lambda$$

where Λ is a diagonal matrix composed of the eigenvalues of $\tilde{Q}$ and superscript "-1" denotes the inverse matrix. If from $\tilde{Z}$ we construct a unitary matrix $\tilde{R}$ given by

$$(100)\quad \tilde{R} = \frac{1}{\sqrt{k}}\ \tilde{Z}$$

then

$$(101)\quad \tilde{R}^{\dagger} = \tilde{R}^{-1}$$

where "†" represents transpose conjugation. We require

$$(102)\quad \tilde{R}^{\dagger}\tilde{R} = I$$

where I is again the identity matrix. From Eqs. (96) - (98) and Eq. (100), we obtain for $\tilde{R}$:

$$(103)\quad \tilde{R} = \frac{1}{\sqrt{k}}\begin{pmatrix} \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{1/2} & \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}} \\ \frac{1}{2}\sum_i \sigma_{s_i}^2 e^{-j\alpha_i} & -\frac{1}{2}\sum_i \sigma_{s_i}^2 e^{-j\alpha_i} \end{pmatrix}$$

Performing the operation indicated in Eq. (102) yields for the normalization constant k:

$$(104)\quad k = \frac{1}{8}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right).$$

From Eqs. (103) and (104) the final expression for the coordinate rotation matrix becomes

$$(105)\quad \tilde{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ \dfrac{\sum_i \sigma_{s_i}^2 e^{-j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}} & \dfrac{-\sum_i \sigma_{s_i}^2 e^{-j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}} \end{pmatrix}$$

If we now multiply Eq. (74) on the left by $\tilde{R}^{\dagger}$ and insert $\tilde{R}\,\tilde{R}^{\dagger}$ into the second term, we obtain

$$(106)\quad \frac{d}{dt}\tilde{R}^{\dagger}\tilde{W} + 2k[\tilde{R}^{\dagger}\tilde{Q}\,\tilde{R}]\tilde{R}^{\dagger}\tilde{W} = 2k\,\tilde{R}^{\dagger}\tilde{S}\,.$$

By defining the relations

$$(107)\quad \tilde{\Gamma} = \tilde{R}^{\dagger}\tilde{W} = \begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix}$$

and

$$(108)\quad \tilde{V} = \tilde{R}^{\dagger}\tilde{S} = \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}$$

and using the correspondence of Eq. (99), we can rewrite Eq. (106) conveniently as

$$\frac{d}{dt}\tilde{\Gamma} + 2k\Lambda\tilde{\Gamma} = 2k\tilde{V}. \tag{109}$$

The elements of $\tilde{\Gamma}$ will be referred to as the "unitary weights" of the array. This is analogous to the "normal weights" of the array represented by η in Eq. (49), when all the elements are real.

Recall that Λ is a real diagonal matrix and is composed of the eigenvalues of $\tilde{Q}$. Specifically from Eqs. (87), (88) and (99), we find

(110)

$$\Lambda = \begin{pmatrix} \frac{1}{2}\sum_i \sigma_{s_i}^2 + \sigma_n^2 + \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j)\right)^{\frac{1}{2}} & 0 \\ 0 & \frac{1}{2}\sum_i \sigma_{s_i}^2 + \sigma_n^2 - \frac{1}{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j)\right)^{\frac{1}{2}} \end{pmatrix}.$$

Since $\tilde{\Gamma}$ and $\tilde{V}$ represent column vectors, the equations are no longer coupled. If we now carry out the indicated operation of Eq. (108), we obtain for $\tilde{V}$

(111)

$$\tilde{V} = \tilde{R} + \tilde{S} = \frac{A}{2\sqrt{2}} \begin{pmatrix} \sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i} + \dfrac{\sum_{i=1}^{m} \sigma_{s_i} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j)\right)^{1/2}} \\ \sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i} - \dfrac{\sum_{i=1}^{m} \sigma_{s_i} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j)\right)^{1/2}} \end{pmatrix}.$$

We are now in a position to obtain the equations describing the unitary weights of the array. Making the appropriate substitutions into Eq. (109) yields a set of separate equations which may be written as

(112)

$$\frac{d}{dt}\tilde{\gamma}_1(t)+2k\left[\sigma_n^2+\frac{1}{2}\sum_i\sigma_{s_i}^2+\frac{1}{2}\left(\sum_i\sigma_{s_i}^4+2\sum_{i\neq j}\sum\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}\right]\tilde{\gamma}_1(t)=$$

$$k\cdot A\cdot\sum_{i=1}^{m}\sigma_{s_i}e^{j\alpha_i}+\frac{k\sum_{i=1}^{m}\sigma_{s_i}\sum_{i=1}^{n}\sigma_{s_i}^2e^{j\alpha_i}}{\left(\sum_i\sigma_{s_i}^4+2\sum_{i\neq j}\sum\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}}$$

and

(113)

$$\frac{d}{dt}\tilde{\gamma}_2(t)+2k\left[\sigma_n^2+\frac{1}{2}\sum_i\sigma_{s_i}^2-\frac{1}{2}\left(\sum_i\sigma_{s_i}^4+2\sum_{i\neq j}\sum\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}\right]\tilde{\gamma}_2(t)=$$

$$k\cdot A\cdot\sum_{i=1}^{m}\sigma_{s_i}e^{j\alpha_i}-\frac{k\sum_{i=1}^{m}\sigma_{s_i}\sum_{i=1}^{n}\sigma_{s_i}^2e^{j\alpha_i}}{\left(\sum_i\sigma_{s_i}^4+\sum_{i\neq j}\sum\sigma_{s_i}^2\sigma_{s_j}^2\cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}}.$$

The solution of Eq. (57) has essentially been reduced to solving two first-order differential equations of the form

$$(114)\quad \frac{d}{dt}\tilde{\gamma}_i(t)+\lambda_i\tilde{\gamma}_i(t)=k\tilde{v}_i.$$

Equations (112) and (113) are then easily solved, yielding as solutions:

(115)

$$\tilde{\gamma}_1(t) = k_1 e^{-2k\lambda_1 t} + \frac{A}{\lambda_1 \cdot 2 \cdot 2}\left(\sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i} + \frac{\sum_{i=1}^{m} \sigma_{s_i} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2 \sum_{i \neq j} \sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j) \right)^{\frac{1}{2}}} \right)$$

and

(116)

$$\tilde{\gamma}_2(t) = k_2 e^{-2k\lambda_2 t} + \frac{A}{\lambda_2 \cdot 2 \cdot 2}\left(\sum_{i=1}^{m} \sigma_{s_i} \cdot e^{j\alpha_i} - \frac{\sum_{i=1}^{m} \sigma_{s_i} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i}}{\left(\sum_i \sigma_{s_i}^4 + 2 \sum_{i \neq j} \sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j) \right)^{\frac{1}{2}}} \right),$$

where k_1 and k_2 are the constants dependent upon the initial weight values and λ_1 and λ_2 are the eigenvalues of $\tilde{Q}$ given in Eqs. (87) and (88). It is apparent that the complex weights $\tilde{w}_1$ and $\tilde{w}_2$ may be obtained from the unitary weights $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ by the simple transformation

(117) $\tilde{W} = \tilde{R}\,\tilde{\Gamma}$.

Then from Eq. (105), we find

(118) $$\tilde{w} = \frac{\tilde{\gamma}_1 + \tilde{\gamma}_2}{2}$$

and

(119)

$$\tilde{w}_2 = \frac{1}{\sqrt{2}\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}} \cdot \left(\sum_i \sigma_{s_i}^2 e^{-j\alpha_i} \tilde{\gamma}_1 - \sum_i \sigma_{s_i}^2 e^{-j\alpha_i} \tilde{\gamma}_2\right).$$

Substituting Eqs. (115) and (116) into Eq. (118) results in

(120)

$$\tilde{w}_1(t) = \frac{1}{2}\left(k_1 e^{-2k\lambda_1 t} + k_2 e^{-2k\lambda_2 t}\right) + \frac{A}{4\cdot\lambda_1\cdot\lambda_2}\left((\lambda_1+\lambda_2)\cdot \sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i} + \frac{\sum_{j=1}^{m} \sigma_{s_j} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i} \cdot (\lambda_2-\lambda_1)}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}}\right)$$

and from Eq. (119)

(121)

$$\tilde{w}_2(t) = \frac{\sum_i \sigma_{s_i}^2 e^{-j\alpha_i}}{2\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}}\left[\left(k_1 e^{-2k\lambda_1 t} - k_2 e^{-2k\lambda_2 t}\right) + \frac{A}{4\cdot\lambda_1\cdot\lambda_2}\left(\sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i}\cdot(\lambda_2-\lambda_1) + \frac{\left(\sum_{i=1}^{m} \sigma_{s_i} \sum_{i=1}^{n} \sigma_{s_i}^2 e^{j\alpha_i}\right)(\lambda_1+\lambda_2)}{\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}}\right)\right].$$

The following relations are easily verified:

$$(122)\quad \lambda_2-\lambda_1 = -\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \cos(\alpha_i-\alpha_j)\right)^{\frac{1}{2}}$$

$$(123)\quad \lambda_1+\lambda_2 = 2\sigma_n^2 + \sum_i \sigma_{s_i}^2$$

and

$$(124)\quad 4\cdot\lambda_1\cdot\lambda_2 = 4\sigma_n^2(\sigma_n^2 + \sum_i\sigma_{s_i}^2) + 2\sum_i\sum_j \sigma_{s_i}^2\sigma_{s_j}^2\,(1-\cos(\alpha_i-\alpha_j))\ .$$

Using the foregoing results, Eq. (120) readily reduces to

$$(125)\quad \tilde{w}_1(t) = \frac{1}{2}\left[k\,e^{-2k\lambda_1 t} + k\,e^{-2k\lambda_2 t}\right]$$

$$+ A\cdot\frac{\left(\sum_{i=1}^m \sigma_{s_i} e^{j\alpha_i}\right)\left(\sum_{i=1}^n \sigma_{s_i}^2 + 2\sigma_n^2\right) - \left(\sum_{i=1}^m \sigma_{s_i}\right)\left(\sum_{i=1}^n \sigma_{s_i}^2 e^{j\alpha_i}\right)}{4\sigma_n^2(\sigma_n^2 + \sum_i\sigma_{s_i}^2) + 2\cdot\sum_i\sum_j\sigma_{s_i}^2\sigma_{s_j}^2(1-\cos(\alpha_i-\alpha_j))}\ .$$

Eq. (121) may be simplified in a similar manner. We obtain for $\tilde{w}_2(t)$:

$$(126)\quad \tilde{w}_2(t) = \frac{\sum_i \sigma_{s_i}^2 e^{-j\alpha_i}}{2\left[\sum_i \sigma_{s_i}^4 + 2\cdot\sum_{i\neq j}\sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i-\alpha_j)\right]^{\frac{1}{2}}} \cdot \left(k_1 e^{-2k\lambda_1 t} - k_2 e^{-2k\lambda_2 t}\right)$$

$$+ A \cdot \frac{\left(\sum_{i=1}^{m} \sigma_{s_i}\right)\left(\sum_i \sigma_{s_i}^2 + 2\sigma_n^2\right) - \left(\sum_i \sigma_{s_i}^2 e^{-j\alpha_i}\right)\left(\sum_{i=1}^{m} \sigma_{s_i} e^{j\alpha_i}\right)}{4\sigma_n^2(\sigma_n^2 + \sum_i \sigma_{s_i}^2) + 2\sum_i \sum_j \sigma_{s_i}^2 \sigma_{s_j}^2 (1-\cos(\alpha_i-\alpha_j)}\ .$$

From Eqs. (60) and (61) it is seen that the real array weights can simply be found from the complex weighting coefficients by means of the following relations:

$$(127)\quad w_1(t) = \text{Real}\ (\tilde{w}_1(t))$$

$$(128)\quad w_2(t) = -\text{Imag}\ (\tilde{w}_1(t))$$

$$(129)\quad w_3(t) = \text{Real}\ (\tilde{w}_2(t))$$

$$(130)\quad w_4(t) = -\text{Imag}(\tilde{w}_2(t)).$$

To summarize, then, we have developed expressions for the weighting coefficients of a two-element adaptive array with an arbitrary number of jamming and desired CW signals incident upon it.

Let us examine the temporal response of the weights. From Eq. (125) and (126) the transient response of the complex weights are

$$(131)\quad \tilde{w}_{1tr.}(t) = \frac{1}{2}\left(k_1 \cdot e^{-2k\lambda_1 t} + k_2 \cdot e^{-2k\lambda_2 t} \right)$$

and

$$(132)\quad \tilde{w}_{2tr.}(t) = \frac{\sum_i \sigma_{s_i}^2 e^{j\alpha_i}}{2\left(\sum_i \sigma_{s_i}^4 + 2\sum_{i\neq j}\sum \sigma_{s_i}^2 \sigma_{s_j}^2 \cos(\alpha_i - \alpha_j) \right)^{\frac{1}{2}}} \left(k_1 e^{-2k\lambda_1 t} - k_2 e^{-2k\lambda_2 t} \right).$$

We notice the transient response contains decaying exponentials. The rate of decay of these functions is proportional to the loop gain constant k and the eigenvalues of $\tilde{Q}$, λ_1 and λ_2. Recall that these eigenvalues are functions of the signal powers $\sigma_{s_i}^2$, the noise power σ_n^2 and the electrical phase shift between antenna elements α_i. We then observe that as the number of signals increases, the eigenvalues also increase (See Eqs. (92) and (93).) The array response will then become faster as the number of signals increase. Also, a larger element noise power σ_n^2 and loop gain constant k results in a faster array response. The array weights then converge to their final solution in a shorter period of time with many signals present than with few. Unfortunately, as it will be demonstrated, the output SNR degrades as the number of signals increases.

IV. THE PERFECTLY CONSTRAINED ARRAY

The results of the previous section may be used to study the performance of a two-element array when there are two signals present. When an N-element array has N signals incident upon it, the array is said to be "perfectly constrained." It will be shown that with element noise present, the SNR at the array output is degraded. Not only is the desired signal amplitude below that of the reference signal [16] but the interference signal is also present at the array output.

Let us suppose two CW signals are incident upon the array (see Fig. 5). The signal arriving at angle θ_1 off broadside is chosen to be the desired signal and the other is a directional interference signal. From Eq. (125) - (130) the steady-state weights are found to be

$$(133)\quad w_1(\infty) = \frac{A\cdot\sigma_{s_1}((\sigma_{s_2}^2+2\sigma_n^2)\cos\alpha_1-\sigma_{s_2}^2\cdot\cos\alpha_2)}{2\sigma_n^2(\sigma_n^2+\sigma_{s_1}^2+\sigma_{s_2}^2)+2\sigma_{s_1}^2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))}$$

$$(134)\quad w_2(\infty) = \frac{A\cdot\sigma_{s_1}((\sigma_{s_2}^2\sin\alpha_2)-(\sigma_{s_2}^2+2\sigma_n^2)\cdot\sin\alpha_1)}{4\sigma_n^2(\sigma_n^2+\sigma_{s_1}^2+\sigma_{s_2}^2)+2\sigma_{s_1}^2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))}$$

$$(135)\quad w_3(\infty) = \frac{A\cdot\sigma_{s_1}(2\sigma_n^2+\sigma_{s_2}^2\cdot(1-\cos(\alpha_1-\alpha_2)))}{4\sigma_n^2(\sigma_n^2+\sigma_{s_1}^2+\sigma_{s_2}^2)+2\sigma_{s_1}^2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))}$$

and

$$(136)\quad w_4(\infty) = \frac{A\cdot\sigma_{s_1}\sigma_{s_2}^2 \cdot \sin(\alpha_1-\alpha_2)}{4\sigma_n^2(\sigma_n^2+\sigma_{s_1}^2+\sigma_{s_2}^2)+2\sigma_{s_1}^2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))} .$$

The output of a two-element array with n CW signals incident upon it is

$$(137)\quad s(t) = \sum_{i=1}^{n} \sigma_{s_i}((w_4 + w_1 \sin\alpha_i + w_2 \cos\alpha_i)^2 + (w_3 + w_1 \cos\alpha_i - w_2 \sin\alpha_i)^2)^{\frac{1}{2}} \cdot \cos(w_i t - \tan^{-1}\left(\frac{w_4+w_1 \sin\alpha_i + w_1 \cos\alpha_i}{w_3+w_1 \cos\alpha_i - w_2 \sin\alpha_i}\right)) .$$

Substituting Eqs. (133) - (136) into Eq. (137), we find that the desired part of the array output is

$$(138)\quad S_d(t) = \frac{A\cdot\sigma_{s_1}^2 \cdot (4\sigma_n^2+2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2)))\cdot\cos(\omega_1 t)}{4\sigma_n^2(\sigma_n^2+\sigma_{s_1}^2+\sigma_{s_2}^2) + 2\sigma_{s_1}^2\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))} .$$

We notice that because element noise is present, the desired signal output does not match that of the reference signal given in Eq. (50). Similar results were obtained by Compton [16]. However, this is not

the only effect of the element noise. If we calculate the undesired part of the array output, we find

(139)

$$S_j(t) = \frac{A \cdot \sigma_{s_1} \sigma_{s_2}^2 \sigma_n \cdot \cos\left(\frac{\alpha_2 - \alpha_2}{2}\right)}{4\sigma_n^2 (\sigma_n^2 + \sigma_{s_1}^2 + \sigma_{s_2}^2) + 2\sigma_{s_1}^2 + \sigma_{s_2}^2 (1 - \cos(\alpha_1 - \alpha_2)}$$

$$\cdot \cos\left(w_2 t - \tan^{-1}\left(\frac{\sin(\alpha_2 - \alpha_1)}{1 + \cos(\alpha_2 - \alpha_1)}\right)\right) .$$

We see that the jamming signal is also present at the array output. By changing the value of the weights, the element noise causes the spatial null formed by the array weights to no longer exactly be in the jamming signal direction. For example, in Figs. 6 and 7 we see the array pattern with one jamming and one desired signal present. In both cases the jamming and desired signals have equal power. In Fig. 6, the jamming signal-to-element noise power ratio (σ_j^2/σ_n^2) is 20 dB. The element noise has little effect on the array pattern. The jamming signal is well within the null. However, as the element noise power is increased, the null moves farther away from the jamming signal direction. For example, in Fig. 7 we see the pattern when σ_j^2/σ_n^2 = 6 dB (the desired signal power and interference power are unchanged). We notice the jamming signal power is now only 18 dB below the desired signal power. This may be of serious consequence when both the jamming and desired signals are low-power signals (power levels near the element noise power). We notice from Eq. (139)

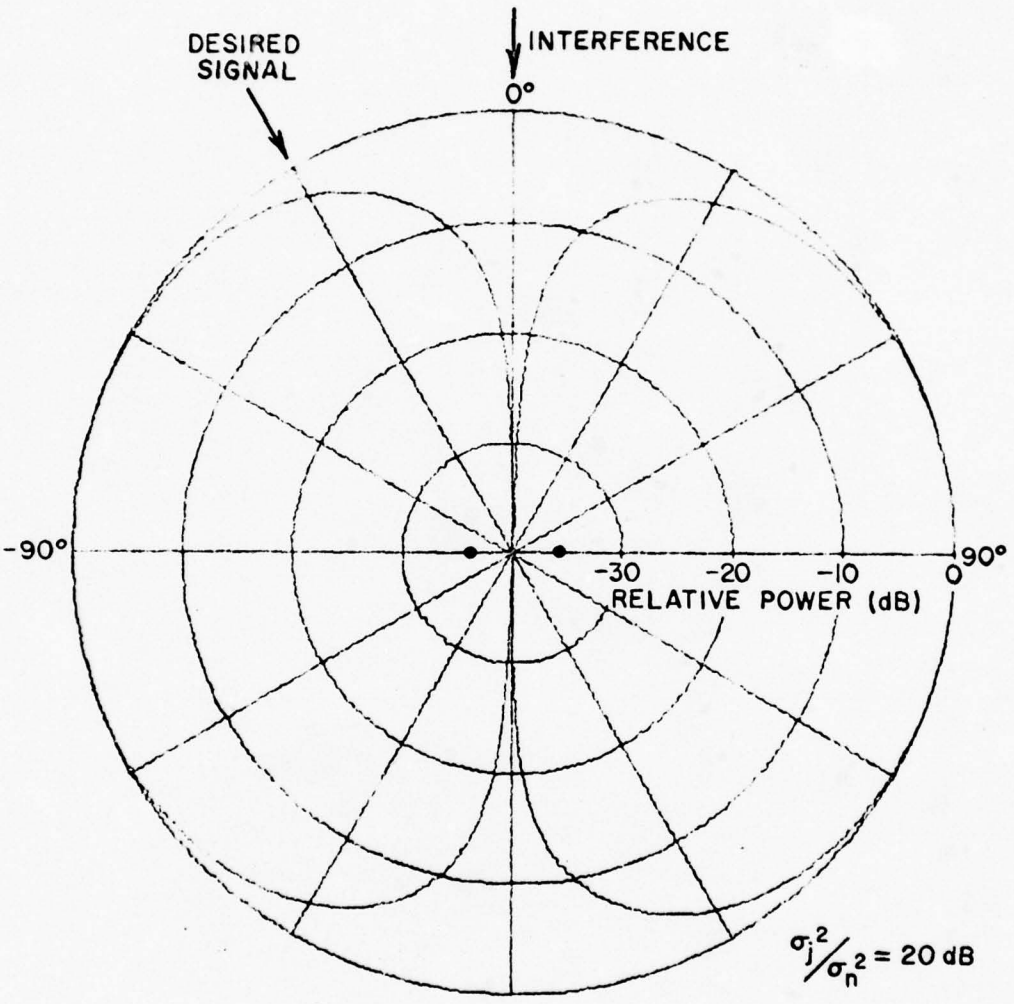

DESIRED
SIGNAL
INTERFERENCE
0°
-90°
90°
-30
-20
-10
0
RELATIVE POWER (dB)
$\sigma_j^2/\sigma_n^2 = 20$ dB

Fig. 6.

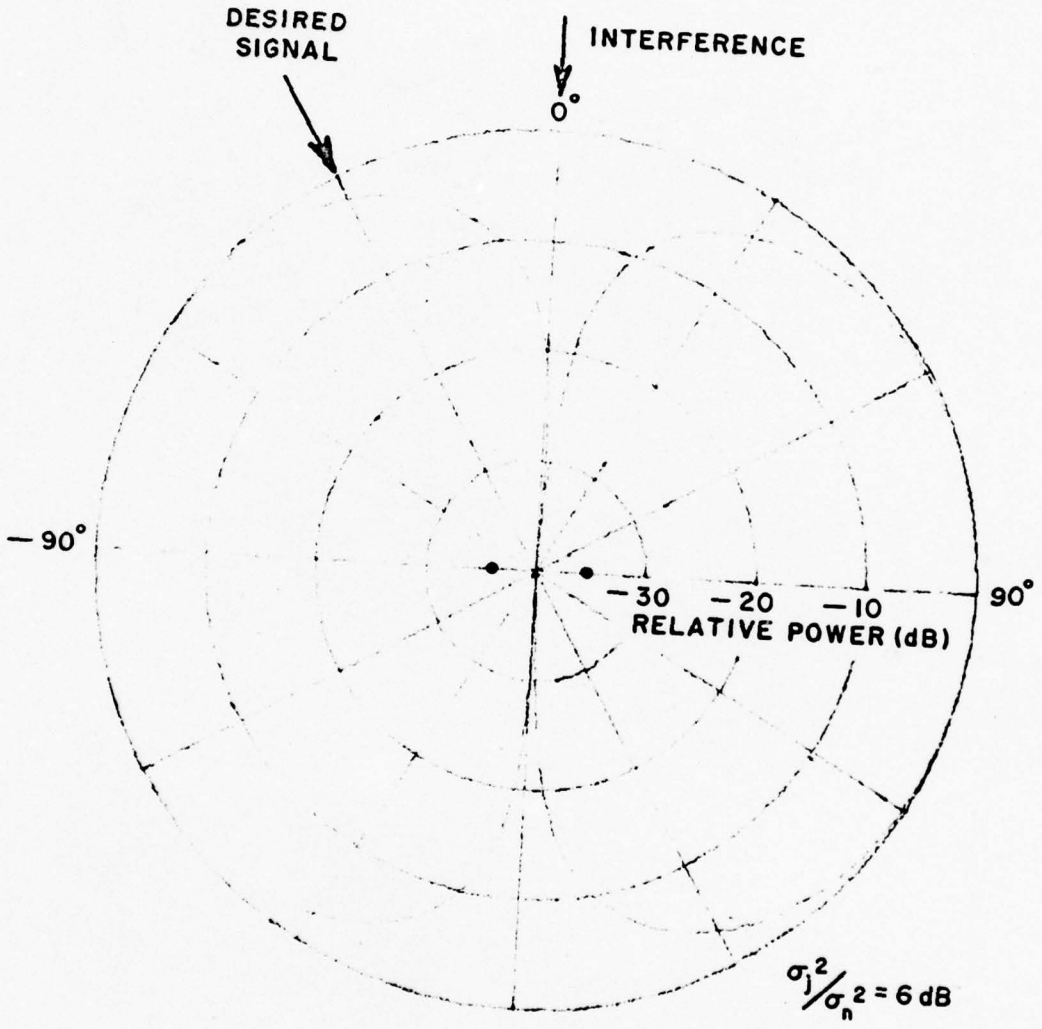

DESIRED SIGNAL
INTERFERENCE
0°
−90°
90°
−30
−20
−10
RELATIVE POWER (dB)
$\sigma_j^2/\sigma_n^2 = 6$ dB

Fig. 7.

$$\lim_{\sigma_n^2 \to 0} S_j(t) = 0 \quad . \tag{140}$$

In other words, as the element noise power becomes small, the jamming signal is no longer present at the array output.

The situation may also arise that both incoming signals are desired [15]. Similar calculations show that the desired signal amplitudes are again below the reference signal amplitude when element noise is present. The phase of the desired signals is correct.

We have seen, then, that the array makes a compromise between the contribution to $\epsilon^2(t)$ due to the element noise and those due to the signals. Since the element noise is uncorrelated between channels, the array is unable to remove its contribution at the output.

V. THE OVER-CONSTRAINED ARRAY

The results of section III may also be used to study the performance of a two-element array when more than two signals are present. When an N-element array has more than N signals incident upon it, the array is said to be "over-constrained."

Let us assume a two-element array has three CW signals incident upon it. In other words, the antenna signals are from Eqs. (71) and (72):

$$(141)\quad y_1(t) = 2\{\sigma_{s_1}\cdot\cos(w_1t-\alpha_1)+\sigma_{s_2}\cdot\cos(w_2t-\alpha_2)+\sigma_{s_3}\cdot\cos(w_3t-\alpha_3)\}$$

and

$$(142)\quad y_2(t) = 2\{\sigma_{s_1}\cdot\cos(w_1t)+\sigma_{s_2}\cdot\cos(w_2t)+\sigma_{s_3}\cdot\cos(w_3t)\}\ .$$

First, consider the response with no element noise present. We assume the signal at frequency ω_1 to be desired and the others to be CW interference signals. From Eqs. (125) - (130) we determine the steady-state weights to be:

$$(143)\quad w_1(\infty) = \frac{A\cdot\sigma_{s_1}\cdot(\sigma_{s_2}^2(\cos\alpha_1-\cos\alpha_2)+\sigma_{s_3}^2(\cos\alpha_1-\cos\alpha_3)}{2(\sigma_{s_1}^2\cdot\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))+\sigma_{s_2}^2\sigma_{s_3}^2(1-\cos(\alpha_2-\alpha_3))+\sigma_{s_1}^2\sigma_{s_3}^2(1-\cos(\alpha_1-\alpha_3)))}$$

$$(144)\quad w_2(\infty) = \frac{A\cdot\sigma_{s_1}(\sigma^2_{s_2}(\sin\alpha_2 - \sin\alpha_1) + \sigma^2_{s_3}(\sin\alpha_3 - \sin\alpha_1))}{2(\sigma^2_{s_1}\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2))+\sigma^2_{s_2}\sigma^2_{s_3}(1-\cos(\alpha_2-\alpha_3))+\sigma^2_{s_1}\sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3)))}$$

$$(145)\quad w_3(\infty) = \frac{A\cdot\sigma_{s_1}\cdot(\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2)) + \sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3)))}{2(\sigma^2_{s_1}\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2))+\sigma^2_{s_2}\sigma_{s_3}(1-\cos(\alpha_2-\alpha_3))+\sigma^2_{s_1}\sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3)))}$$

and

$$(146)\quad w_4(\infty) = \frac{A\cdot\sigma_{s_1}(\sigma^2_{s_2}\cdot\sin(\alpha_1-\alpha_2) + \sigma^2_{s_3}\cdot\sin(\alpha_1-\alpha_2))}{2(\sigma^2_{s_1}\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2))+\sigma^2_{s_2}\sigma^2_{s_3}(1-\cos(\alpha_2-\alpha_3))+\sigma^2_{s_1}\sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3)))}.$$

We can substitute Eqs. (143) - (146) into Eq. (137) and determine the array output. For the desired signal, we have

$$(147)\quad s_d(t) = \frac{A\cdot\sigma^2_{s_1}(\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2)) + \sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3)))\cdot\cos(w_1 t)}{\sigma^2_{s_1}\cdot\sigma^2_{s_2}(1-\cos(\alpha_1-\alpha_2))+\sigma^2_{s_2}\sigma^2_{s_3}(1-\cos(\alpha_2-\alpha_3))+\sigma^2_{s_1}\sigma^2_{s_3}(1-\cos(\alpha_1-\alpha_3))}\ .$$

From Eq. (147) we notice that even though there is no element noise, the amplitude of the desired part of the array output does not match that of the reference signal (recall the reference signal amplitude is A). We do notice that the desired signal phase is correct, however.

A similar calculation for the interference signals at the array output may be made. When this is done, we find that both interference signals are present at the array output. Performing these calculations and using Eq. (147), we determine the signal-to-jam (interference) ratio (S_o/J_o) at the array output to be

(148)

$$S_o/J_o = \frac{4\cdot\sigma_{s_1}^4(\sigma_{s_2}^2(1-\cos(\alpha_1-\alpha_2))+\sigma_{s_3}^2(1-\cos(\alpha_1-\alpha_3)))^2}{4(\sigma_{s_2}^4+\sigma_{s_3}^4)+2\sigma_{s_3}^4\cdot\cos(\alpha_1+\alpha_2-2\alpha_3)+2\sigma_{s_2}^4\cos(\alpha_1+\alpha_3-2\alpha_2)+4(\sigma_{s_2}^4+\sigma_{s_3}^4)\cos(\alpha_2-\alpha_3) \\ -2(\sigma_{s_3}^4+\sigma_{s_2}^4)\cos(\alpha_1-\alpha_3)} .$$

With two signals incident on a two-element array, the signal to-jam ratio (with $\sigma_n^2 = 0$) may be shown from Eq. (138) and (139) to be infinite. The additional jamming signal has caused a degradation in S_o/J_o.

Suppose we now consider the first two signals at the input to be desired signals. By again calculating the array output, we find that not only are the desired signal amplitudes degraded and the jamming signal present at the array output, but the desired signal phase no longer matches the reference signal phase. In an adaptive array for use with more than one desired signal, phase control of the desired signals may not be possible. For example, the four-element array described in [15] for use with sensors as many as four desired signals may be present at the array input.

It has been demonstrated in this section that when an adaptive array is over-constrained, the array performance is degraded. We have found that the desired signal amplitude (and possibly the phase) no longer matches the reference signal. The interference is also present at the output. With element noise, the performance will be further degraded although this degradation will be small when the element noise power is much smaller than the signal powers.

VI. ARRAY PERFORMANCE WITH A CONTINUOUS INTERFERENCE SECTOR

The results of section VI for the array performance with discrete signals will now be extended to the case of a continuous sector of radiation. It has been shown in section II that when the signals incident upon the array are such that their bandwidths are greater than the product of the loop gain constant (k) and the signal powers, the time-varying components may be neglected in the differential equations describing the weights. In this section we assume a continuous sector of spatial radiation is incident upon a two-element array. Moreover, we assume the frequency spectrum of this radiation the same from all angles and the bandwidth of this radiation is wide enough that we may neglect all but the d.c. components in the weight equations. We also assume the phase shift between antennas due to propagation delay will be nearly constant over the entire signal bandwidth.* Then

$$(149)\quad \lambda_{W_i} \sim \lambda_W \text{ for all } i.$$

With this approximation we may proceed directly from the values of $\tilde{Q}$ calculated in Eq. (83). Recall, the terms of $\tilde{Q}$ were of the form

$$(150)\quad q = \sum_i f\left(\frac{2\pi L}{\lambda_W} \sin \theta_i\right) \sigma_{s_i}^2 ,$$

where $\sigma_{s_i}^2$ is the power of each discrete signal. We define $D(\theta_i)$ to

*We have neglected the effect of the wave polarization.

be a spatial power density defined over $\theta_1 < \theta < \theta_2$. Also, define Δ to be a partition of the interval $|\theta_1, \theta_2|$ and $||\Delta||$ to be the "norm" of this partition. Then let us suppose we make the number of signals go to infinity while keeping the total power constant; i.e., as $||\Delta|| \to 0$ as $n \to \infty$,

$$\lim_{||\Delta|| \to 0} \sum_i f\left(\frac{2\pi L}{\lambda_i} \sin(i\Delta\theta)\right) \cdot D(i\Delta\theta) \cdot i\Delta\theta$$

remains finite. The summation may then be replaced by a Riemann integral. In other words, we have replaced the model of many discrete signals of a relatively small difference in frequency ($\Delta\omega$) with respect to the carrier by a continuous spatial distribution of radiation.

Practically this situation arises when a waveform irradiates a rough surface (e.g., surface of the earth, ocean, ionospheric skip, etc.) and the reflected energy appears to be spread in angle with no specific source of origin. Another example would also be sky noise.

With this in mind we write Eq. (150) as

$$(151)\quad q = \int_{\theta_1}^{\theta_2} f\left(\frac{2L}{\lambda_w}\right) \sin\theta \quad D(\theta)\, d\theta$$

Specifically, from Eq. (151) and the relations of Eq. (64) - (67), the components of $\tilde{Q}$ now become:

$$(152)\quad \tilde{B} = \int_{\theta_1}^{\theta_2} D(\theta)\, e^{j \frac{2\pi L}{\lambda_\omega} \sin\theta}\, d\theta$$

$$(153)\quad \tilde{C} = \int_{\theta_1}^{\theta_2} D(\theta) e^{-j \frac{2\pi L}{\lambda_\omega} \sin\theta}\, d\theta$$

and

$$(154)\quad \tilde{A} = \tilde{D} = \int_{\theta_1}^{\theta_2} D(\theta) d\theta\ .$$

The expression for $\tilde{Q}$ is then

$$(155)\quad \tilde{Q} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \int_{\theta_1}^{\theta_2} D(\theta) \begin{pmatrix} 1 & e^{j \frac{2\pi L}{\lambda_\omega} \sin\theta} \\ e^{-j \frac{2\pi L}{\lambda_\omega} \sin\theta} & 1 \end{pmatrix} d\theta.$$

Define the relations:

$$(156)\quad \mu = \sin\theta \quad , \quad \theta = \sin^{-1}(\mu)\ ,$$

$$(157)\quad d\mu = \cos\theta \cdot d\theta, \quad d\theta = \frac{d\mu}{\cos\theta} = \frac{d\mu}{\sqrt{1-\mu^2}}\ ,$$

$$(158)\quad \beta = \frac{2\pi L}{\lambda_\omega},$$

and

$$(159)\quad f(\mu) = \frac{D(\sin^{-1}(\mu))}{\sqrt{1-\mu^2}}.$$

Using the previous definitions, Eqs. (152) - (154) become

$$(160)\quad \tilde{B}(-\beta) = \int_{\mu_1}^{\mu_2} f(\mu) e^{j\beta\mu}\, d\mu$$

$$(161)\quad \tilde{C}(\beta) = \int_{\mu_1}^{\mu_2} f(\mu) e^{-j\beta\mu}\, d\mu$$

and

$$(162)\quad \tilde{A} = \tilde{D} = \int_{\mu_1}^{\mu_2} f(\mu)\, d\mu = \tilde{B}(0)$$

where $\tilde{B}(\beta)$ is the Fourier transform of f(u). The matrix $\tilde{Q}$ may conveniently be written as

$$(163)\quad \tilde{Q} = \begin{pmatrix} B(0) & B(-\beta) \\ B(\beta) & B(0) \end{pmatrix}$$

It is worth noting here that if the incident signal is arriving from a discrete angle θ_i, then we may represent this case as

$$(164)\quad D(\theta) = \sigma_{s_i}^2\ \delta(\theta-\theta_i).$$

Substituting Eq. (164) into Eqs. (160) - (162) results in

$$(165)\quad \tilde{B} = \int_{\theta_1}^{\theta_2} \sigma_{s_i}^2\ e^{j\frac{2\pi L}{\lambda_\omega}\sin\theta} \cdot\delta(\theta-\theta_i)d\theta = \sigma_{s_i}^2 e^{j\frac{2\pi L}{\lambda_\omega}\sin\theta_i},$$

$$(166)\quad \tilde{C} = \int_{\theta_1}^{\theta_2} \sigma_{s_i}^2\ e^{j\frac{2\pi L}{\lambda_\omega}\sin\theta} \cdot \delta(\theta-\theta_i)d\theta = \sigma_{s_i}^2 e^{-j\frac{2\pi L}{\lambda_\omega}\sin\theta_i}$$

and

$$(167)\quad \tilde{A} = \tilde{D} = \int_{\theta_1}^{\theta_2} \sigma_{s_i}^2\ \delta(\theta-\theta_i)d\theta = \sigma_{s_i}^2 .$$

Then by allowing discrete signals to be represented by Dirac Delta functions, the continuous channel representation readily reduces to the discrete case previously investigated.

If we assume the impinging radiation sector to be a jamming source, then a desired signal source must also be present in order for the weights to have non-zero solutions. We define

$$(168)\quad B(\beta) = B_j(\beta) + B_d(\beta)$$

where $B_j(\beta)$ is due to the jammer sector of radiation and $B_d(\beta)$ to the desired sector. Since the interference and desired signals are uncorrelated, we may add the contributions of the jamming and desired signals in $B(\beta)$. Using Eq. (168) we are now able to write $\tilde{Q}$ as

$$(169)\quad \tilde{Q}_{d+j} + \tilde{Q}_d + \tilde{Q}_j = \begin{pmatrix} B(0) & B(-\beta) \\ B(\beta) & B(0) \end{pmatrix}.$$

If we now assume the desired signal to be a CW signal at frequency ω_d located at a discrete spatial angle θ_d, then

$$(170)\quad \tilde{Q}_d = \begin{pmatrix} \sigma_{s_d}^2 & \sigma_{s_d}^2 \cdot e^{j\beta\mu_d} \\ \sigma_{s_d}^2 \cdot e^{-j\beta\mu_d} & \sigma_{s_d}^2 \end{pmatrix}$$

where from Eq. (156) we have

$$(171)\quad \mu_d = \sin\theta_d$$

and $\sigma_{s_d}^2$ is the desired signal power. The reference signal R(t) will again be of the form of the desired signal, then

(172) $R(t) = A \cdot \cos \omega_d t.$

From Eq. (80) the vector $\tilde{S}$ is

(173) $$\tilde{S} = \frac{A}{2} \begin{pmatrix} \sigma_{s_d} \cdot e^{j\beta\mu_d} \\ \sigma_{s_d} \end{pmatrix}$$

Figure 8 shows a two-element array with a sector of jamming radiation and a single desired signal.

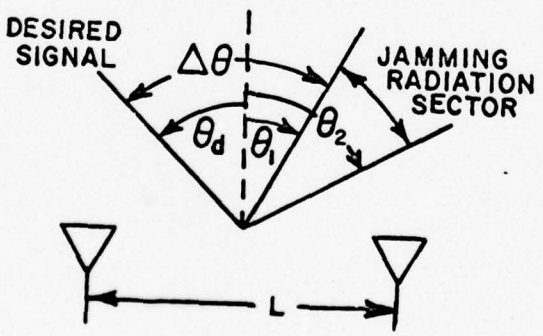


Fig. 8--Array signals.

We notice that if f(u) in Eq. (156) is real, then

$$B(-\beta) = B^*(\beta) \tag{174}$$

and if f(u) is purely imaginary

$$B(-\beta) = -B^*(\beta) \ . \tag{175}$$

By substituting Eqs. (170) and (173) into Eq. (69), we find that the complex weight response is determined by

$$\frac{d}{dt}\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} + 2k \begin{pmatrix} B(0) & B(-\beta) \\ B(\beta) & B(0) \end{pmatrix} \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = K \cdot A \cdot \sigma_{s_d} \cdot \begin{pmatrix} e^{j\beta\mu_d} \\ 1 \end{pmatrix} . \tag{176}$$

The method of solution is the same as that of the previous sections. We first determine the eigenvalues of $\tilde{Q}$ to be

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} B(0) + (B(\beta)B(-\beta))^{\frac{1}{2}} & \\ 0 & B(0)-(B(\beta)B(-\beta))^{\frac{1}{2}} \end{pmatrix} . \tag{177}$$

Since $f(u)$ is a power spatial density, then it is necessary real and $\tilde{Q}$ is Hermitian with real eigenvalues. Moreover, we note from Eq. (177) that

$$(178)\quad (B(\beta)B(-\beta))^{\frac{1}{2}} = (B(\beta)B^*(\beta))^{\frac{1}{2}} = |B(\beta)| \ .$$

Using the results of Eqs. (168), (177) and (178), the coordinate rotation matrix becomes

$$(179)\quad \tilde{\tilde{R}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \dfrac{B(\beta)}{|B(\beta)|} & \dfrac{-B(\beta)}{|B(\beta)|} \end{pmatrix}.$$

Again we transform the coordinates into the principal axes of $\tilde{Q}$. The expressions describing the unitary weights are obtained by evaluating

$$(180)\quad \tilde{\tilde{V}} = \tilde{\tilde{R}}^{\dagger}S$$

and

$$(181)\quad \tilde{\tilde{\Gamma}} = \tilde{\tilde{R}}^{\dagger}\tilde{\tilde{W}}.$$

Substituting Eqs. (177), (180) and (181) into Eq. (69) yields

$$(182)\quad \frac{d}{dt}\begin{pmatrix} \tilde{\gamma}_1(t) \\ \tilde{\gamma}_2(t) \end{pmatrix} + 2k\begin{pmatrix} B(0)+|B(\beta)| & 0 \\ 0 & B(0)-|B(\beta)| \end{pmatrix}\begin{pmatrix} \tilde{\gamma}_1(t) \\ \tilde{\gamma}_2(t) \end{pmatrix} + \frac{K\cdot A\cdot \sigma_{s_d}}{\sqrt{2}}\begin{pmatrix} e^{j\beta\mu_d} + \dfrac{B^*(\beta)}{B(\beta)} \\ e^{j\beta\mu_d} - \dfrac{B^*(\beta)}{|B(\beta)|} \end{pmatrix}.$$

the system of Eq. (182) is easily solved yielding as solutions

$$(183)\quad \tilde{\gamma}_1(t) = k_1 e^{-2k[B(0)+|B(\beta)|]t} + \frac{A\cdot\sigma_{s_d}\left(e^{j\beta\mu_d} + \frac{B^*(\beta)}{|B(\beta)|}\right)}{2\cdot\sqrt{2}(B(0) + |B(\beta)|)}$$

and

$$(184)\quad \tilde{\gamma}_2(t) = k_2 e^{-2k[B(0)-|B(\beta)|]t} + \frac{A\cdot\sigma_{s_d}\left(e^{j\beta\mu_d} - \frac{B^*(\beta)}{|B(\beta)|}\right)}{2\cdot\sqrt{2}[B(0)-|B(\beta)|]}$$

where k_1 and k_2 are the constants dependent upon the initial conditions of the system. From the inverse transformation given by Eq. (116), the complex array weights may be obtained from the above expressions. Again

$$(185)\quad \tilde{w} = \tilde{\tilde{R}}\tilde{\Gamma}$$

Making the necessary substitutions we find the expressions for the complex weight response are

$$(186)\quad \tilde{w}_1(t) = \frac{1}{\sqrt{2}}\left(k_1 e^{-2k\lambda_1 t} + k_2 e^{-2k\lambda_2 t}\right) + \frac{A\cdot\sigma_{s_d}[B(0)e^{j\beta\mu_d} - B^*(\beta)]}{4[B^2(0) - |B(\beta)|^2]}$$

and

$$(187)\quad \tilde{w}_2(t) = \frac{B(\beta)}{2|B(\beta)|}\left(k_1 e^{-2k\lambda_1 t} - k_2 e^{2k\lambda_2 t}\right) + \frac{A\cdot\sigma_{s_d}\left(B(0)-B(\beta)e^{j\beta\mu_d}\right)}{4\left(B^2(0) - |B(\beta)|^2\right)} .$$

These specific general expressions are given as a function of $B(\beta)$, the Fourier transform of $f(u)$. $f(u)$ is directly related to the spatial power density (Eq. (156)). The array weights given in Eqs. (186) and (187) then may be found from the spatial power density $D(\theta)$ and the resulting adaptive array performance may then be examined analytically.

In the next section we treat several specific examples of continuous jamming radiation. For convenience we have used the results for many discrete signals to represent this continuous sector.

VII. RESULTS

In the previous sections the values for the array weights of a two-element adaptive array were calculated. For the conditions of multiple signal interference or a continuous sector of interference, the steady-state array weights may be found. Digital computer programs have been written to carry out these computations. The programs calculate not only the final weights, but also the resultant array pattern and the signal-to-noise ratio SNR at the array output. The programs are given in Appendix II. In this section we discuss the results of these computations.

For a continuous interference sector it is found that when the desired signal is separated from the interference by more than 7 degrees, the SNR degrades as the interference sector becomes larger. When the separation between the desired signal becomes small ($< 7^{o}$), the SNR surprisingly increases as the interference sector becomes wider. This result is especially interesting and will be discussed again later.

Figures 9 through 15 show some typical plots of the final array patterns. In these plots the desired signal amplitude (σ_{S_1}) is 1. The total interference power is then 10dB greater than the desired signal power at each element. The element noise amplitude is .01. From Eqs. (125) and (126) for the final weight values, the antenna patterns are generated. The element spacing is chosen as $\lambda/2$ in each case.

In Fig. 9 the desired signal arrives at -30° off broadside. The interference sector is from 0° to 1°. We notice most of the interference is well within the null. In Fig. 10 the interference is increased to 5° and in Fig. 11 to 15°. (The interference power remains the same in each case.) We notice the center of the null remains at the center of the interference in each figure.

In Fig. 12 the interference sector remains 0° to 15° while the desired signal is now at -15°. We notice no pattern change. In Fig. 13 the desired signal arrives at -5°. A close observation shows that the null has moved slightly to the right and the pattern remains beamed upon the desired signal. However, as the interference sector is decreased to 5°, the desired signal suffers an attenuation of 5 dB as may be seen by comparing Fig. 13 with Fig. 14. A further attenuation of 2 dB is also evident in Fig. 15 when the interference sector is decreased to 1°.

Figures 16 through 19 show the signal-to-noise ratio SNR at the array output as a function of the interference sector width for various desired signal positions. In these figures, the interference sector varies in width from 0° to ANG°. In Fig. 16 the desired signal varies from -60° to -5° and in Fig. 17, -10° to 3°. When the desired signal is within 7° of the interference sector, we notice an increase in the SNR as the interference angle increases. The SNR increases because, as we have seen in Figs. 13 and 15, the array causes the null to turn away from the desired signal as the interference sector increases.

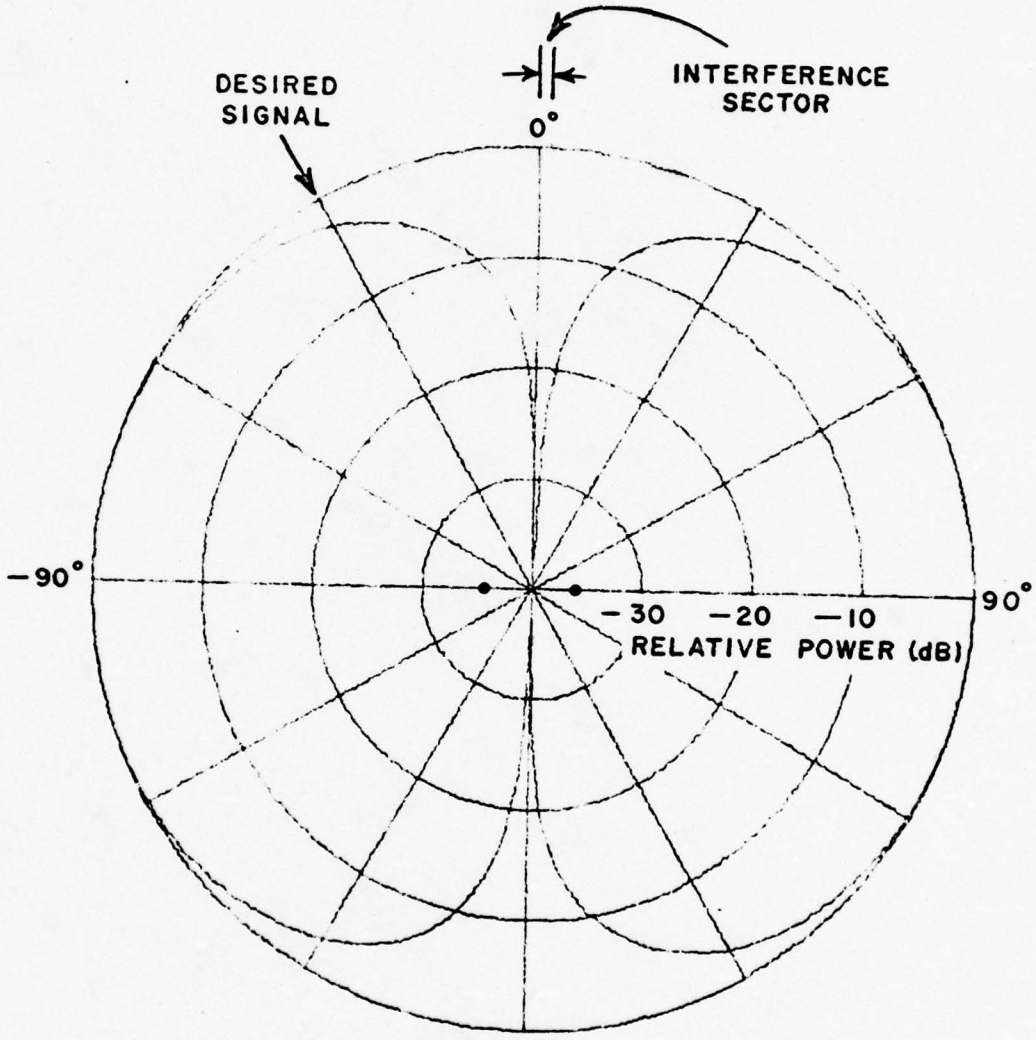

DESIRED
SIGNAL
INTERFERENCE
SECTOR
0°
-90°
90°
-30
-20
-10
RELATIVE POWER (dB)

Fig. 9.

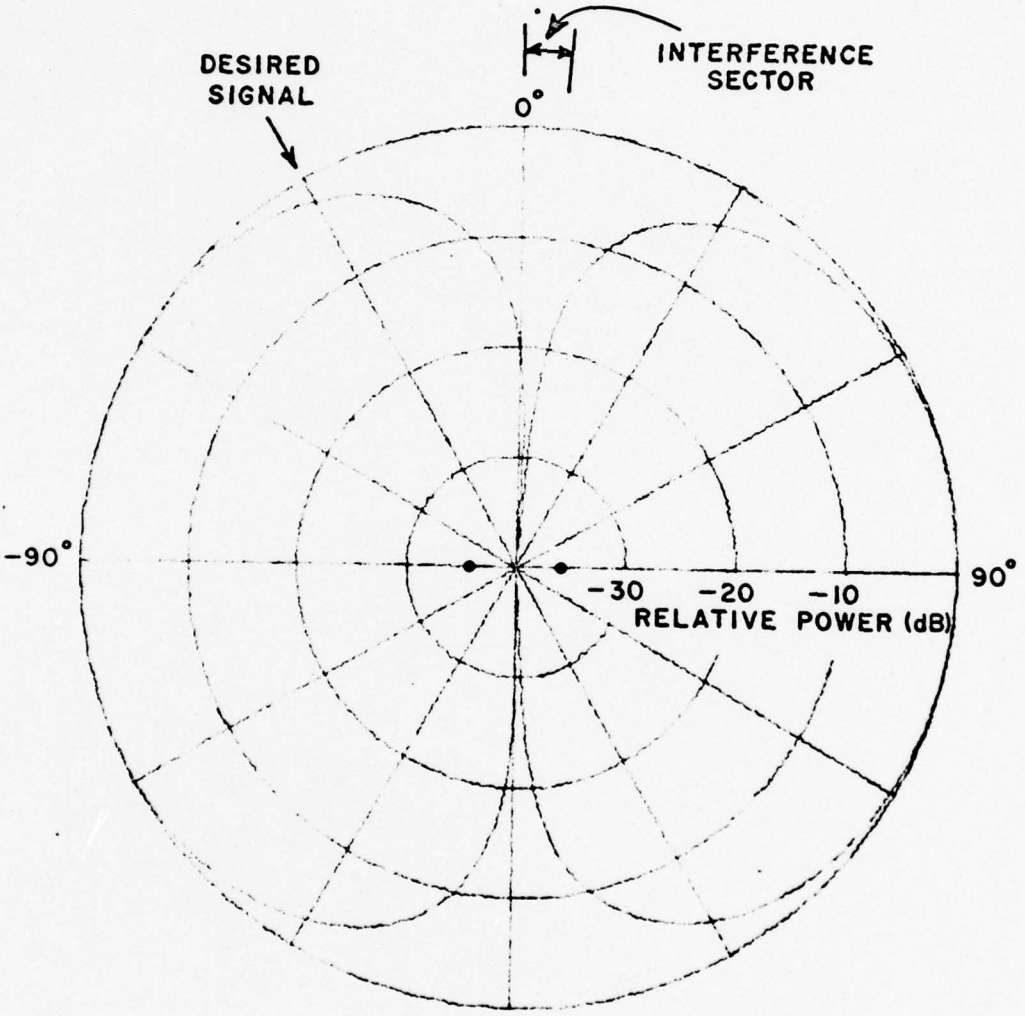

DESIRED
SIGNAL
INTERFERENCE
SECTOR
0°
-90°
90°
-30
-20
-10
RELATIVE POWER (dB)

Fig. 10.

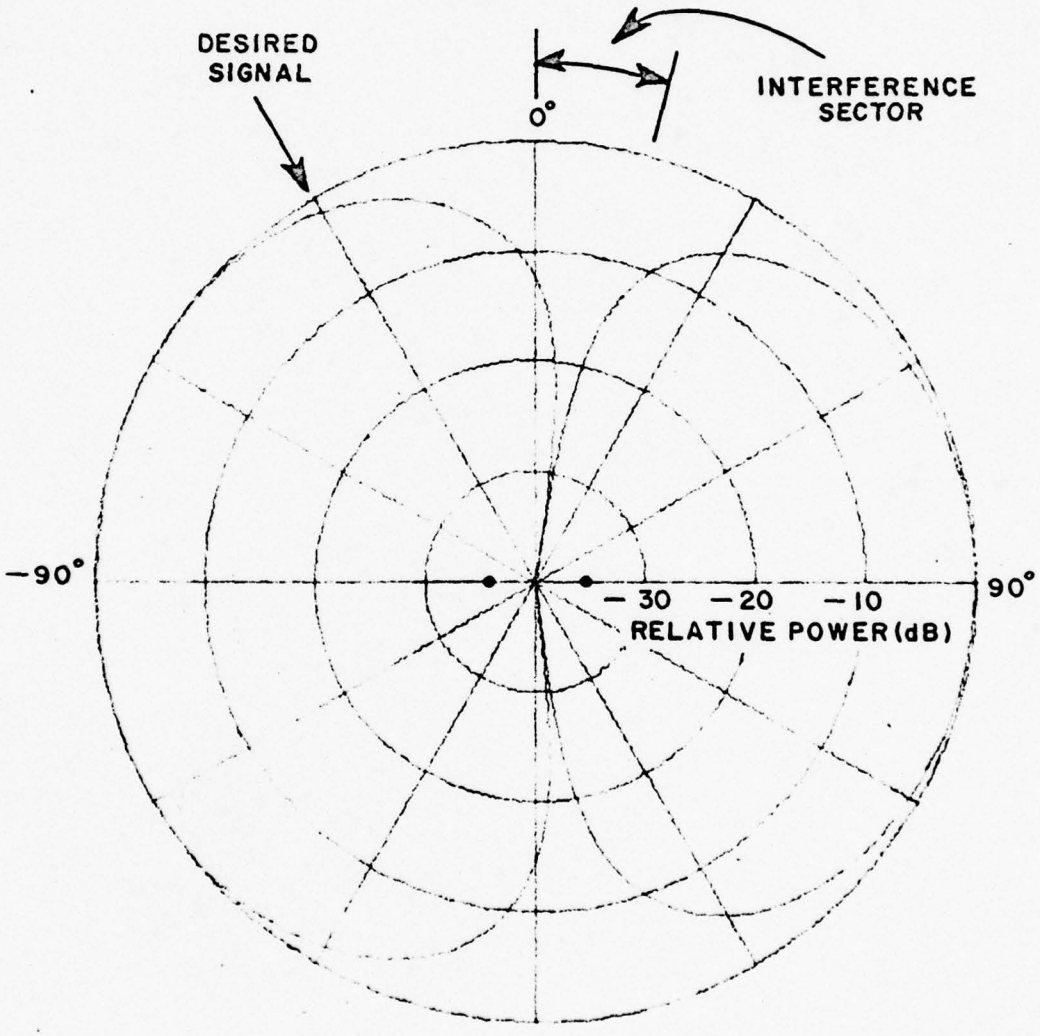

DESIRED
SIGNAL
0°
INTERFERENCE
SECTOR
−90°
90°
−30
−20
−10
RELATIVE POWER(dB)

Fig. 11.

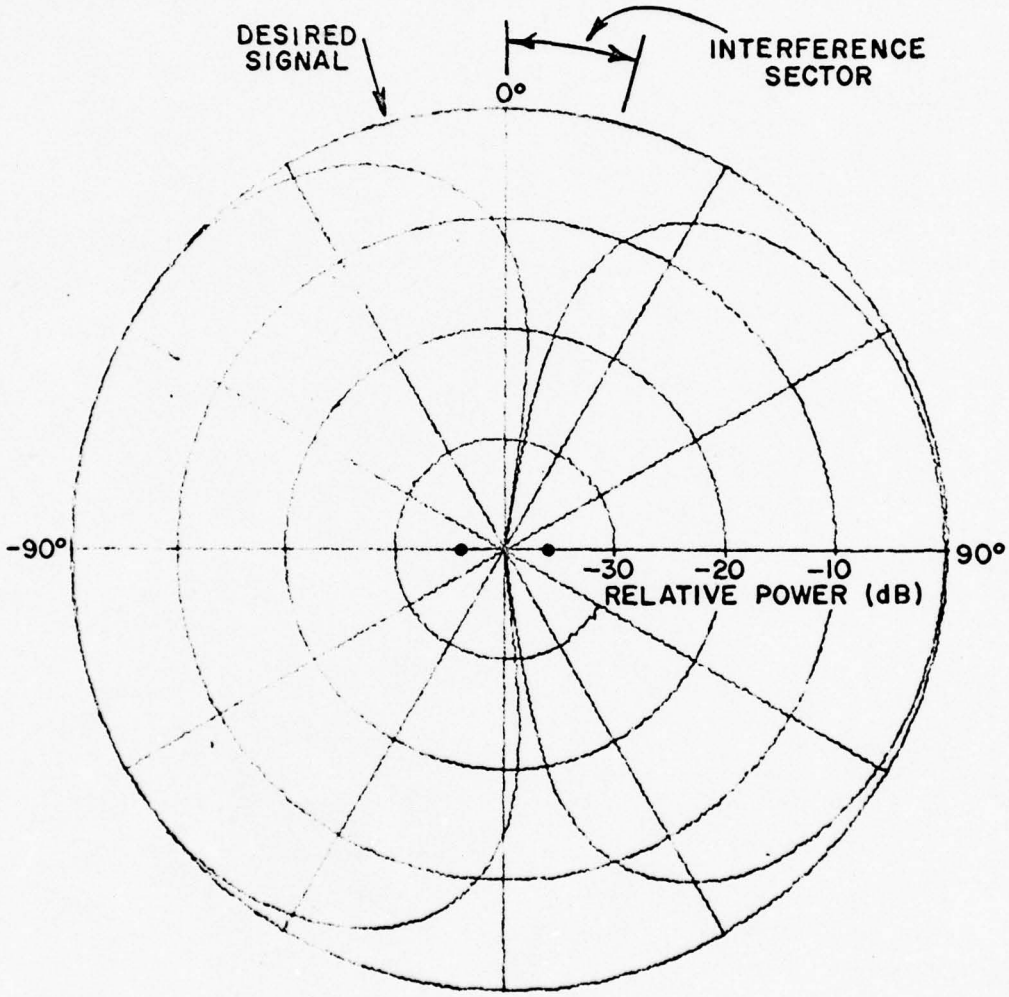

DESIRED
SIGNAL
0°
INTERFERENCE
SECTOR
-90°
90°
-30
-20
-10
RELATIVE POWER (dB)

Fig. 12.

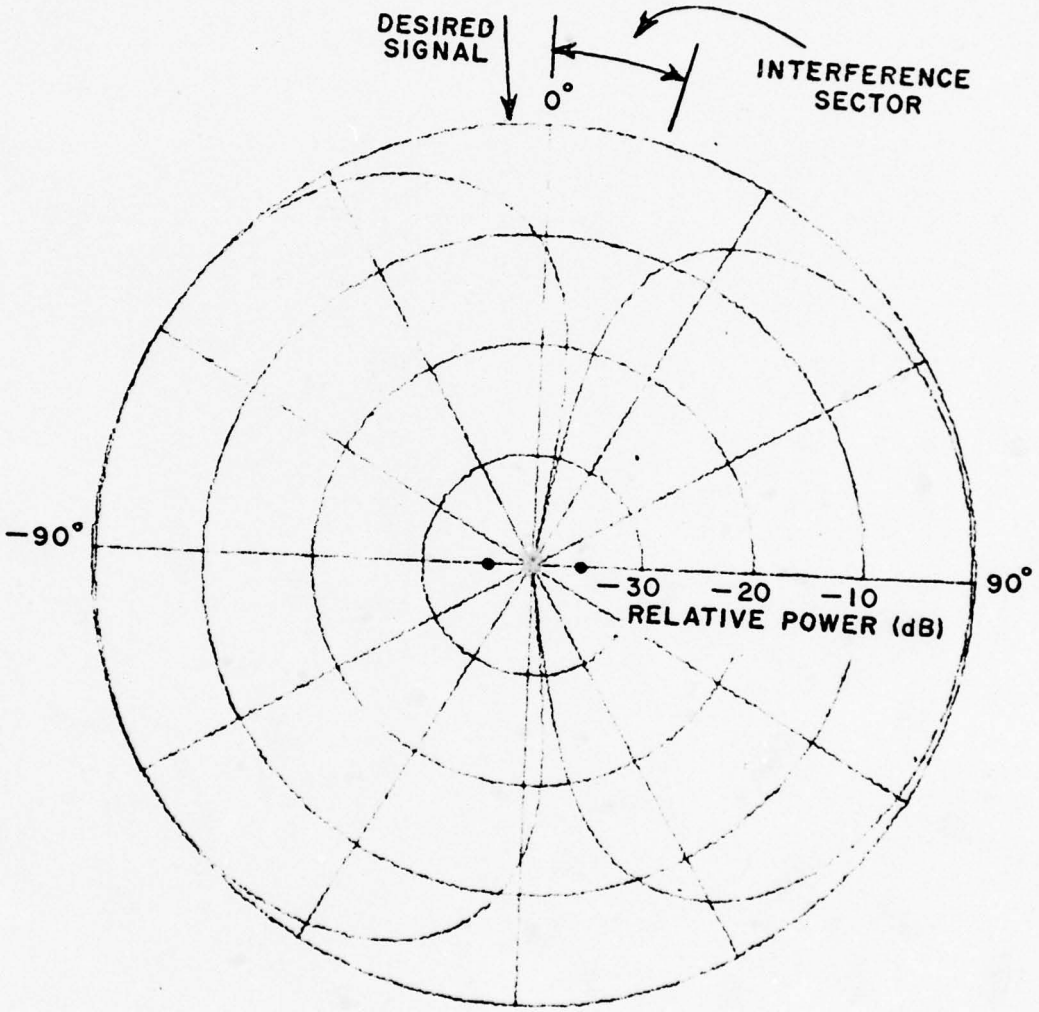

DESIRED SIGNAL
0°
INTERFERENCE SECTOR
−90°
90°
−30
−20
−10
RELATIVE POWER (dB)

Fig. 13.

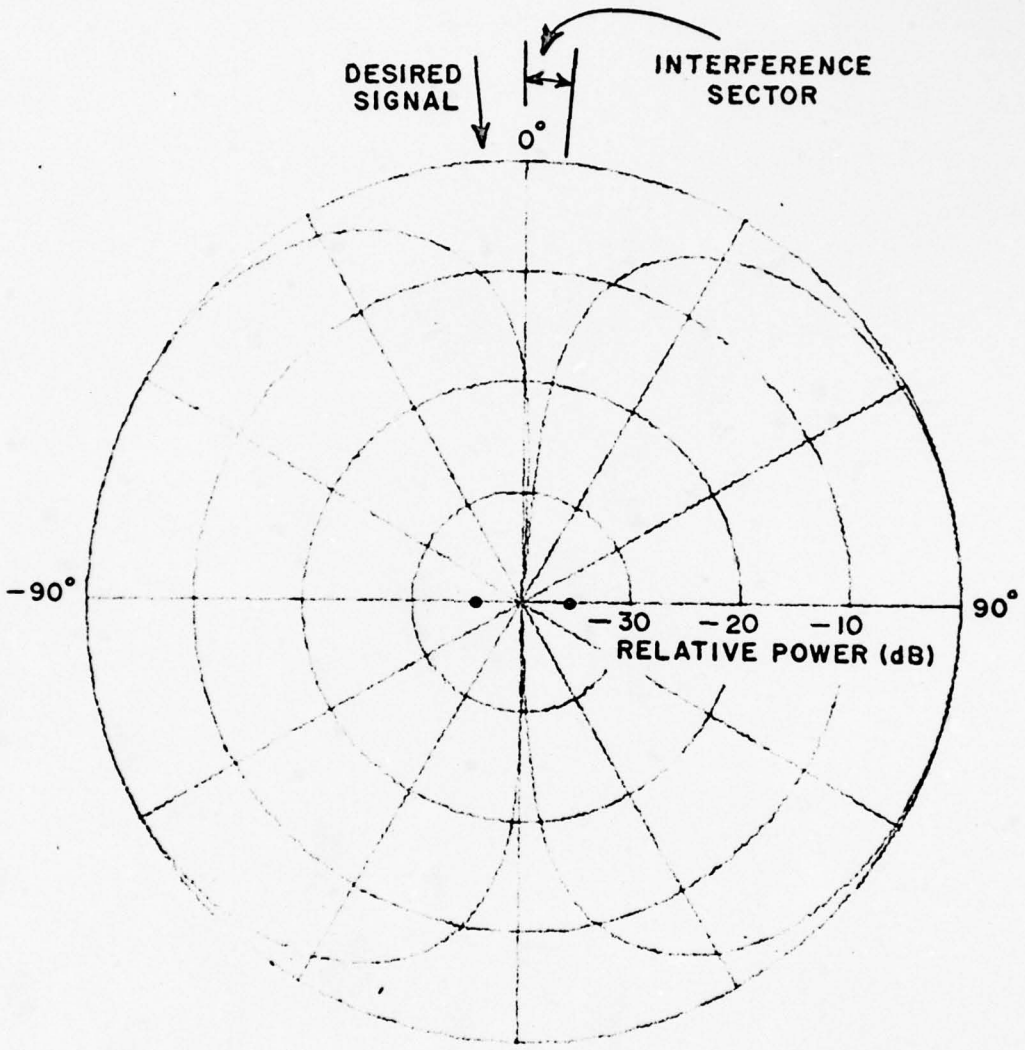

DESIRED SIGNAL
INTERFERENCE SECTOR
0°
-90°
90°
-30
-20
-10
RELATIVE POWER (dB)

Fig. 14.

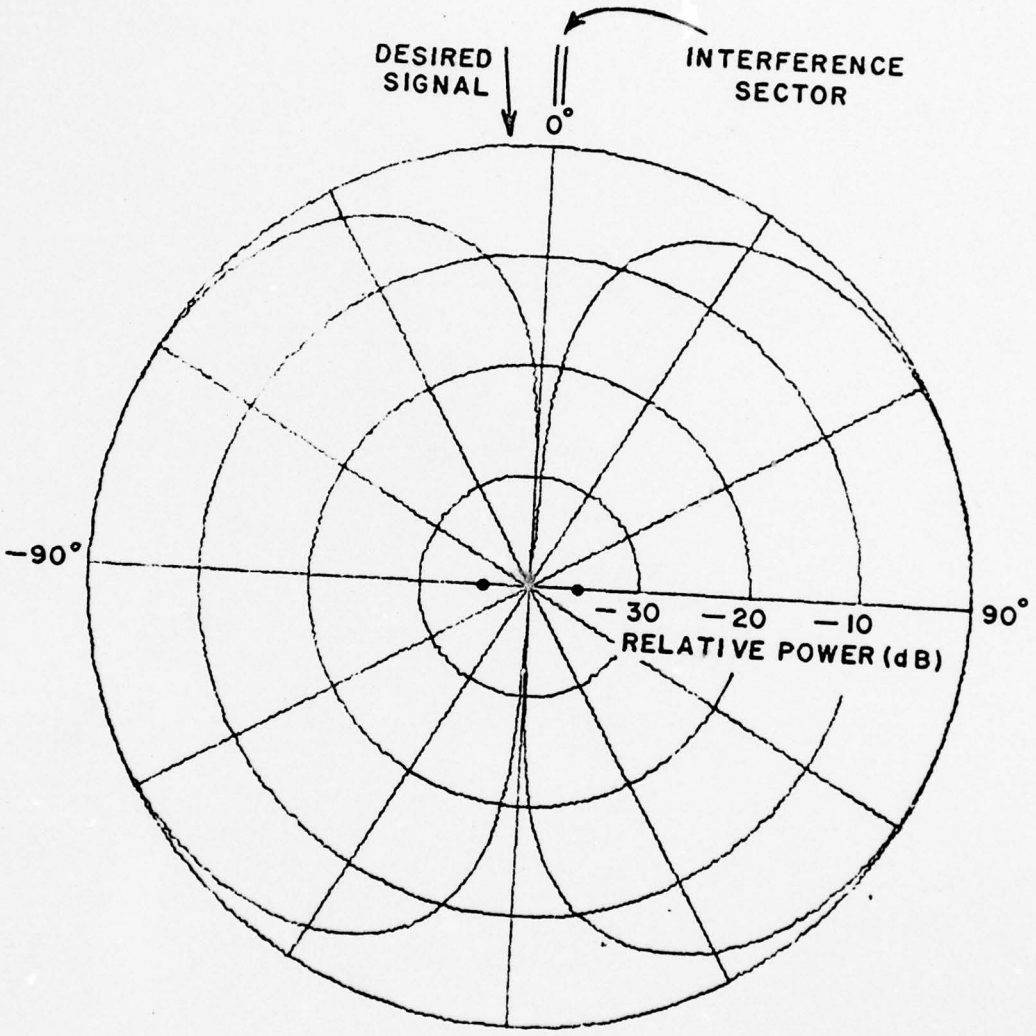

DESIRED SIGNAL
INTERFERENCE SECTOR
0°
-90°
90°
-30
-20
-10
RELATIVE POWER (dB)

Fig. 15.

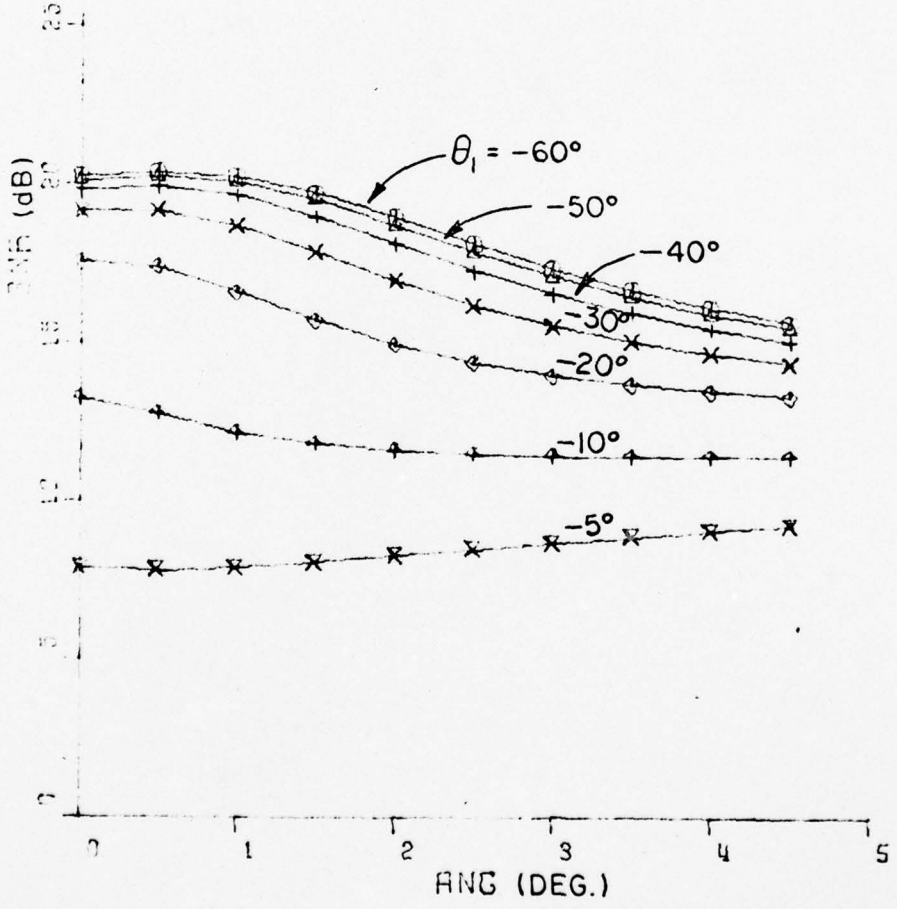


Fig. 16.

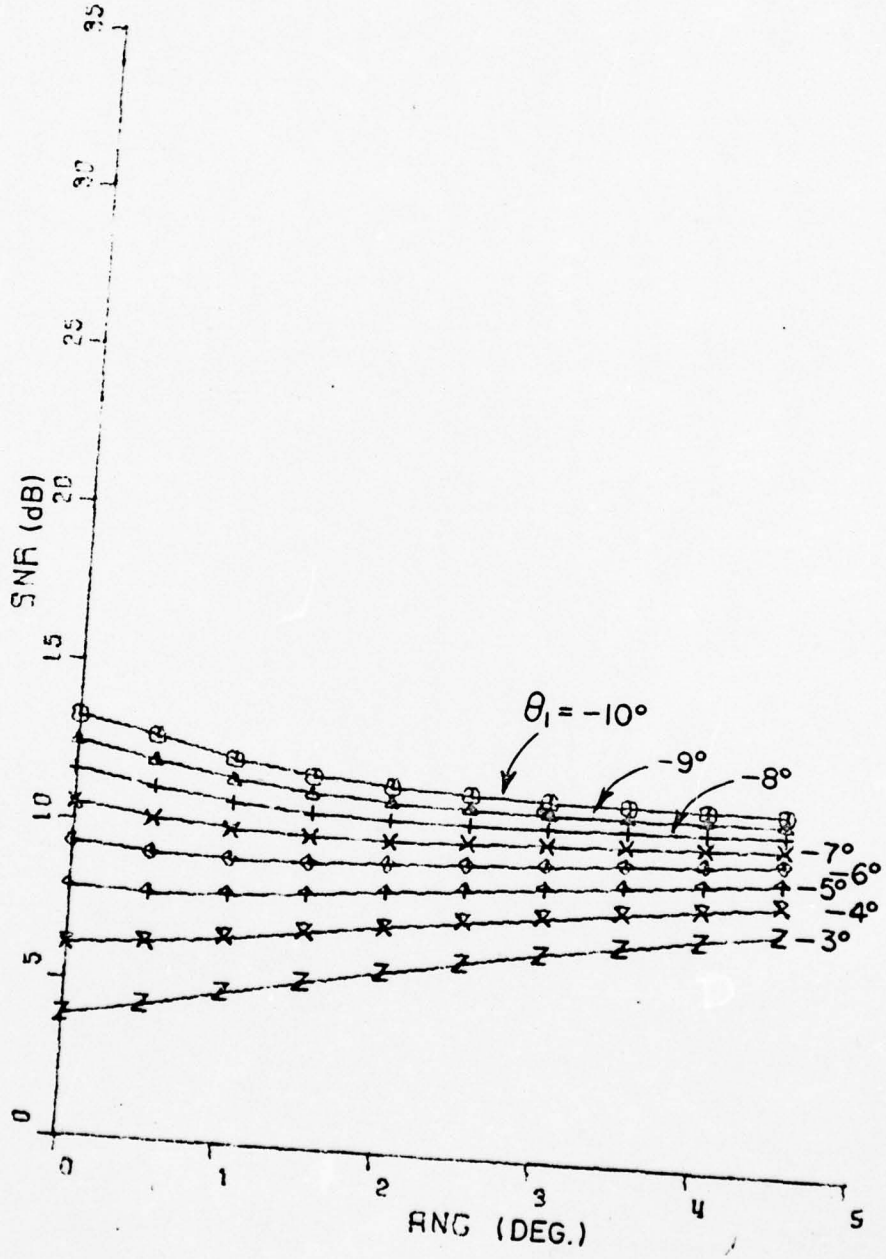


Fig. 17.

In Fig. 18 we notice the change in SNR is small beyond (AN6=) 5°.

In Fig. 19 we notice that beyond 15° there is no change in the SNR, regardless of the desired signal angle. For a jamming sector beyond 15°, an SNR degradation of 8 dB is apparent in Fig. 19 when the desired signal is at -60°. We, however, notice an SNR improvement of 4 dB when the radiation sector is increased from $.1^{\circ}$ to 30° for a desired signal arriving at -5°.

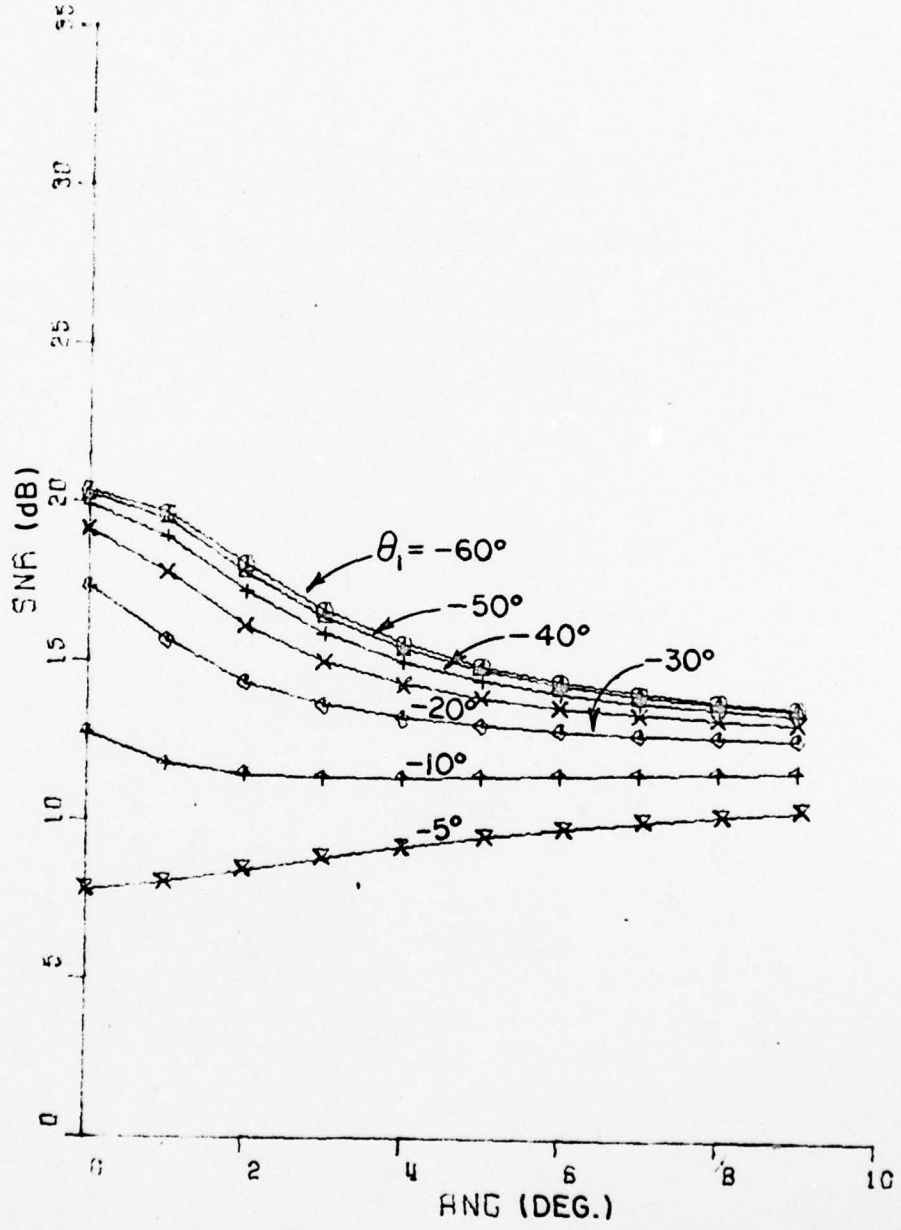

SNR (dB)
0
5
10
15
20
25
30
35
0
2
4
6
8
10
RNG (DEG.)
θ_i = -60°
-50°
-40°
-30°
-20°
-10°
-5°

Fig. 18.

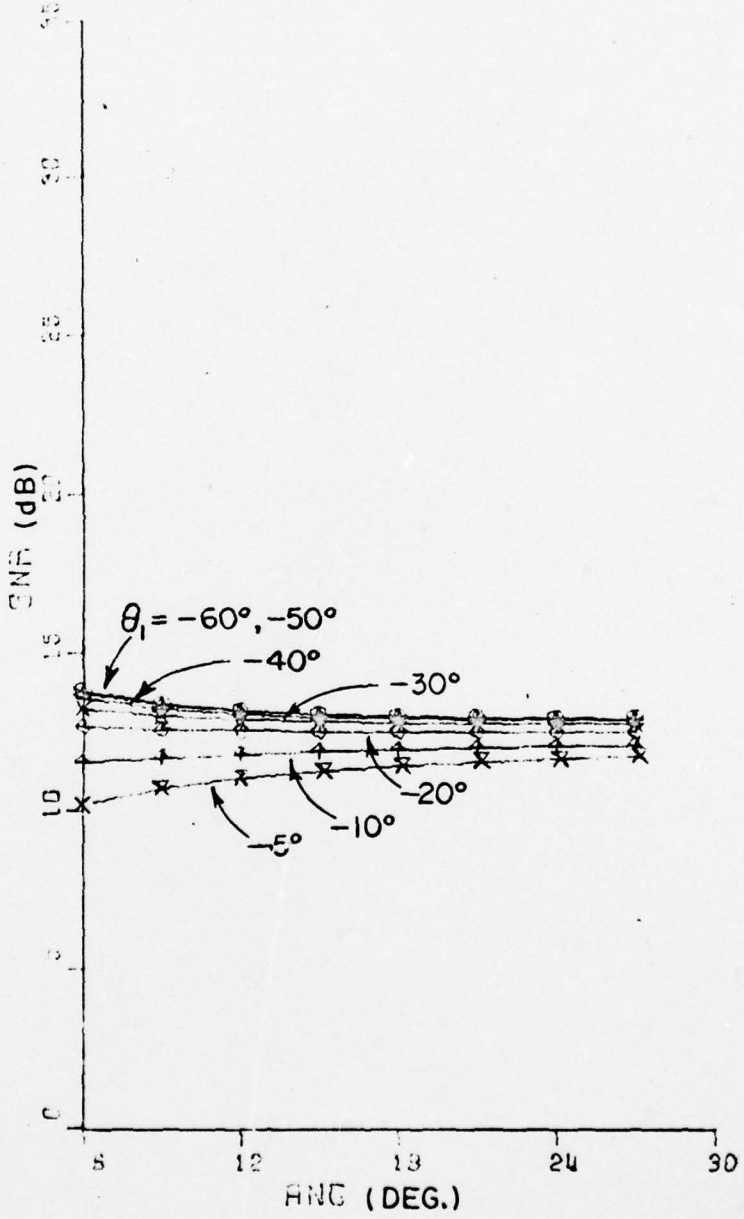


Fig. 19.

VIII. CONCLUSIONS

This report has presented a study of the behavior of a two-element adaptive array with multiple discrete signals and a contiuous interference sector. It has been determined mathematically that when the signal bandwidths are large compared to the feedback loop bandwidth, only the d.c. terms need be considered in the weight equations. Using this assumption, the weight solutions for a two-element adaptive array were derived by reducing the order of the equations describing the system. An analytical method for determinign the weight values with a continuous spatial distribution of interference is also presented.

From the weight solutions, we have observed that the element noise enables the jamming signal to be present at the array output when the array is fully constrained. When the array is over-constrained, the interference is always present at the array output. Also, control of the desired signal phase may be lost when many desired signals are present.

With a continuous interference sector present, we have seen that the SNR degrades as this sector increases, when the desired signal is separated from the interference by more than 7°. When this separation becomes small ($<7^{\circ}$), we find that the SNR improves as the jamming sector width increases.

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APPENDIX I

In this appendix we discuss a method for determining the coordinate rotation matrix R given in Eq. (57) from its reduced form $\tilde{R}$ given by Eq. (100).

The equations describing the response of a two-element adaptive array are given in Eq. (57). It was noted by Compton |21| that when Φ exhibits the type of symmetry shown in Eq. (58), the four coupled equations of Eq. (57) may be reduced to a pair of complex equations and the eigenvalues of Φ may be determined from the reduced form of Φ. It will be shown in this appendix that the eigenvectors and hence the coordinate rotation matrix may also be found from the reduced equations.

It was noted in section III that by allowing

(A-1) $\tilde{A} = \phi_{11} + j\phi_{12}$

(A-2) $\tilde{B} = \phi_{13} + j\phi_{14}$

(A-3) $\tilde{C} = \phi_{31} + j\phi_{32}$

(A-4) $\tilde{D} = \phi_{33} + j\phi_{34}$

a new matrix $\tilde{Q}$ can be formed given by

$$(A\text{-}5)\quad \tilde{\tilde{Q}} = \begin{pmatrix} \tilde{\tilde{A}} & \tilde{\tilde{B}} \\ \tilde{\tilde{C}} & \tilde{\tilde{D}} \end{pmatrix}.$$

Then by finding the eigenvalues of $\tilde{Q}$, the eigenvalues of Φ may be determined |21|. Specifically, if the two eigenvalues of $\tilde{Q}$ are

$$(A\text{-}6)\quad \lambda_{Q_i} \cdot \ell_1 \pm j\ell_2 = L_1,\ \ell_3 \pm j\ell_4 = L_2$$

then the eigenvalues of Φ are

$$(A\text{-}7)\quad \lambda_{\Phi_i} = L_1, L_1^*, L_2, L_2^* \ .$$

Recall that it was also shown in section II that the equation of a two-element array may be reduced to the form

$$(A\text{-}8)\quad \frac{d}{dt}\begin{pmatrix} w_1 - jw_2 \\ w_3 - jw_4 \end{pmatrix} + 2k \begin{pmatrix} \phi_{11}+j\phi_{12} & \phi_{13}+j\phi_{14} \\ \phi_{31}+j\phi_{32} & \phi_{33}+j\phi_{34} \end{pmatrix} \begin{pmatrix} w_1 - jw_2 \\ w_3 - jw_4 \end{pmatrix} = 2k \begin{pmatrix} s_1 - js_2 \\ s_3 - js_4 \end{pmatrix}.$$

In order to solve Eq. (A-8) we make a rotation of coordinates. Let

$$(A\text{-}9)\quad \tilde{\tilde{W}} = \begin{pmatrix} w_1 - jw_2 \\ w_3 - jw_4 \end{pmatrix} = \begin{pmatrix} \tilde{\tilde{\Gamma}}_{11} & \tilde{\tilde{\Gamma}}_{12} \\ \tilde{\tilde{\Gamma}}_{21} & \tilde{\tilde{\Gamma}}_{22} \end{pmatrix} \begin{pmatrix} \tilde{\tilde{\gamma}}_1 \\ \tilde{\tilde{\gamma}}_2 \end{pmatrix} = \tilde{\tilde{R}}\tilde{\tilde{\gamma}} \ .$$

Substituting Eq. (A-9) into Eq. (A-8) and multiplying the result on the left by R^{-1} gives

(A-10)
$$\frac{d}{dt}\begin{pmatrix}\tilde{\gamma}_1\\ \tilde{\gamma}_2\end{pmatrix} + 2k\left[\begin{pmatrix}\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12}\\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{pmatrix}\begin{pmatrix}\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12}\\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22}\end{pmatrix}\right]\begin{pmatrix}\tilde{\gamma}_1\\ \tilde{\gamma}\end{pmatrix} = 2k\begin{pmatrix}\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12}\\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{s}_1\\ \tilde{s}_2\end{pmatrix}$$

where

(A-11)
$$\begin{pmatrix}\tilde{s}_1\\ \tilde{s}_2\end{pmatrix} = \begin{pmatrix}s_1 - j\,s_2 & \\ s & -j\;s_4\end{pmatrix}.$$

If the rotation matrix $\tilde{R}$ is chosen so the bracketed matrix product is diagonal,

(A-12)
$$\begin{pmatrix}\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12}\\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22}\end{pmatrix}^{-1}\begin{pmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{pmatrix}\begin{pmatrix}\tilde{\Gamma}_{11} & \tilde{\Gamma}_{12}\\ \tilde{\Gamma}_{21} & \tilde{\Gamma}_{22}\end{pmatrix} = \begin{pmatrix}L_1 & 0\\ 0 & L_2\end{pmatrix}$$

then Eq. (A-8) are uncoupled and the solution is easily obtained. The values L_1 and L_2 are the eigenvalues of Q. The unitary rotation matrix R is composed of the eigenvectors E_1 and E_2 corresponding to the eigenvalues L_1 and L_2,

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(A-13) $\tilde{\tilde{R}} = \frac{1}{\sqrt{r}} (E_1, E_2)$.

Since $\tilde{R}$ is unitary

(A-14) $\tilde{\tilde{R}}^{-1} = \tilde{\tilde{R}}^{\dagger}$.

We can then rewrite Eq. (A-12) as

(A-15)

$$\frac{1}{k}\begin{pmatrix} x_{11}-jy_{11} & x_{21}-jy_{21} \\ x_{12}-jy_{12} & x_{22}-jy_{22} \end{pmatrix}\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}\begin{pmatrix} x_{11}+jy_{11} & x_{12}+jy_{12} \\ x_{21}+jy_{21} & x_{22}+jy_{22} \end{pmatrix} = \begin{pmatrix} \ell_1+j\ell_2 & 0 \\ 0 & \ell_3+j\ell_4 \end{pmatrix},$$

where the elements of $\tilde{R}$ are now defined as

(A-16) $$\tilde{\tilde{\Gamma}}_{ij} = \frac{x_{ij}+jy_{ij}}{\sqrt{k}}$$

From Eq. (47) we have

(A-17)

$$R^T \Phi R = \frac{1}{k}\begin{pmatrix} \Gamma_{11} & \Gamma_{21} & \Gamma_{31} & \Gamma_{41} \\ \Gamma_{12} & \Gamma_{11} & \Gamma_{32} & \Gamma_{42} \\ \Gamma_{13} & \Gamma_{23} & \Gamma_{33} & \Gamma_{43} \\ \Gamma_{14} & \Gamma_{24} & \Gamma_{34} & \Gamma_{44} \end{pmatrix}\begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} \\ \phi_{21} & \phi_{22} & \phi_{23} & \phi_{24} \\ \phi_{31} & \phi_{32} & \phi_{33} & \phi_{34} \\ \phi_{41} & \phi_{42} & \phi_{43} & \phi_{44} \end{pmatrix}\begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{pmatrix} =$$

$$\begin{pmatrix} \ell_1 & \ell_2 & 0 & 0 \\ -\ell_2 & \ell_1 & 0 & 0 \\ 0 & 0 & \ell_3 & \ell_4 \\ 0 & 0 & -\ell_4 & \ell_3 \end{pmatrix}$$

(See [21] pp. 90-91.)

From Eq. (A-17) we notice

$$\text{(A-18)} \quad \text{Real}(L_1) = \ell_1 = \frac{1}{k} \sum_{i=1}^{4} \sum_{j=1}^{4} \Gamma_{i1} \phi_{ij} \Gamma_{j1}$$

$$\text{(A-19)} \quad \text{Imag}[L_1] = \ell_2 = \frac{1}{k} \sum_{i=1}^{4} \sum_{j=1}^{4} \Gamma_{i1} \phi_{ij} \Gamma_{j2},$$

etc.

The following relations are evident from the symmetry shown in Eq. (58):

$$\text{(A-20a)} \quad \phi_{11} = \phi_{22}$$

$$\text{(A-20b)} \quad \phi_{12} = \phi_{21}$$

$$\text{(A-20c)} \quad \phi_{13} = \phi_{24}$$

(A-20d) $\phi_{14} = -\phi_{23}$

(A-20e) $\phi_{31} = \phi_{42}$

(A-20f) $\phi_{32} = -\phi_{41}$

(A-20g) $\phi_{33} = \phi_{44}$

(A-20h) $\phi_{34} = -\phi_{43}$

Now if we expand Eqs. (A-15) and (A-17) into the form of Eqs. (A-18) and (A-19) and make a term-by-term comparison imposing the relationships of Eqs. (A-20), we find

(A-21a) $\Gamma_{11} = x_{11}$

(A-21b) $\Gamma_{21} = -y_{11}$

(A-21c) $\Gamma_{31} = x_{21}$

(A-21d) $\Gamma_{41} = -y_{21}$

(A-22a) $\Gamma_{12} = y_{11}$

(A-22b) $\Gamma_{22} = x_{11}$

(A-22c) $\Gamma_{32} = y_{21}$

(A-22d) $\Gamma_{42} = x_{21}$

(A-23a) $\Gamma_{43} = x_{12}$

(A-23b) $\Gamma_{23} = -x_{12}$

(A-23c) $\Gamma_{33} = x_{22}$

(A-23d) $\Gamma_{43} = -y_{22}$

(A-24a) $\Gamma_{14} = y_{12}$

(A-24b) $\Gamma_{24} = x_{12}$

(A-24c) $\Gamma_{34} = y_{22}$

(A-24d) $\Gamma_{44} = x_{22}$.

In other words

$$(A\text{-}25)\quad R = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{pmatrix} = \begin{pmatrix} x_{11} & y_{11} & x_{12} & y_{12} \\ -y_{11} & x_{11} & -y_{12} & x_{12} \\ x_{21} & y_{32} & x_{22} & y_{22} \\ -y_{21} & x_{21} & -y_{22} & x_{22} \end{pmatrix}$$

where x_{ij} and y_{ij} are the term of $\tilde{R}$ given in Eq. (A-16). Then by finding the complex rotation matrix $\tilde{R}$, we have also determined the real rotation matrix R. This is also to say that by finding the eigenvectors of $\tilde{Q}$, we have found the eigenvectors of Φ.

APPENDIX II

```
      DIMENSION ALPH(5)
      COMPLEX ALPH
C********PATTERN ROUTINE**********
      DIMENSION A(12), THETA(12), SIG(12), SIP(12)
      COMPLEX TER, XYZ
      COMPLEX DESO, AJMC
      COMPLEX XSJ, XS, W1C ,W2C
      COMPLEX SIP,X,T,U,SRP,XX,TT,UU
      COMPLEX YY,ZZ,SS,AAN
      DATA PI, TPI/3.14159265, 6.28318531/
      WRITE(8,61)
   61 FORMAT('TOTAL NUMBER OF SIGNALS')
      READ(8,-) N
      WRITE(8,77)
   77 FORMAT(' TOTAL NUMBER OF DESIRED SIGNALS')
      READ(8,-) M
      WRITE(8,62)
   62 FORMAT('SIGNAL LOCATIONS')
      DO 2 J=1,N
      READ(8,-) THETA(J)
      SIG(J)=1.
      NOR=N-M
      TY=THETA(J)/(57.2957758)
      A(J)=PI*SIN(TY)
    2 CONTINUE
      WRITE(8,63)
   63 FORMAT(' NOISE AMPLITUDE')
      READ(8,-) AN
      DESO=CMPLX(0.,0.)
      AJMC=CMPLX(0.,0.)
      RNT=0.
      DO 21 I=1,N
      DO 21 J=1,N
   21 RNT=RNT+(1.-COS(A(I)-A(J)))
      X=CMPLX(0.,0.)
      XS=X
      DO 5 K=1,N
      WRITE(6,-) THETA(K)
      XX=CMPLX(0.,-A(K))
      XSJ=CMPLX(0.,A(K))
      XS=XS+SIG(K)*SIG(K)*CEXP(XX)
    5 CONTINUE
```

```
      DO 7 K=1,M
      XX=CMPLX(0., +A(K))
    7 X=X+SIG(K)*CEXP(XX)
      Y=0.
      MM=M+1
      DO 8 K=1,N
      Y=Y+SIG(K)*SIG(K)
      Z=0.
      CONTINUE
    8 CONTINUE
      Y=Y+2.*AN*AN
      DO 9 K=1,M
    9 Z=Z+SIG(K)
      T=CMPLX(0.,0.)
      S=0.
      DO 10 K=1,N
      TT=CMPLX(0.,+A(K))
      S=S+SIG(K)*SIG(K)
   10 T=T+SIG(K)*SIG(K)*CEXP(TT)
      DEN=4.*AN*AN*(AN*AN+S)+RNT
      YYZ=CMPLX(1.,0.)
      YY=CMPLX(Y,0.)
      ZZ=CMPLX(Z,0.)
      AAN=CMPLX(AN,0.)
      W1C=(X*YY-ZZ*T)/DEN
      W2C=(ZZ*YY-XS*X)/DEN
      ALPH(1)=W1C
      ALPH(2)=W2C
      WRITE(6,*) ALPH(1)
      CALL EPLT(ALPH)
  901 CONTINUE
      END
      SUBROUTINE EPLT(ALPH)
      COMPLEX FIELD, ART, ALPH
      DIMENSION ALPH(10), FM(361),ARR(10), ART(10)
      DIMENSION W(10)
      DIMENSION IBUF(100)
      DATA PI,TPI,DPR/3.14159265,6.2831853,57.2957795
      CALL PLOTS(IBUF,100,3)
      CALL FACTOR(.5)
      FMX=0.
      LI=2
      IQ=LI-1
      WRITE(1,777) IPEQ, ISS
  777 FORMAT(' IPEQ=',I2,' ISS=',I2//)
      DO 29 J=1,361,1
      FIELD=(0.,0.)
      DO 30 IJ=1,LI,1
      IQ=IJ-1
```

```
      ARR(IJ)=PI*(1./1.)*COS(J/DPR)
      ART(IJ)=CMPLX(0.,ARR(IJ))
      FIELD=FIELD+ALDR(IJ)*CEXP(IO*ART(IJ))
30    CONTINUE
      FIELDR=CABS(FIELD)
      IF(FIELDR.GE.FMX)FMX=FIELDR
      FM(J)=FIELDR
29    CONTINUE
      CALL POLPLT(FM,FMX)
      RETURN
      END
      SUBROUTINE POLPLT(FM,FMX)
      DIMENSION FM(361)
      DATA PI,TPI,DPR/3.14159265,6.2831853,57.2957795
      CALL PLOT(5.5,5.5,-3)
      DO 110 I=1,4
      RAD=5.*(5-I)/4.
      CALL PLOT(RAD,0.,3)
      DO 11 J=1,181
      ANG=(J-1)*PI/90.
      XX=RAD*COS(ANG)
      YY=RAD*SIN(ANG)
110   CALL PLOT(XX,YY,2)
      DO 112 I=1,6
      ANG=(I-1)*PI/6.
      ANGS=ANG+PI
      ANGE=ANG
      IF(I.EQ.2*(I/2)) GO TO 111
      ANGS=ANG
      ANGE=ANG+PI
111   CONTINUE
      XX=5.*COS(ANGS)
      YY=5.*SIN(ANGS)
      CALL PLOT(XX,YY,3)
      XX=5.*COS(ANGE)
      YY=5.*SIN(ANGE)
112   CALL PLOT(XX,YY,2)
      DO 120 I=1,361
      RAD=20.*ALOG10(FM(I)/FMX)
      RAD=(RAD+40.)/8.
      IF(RAD.LT.0.)RAD=0
      ANG=(I-1)/DPR
      YY=RAD*SIN(ANG)
      XX=RAD*COS(ANG)
      IPEN=2
      IF(I.EQ.1) IPEN=3
120   CALL PLOT(XX,YY,IPEN)
      CALL PLOT(7.5,-5.5,-3)
      CALL PLOT(0.,0.,999)
      RETURN
      ENDS
```

```
      DIMENSION ALPH(5)
      COMPLEX ALPH
C*********S/J ROUTINE*******
      DIMENSION XAT(8)
      DIMENSION DTS(11)
      DIMENSION A(13), THETA(13), SIG(13), SIP(13)
      DIMENSION IBUF(100)
      DIMENSION XIT(13), SNR(13)
      COMPLEX TER, XYZ
      COMPLEX XY1,ECM,YS1,SJ,EENP
      COMPLEX DESO, AJMC
      COMPLEX XSJ, XS, W1C ,W2C
      COMPLEX SIF,X,T,U,SRP,XX,TT,UU
      COMPLEX YY,ZZ,SS,AAN
      DATA PI, TPI/3.14159265, 6.2831853/
      N=1
      SIG(1)=.1
      XAT(1)=-10.
      XAT(2)=-9.
      XAT(3)=-8.
      XAT(4)=-7.
      XAT(5)=-6.
      XAT(6)=-5.
      XAT(7)=-4.
      XAT(8)=-3.
      READ(8,-) THETA(2)
      READ(8,-) RRR
      CONTINUE
      CALL PLOTS(IBUF,100,3)
      CALL FACTOR(.9)
      T2=THETA(2)
      XRX=2.*RRR
      CALL AXIS(0.,1.,3HANG,-3,5.,0.,0.0,XRX,1.,-1)
      CALL AXIS(0.,1.,3HSNR,3,7.,90.,0.,5.,1.,-1)
      CALL PLOT(0.,2.,-3)
  200 CONTINUE
      DO 119 JA=1,8
      THETA(1)=XAT(JA)
      DTS(1)=.1
      DTS(2)=.2
      DTS(3)=.3
      DTS(4)=.4
      DTS(5)=.5
      DTS(6)=.6
      DTS(7)=.70
      DTS(8)=.80
      DTS(9)=.90
      DTS(10)=1.0
      DTS(11)=0.
      DO 100 IJ=1,10
      N=12
      DO 55 I=2,N
      IR=I+1
   55 THETA(IR)=THETA(I)+RRR*DTS(IJ)
      NM=N+1
```

```
      DO 15 J=1,M
      TY=THETA(J)/(57.29577951)
      A(J)=PI*SIN(TY)
15    CONTINUE
      AN=.01
      DESO=CMPLX(0.,0.)
      AUMC=CMPLX(0.,0.)
      DO 42 I=2,M
      ENOE=D-1
42    SIG(I)=1.
      RNT=0.
      DO 21 I=1,M
      DO 21 J=1,M
      SI2=SIG(I)*SIG(I)
      SJ2=SIG(J)*SIG(J)
      SIJ2=SI2*SJ2
21    RNT=RNT+SIJ2*(1.-COS(A(I)-A(J)))
      X=CMPLX(0.,0.)
      XS=X
      DO 5 K=1,M
      XX=CMPLX(0.,-A(K))
      XSJ=CMPLX(0.,A(K))
      XS=XS+SIG(K)*SIG(K)*CEXP(XX)
5     CONTINUE
      DO 7 K=1,M
      XX=CMPLX(0., A(K))
7     X=X+SIG(K)*CEXP(XX)
      Y=0.
      MM=M+1
      DO 8 K=1,M
      Y=Y+SIG(K)*SIG(K)
      Z=0.
      CONTINUE
8     CONTINUE
      Y=Y+2.*AM*AM
      DO 9 K=1,M
9     Z=Z+SIG(K)
      T=CMPLX(0.,0.)
      S=0.
      DO 10 K=1,M
      TT=CMPLX(0.,+A(K))
      S=S+SIG(K)*SIG(K)
10    T=T+SIG(K)*SIG(K)*CEXP(TT)
      DEN=4.*AM*AM*(AN*AN+S)+RNT
      XX2=CMPLX(1.,0.)
      YY=CMPLX(Y,0.)
      ZZ=CMPLX(Z,0.)
      AM=CMPLX(AM,0.)
      WIC=(X*YY-ZZ*T)/DEN
      W2C=(ZZ*YY-XS*X)/DEN
      ALPH(1)=W1C
      ALPH(2)=W2C
      ECM=CMPLX(1.,1.)
      TI=M+1
      BOJ=0.
```

```
      DO 36 J=2,N
      W1=REAL(W1C)
      W2=-AIMAG(W1C)
      W3=REAL(W2C)
      W4=-AIMAG(W2C)
      RIN=W4+(W1*SIN(A(J)))+(W2*COS(A(J)))
      RQN=W3+(W1*COS(A(J)))-(W2*SIN(A(J)))
      SJP=((RIN*RIN)+(RQN*RQN))*(STG(J)**2)
      PJJ=PJJ+SJP*SJP
   36 CONTINUE
      RDI=W4+(W1*SIN(A(1)))+(W2*COS(A(1)))
      RDQ=W3+(W1*COS(A(1)))-(W2*SIN(A(1)))
      SDP=((RDI*RDI)+(RDQ*RDQ))*(STG(1)**2)
      PD=10.*ALOG10(SDP)
      ENP=2.*AN*AN*(W1**2+W2**2+W3**3+W4**2)
      PNN=PJJ+ENP
      PN=10.*ALOG10(PNN)
      NT=N-1
      SNR(IJ)=PD-PN
      WRITE(6,-) PD, PN, SNR(IJ)
   22 FORMAT(F10.6//)
  100 CONTINUE
      CONTINUE
      CALL LINE(DTS,.1,.2,SNR,0.,5.,10.1,JA)
      WRITE(6,91)
   91 FORMAT(///)
      READ(8,-)
  119 CONTINUE
      CALL PLOT(0.,0.,999)
      CALL EXIT
      END
```