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MATCHED FILTERING AND OPTIMAL USE OF AN ANTENNA,(U)
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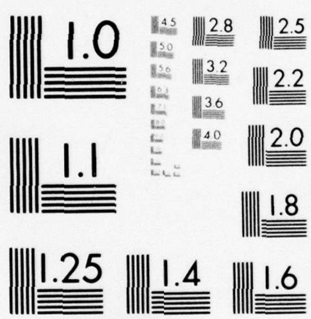

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ABSTRACT

Originally published in the French language in "Signal Processing — with Emphasis on Underwater Acoustics," by the NATO Advanced Study Institute, 14-26 September 1964, pp. 162-299, this study has been translated to serve as a ready reference work. The study itself lies in the area of research that is concerned with the improvement of long-range-detection techniques. Subjects covered include (1) a formal solution to matched filtering with N inputs, (2) the case of identical signals — proper filtering — orthogonal images of a system with N noise inputs, (3) narrow-band approximation — variation of proper filtering, (4) case of two inputs — coherent noise, (5) matched filtering and directivity, and (6) generation of proper filtering — autoadaptive systems — practical stationarity conditions.

ADMINISTRATIVE INFORMATION

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GLOSSARY OF SYMBOLS

a	separation of receiver elements
α	column matrix of ones
$\alpha(t)$	unitary signal
$B(t), b(t)$	noise inputs
b	ratio of $S/N(NPF)$ to S/N (simple summation)
b_j^{ω}	elementary noise component from direction ω at j^{th} receiving element
$\beta(t), \beta_p(t)$	noise at filter output
$C_{jk}(\tau)$	crosscorrelation of j^{th} and k^{th} noise components
$C_{jk}(\nu)$	cross power spectrum of j^{th} and k^{th} noise components
$C_n(\tau)$	autocorrelation of noise $n(t)$
$C_n(\nu)$	power density spectrum of noise $n(t)$
C	matrix of C_{jk}
c	velocity of propagation
Γ_p	normalized proper filter gain
$\gamma_{jk}(\tau)$	crosscorrelation of filtered noise components
$\gamma_{jk}(\nu)$	cross power spectrum of filtered noise components
$\gamma_{\Sigma_0}(\nu)$	spectral density of noise at output $\sum$
$D(\omega, \nu)$	complex directivity coefficient
d_{Σ}	spectral density of summed noises
Δ	determinant of matrix of C_{jk}
$\delta(t)$	unit impulse function
$E\ \chi$	mathematical expectation of x
E_j, E_k	receiver elements (j^{th}, k^{th})
E_s	signal energy
F	frequency (cps) of reference voltage
$F(a)$	spatial correlation of noise inputs to receiver elements with separation a

f	frequency (cps)
$G_p(\nu)$	system function of proper filter
$h_k(\nu)$	system function of k^{th} filter
$h_k(t)$	impulse response of k^{th} filter
h	matrix of $h_k(\nu)$
$I(x)$	imaginary part of complex number x
i	$\sqrt{-1}$
Θ	complex correlation coefficient
θ	time constant
θ_s	average duration of stationarity
j,k	(subscripts) indices
k,K	constants
λ_j	eigenvalue of matrix C
λ	matrix of eigenvalues λ_j
M_{jk}	cofactor of determinant Δ
M	matrix of cofactors M_{jk}
$M_{jk}^{\omega}(\tau)$	crosscorrelation of noise components from direction ω at j^{th} and k^{th} receiving elements
$M_{jk}^{\omega}(\nu)$	cross power spectrum derived from $M_{jk}^{\omega}(\tau)$
μ	constant
N	number of elements or channels
$N_{jk}(\tau)$	crosscorrelation of j^{th} and k^{th} noise components
$N_{jk}(\nu)$	cross power spectrum of j^{th} and k^{th} noise components
$n(t)$	arbitrary noise input
ν	frequency (cps)
P_β	mean noise power
P_m	noise power minimized by optimal filtering
$p_k(\nu)$	system function of filter

p	matrix of p_k
π_k	k^{th} proper filter
Q	homogeneity factor (constant)
q	$\sqrt{\frac{C_{11}}{C_{12}}}$ for 2-input system
$R_j(t)$, $r_j(t)$	impulse response of filters
$R_j(\nu)$, $r_j(\nu)$	system functions of filters
R, r	matrices of R_j, r_j
$R(x)$	real part of complex number x
ρ	signal-to-noise ratio (S/N)
ρ_Σ	S/N for simple summation
ρ_m	optimized (maximized) S/N
$S(t)$	signal
$\sigma(\nu)$	Fourier transform of output signal
T	time duration of signal
t	time
τ	time lag
$V^*_{jk}(\nu)$	normalized cross power spectrum of j^{th} and k^{th} noise inputs after signal compensation - case of omnidirectional noise
V^*	matrix of V^*_{jk}
$\Phi(\nu)$	system function of filter with unit gain
Φ	matrix of $\Phi(\nu)$
ϕ	bandwidth of narrow-band filter
$x_j(\nu)$	system function of j^{th} filter
X	column matrix of x_j

$Y_{jk}(\tau)$ — crosscorrelation of noises at j^{th} and k^{th} receiving elements in omnidirectional noise

$Y_{jk}(\nu)$ — cross power spectrum derived from $y_{jk}(\tau)$

ω — spatial parameter defining direction of a plane wave

$\overline{\omega}$ — unit vector in direction ω

(4) — numbers in parentheses correspond to references

CHAPTER I.

SOME PRACTICAL PROBLEMS AT THE BEGINNING OF THE STUDY

Summary

This study lies in that area of research that is concerned with the improvement of long-range detection techniques. By extending the well-known concept of a matched filter to include a receiver with several elements and by using a few simple examples, it is shown that current ideas about the optimum use of an antenna provide answers to only a few special aspects of the general problem.

I-1. General Outline

The problem of extracting weak signals from interfering noise may be found in the general area of Information Theory. We may consider two different approaches to the problem, each arising from a different practical situation. The first approach is concerned with the faithful reproduction of an input signal at the output of a communication channel. Here, the applicable theory is that of information *transmission*; the signal is unspecified except for the variety of forms or values which it may assume and the probability of occurrence of each. Optimization criteria are based upon the reconstruction of the most probable form of the signal, that is, the best "guess" at the original form and content of the transmitted signal. This first approach is that of optimizing the reconstruction of a message that has been deteriorated by noise.

The second aspect of the processing of weak signals - the aspect with which this paper is concerned - has its origins in practical problems associated with the development of long-range detection methods, using both electromagnetic (radar) and acoustic (sonar) techniques. Here, we wish to be informed of the presence of an obstacle or target as soon as possible and at as great a distance as possible. The "echo" returned by such a target or obstacle is weak, immersed in the inevitable interfering noise, but it is assumed to be of a specified or known form - the same form as a specific model (i.e., identical to the model except for some constant amplitude factor and time lag). The model might be the original emitted signal, which may be controlled or even derived from it by a known transformation (e.g. doppler shift). Conceivably each possible doppler shift value may yield a different "model." We may group the variables in the received signal in order to form a reasonable number of specific models and treat all models separately but simultaneously. The basic problem is reduced to the question: what are the criteria which optimize the detection of the presence of a signal of a given form? It is not required that the signal be accurately reproduced; it is even permissible to transform and alter it. What is required is the best possible indication of its presence; this indication might be, for example, a maximum contrast in the output level of the receiver.

I-2. Matched Filtering and Multiple Antennas

A particular, but essential, solution to the problem stated in the preceding paragraph may be found using the Matched Filter Theorem. This study deals with the scope of that theorem and with the criteria which it utilizes.

In its classical form, the Matched Filter Theorem treats a single signal and a single noise and does not deal with the multi-parameter aspects of the detection of weak signals using an antenna. An antenna is an ensemble of N sensors whose outputs are collected in some more or less complex fashion. For each sensor there is a corresponding noise. When a signal reaches an antenna, it is distributed in some specific manner into N particular signal outputs.

What, then, is the best possible utilization of the N signals and the N noise outputs for optimal detection of the presence of an incident signal? What is the best arrangement of the sensors? And what gain might be expected from using N sensors as compared to using a single sensor? It may be noted that the usual ways of evaluating antenna gain answer the last question in special cases only. In some cases, this gain is expressed simply by the number N itself.

Classical reasoning assumes that at the output the signal power increases as N^2 while that of the noise increases as N. This reasoning, however, postulates *statistical independence among the noise* outputs from the N elements or sensors and is therefore dependent upon a particularly simple hypothesis about the statistical "relationships" of the N noise outputs.

In another case, antenna gain is computed in terms of "directivity;" thereby introducing the more or less implicit hypothesis of a noise field with particular spatial properties, known as "omnidirectional noise," which completely defines the statistical relationships of the N noise components.

Two methods of improving an antenna are linked to these two means of evaluating antenna gain. In the case of independent noise components, N is made very large and the outputs of the N elements are summed. This "direct sum" is the <u>antenna processing</u>, that is, the way in which the outputs of the elements are assembled. In the case of omnidirectional noise, there are more subtle solutions, consisting, for example, of assigning to each element a suitable <u>weighting factor</u> before summing the outputs, such that the "directivity" is optimized.

Methods for the calculation of directivity, derivation of directivity patterns for specific arrays of elements, and methods for reducing or equalizing "secondary lobes" have provided abundant literature (references (6), (15) to (22)) because of the usefulness of the directivity concept in both transmission and reception. The underlying hypothesis of omnidirectional noise is, however, always present when this concept is applied to a receiving antenna.

The variety of techniques for estimating antenna gain suggests that optimal antenna processing depends upon the statistical relationships between the noise components associated with the elements of the antenna.

For a given antenna, the noise outputs are known. Their properties are, in general, stable (stationary), at least for a time duration on the order of the signal duration. Thus, they may be <u>measured</u> and the best methods of handling them may be determined. These methods may include direct summation or optimization of directivity, if the nature of the noise corresponds to these special cases, or they may be something quite different and may not even require that N be large. Such is the case, for instance, with "coherent" noise interference, which, like the signal, propagates in a single plane wave (from a different direction from that of the signal). This coherent interference is a case of still another statistical relationship: the noise components in the antenna elements differ from one another by a time delay only. Using only two elements, it is possible to null the noise without losing the signal, which is obviously the optimum procedure. All that is required is to take the difference between the outputs of the two elements after having introduced into one of them a delay corresponding to the difference in path length for the noise itself. A limited case such as this illustrates the way in which optimum antenna processing may occasionally be achieved when the statistical relationships between the noise components are taken into account. Hence, if the N noise components are known, the problem is to define and construct the optimal processing system.

A logical second step is to allow the statistical relationships between noise components to vary slowly with time in an irregular, unpredictable manner. We must then look for autoadaptive systems that evolve the optimum antenna processing as a function of the statistical relationships - relationships which are continually tested in the course of that evolution.

A great deal of work to date has been devoted to autoadaptive systems, but in a rather different theoretical context. In general, it has been concerned with the construction of receivers that progressively "adjust" themselves to the carrier frequency of a repetitive signal (as in the case of radar). The evolution of the receiver is thus dictated by the signal itself (references (25), (26), (27)).

Here, on the other hand, we want to modify the receiver as a function of the noise, which is always present, in such a way that reception will be optimized for the signal whenever it arrives. Note that the idea of adaptation does not have the same meaning in a "matched filter" as it does in an "adaptive system". In the first case, it has a "spectral" sense and corresponds to a well-defined criterion; in the second, it takes on a "temporal" meaning and is found to be tied in with a practical conception of stationarity.

These practical problems are mentioned here because they will be considered in the course of the first few chapters, employing more theoretical techniques. They will not be resolved in all their generality, but only in those special cases which are of particular interest. Thus, the ensuing study is developed in the direction of decreasing generality, from the theoretical basis formed by generalizing the Matched Filter Theorem to technological methods and principles useful in the construction of equipment. Meanwhile, some general properties of the statistical relationships of the noise components and their illustration through the important concept which we have called proper antenna filtering will be examined, especially with respect to matched filtering. The cases in which it is possible to obtain very large antenna gain will be stated precisely, and it will be shown that optimization of "directivity" is actually a special case of matched filtering.

I-3. Some Comments on Notation

a. As a general rule, two functions related by a Fourier transformation are designated by the same letter, for example[1]

$$h(t) \rightleftarrows h(\nu)$$

This convention does not lead to any ambiguity, since the variable is always indicated. It has the advantage of economy of symbols.

b. The symbol $\int$ indicates an integral between the limits $-\infty$ and $+\infty$.

c. We have made use of some results of harmonic analysis of unspecified functions (references (1) and (7)).

Generally the variable τ is indicated for auto- and crosscorrelation functions. The variable ν may be omitted from the notation for the Fourier transforms of the correlation functions, especially when dealing with matrices. The complex number C_{jk} designates either $C_{jk}(\nu)$ or $C_{jk}(\nu_o)$ for a particular value ν_o of ν The convention, in this case, is always defined in this text.

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[1]To state it more precisely:

$$h(t) = \int h(\nu)\, e^{+2\pi i\nu t}\, d\nu$$

$$h(\nu) = \int h(t)\, e^{-2\pi i\nu t}\, dt .$$

CHAPTER II

RESTATEMENT OF THE CLASSICAL MATCHED FILTER THEOREM

Summary

This theorem, taken from the area of prediction theory, is concerned with specific signals (of known form) mixed with stationary noise. It defines the linear filtering process that optimizes a certain parameter called "signal-to-noise ratio." This parameter summarizes the best possible information about the presence or absence of the signal without being concerned with preserving its "shape" or form at the output. Although this "best possible information" is limited to the case of Gaussian noise, it is frequently found in practice. The optimizations derived during the course of the discussion are in agreement with matched filter theory.

II-1. History of the Concept of Matched Filtering

The concept of matched filters was introduced in the technical literature around 1943 by D. O. North (10) in a study of the detection of weak signals of some known form $S(t)$ in noise that is stationary and has a uniform spectral density. North's essential result is that the filter which maximizes at its output a certain parameter known as the "signal-to-noise ratio" is the filter whose impulse response is the image of the signal: $S(-t)$. This result justifies the expression "matched filter" later used by Van Vleck and Middleton (11), who obtained the same result independently. The extension of this theorem to the case of noise having a nonuniform spectral density is found in the work of B. M. Dwork (5), as well as in that of L. A. Zadeh and J. R. Ragazzini (12).

These latter authors, in particular, demonstrated that the "signal-to-noise ratio" criterion is, in fact, a criterion of the "separation" between the signal and the noise. This observation connects matched filtering with the Wiener-Hopf equation (8).

Criteria of "separation," however, are non-statistical criteria, which depend only upon the spectral density or autocorrelation of the noise, that is, on a single moment of the probability distribution. The noise is entirely specified by this moment only in the case of a Gaussian distribution.

Another approach to the problem of optimizing weak signal detection is through statistical decision theory. The literature on the subject is abundant, and we intend here only to skim its surface in order to specify the criteria for a matched filter (references (3), (4), (9), (28), (29), (30), (31)). This theory takes into account the ensemble of statistical properties of the combination of signal and noise in order to deal with, especially, the a posteriori probabilities (for a given combination) of the presence or absence of signal. The ideal receiver must, then, construct the "probability ratio" connected with these probabilities. It may be shown that if the noise is Gaussian and has a uniform density, this probability ratio is completely described by the convolution of the "mixture" (of signal and noise) with the "pure" signal. This convolution is nothing more than the output of the matched filter (having impulse response $S(-t)$). Thus, matched filtering appears as a "statistical optimum" in the Gaussian case. This case is especially important in practice. The signals to be processed are often "narrow band", that is, lying in a small band around a center frequency. It may then be assumed that the spectral density is uniform in the band.

On the other hand (reference (13)), for a very large class of non-Gaussian noise, narrow-band filtering tends to restore some of the Gaussian characteristics. It may be said that optimization in the sense of matched filter criteria is often very close to optimization in the statistical sense. One may then think of matched filtering as being very close to an ideal receiver.

These considerations underline the importance of the matched filter criterion that may at first seem a little arbitrary. This criterion is the maximizing of the signal-to-noise ratio defined in the following way:

$$\rho = \frac{\text{instantaneous power of the signal at arbitrary time } t}{\text{average power of the noise}} .$$

As stated more precisely in the succeeding pages, this criterion corresponds to the requirement of producing at the filter output maximum contrast between the presence and the absence of a signal. This is to be accomplished by collecting all the energy of the signal in order to produce a peak that is as narrow and as high above the average noise power as possible.

II-2. Matched Filtering With One Input

Let us recall that the theoretical solution of the matched filter may be presented in two equivalent forms related by a Fourier transformation: the temporal form, which furnishes the impulse response $h(t)$ of the filter; and the spectral form, which gives its transfer function $h(\nu)$

$$h(t) \rightleftarrows h(\nu) . \qquad \text{(II-1)}$$

Let $S(t)$ be the signal having a spectrum $S(\nu)$, let $B(t)$ be the stationary noise whose autocorrelation is $C(\tau)$ derived from its spectral density $C(\nu)$, and let t_0 be an arbitary time.

a. The impulse response h(t) is specified by the integral equation

$$\int h(\theta) \cdot C(t-\theta)\, d\theta = KS(t_0-t) \ , \qquad \text{(II-2)}$$

where K is an arbitrary real factor.

b. The transfer function $h(\nu)$ is given by

$$h(\nu) = Ke^{-2\pi i\nu t_0} \cdot \frac{S^*(\nu)}{C(\nu)} \ , \qquad \text{(II-3)}$$

where * designates the complex conjugate.

The factor $e^{-2\pi i\nu t_0}$ of Eq.(II-3) corresponds to a time lag of t_0. Since t_0 and K are arbitrary, it may be said that the matched filter is defined except for a real factor and a time lag. This is physically obvious. The real factor acts in the same way upon both signal and noise and can not alter their behavior. As for the time lag t_0, if it varies, the time of the appearance of the signal "peak" is more or less displaced, but its height is not changed. It is preferable generally to have the time lag as small as possible in order that the observer may be informed of the presence of the signal as soon as possible after it reaches the receiver input. However, it may happen that considerations relating to the realization of the filter will lead to a compromise in this area.

Nevertheless, we will, in the course of this dissertation, again encounter the factor $Ke^{-2\pi i\nu t_0}$, which, as we shall see, has no effect upon the filtering processes. On the basis of the preceding considerations, it is not difficult to find that, in the presence of white noise, the matched filter is defined by (except for the above factor)

$$h(\nu) = S^*(\nu)$$
$$h(t) = S(-t)$$

and that, as a result, it performs the convolution of its input with the signal (more precisely with the signal "reflected" in time—the image of the signal).

Having reviewed these results, it will be possible to deduce them as well as all the other properties of matched filtering, from the generalization of the theorem to a system of N inputs - a generalization which we shall now consider.

CHAPTER III

MATCHED FILTERING WITH N INPUTS - FORMAL SOLUTION

Summary

This chapter treats the problem of generalizing matched filtering to N inputs (multiple filtering). An antenna with N sensitive elements exhibits, at all of these elements, N parasitic noise components whose statistical relationships are defined by the crosscorrelation functions of the noises taken two at a time. It also delivers a "multiple signal" consisting of N particular time functions. What is the multiple filtering of these data - the ensemble of N filters - such that, when the N outputs are summed, the characteristic parameter of matched filtering (S/N) is optimized?

The formal answer to this question is presented in two equivalent forms - the list of N impulse responses of the desired filters, and their N complex gains (system functions).

In the second form, the solution is seen to be unique (except for a real factor and a time delay) and suitable for expression in convenient matrix notation.

Two characteristic properties are established:

a. The optimized parameter is independent of the reference time t_0.

b. Except for a real factor and a time delay (representing an arbitrary choice of the time origin), the spectrum of the signal and the spectral density of the noise are identical at the output of the matched filter.

III-1. Status of the Problem

Let us consider a system of N inputs $E_1, E_2 \ldots . E_N$ where

a. the signal is represented by N specific real functions of time,

$$S_1(t),\ S_2(t),\ \ldots .\ S_N(t)\ ,$$

b. the interference (parasitic noise) is represented by N unspecified real functions, $B_1(t),\ B_2(t),\ \ldots ..\ B_N(t)$,
which are stationary and whose correlations are stationary.

By definition, let[2]

$$C_{jk}(\tau) = E\left\{B_j(t)\ B_k(t+\tau)\right\} , \qquad \text{(III-1)}$$

where E designates the mathematical expectation (of the function in the brackets).

The real functions $C_{jk}(\tau)$ are autocorrelations (even functions) when $j = k$ and crosscorrelations when $j \neq k$. The N^2 equations of the type (III-1) express the statistical relationships between the N noise inputs. These noise inputs are independent of the signals $S_1(t)$, $S_2(t)$, ... $S_N(t)$.

We then ask:

What is the linear filtering process represented by N filters having impulse responses $R_1(t), R_2(t), \ldots R_N(t)$ which, when applied to the inputs $E_1, E_2 \ldots E_N$, respectively, have the following property: by taking the sum of their N outputs, a common output $\sum$ is formed in which the following ratio is optimized (maximized):

$$\rho = \frac{\text{instantaneous power of the signal at arbitrary } t_0}{\text{average noise power}} . \qquad \text{(III-2)}$$

Thus, the characteristic criterion of matched filtering is utilized.

III-2. Review of the Classical Properties of the Crosscorrelation of Two Noise Inputs

Before proceeding to the proof, let us recall a few classical properties of the functions $C_{jk}(\tau)$ and $C_{jk}(\nu)$.

a. First of all, according to the definition,

$$C_{jk}(\tau) = C_{kj}(-\tau) . \qquad \text{(III-3)}$$

b. Expressing the fact that the power of a real noise

$$B_j(t) + \lambda B_k(t) \quad (\lambda \text{ real})$$

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[2]Other authors adopt a convention which corresponds to exchanging τ and $-\tau$ in the relation III-1. This results, for the remainder of this paper, in the exchanging of the correlation matrix with its transpose.

is a positive quantity regardless of the value of λ, it may be shown (Schwartz inequality) that

$$\left[C_{jk}(\tau) \right]^2 \leq C_{jj}(0) \cdot C_{kk}(0), \text{ for all } \tau, \qquad \text{(III-4)}$$

the right side of the inequality being, moreover, the product of the powers of $B_j(t)$ and of $B_k(t)$.

c. The existence of $C_{jj}(\nu)$, the Fourier transform of $C_{jj}(\tau)$, will be assumed. It is the spectral density of $B_j(t)$, a real, even, non-negative function of ν for any value of ν.

d. The existence of $C_{jk}(\nu)$, the Fourier transform of $C_{jk}(\tau)$, will be assumed. As the transform of a real function, it satisfies the equation

$$C_{jk}(\nu) = C^*_{jk}(-\nu) , \qquad \text{(III-5)}$$

which expresses its Hermitian symmetry with respect to the variable ν. Moreover, these functions have Hermitian symmetry with respect to their indices since Eq.(III-3) implies that

$$C_{jk}(\nu) = C^*_{kj}(\nu) . \qquad \text{(III-6)}$$

Since the $C_{jj}(\nu)$ are obviously spectral densities we will call the $C_{jk}(\nu)$ crosscorrelation spectra.

e. If $B_j(t)$ and $B_k(t)$ are filtered by filters having impulse responses $R_j(t)$ and $R_k(t)$, respectively, two new noises $b_j(t)$ and $b_k(t)$ will be obtained whose spectral densities are

$$\gamma_{jj}(\nu) = C_{jj}(\nu) \; |R_j(\nu)|^2$$

and

$$\gamma_{kk}(\nu) = C_{kk}(\nu) \; |R_k(\nu)|^2. \qquad \text{(III-7)}$$

The crosscorrelation function of $b_j(t)$ and $b_k(t)$ is, by definition,

$$\gamma_{jk}(\tau) = E\left\{ b_j(t) \, . \, b_k(t+\tau) \right\} \quad . \qquad \text{(III-8)}$$

A classical calculation (Appendix I) shows that

$$\gamma_{jk}(\tau) = \int R_j^*(\nu) \, . \, R_k(\nu) \, . \, C_{jk}(\nu) \, e^{2\pi i\nu\tau} d\nu \ , \qquad \text{(III-9)}$$

from which we derive the Fourier *transform of* $\gamma_{jk}(\tau)$:

$$\gamma_{jk}(\nu) = R_j^*(\nu) \, . \, R_k(\nu) \, . \, C_{jk}(\nu) \quad . \qquad \text{(III-10)}$$

f. Let two filters $R_j(\nu)$ and $R_k(\nu)$ be <u>identical</u> with real impulse responses (with Hermitian transfer functions) having gains equal to unity in a band ϕ around ν_0 (and $-\nu_0$) and zero elsewhere. Let us assume that this band is narrow enough so that $C_{jk}(\nu)$ does not differ from $C_{jk}(\nu_0)$ within the band.

The crosscorrelation function obtained for the two noise inputs is, according to (III-9) and (III-6),

$$2\phi \, \frac{\sin \pi\phi\tau}{\pi\phi\tau} \, . \, |C_{jk}(\nu_0)| \cos\left\{ 2\pi\nu_0\tau + \text{Arg}\left[C_{jk}(\nu_0)\right]\right\} \quad ,$$

which, as $\phi \to 0$, approaches the limit

$$\gamma_{jk}(\tau) = 2\phi |C_{jk}(\nu_0)| \cos\left\{ 2\pi\nu_0\tau + \text{Arg}\left[C_{jk}(\nu_0)\right]\right\} \qquad \text{(III-11)}$$

(see application in Chapter IX);

that is, it approaches a real, sinusoidal function of τ whose phase is equal to the argument of $C_{jk}(\nu_0)$ and whose amplitude is proportional to $|C_{jk}(\nu_0)|$.

The Fourier transform is a spectral "line" at the frequency ν_0 characterized by the complex number $C_{jk}(\nu_0)$. Thus, <u>$C_{jk}(\nu_0)$ is a complex number which describes *the crosscorrelation of noises* $B_j(t)$ and $B_k(t)$ by two identical narrow-band filtering processes at the frequency ν_0.</u>

The function $\gamma_{jk}(\tau)$ given by (III-11) is justified by the inequality (III-4), which becomes here

$$\left[\gamma_{jk}(\tau) \right]^2 \leq \gamma_{jj}(0) \, . \, \gamma_{kk}(0) \ , \qquad \text{for all } \tau$$

from which may be deduced

$$|C_{jk}(\nu_0)|^2 \leq C_{jj}(\nu_0) \, . \, C_{kk}(\nu_0) \ . \qquad \text{(III-12)}$$

The right-hand side is the product of spectral densities, for $\nu = \nu_0$, of the noises $B_j(t)$ and $B_k(t)$. Since the preceding equation is true for all ν_0, we may write

$$|C_{jk}(\nu)|^2 \leq C_{jj}(\nu)\ C_{kk}(\nu), \text{ for all } \nu, \tag{III-13}$$

or in determinant form ,

$$\begin{vmatrix} C_{jj} & C_{jk} \\ C_{kj} & C_{kk} \end{vmatrix} \geq 0 \ , \quad \text{for all } \nu. \tag{III-13a}$$

III-3. Review and Physical Interpretation of a Fundamental Property of the Crosscorrelation of Two Noises

Let us consider, for the present, the problem posed in Paragraph III-1 and illustrated by Fig. III-1 (multiple filtering). The signal at the output of filter $R_j(t)$ is

$$\int R_j(\theta)\ S_j(t-\theta)\ d\theta \ ,$$

and as a result the signal at the output $\sum$ is

$$\sigma(t) = \sum_j \int R_j(\theta)\ S_j(t-\theta)\ d\theta \ . \tag{III-14}$$

The noise at the output of filter $R_j(t)$ is

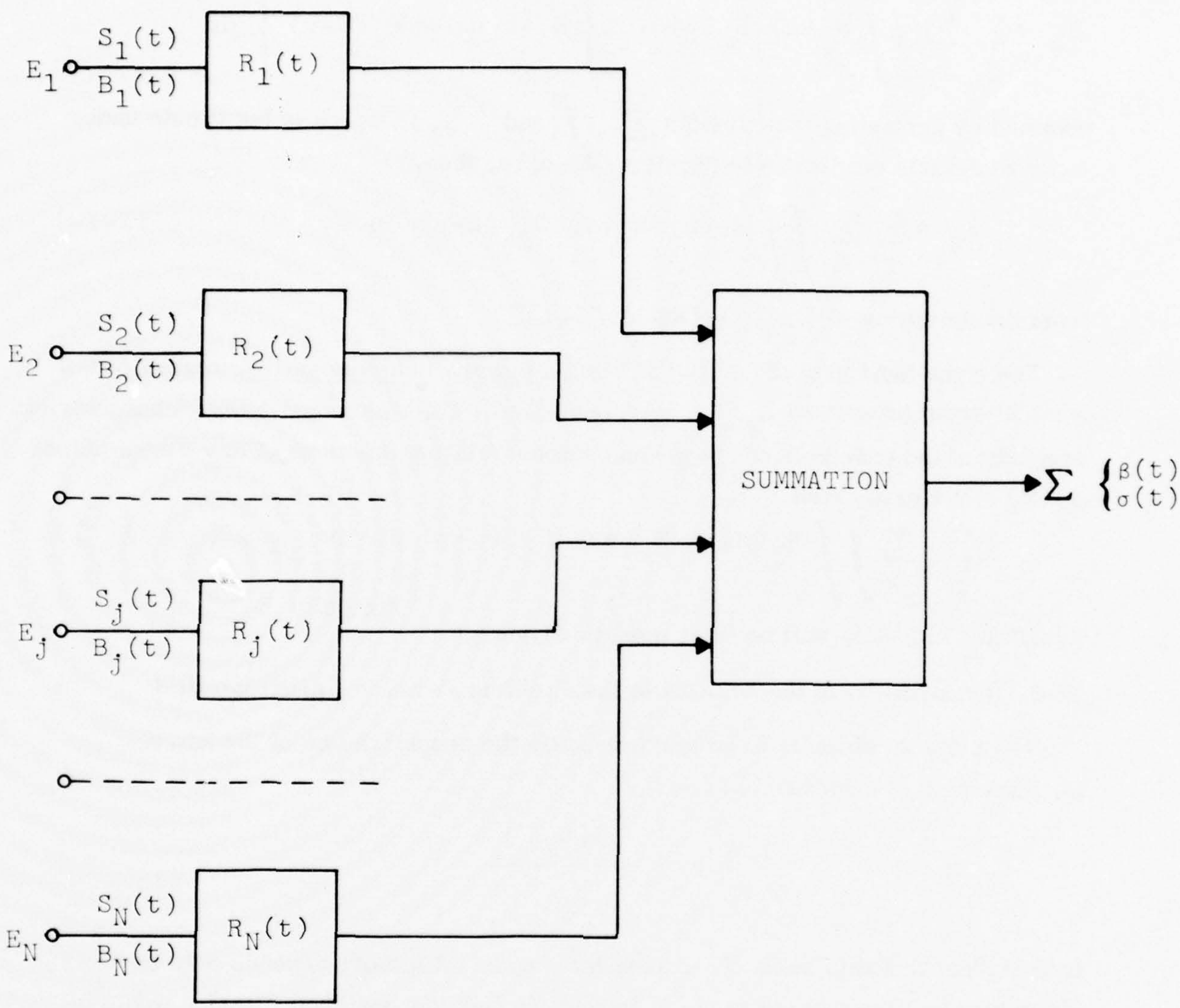
$$\int R_j(u)\ B_j(t-u)\ du \ ,$$

and the noise at $\sum$ is

$$\beta(t) = \sum_j \int R_j(u)\ B_j(t-u)\ du \ . \tag{III-15}$$

Its average power is

$$P_\beta = E\left\{ \beta^2(t) \right\} = E\left\{ \left[\sum_j \int R_j(u)\ B_j(t-u)\ du \right]^2 \right\} . \tag{III-16}$$



$S_1(t)$
E_1
$B_1(t)$
$R_1(t)$
$S_2(t)$
E_2
$B_2(t)$
$R_2(t)$
$S_j(t)$
E_j
$B_j(t)$
$R_j(t)$
$S_N(t)$
E_N
$B_N(t)$
$R_N(t)$
SUMMATION
$\Sigma \begin{cases} \beta(t) \\ \sigma(t) \end{cases}$

Fig. III-1

The $\left[\sum\right]^2$ in Eq (III-16) may be put in the form of a double $\sum$:

$$P_\beta = \sum_j \sum_k \iint R_j(u)\ R_k(v) \cdot E\left\{ B_j(t-u) \cdot B_k(t-v) \right\} du\ dv \ , \qquad \text{(III-17)}$$

obtained by permuting the symbols $\sum$, $\int$, and E, which is legitimate under many realizable conditions in physics. We have, then,

$$P_\beta = \sum_j \sum_k \iint R_j(u)\ R_k(v)\ C_{jk}(u-v)\ du\ dv \qquad \text{(III-18)}$$

from the definition of $C_{jk}(t)$ (Eq. (III-1)).

The right-hand side of (III-18) being a <u>power</u>, is <u>non-negative</u>, regardless of what filtering processes $R_j(t)$ are considered. This fact constitutes a characteristic property of the ensemble of cross- and autocorrelation functions of any N real noises $B_j(t)$. We have, then,

$$\sum_j \sum_k \iint R_j(u) \cdot R_k(v)\ \ C_{jk}(u-v)\ du\ dv \geq 0 \ , \qquad \text{(III-19)}$$

for any R_j .

Equation (III-19) will be later used to advantage.

III-4. Investigation of the Solution to the Problem Posed in Paragraph III-1.

The ratio ρ which is to be maximized by the proper choice of the filters $R_j(t)$ is, according to equation (III-2),

$$\rho = \frac{\left[\sigma(t_0)\right]^2}{P_\beta} \ . \qquad \text{(III-20)}$$

It is evident that this ratio ρ, except for a constant factor, depends only on $R_j(t)$. We may recall the remark made in Paragraph II-2 that matched filtering would be defined except for a constant factor. In order to maximize ρ, this arbitrary factor must be chosen in such a way that the numerator of ρ, that is, $\left[\sigma(t_0)\right]^2$, is given an arbitrary non-zero value k^2 and then, taking this constraint into consideration, the denominator P_β is minimized.

The problem is now reduced to finding filters $R_j(t)$ such that

$$\sigma(t_0) = \sum_j \int R_j(\theta)\ S_j(t_0-\theta)\ d\theta = k \ , \qquad \text{(III-21)}$$

and such that the quantity P_β in Eq. (III-8) is minimized.

Let us consider the ensembles of N impulse responses $r_1(t)$, $r_2(t)$, .. $r_N(t)$ such that

$$\sum_j \int r_j(\theta)\ S_j(t_0-\theta)\ d\theta = 0 \quad ; \qquad \text{(III-22)}$$

that is, filtering processes which null (or reduce to zero) the signal at the output Σ at time t_0.

It is clear that all filtering of the form

$$R_j(t) + \alpha\ r_j(t)$$

with some given real α satisfies Eq. (III-21), as does $R_j(t)$ itself. By varying α and $r_j(t)$ (constrained only by Eq. (III-22)) we may associate with any filtering process $R_j(t)$ an infinite number of filtering processes

$$R_j(t) + \alpha r_j(t) \quad ,$$

each one of which assigns the same value k^2 to the numerator of ρ and different values to the denominator P_β.

If the chosen filtering $R_j(t)$ is indeed a solution to the problem, that is, if it minimizes P_β, it means that the value of P_β for that $R_j(t)$ is less than or equal to all other values obtained for filtering processes $R_j(t) + \alpha r_j(t)$ with arbitrary real α and $r_j(t)$ constrained by (III-22). Now, calling

$h_j(t)$, one of the desired filtering processes,

D_h, the value of P_β for $R_j(t) = h_j(t)$, and

$D_{h,r}$ the value of P_β for $R_j(t) = h_j(t) + \alpha r_j(t)$,

the desired condition may be written

$$D_h \leq D_{h,r} \qquad \text{(III-23)}$$

for arbitrary α and for an arbitrary value of r_j constrained only by (III-22).

Substituting from Eq. (III-18), the inequality (III-23) becomes

$$\sum_{jk} \iint h_j(u)\ h_k(v)\ C_{jk}(u-v)\ du\ dv \qquad \text{(III-24)}$$

$$\leq \sum_{jk} \iint \Big[h_j(u) + \alpha r_j(u) \Big] \Big[h_k(v) + \alpha r_k(v) \Big]\ C_{jk}(u-v)\ du\ dv$$

for any α and any r constrained by (III-22), or rearranging the above in terms of α,

$$\alpha^2 \sum_{jk} \iint r_j(u) \cdot r_k(v)\, C_{jk}(u-v)\, du\, dv$$
$$+ \alpha \sum_{jk} \iint \left[r_j(u)\, h_k(v) + h_j(u)\, r_k(v) \right] C_{jk}(u-v)\, du\, dv \geq 0 \tag{III-25}$$

for any α and any r constrained by (III-22).

The coefficient of α^2 in (III-25) is of the form found in Eq. (III-19); hence it is non-negative. It represents, moreover, the noise power at the output $\sum$ for the filtering process $r_j(t)$. Since the first term of Eq. (III-25) remains positive for all α, in order to prevent the sign of the second term from prevailing for small values of $|\alpha|$, it is necessary to set the coefficient of α equal to zero. This condition is then

$$\sum_{jk} \iint \left[r_j(u) \cdot h_k(v) + h_j(u)\, r_k(v) \right] C_{jk}(u-v)\, du\, dv = 0\ , \tag{III-26}$$

for any r constrained by (III-22).

III-5. Demonstration - Temporal Form of the Solution

The first term (left side) of (III-26) is composed of two terms. Because of Eq. (III-3), which implies

$$C_{jk}(u-v) = C_{kj}(v-u)\ ,$$

these two terms may be written in symmetric form with their two indices, as well as u and v permuted.

It is evident, in this form, that they are equal. Condition (III-26) is then reduced to

$$\sum_{jk} \iint r_j(u)\, h_k(v)\, C_{jk}(u-v)\, du\, dv = 0\ , \tag{III-27}$$

for any r constrained by (III-22).

Equation (III-27) may be written

$$\sum_j \int r_j(u) \left[\sum_k \int h_k(v)\, C_{jk}(u-v)\, dv \right] du = 0 \,, \qquad \text{(III-28)}$$

and from (III-22)

$$\sum_j \int r_j(u)\, S_j(t_0-u)\, du = 0 \,. \qquad \text{(III-22)}$$

In order to simplify the notation for the moment, let us set the bracketed term of (III-28) equal to $f_j(u)$. Among all the possible systems of functions $r_j(u)$, let us consider those which possess only a single function not identically equal to zero, for example $r_1(u)$. This function is constrained by Eq. (III-22), which is expressed as

$$\int r_1(u)\, S_1(t_0-u)\, du = 0 \,. \qquad \text{(III-22a)}$$

Condition (III-28) defines only $f_1(u)$; for any function $r_1(u)$ constrained by (III-22a), $f_1(u)$ must satisfy

$$\int r_1(u)\, f_1(u)\, du = 0 \,. \qquad \text{(III-28a)}$$

Equations (III-22a) and (III-28a) are seen to result in

$$f_1(u) = K_1\, S_1(t_0-u), \qquad \text{(III-29)}$$

where K_1 is a factor independent of u (Appendix II). The preceding relation is valid for all values of the index j, and as a result

$$f_j(u) = K_j\, S_j(t_0-u) \,.$$

Applying these necessary conditions to (III-28), we obtain

$$\sum_j K_j \int r_j(u)\, S_j(t_0-u)\, du = 0.$$

Comparing with (III-22) we observe that all K_j must be equal to one another and to the same factor K, which is independent of the index j.

Finally, the system of equations which gives the impulse responses $h_k(t)$, the solution to the problem, is the ensemble of equations of the type

$$\sum_k \int h_k(v)\, C_{jk}(u-v)\, dv = K\, S_j(t_0-u) \quad , \qquad \text{(III-30)}$$

$$\text{for } j = 1, 2, \ldots\ldots N .$$

These N integral equations define the N impulse responses $h_1(t), h_2(t), \ldots\ldots h_N(t)$ and give the desired solution to the problem in its temporal form (impulse responses).

Optimal filtering comprised of the filters $h_k(t)$ is defined by Eq. (III-30), except for a factor K. This factor is arbitrary but it is still related to the other arbitrary factor $k = \sigma(t_0)$ by Eq. (III-21) by which the numerator of ρ was normalized; it is related as well to the noise power at the output $\sum$ (denominator of ρ), which is minimized for a given K but which is nevertheless proportional to K^2.

In fact, Eq. (III-21) is written, for the filtering process $h_j(t)$,

$$k = \sigma(t_0) = \sum_j \int h_j(u)\, S_j(t_0-u)\, du \quad , \qquad \text{(III-31)}$$

which is, multiplying both sides by K,

$$Kk = \sum_j \int h_j(u) \left[K\, S_j(t_0-u) \right] du \qquad \text{(III-32)}$$

and, replacing the bracketed term of (III-32) by its value derived from (III-30),

$$Kk = \sum_j \sum_k \iint h_j(u)\, h_k(v)\, C_{jk}(u-v)\, du\, dv . \qquad \text{(III-33)}$$

Thus, the second term (right side) of (III-33) is the value of power P_β for the optimum filtering process $h_j(t)$, that is, the noise power at output $\sum$ -power which is minimized by that optimum filtering. The minimum power is designated by P_m.

We have then

$$Kk = P_m,$$

or

$$K\sigma(t_0) = P_m .$$

As for the value of the ratio ρ, which is made a maximum ρ_M by the optimal filtering $h(t)$, it is equal by definition to

$$\rho_M = \frac{\left[\sigma(t_0)\right]^2}{P_m} = \frac{\sigma(t_0)}{K} = \frac{k}{K} \quad .$$

These equations may be summarized by

$$K = \frac{\sigma(t_0)}{\rho_M} = \frac{P_m}{\sigma(t_0)} \quad , \tag{III-34}$$

and the proportionality of P_m to K^2 may be made obvious by

$$P_m = \rho_M K^2 \quad . \tag{III-35}$$

The coefficient K has the dimensions of signal amplitude.

III-6. Spectral Form of the Solution - Value of the Optimized Parameter

By replacing both sides of Eq. (III-30) by their Fourier transforms, we obtain the solution to the optimum filtering problem in "spectral" form. The left side of (III-30) contains the convolution of $h_k(t)$ with $C_{jk}(t)$. Its transform is the product of the Fourier transforms $h_k(\nu)$ and $C_{jk}(\nu)$. The transform of the first term is then

$$\sum_k h_k(\nu)\, C_{jk}(\nu) \quad .$$

If $S_j(\nu)$ is the Fourier transform of $S_j(t)$,, the Fourier transform of the right side of (III-30) is

$$S_j^*(\nu)\, e^{-2\pi i \nu t_0} \quad ,$$

where the asterisk denotes the complex conjugate. Accordingly, the system of Eqs. (III-30) yields, through transformation, the equations

$$\sum_k C_{jk}(\nu)\, h_k(\nu) = K \quad e^{-2\pi i \nu t_0} \quad S_j^*(\nu), \text{ for } j=1,2...N \tag{III-36}$$

In Eq. (III-36) the quantities $h_k(\nu)$ (system functions of the filters) are dimensionless since they are ratios of two amplitude spectra. The $C_{jk}(\nu)$

are homogeneous in spectral density, that is, in power per cycle (power x time); hence, in <u>energy</u>.

The $S(\nu)$ are in terms of <u>amplitude per cycle</u> (Fourier transforms of amplitudes), and K is the unit of amplitude of the signal.

One may write, introducing the unit of time T,

$$\sum_k C_{jk}(\nu)\ h_k(\nu) = K^2 T\ e^{-2\pi i\nu t_0}\ \frac{S_j(\nu)}{KT},$$

where K^2T is the unit of energy or spectral density.

In terms of dimensionless quantities, Eq. (III-36) may be written

$$\sum_k \frac{C_{jk}(\nu)}{K^2T}\ h_k(\nu) = e^{-2\pi i\nu t_0}\ \frac{S_j^*(\nu)}{KT}. \qquad \text{(III-36a)}$$

However, this form is less concrete, and we will retain the prerogative to speak of system functions, densities, and signals.

The system (III-36) is a system of N linear equations in N unknowns consisting of the $h_k(\nu)$.. Solving explicitly, we have the N equations

$$h_k(\nu) = Ke^{-2\pi i\nu t_0}\ \frac{1}{\Delta(\nu)}\ \sum_j S_j^*(\nu)\ M_{jk}(\nu), \qquad \text{(III-37)}$$

$$\text{for } k = 1, 2, \ldots\ldots N,$$

where $\Delta(\nu)$ denotes the determinant of the $C_{jk}(\nu)$ (j is the line index and k the column index), and where $M_{jk}(\nu)$ denotes the cofactor of the element(j, k) of $\Delta(\nu)$ (that is, the determinant obtained by suppressing the j^{th} line and the k^{th} column and assigning the sign $(-1)^{j+k}$).

The system of equations given by (III-37) constitutes the "spectral" form of the solution. It defines the optimum filtering process by the system functions of the N filters, , $h_1(\nu)$, $h_2(\nu)$, ... $h_N(\nu)$.

Let us note that the solution of the linear system (III-36) is generally unique provided that $\Delta(\nu) \neq 0$).

Thus, the $h_k(\nu)$ are, in general, completely determined, as are, consequently, the $h_k(t)$. One may then speak of the solution to the problem of matched filtering with N inputs - a unique solution defined by (III-36).

The case of $\Delta(\nu) = 0$ will be investigated later. The $C_{jk}(\nu)$ are Hermitian with respect to the variable ν (Eq. (III-5)). The $S_j(\nu)$ are also, as transforms of the real signals $S_j(t)$. Thus, all the system functions $h_k(\nu)$ are Hermitian with respect to ν, and the impulse responses $h_k(t)$ are real. *Driven by real signals or* noise, an $h_k(t)$ delivers a real signal or a real noise.

An interesting expression for the maximum value of ρ_M of the ratio ρ, optimized by the filtering process $\left\{h_k(\nu)\right\}$, may be drawn from the Eqs. (III-37). In fact, Eq. (III-31) may be written

$$\sigma(t_0) = \sum_k \int S_k(u)\; h_k(t_0-u)\; du\ , \qquad \text{(III-31)}$$

and when transformed by Parseval's identity gives the result

$$\sigma(t_0) = \sum_k \int S_k(\nu)\; h_k(\nu)\; e^{2\pi i\nu t_0}\; d\nu \qquad \text{(III-38)}$$

(the Fourier transform of $h_k(t_0-t)$ being $h_k(-\nu)\; e^{2\pi i\nu t_0}$, , that is, $h_k^*(\nu)\; e^{-2\pi i\nu t_0}$, since $h_k(t)$ is real, as we have seen previously).

Substituting the value of $h_k(\nu)$ given by (III-37) into Eq. (III-38), we have

$$\sigma(t_0) = \sum_k \int S_k(\nu)\; \frac{K}{\Delta(\nu)} \left[\sum_j S_j^*(\nu)\; M_{jk}(\nu)\right] d\nu\ ,$$

or from (III-34)

$$\rho_M = \frac{\sigma(t_0)}{k} = \sum_j \sum_k \int \frac{S_j^*(\nu)\; S_k(\nu)\; M_{jk}(\nu)}{\Delta(\nu)}\; d\nu\ . \qquad \text{(III-39)}$$

Note that since ρ_M is a ratio of powers the right side of (III-39) is always non-negative for any noise $B_j(t)$ and any signal $S_j(t)$.

The value ρ_M is the integral of a function of ν:

$$D(\nu) = \sum_j \sum_k S_j^*(\nu)\; S_k(\nu)\; \frac{M_{jk}(\nu)}{\Delta(\nu)}\ , \qquad \text{(III-40)}$$

whose real, non-negative character for all ν will become evident shortly (Eq. III-52)).

The output signal $\sigma(t)$ has an amplitude spectrum given by

$$\sigma(\nu) = \sum_k S_k(\nu)\; h_k(\nu) \qquad \text{(III-41)}$$

or, taking (III-37) and III-40) into account,

$$\sigma(\nu) = \left(Ke^{-2\pi i\nu t_0}\right) . D(\nu) \quad . \qquad \text{(III-42)}$$

III-7. Matrix Notation - Characteristic Property of Matched Filtering

We will now express the preceding equations in a particularly convenient matrix notation that is suggested quite naturally by Eq. (III-36).

In this notation, the table of $C_{jk}(\nu)$ constitutes a square matrix of order N, , which, according to Eq. (III-6),, is Hermitian. The ensemble of system functions $R_j(\nu)$ or $h_j(\nu)$ may be represented by column matrix R or h. The ensemble of signals may be designated by a column matrix S.

It is obvious that the matrices S and R represent different physical quantities (signals and system functions). Thus, all operations between matrices S and R are not justified from a physical point of view. For example, a linear combination $\alpha S+\beta R$ has no meaning and will not be used. On the other hand, a matrix multiplication such as $\tilde{S}h$ or $\tilde{h}S$ represents the right-hand side of (III-41)[3], that is, the amplitude spectrum of the output signal of the matched filter. More generally,

$$\tilde{S}R = \tilde{R}S$$

designates the signal output of the multiple filtering process of Fig. (III-1). Moreover, using only dimensionless quantities (Eq. (III-36a)), only physical quantities of the same nature may be manipulated.

Anticipating its justification given in Paragraph V-4, let us use, for the moment, Eq. (V-21), which gives the spectral density of the multiple filtering process R at the output $\sum$.

For any arbitrary $R_j(t) \rightleftarrows R_j(\nu)$, we have

$$\gamma_\Sigma(\nu) = \sum_j \sum_k R_j^*(\nu)\, R_k(\nu)\, C_{jk}(\nu) \quad , \qquad \text{(V-21)}$$

which is expressed in matrix notation as

[3]The symbols of matrix calculations are those recommended by Standard NFX02-110 of April 1962.

$$\gamma_\Sigma(\nu) = R^+ CR \quad . \tag{III-43}$$

The right-hand side is the Hermitian form of the matrix C. In the case of matched filtering, the spectral density of the output noise is

$$\gamma_{\Sigma_0}(\nu) = h^+ Ch \; . \tag{III-44}$$

Equation (III-36) is written

$$Ch = \left[Ke^{-2\pi i \nu t_0}\right] S^* \quad . \tag{III-45}$$

(In matrix expressions, coefficients will be placed in brackets for greater clarity).

Equation (III-37) gives, when $\Delta(\nu) \neq 0$,

$$h = \left[Ke^{-2\pi i \nu t_0}\right] C^{-1} S^* \quad . \tag{III-46}$$

Matrix C being Hermitian, we have

$$C^+ = C \text{ and } C^{-1+} = C^{-1}. \tag{III-47}$$

Setting the matrices associated with both sides of (III-46) equal to one another, we have

$$h^+ = \left[Ke^{+2\pi i \nu t_0}\right] S^{*+} C^{-1+} \; . \tag{III-48}$$

Since $S^{*+} = \tilde{S}$, the transpose of S, we have

$$h^+ = \left[Ke^{+2\pi i \nu t_0}\right] \tilde{S} C^{-1} \; . \tag{III-49}$$

Equation (III-44) is then written

$$\gamma_{\Sigma_0}(\nu) = h^+ Ch = \left[K^2\right] \tilde{S} C^{-1} S^* \quad . \tag{III-50}$$

Substituting in (III-40) and noting that $\dfrac{M_{jk}(\nu)}{\Delta(\nu)}$ is the term corresponding to "line k, column j" of C^{-1}, we may write

$$D(\nu) = \tilde{S}\, C^{-1}\, S^{*} , \qquad \text{(III-51)}$$

and, consequently, from (III-50)

$$\gamma_{\Sigma_0}(\nu) = K^2\, D(\nu) \quad . \qquad \text{(III-52)}$$

As a result of (III-52), $D(\nu)$ is indeed a real and non-negative function of ν and is homogeneous at a given time.

Comparing (III-42) and (III-52), one is led to the characteristic rule of matched filtering, generalized here to the case of N inputs: except for the factor $\left[\dfrac{e^{-2\pi i\nu t_0}}{K}\right]$, , the spectrum of the output signal is identical to the spectral density of the noise at the output.

$$\frac{\gamma_{\Sigma_0}(\nu)}{K^2} = \frac{\sigma(\nu)}{Ke^{-2\pi i\nu t_0}} = D(\nu) \quad . \qquad \text{(III-53)}$$

The identity becomes complete when one adopts the conventions

$$K = 1 \text{ and } t_0 = 0 \ .$$

The first convention consists of defining the unit of amplitude; it destroys the homogeneity of the preceding equations. The second expresses the fact that the time origin is taken to be the instant when the "peak" of the output signal appears. This convention of notation, which does not alter the generality of the problem, will ultimately be adopted.

A convenient integration of (III-42) yields the result

$$\sigma(t_0) = \int \sigma(\nu)\, e^{2\pi i\nu t_0}\, d\nu = \int KD(\nu)\, d\nu = K\, \rho_M ,$$

and integration of (III-52) gives

$$P_m = K^2 \rho_M \ .$$

Thus, we again encounter Eqs. (III-34) and (III-35).

Note that the matrix form of Eq. (III-41) is

$$\sigma(\nu) = \tilde{S}h \tag{III-54}$$

and that the comparison of (III-44), (III-53), and (III-54) gives

$$\gamma_{\Sigma_0}(\nu) = h^+ Ch = \left[Ke^{2\pi i\nu t_0}\right] \tilde{S}h \quad . \tag{III-55}$$

This equation may be demonstrated directly by beginning with Eq. (III-45), setting the two matrices associated with the two sides equal to each other, and multiplying on the right by h.

III-8. Properties of the Output Signal

Equation (III-34) shows that $\sigma(t_0)$ has necessarily the same sign as K, , which we will assume to be positive in order to conform to the current practice which displays the "peak" of the signal "upward"; however, the opposite convention would be just as useful. Let us justify the qualitative term "peak" showing that $\sigma(t_0)$ is indeed the maximum value of the output signal $\sigma(t)$. This is equivalent to showing that $\sigma'(t) = (t_0+t)$ is less than or equal in magnitude to $\sigma'(0) = \sigma(t_0)$,

or

$$|\sigma'(t)| \leq \sigma'(0) \quad . \tag{III-56}$$

Then,

$$\sigma'(t) = \sigma(t_0+t) = \int \sigma(\nu)\ e^{2\pi i\nu(t_0+t)}\ d\nu \ ,$$

and using equation (III-42),

$$\sigma'(t) = K\int D(\nu)\ e^{2\pi i\nu t}\ d\nu \ ,$$

which is, as we know, real and non-negative, and the obvious inequality

$$\left|\int D(\nu)\ e^{2\pi i\nu t}\ d\nu\right| < \int D(\nu)\ d\nu$$

precisely demonstrates Eq. (III-56) . Note that $KD(\nu)$, being even in ν, has a Fourier transform $\sigma'(t)$ that is even in t . Thus, $\sigma(t)$ is symmetric about time $t = t_0$.

APPENDIX I

We have, in fact

$$b_j(t) = \int B_j(\theta)\ R_j(t-\theta)\ d\theta\ ,$$

$$b_k(t) = \int B_k(\theta')\ R_k(t-\theta')\ d\theta',$$

from which we get

$$\gamma_{jk}(\tau) = \iint E\ \Big\{ B_j(\theta)\ B_k(\theta') \Big\}\ R_j(t-\theta)\ R_k(t+\tau-\theta')\ d\theta\ d\theta'\ ,$$

$$\gamma_{jk}(\tau) = \iint\ C_{jk}(\theta'-\theta)\ R_j(t-\theta)\ R_k(t+\tau-\theta')\ d\theta\ d\theta'\ .$$

Letting

$$\theta' - \theta = \sigma\ ,$$
$$t - \theta = \omega\ ,$$

we have

$$\gamma_{jk}(\tau) = \iint\ C_{jk}(\sigma)\ R_j(\omega)\ R_k(\tau+\omega-\sigma)\ d\sigma\ d\omega\ .$$

Then,

$$R_k(\tau) = \int R_k(\nu)\ e^{2\pi i\nu\tau}\ d\nu\ ,$$

from which we get

$$\gamma_{jk}(\tau) = \int_\sigma \int_\omega \int_\nu C_{jk}(\sigma)\ R_j(\omega)\ R_k(\nu)\ e^{2\pi i\nu(\tau-\omega-\sigma)}\ d\omega\ d\sigma\ d\nu$$

$$\gamma_{jk}(\tau) = \int_\nu R_k(\nu)\ e^{2\pi i\nu\tau}\left[\int R_j(\omega)\ e^{2\pi i\nu\omega}\ d\omega\right]\left[\int C_{jk}(\sigma)\ e^{-2\pi i\nu\sigma}\ d\sigma\right]\ d\nu\ .$$

The first bracketed term is $R_j^*(\nu)$.

The second bracketed term is $C_{jk}(\nu)$.

Then,

$$\gamma_{jk}(\tau) = \int R_j^*(\nu)\ R_k(\nu)\ C_{jk}(\nu) \quad \left[e^{2\pi i \nu \tau}\right] \ d\nu \ .$$

This example is analogous to that of reference (7), page 39.

APPENDIX II

$S(u)$ being a given function, let us determine a function $f(u)$ such that, for any function $r(u)$ constrained by the relation $\int r(u)\ S(u)\ du = 0$,

we have $\int r(u)\ f(u)\ du = 0$.

There are two points $u = u_1$ and $u = u_2$ for which the values of $S(u)$ are $S(u_1)$ and $S(u_2)$ (which will be assumed to be different from zero). Consider the function

$$r(u) = \frac{\delta(u-u_1)}{S(u_1)} - \frac{\delta(u-u_2)}{S(u_2)} \ .$$

This satisfies the first equation since $\int \delta(u-u_1)\ S(u)\ du = S(u_1)$.

We must have therefore

$$\frac{1}{S(u_1)} \int \delta(u-u_1)\ f(u)\ du - \frac{1}{S(u_2)} \int \delta(u-u_2)\ f(u)\ du = 0$$

and

$$\frac{f(u_1)}{S(u_1)} = \frac{f(u_2)}{S(u_2)} \ .$$

Considering u_2 as an arbitrary point, it is clear that $\frac{f(u)}{S(u)}$ is a constant, independent of the point considered and thus independent of u. Therefore, $f(u) = K\ S(u)$.

CHAPTER IV

REMARKS CONCERNING THE FORMAL SOLUTIONS OF SOME SPECIAL CASES

Summary

The preceding formal solution may, of course, be applied to the classical problem of a single input.

a. In the special case where the N noise inputs are uncorrelated, it is shown that multiple matched filtering is achieved by summing the outputs of separate matched filters in each channel.

b. If the N noise inputs are both uncorrelated and of the same uniform spectral density, the generalization of a well-known, essential property of the classical theorem establishes the fact that the value of the optimized parameter depends only upon the total energy of the multiple signal (of the N signal inputs).

c. The parameter ρ_m expresses the performance (S/N) of the matched filter for given signal and noise inputs. If the noise is defined and the choice of a signal is arbitrary (a case frequently found in practice), the value of ρ_m depends upon that choice.

Examination of the expression for ρ_m shows that it is advantageous to choose signals in selected frequency bands whose specification is based upon the correlation matrix of the N noise inputs. This qualitative observation introduces the idea of "eliminable interference" or "infinite signal-to-noise ratio," which will be developed later.

IV-1. Limitations of the Cases Considered in This Chapter

The systems of Eqs. (III-30) and (III-37) provide, in temporal form and spectral form, respectively, the solution to the problem posed in Paragraph III-1.

First, let us note that according to (III-37) the system functions of the filters $h_k(\nu)$ are defined only for $\Delta(\nu) \neq 0$. Therefore, it is a good idea to examine the case in which this condition may not be fulfilled. This, however, is part of the study of the properties of $\Delta(\nu)$ that will be undertaken later.

IV-2. The Case of a Single Input

With the aid of Eqs. (III-30) and (III-37), the solution of a matched filter with a single input may be found - a solution summarized in Paragraph II-2. Equation (III-3) yields Eq. (II-2) directly for the case of $N = 1$, by letting

$$C_{11}(\tau) = C(\tau) ,$$

the autocorrelation function of the single noise input.

Equation (III-37) no longer has any meaning in the case where $N = 1$ since the cofactor $M_{11}(\nu)$ is not defined, but Eq. (III-36) is reduced to

$$h_1(\nu) \; C_{11}(\nu) = K e^{-2\pi i \nu t_0} \; S_1^*(\nu) ,$$

or

$$h_1(\nu) = K e^{-2\pi i \nu t_0} \cdot \frac{S_1^*(\nu)}{C_{11}(\nu)} , \qquad \text{(IV-1)}$$

which is identical to (II-3) since $C_{11}(\nu) = C(\nu)$ is the spectral density of the single noise input considered here.

The rule given in Paragraph III-7 is also valid since the spectrum of the output signal is

$$\sigma(\nu) = h(\nu) \; S(\nu) = K e^{-2\pi i \nu t_0} \cdot \frac{|S(\nu)|^2}{C(\nu)} ,$$

while the spectral density of the output noise is

$$\gamma_{\Sigma_0} = C(\nu) \,.\, |h(\nu)|^2 = K^2 \cdot \frac{|S(\nu)|^2}{C(\nu)}$$

with $$D(\nu) = \frac{|S(\nu)|^2}{C(\nu)} .$$

IV-3. Matched Filter Rule for Uncorrelated Noise. Signal Energy and Signal-to-Noise Ratio

Let us consider the simple case where the N noise components are not correlated. In this case, since the crosscorrelation functions are identically zero, all the elements $C_{jk}(\nu)$ of the determinant $\Delta(\nu)$ are zero for $j \neq k$. Thus, $\Delta(\nu)$ becomes diagonal and takes on the value of the product of the N spectral densities $C_{jj}(\nu)$.

The cofactors $M_{jk}(\nu)$ are themselves zero for $j \neq k$, and one cofactor $M_{jj}(\nu)$ is the product of $N - 1$ spectral densities obtained by omitting $C_{jj}(\nu)$. As a result, Eqs. (III-37) are reduced to

$$h_k(\nu) = Ke^{-2\pi i\nu t_0} \cdot \frac{S_k^*(\nu)\, M_{kk}(\nu)}{\Delta(\nu)} ,$$

that is, to the N equations

$$h_k(\nu) = Ke^{-2\pi i\nu t_0} \cdot \frac{S_k^*(\nu)}{C_{kk}(\nu)} , \qquad \text{(IV-2)}$$

for $k = 1, 2, \ldots . N$.

The system of Eqs.(IV-2) points out that for each input the optimum filter is the same as the case in which that input is considered to be isolated. The following rule may then be stated (referring to Fig. III-1):

A matched filtering process with N inputs, whose noise inputs are uncorrelated, may be obtained by taking the sum of the outputs of the matched filters for each input.

In the special case considered here, the value of ρ_m given by expression (III-39) is reduced to

$$\rho_m = \sum_j \int \frac{|S_j(\nu)|^2}{C_{jj}(\nu)}\, d\nu . \qquad \text{(IV-3)}$$

Furthermore, if, the N spectral densities are uniform and equal ,

$$C_{jj}(\nu) = d ,$$

and we have

$$\rho_m = \frac{1}{d} \sum_j \int |S_j(\nu)|^2 d\nu . \qquad \text{(IV-4)}$$

The integral $\int |S_j(\nu)|^2 d\nu$ represents the energy of the signal $S(t)$

(Parseval's identity). Then, the ratio ρ_m depends only upon the sum of the energies of the N signals, that is, upon the total signal energy at the inputs.

This statement generalizes the equivalent rule valid for the case of one input as long as the noise has a uniform density. We have then

$$\rho_m = \frac{E_s}{d} ,$$

where E_s is the energy of the signal (see reference (8), Chapter IV).

IV-4. General Case - Usefulness of the Arbitrary Parameter

A matched filter with N inputs is defined except for a constant factor - designated here by K - for all filters h_k, and an arbitrary time lag represented by the term $e^{-2\pi i \nu t_0}$, a factor in the N system functions $h_k(\nu)$. This time lag depends upon the instant of time t_0 chosen for the appearance of the "peak" output signal. As has been already suggested in II-2, the performance of the system represented by ρ_m – which may be defined as the signal-to-noise ratio of the process – is independent of K and of t_0, which is evident from Eq. (III-39). It is equally clear physically that no change in the detection ability of the system is brought about by inserting the same ideal delay line in series with each input. The appearance of the signal peak is only changed in time without modifying the height of that peak above the mean noise level. By this observation, we touch upon the question of the possibility of achieving filters defined by Eqs. (III-30) or (III-37). We know that, in order to be realizable, a filter must have an impulse response which is zero for $t \leq 0$, bounded, and unconditionally integrable. If a theoretical filter satisfies only the two latter conditions, its unit impulse response goes to zero at the limits $t = +\infty$ and $t = -\infty$. It is possible then to satisfy the first condition in an approximate way by a translation of the unit impulse response to the right on the time axis, that is, by a supplementary time delay. This amounts to saying that the physical approximation of a filter is often facilitated by adding adequate time delay. The approximate realization of the matched filter will thus be facilitated by permitting an appropriate value of t_0; this time delay, although important, certainly constitutes an inconvenience since it postpones the instant when the observer is informed of the presence of the signal, but it allows us to obtain a good approximation of the ideal performance expressed by ρ_m.

IV-5. Limited Possibilities for Elimination of Noise Having a Particular Structure

Let us consider Eq. (III-39).

$$\rho_m = \int \sum_j \sum_k \frac{S_j^*(\nu)\ S_k(\nu)\ M_{jk}(\nu)}{\Delta(\nu)} \quad . \qquad \text{(III-39)}$$

The possibility appears here of having ρ_m become "infinite" if there exist frequencies for which $\Delta(\nu)$ becomes zero (provided that the signals indeed possess spectral components at these frequencies and that the sum of the cofactors does not go to zero as well).

Such a possibility, which corresponds to the total elimination of noise, naturally constitutes a limiting case. It is apparent, however, that it may be interesting in practice to concentrate the signal energy in those frequency bands where $\Delta(\nu)$ is very small, if such bands exist.

Formally, structures having "eliminable noise" may be defined in terms of matched filtering. The case of "coherent interference" which belongs to this category will be treated later (Paragraph VII-5). It can be immediately seen that no such situation ever arises in the case of N uncorrelated noises; in fact, as we have seen, $\Delta(\nu)$ is reduced, in such a case, to the product of spectral densities; that is, to a non-negative quantity. Further, the spectral density of noise at an input is never zero in practice (because of thermal noise). Thus, $\Delta(\nu)$ may not be zero at any frequency (Paragraph VI-8).

Formally, the case $\Delta(\nu)=0$ may be encountered if the noise inputs are statistically related. However, in practice, there will always be one part of the total interference - thermal noise - which, because of its independence from one input to another, will prevent $\Delta(\nu)$ from being reduced to zero. Thus, $\Delta(\nu)=0$ appears as a limiting case, interesting in its formalism but physically inaccessible, as is the "infinite" signal-to-noise ratio.

This restriction does not take away any of the practical interest from the following statement: the signal-to-noise ratio tends to become very large when $\Delta(\nu)$ becomes very small. In order to use this fact, we must, given the choice, use signals in the band or bands where $\Delta(\nu)$ is the smallest; as a result, we must know $\Delta(\nu)$.

This preceding statement is obviously a generalization of the very simple case of one input for which the expression in (III-39) becomes

$$\rho_m = \int \frac{|S(\nu)|^2}{C(\nu)} \, d\nu \ . \qquad \text{(IV-5)}$$

Applying the theorem of the mean, it may be shown that for a given signal energy,

$$E_s = \int |S(\nu)|^2 \, d\nu \ .$$

The maximum value for ρ_m will be obtained by placing this energy in that band - or in the limiting case, at that frequency - where $C(\nu)$ is minimized. Since it is a question of a limiting case, it is particularly interesting to know the nature of the statistical relationships - let us say, the spatial structure of the noise at the inputs - which will result in $\Delta(\nu)$ being zero. In fact, the performance of matched filtering is then considerable since it permits a theoretically infinite signal-to-noise ratio. In other words, it allows us - in this case only - to suppress the noise. A more complete discussion of this case will appear later.

At present, let us be satisfied to take note of the formal difficulty involved in practical use of the case $\Delta(\nu) = 0$. In fact, K being considered an arbitrary, fixed constant - easily normalized to the value 1 - in Eq. (III-37) , the $h_j(\nu)$ are found to be unbounded and thus unrealizable in just that case where they yield the best results. The output signal $\sigma(t_0)$ from filters with "infinite" gain is itself infinite. Its observation would thus be inconvenient. We will see in Chapter VI how the concept of normalized matched filtering permits us to avoid this difficulty in the case of a narrow band.

CHAPTER V

THE CASE OF IDENTICAL SIGNALS - PROPER FILTERING-ORTHOGONAL IMAGES OF A SYSTEM WITH N NOISE INPUTS

Summary

To assume N identical signals does not reduce the generality of the problem. This is a case to which we may always return and one to which, in general, even the most advanced techniques are effectively reduced. In this case, the following may be established:

a. Matched filtering is divided into (1) a proper antenna filtering (PF), which depends only on the N noises and involves one filter per channel, and (2) a unique filter defined by the signal alone, which is nothing more than the matched filter for the signal in noise with uniform density.

b. The direct sum of the N elements of an antenna **forms** a part of a PF - now an optimal process - only if the N noises are uncorrelated and have the same spectral density.

The concept of proper filtering is very important. The properties of a PF are associated with the noise alone. The "system function" (complex gain) of a PF may be defined for the signal between the input and output. It is always real and non-negative, and the first characteristic property of the PF is the following:

The gain of the PF is equal (except for a homogeneity factor) to the spectral density of the output noise.

The study of a PF reduces naturally to a study of the correlation matrix, whose known classical properties will first be reviewed. The eigenvalues of this matrix are functions of frequency and have the properties of a spectral density.

The initial argument assuming N correlated noise inputs $B_j(t)$ may be replaced by a simpler argument assuming N uncorrelated noise inputs derived from the first, which represent the ensemble $B_j(t)$ equally well. This transposition offers the same convenience as the reduction of a surface to principal axes. Thus it is, in itself, interesting and will be used effectively in the remainder of the study.

The N "reduced" noise inputs, which we will call the "orthogonal images" of the $B_j(t)$, are obtained from the latter with the aid of a collection of linear filters comprising a "matrix σ ". This means of generation illustrates the abstract operation of diagonalizing the correlation matrix C .

The "orthogonal images" are uncorrelated, have for their spectral densities the eigenvalues of the correlation matrix, and are such that to any multiple filtering of the $B_j(t)$ there corresponds a multiple filtering of the images that yields the same signal and the same noise spectral density. The latter, however, is easier to study.

In particular, to the matched filtering of a signal in the $B_j(t)$ there corresponds the matched filtering of the signal transformed (through the matrix σ) to "orthogonal image" form. Proper filtering of the $B_j(t)$ corresponds to matched filtering of unlike signals in uncorrelated noise.

Thus, from now on, we will be able to transpose the entire study into the realm of "orthogonal images", which will permit us to establish, in a simple, "physical" way, certain properties of matched filters in the limiting case where the correlation matrix becomes singular.

Finally, the second characteristic property of proper filtering is established; it is a multiple filtering process where the crosscorrelation between a single noise input and the output noise is the same for all inputs. Moreover, this mutual crosscorrelation is zero for any value of τ except for one $(\tau=0)$ but dependent upon the choice of the time origin. This is a "microscopic" crosscorrelation.

V-1. Non-Restrictive Character of the Case of Identical Signals

We will consider, for the present, the case in which the N signals are identical, first emphasizing the practical interest of this special case; it is immediately clear that we may always come back to it. Let us assume, as we have done before, that the N inputs are the N elements of a receiving antenna with fixed elements arranged in some arbitrary manner. The signal to be received is carried by a plane wave from a known direction. Let us put on each element an ideal delay line of such a value that the relative time lags of the elements for that particular direction are equalized. The output signals of the N delay lines become identical. Naturally the statistical relationships between the noise components are not the same at the outputs as at the inputs, but this does not restrict us much since we have assumed them to be arbitrary. We have thus constructed a new system with N inputs (the delay line outputs) at which the N signals are identical. The operation may be repeated for each direction of the signal plane wave. This is exactly what is done, in general, in a technique aptly named "preformation of beams", designed to provide for each direction an output whose signal-to-noise ratio is improved by putting all the signals in phase. Let us say "improved" and not "optimized" since it will be seen later that it is necessary to consider this simple "putting in phase" and the limits of its efficacy. (See Paragraph V-2.)

Be that as it may, making signals at antenna elements identical is already in current practice and is related to the angular separation of the different plane waves which are received.

Strictly speaking, the process of delay compensation is sufficient to render the signals identical only if one is dealing with point antenna elements, which do not alter the incident sound field by their presence.

If there exists a rigid structure around which sound waves are diffracted, delays computed from the geometry are not adequate. For a given element, however, this diffraction appears as a linear filtering which modifies, at the input, the type of signal carried by the incident wave. This filtering is defined by the relative geometry of the element and the structure and by the boundary conditions of the acoustic field at the structure. At any rate, one may at least formally assume each input to be affected by an inverse filtering which nullifies the diffraction and restores identical signals to all the inputs. The compensation process is complex but still possible.

Let us summarize:

a. At the formal level, we may always reduce the problem to the case of identical signals. Properly speaking, there is no restriction here upon the general study of matched filtering with N inputs.

b. From a practical standpoint, it is often very close to the situation of N identical signals.

V-2. Formal Solution for the Case of Identical Signals

Assume that the N signals are identical:

$$S_1(t) = S_2(t) = \ldots\ldots. S_N(t) = S(t) \;,$$
$$S_1(\nu) = S_2(\nu) = \ldots\ldots. S_N(\nu) = S(\nu) \;. \qquad \text{(V-1)}$$

Equation (III-37) is then reduced to the following N equations:

$$h_k(\nu) = Ke^{-2\pi i\nu t_0} \cdot S^*(\nu) \cdot \frac{1}{\Delta(\nu)} \sum_j M_{jk}(\nu) \;, \qquad \text{(V-2)}$$

$$\text{for } k = 1, 2, \ldots. N \;.$$

As a result of (V-2) each filter $h_k(\nu)$ is obtained by placing in series a filter whose gain[4] is

$$Ke^{-2\pi i\nu t_0} S^*(\nu) \;,$$

which depends only on the signal and arbitrary time t_0, with a filter

$$p_k(\nu) = \frac{\sum_j M_{jk}(\nu)}{\Delta(\nu)} \;, \qquad \text{(V-3)}$$

which depends only upon the noise at the N inputs.

The filtering defined by $Ke^{-2\pi i\nu t_0} S^*(\nu)$ is nothing more than the matched filter for the signal $S(t)$ in noise having a uniform density. It is common to all N channels of the system. If we refer to Fig. III-1 we will see that, in the

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[4]Hereafter, except for a homogeneity factor, the expression "complex gain" is understood.

case of identical signals, it is reduced to Fig. V-1, in which the filtering of the signal is accomplished after summing the filtered output of each channel $p_k(\nu)$.

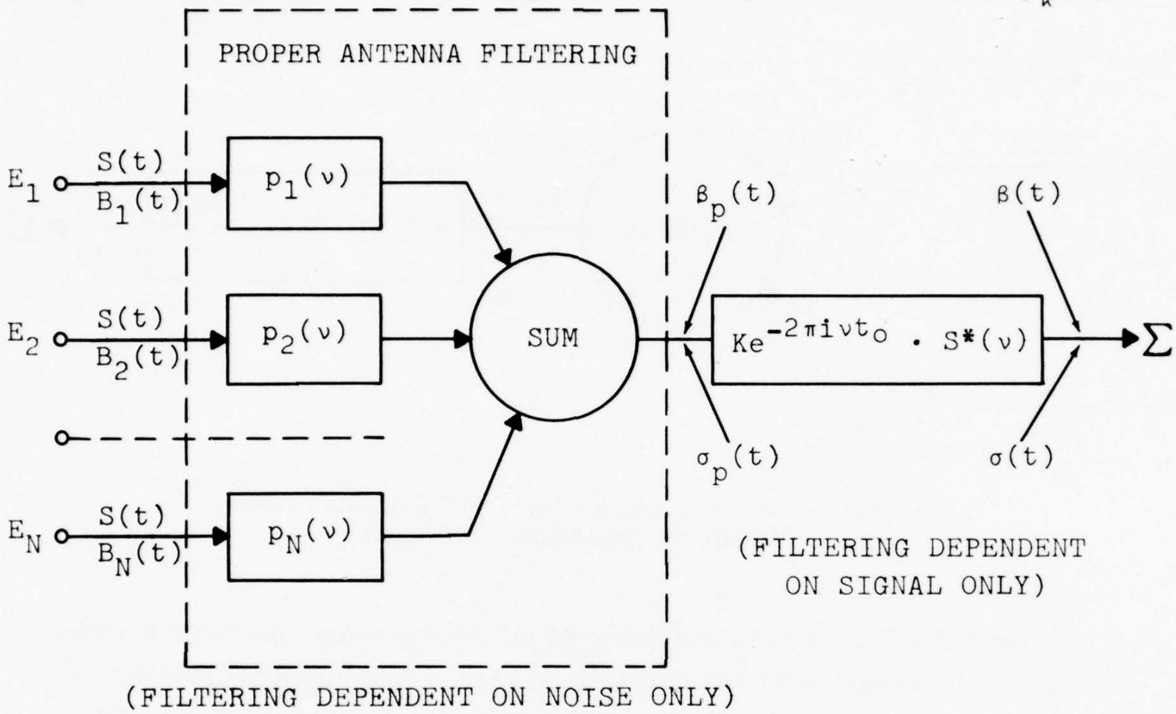

Fig. V-1. Identical Signals – Arbitrary Noise Inputs

Let us consider, for the moment, the special case where the N noise inputs are uncorrelated (see Eq. IV-2) . Equation (V-2) becomes

$$h_k(\nu) = Ke^{-2\pi i\nu t_0} \cdot \frac{S^*(\nu)}{C_{kk}(\nu)} , \qquad \text{(V-4)}$$

and Eq. (V-3) becomes

$$p_k(\nu) = \frac{1}{C_{kk}(\nu)} . \qquad \text{(V-5)}$$

If, moreover, the N spectral densities are identical to $C(\nu)$, all the $p_k(\nu)$ are identical to $\frac{1}{C(\nu)}$. Figure V-1 becomes Fig. V-1(a) in which all filtering processes are reduced to a single filter $Ke^{-2\pi i\nu t_0} \cdot \frac{S^*(\nu)}{C(\nu)}$ placed after the direct summation of the inputs. This single filtering process is, moreover, the matched filter for an arbitrary input.

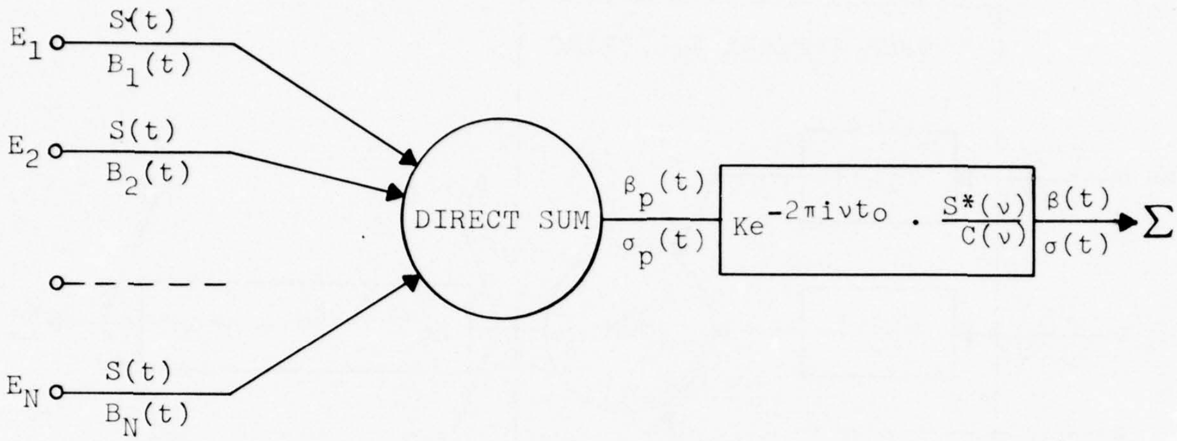

Fig. V-1(a). Identical Signals – Uncorrelated Noise Inputs Having the Same Spectral Density

The preceding line of reasoning presents all the necessary conditions in order that the direct summation of antenna elements may be interposed in the matched filtering process (see Chapter I). As attractive as the replacement of N filters by a single one may be, it is well to realize that an optimal process may be realized in this way only under the following conditions:

For identical signals, with uncorrelated noise inputs having the same spectral density, the matched filtering process is reduced to direct summation of inputs followed by the matched filtering for an arbitrary input signal.

V-3. Proper Filtering of an Antenna and Its Gain. First Characteristic Property

We will now define the important concept of proper antenna filtering. Let us recall the general Eqs. (V-2) and (V-3), referring to identical signals and arbitrary noise (Fig. V-1). The ensemble of filters $p_k(\nu)$ and the subsequent summation will be called the proper filtering for the system having N inputs in the presence of noises $B_1(t)$, $B_2(t)$, ... $B_N(t)$.

The letter P designates the output of the proper filtering process where the signal will be $\sigma_p(t) \rightleftarrows \sigma_p(\nu)$ and where the noise will be $\beta_p(t)$. The numerator of $p_k(\nu)$ in Eq. (V-3) is the sum of the cofactors of the terms of the k^{th} column of the determinant $\Delta(\nu)$. In other words, it is a determinant obtained by starting with $\Delta(\nu)$ and replacing the terms of the k^{th} column by ones.

Proper filtering has been defined from the concept of identical signals, that is, by a single unique signal applied to all elements of the antenna. For such a unique signal, it is equivalent to a linear filter whose system function (complex gain) is $\sum_k p_k(\nu)$ since the spectrum of the signal at P is $S(\nu) \cdot \sum_k p_k(\nu)$.

The system function $G_p(\nu)$ of the proper filtering process will be defined as

$$G_p(\nu) = \sum_k p_k(\nu) = \sum_j \sum_k M_{jk}(\nu) \Big/ \Delta(\nu) \quad . \qquad \text{(V-6)}$$

This is the spectrum of the response obtained at point P when the same unit impulse $\delta(t)$ (Dirac delta function), whose spectrum $S(\nu) = 1$, is applied at the inputs.

The ratio ρ_m (optimum signal-to-noise ratio) at the output becomes, in the case of identical signals (see Eq. (III-39)),

$$\rho_m = \int \frac{|S(\nu)|^2}{\Delta(\nu)} \sum_j \sum_k M_{jk}(\nu)\, d\nu \ , \qquad \text{(V-7)}$$

or

$$\rho_m = \int |S(\nu)|^2 \, G_p(\nu)\, d\nu \ . \qquad \text{(V-7 a)}$$

The density $D(\nu)$ defined by (III-40) becomes in this case

$$D(\nu) = |S(\nu)|^2 \, G_p(\nu) \ , \qquad \text{(V-8)}$$

and as a result $G_p(\nu)$ is a real, non-negative function that is even in ν .

Thus, the proper gain of an antenna is real and non-negative; that is,an arbitrary spectral component of the input signal appears at point P (Fig. V-1) multiplied by a real, positive number and therefore is not changed in phase.

The matrix notation of Paragraph III-7 is modified because column matrix S may be replaced by $\left[S(\nu)\right]\alpha$, where α is the column matrix whose N elements are equal to 1. Equation (III-46) becomes

$$h = \left[K e^{-2\pi i \nu t_0} \cdot S^*(\nu) \right] C^{-1} \alpha \ . \qquad \text{(V-9)}$$

Equation (III-50) (spectral density at Σ) becomes

$$\gamma_{\Sigma_0}(\nu) = h^+ Ch = \left[K^2|S(\nu)|^2\right]\tilde{\alpha}\, C^{-1}\alpha \;, \qquad \text{(V-10)}$$

from which we get

$$D(\nu) = \left[\,|S(\nu)|^2\right]\tilde{\alpha}\, C^{-1}\alpha \qquad \text{(V-11)}$$

and

$$G_p(\nu) = \tilde{\alpha}\, C^{-1}\alpha \;. \qquad \text{(V-12)}$$

The spectrum of the signal at output P is obviously $G_p(\nu)\, S(\nu) = \sigma_p(\nu)$, but at the output Σ , this spectrum is (see Eq. (III-54))

$$\sigma(\nu) = \left[S(\nu)\right]\tilde{\alpha}\, h \;,$$

or

$$\sigma(\nu) = \left[Ke^{-2\pi i\nu t_0}|S(\nu)|^2\right]\tilde{\alpha}\, C^{-1}\alpha \;, \qquad \text{(V-13)}$$

and, of course, Eq. (III-53) remains valid in the form

$$\frac{\gamma_{\Sigma_0}(\nu)}{K} = \frac{\sigma(\nu)}{Ke^{-2\pi i\nu t_0}} = D(\nu) = |S(\nu)|^2\tilde{\alpha}\, C^{-1}\alpha \;. \qquad \text{(V-14)}$$

The concept of <u>proper filtering</u> and especially of antenna gain $G_p(\nu)$ effectively expresses the properties of the ensemble of N noises at the n inputs. This is a result of Eq. (V-12). The spectral density of the noise at the output is given by (V-10), which from (V-12) may be written

$$\gamma_{\Sigma_0}(\nu) = K^2|S(\nu)|^2\; G_p(\nu) \;. \qquad \text{(V-15)}$$

Now, this density is the product of the spectral density at P or $\gamma_{P_0}(\nu)$, with the square of the magnitude of filtering gain, with the filter placed between P and Σ (Fig. V-1). It may be seen that

$$\gamma_{P_0}(\nu) = G_p(\nu) \quad . \qquad \text{(V-16)}$$

Thus, the system function (complex gain) of proper filtering is equal to the spectral density of the noise at the output of that filtering. (See the note of Paragraph V-2.) This property is nothing more than the characteristic rule of matched filtering (Paragraph III-7) applied to proper filtering.

This is the same as saying that everything takes place as if the antenna were equivalent to a single input at which the noise has a spectral density of

$$A(\nu) = \frac{1}{G_p(\nu)} = \frac{\Delta(\nu)}{\sum_j \sum_k M_{jk}(\nu)} \quad , \qquad \text{(V-17)}$$

the gain of the antenna being $G_p(\nu)$, and the input signal, $S(t)$. The matched filtering of such a system (see Eq. (II-3)) is then comprised of

$$Ke^{-2\pi i\nu t_0} S^*(\nu)\, G_p(\nu) \quad ,$$

that is, precisely as in Fig. V-1,where the antenna was replaced by a single filter with a gain $G_p(\nu)$. The preceding property may also be expressed as follows:

Proper filtering is such that the spectral density of the noise at the output is equal to the amplitude spectrum of the impulse response (signal $S(t)$ at all N inputs), which is nothing more than the system function. We are dealing here with an intrinsic property of the antenna in the presence of N given noise inputs.

V-4. Review of the Known Properties of the Correlation Matrix

Since proper filtering of a system is independent of the signal considered, its study is reduced to the study of $\Delta(\nu)$ and its cofactors and, more generally, of the N statistically related noise inputs. Hence, for the remainder of the chapter, we will depart somewhat from the viewpoint of optimal signal detection - to which we will return in the following chapters - in order to review the mathematical properties of $\Delta(\nu)$, and we will relate them, by their physical interpretation, to the properties of N statistically related noises.

Let us first recall that each term of the determinant $\Delta(\nu)$ is a $C_{jk}(\nu)$ having Hermitian symmetry with respect to ν and with respect to its indices, according to Eqs. (III-5) and III-6):

$$C_{jk}(\nu) = C^{*}_{jk}(-\nu) \quad , \tag{III-5}$$

$$C_{jk}(\nu) = C^{*}_{kj}(\nu) \quad . \tag{III-6}$$

As a result, the determinant $\Delta(\nu)$ itself, being a polynomial in C_{jk} of degree N, has Hermitian symmetry with respect to ν:

$$\Delta(\nu) = \Delta^{*}(-\nu) \quad .$$

Furthermore, thanks to Eq. (III-6), reflection about its principal diagonal leaves $\Delta(\nu)$ unchanged and equal to its complex conjugate. Thus,

$$\Delta(\nu) = \Delta^{*}(\nu) \quad . \tag{V-19}$$

As a result, $\Delta(\nu)$ is real and an even function of ν. We know that it is also non-negative, as are the eigenvalues of the matrix C.

In the following discussion, it is intended to define exactly how this property expresses an obvious physical fact; namely, the sum of arbitrary filtering of the N noise inputs under consideration is itself a noise and by virtue of this fact possesses a necessarily non-negative spectral density.

Equation (III-18),

$$P_{\beta} = \sum_{j} \sum_{k} \iint R_j(u)\, R_k(v)\, C_{jk}(u-v)\; du\, dv \quad , \tag{III-18}$$

gives the noise power of the sum of the arbitrary filtering $R_1(t)$, $R_2(t)$, ... $R_N(t)$ of the N noise inputs $B_1(t)$, $B_2(t)$, ... $B_N(t)$ (multiple filtering).

This relation may be written in spectral form. Let us first put it in the form

$$P_{\beta} = \sum_{j} \sum_{k} \int R_j(u) \left[\int R_k(v)\, C_{jk}(u-v)\; dv \right] du \quad .$$

The bracketed term is the convolution of $R_k(t)$ with $C_{jk}(t)$. Its Fourier transform is then $R_k(\nu)\ C_{jk}(\nu)$, and it may be written

$$\int R_k(\nu)\ C_{jk}(\nu)\ e^{2\pi i\nu u}\ d\nu\ ,$$

or

$$P_\beta = \sum_j \sum_k \int_u \int_\nu R_k(\nu)\ C_{jk}(\nu) \left(R_j(u)\ e^{2\pi i \nu u}\right) du\ d\nu\ .$$

In the last expression, integration in u results in $R_j^*(\nu)$, and we have finally

$$P_\beta = \int \left[\sum_j \sum_k R_j^*(\nu)\ R_k(\nu)\ C_{jk}(\nu) \right] d\nu\ . \qquad \text{(V-20)}$$

We would also have been able to obtain Eq. (V-20) from the autocorrelation of the sum of the filtering processes, that is, by generalizing to N noise inputs Eqs. (III-9) and (III-10), which were established for two noise inputs.

Thus, it would appear, according to (V-20), that the spectral density of the multiple filter output is

$$\gamma_\Sigma(\nu) = \sum_j \sum_k R_j^*(\nu)\ R_k(\nu)\ C_{jk}(\nu) = R^+CR\ , \qquad \text{(V-21)}$$

that is, the Hermitian form of the correlation matrix C. It is non-negative for any multiple filtering R (positive definite matrix). Let us recall that, in a linear transformation which diagonalizes the matrix C, the expression of the Hermitian form becomes

$$\gamma_\Sigma(\nu) = \sum_j \lambda_j(\nu)\ |\chi_j(\nu)|^2\ , \qquad \text{(V-22)}$$

where the $\lambda_j(\nu)$ are the eigenvalues of matrix C – real eigenvalues since C is Hermitian – and where the $\chi_j(\nu)$ correspond to the $R_j(\nu)$ in the transformation (reference (14), Chapter III).

The matrix notation for (V-22) is

$$\gamma_\Sigma(\nu) = \chi^+\lambda\chi\ , \qquad \text{(V-23)}$$

where X is the column matrix of $X_j(\nu)$ and where λ (without indices) is the diagonal matrix of eigenvalues.

Equation (V-23) is true for any arbitrary $X_j(\nu)$ (resulting from any arbitrary $R_j(\nu)$. Thus, it is evident that

$$\lambda_j(\nu) \geq 0 \quad \text{for any} \quad j \text{ and } \nu . \tag{V-24}$$

The eigenvalues of the matrix are non-negative. Their product is non-negative,

$$\Delta(\nu) \geq 0 \quad , \tag{V-25}$$

all these properties being consequences of the fact that $\gamma_\Sigma(\nu) \geq 0$ for an arbitrary multiple filtering process.

Any cofactor $M_{jj}(\nu)$ plays the same role as $\Delta(\nu)$ for $N-1$ noises among the N noises considered. Hence

$$M_{jj}(\nu) \geq 0 \quad . \tag{V-26}$$

More generally, every minor of $\Delta(\nu)$ obtained by suppressing a certain number of lines and columns of the same rank, is non-negative. If the minor is of order two we get Eq. (III-13).

To finish this review, note that

$$M_{jk}(\nu) = M^*_{kj}(\nu) \tag{V-27}$$

because of the Hermitian symmetry of $\Delta(\nu)$. Thus, the sum of all $M_{jk}(\nu)$ is real.

We have seen that the gain for proper filtering defined by (V-6),

$$G_p(\nu) = \sum_j \sum_k \frac{M_{jk}(\nu)}{\Delta(\nu)} \quad , \tag{V-6}$$

is, according to (V-8), real, even in ν, and non-negative. Since this is also the case for $\Delta(\nu)$, we have

$$\sum_j \sum_k M_{jk}(\nu), \text{ which is real, non-negative} \tag{V-28}$$

and even in ν.

V-5. Second Characteristic Property of Proper Filtering

Let us make a final observation concerning the physical interpretation of the equation defining proper filtering, which may be written (from (V-9))

$$CP = \alpha \ , \qquad \text{(V-29)}$$

which summarizes N equations of the form

$$C_{j1}P_1 + C_{j2}P_2 + \ldots\ldots C_{jN}\,P_N = 1 \ , \quad \text{for } j = 1, 2, \ldots N. \qquad \text{(V-30)}$$

An arbitrary term of the sum (V-30) is in the form

$$C_{jk}P_{\kappa} \ , \text{ or } \ \Phi^{*}\ C_{jk}P_{\kappa} \ ,$$

where $\Phi(\nu)$ designates the system function of a filter of gain 1 over the whole spectrum. It may be seen by analogy with Eq. (III-10) that $C_{jk}\ P_k$ (a function of ν) is the crosscorrelation spectrum [5] of two noises, the first resulting from the filtering of $B_j(t)$ by $\Phi(\nu)$ (that is, unfiltered $B_j(t)$) and the second resulting from the filtering of $B_k(t)$ by $P_k(\nu)$.

The sum of (V-30) represents, therefore, the crosscorrelation of $B_j(t)$ with the sum of

$B_1(t)$ filtered by $P_1(\nu)$,

$B_2(t)$ filtered by $P_2(\nu)$, etc, ... ,

in other words, the crosscorrelation of $B(t)$ with the output of the proper filter.

Thus, Eq. (V-29) expresses the fact that the crosscorrelation spectrum of any one of the noise inputs $B_j(t)$ with the output of the filter is equal to 1, or rather to Q where Q represents a homogeneity factor. The corresponding crosscorrelation function is thus in the form $Q\ \delta(\tau)$.

Hence, proper filtering is characterized by the following property: it is a multiple filtering process such that the crosscorrelation between any one input and the output is the same and is zero for all τ except $\tau = 0$. This is a "microscopic" crosscorrelation.

[5] That is, as we will recall, the Fourier transform of the crosscorrelation function.

In the special case of a single input, the proper filter has a gain proportional to

$$\frac{1}{\gamma(\nu)} = p(\nu) \quad , \qquad \text{(V-31)}$$

where $\gamma(\nu)$ is the spectral density of the input noise (see Eq. (IV-1)).
The crosscorrelation spectrum of noise between input and output is that of $B(t)$ unfiltered with $B(t)$ filtered by $p(\nu)$,

or according to (III-10), $$[1] \; [\gamma(\nu)] \; [p(\nu)] = 1. \qquad \text{(V-32)}.$$

The corresponding crosscorrelation function is thus equal to $\delta(t)$ except for a homogeneity factor.

CHAPTER VI

NARROW-BAND APPROXIMATION - VARIATION OF PROPER FILTERING (NPF)

Summary

This chapter describes applications and develops the realization of matched filters. We will be limited here to the important and practical case of narrow-band signals. It will be assumed that in the frequency band of interest, properties of noise are independent of frequency. The useful gain of a linear filter is reduced to a complex number that represents a "phase shift" accompanied by a "weighting" of amplitude; these operations are always realizable in the sense of Network Theory, provided that the weighting remains bounded, that is, provided that the gains remain finite.

Thus, the narrow-band approximation offers new possibilities of normalization since the parameters that depend upon the spectral properties of the noise are reduced to constants. These possibilities permit us to avoid a previously described difficulty (Paragraph IV-5): for broad bands, the gains of the filters are not finite (constant K being chosen) if the correlation matrix becomes singular, that is, in the limiting case where the signal-to-noise ratio might be "infinite." For a narrow band, the same difficulty may occur, but it is avoidable to some degree by using a convenient normalization. Thus, we are led to define the variations of proper filtering which perform in the same way for all ordinary cases but offer additional, useful behavior in the limiting cases. These variations are characterized by the disappearance, in the expression for filter gains, of the determinant $\Delta(\nu_0)$ of the correlation matrix.

It is found, moreover, that this disappearance provides an economic advantage. The technology of proper filters, as will be outlined in Chapter IX, consists, in fact, of constructing the C_{jk} and then the M_{jk}, the latter being polynomials of degree $(N-1)$ in C_{jk}. But Δ is, itself, of degree N. By avoiding the construction of Δ, we reduce the technological difficulty by one degree.

Here, consequently, are the points which are to be considered. Calling the previously defined proper filtering simple proper filtering (SPF), let us consider a variation of it, normalized proper filtering (NPF), which has the following properties:

a. It is always defined, even when the correlation matrix becomes singular, unlike SPF.

b. In this latter case it may act in two ways:

- in the case of eliminable noise, the noise is cancelled at the output without losing the signal.
- in the case of cutoff, all the filter gains go to zero simultaneously.

c. A special case of eliminable noise is that in which the noise goes to zero at a single input; all channels cut off except the single input.

d. These cases are physical limits. The inevitable presence of thermal noise prevents their strict realization.

e. Similarly, it will be established, in the case of NPF, that no change occurs if the same uncorrelated noise is superimposed upon the initial noise at each input.

VI-1. Narrow-Band Hypothesis. Physical Meaning of the $C_{jk}(\nu_0)$ and of the Matrix Notation

We have seen that the hypothesis of identical signals does not, strictly speaking, constitute a restriction on the general study of matched filtering, with which we will be concerned from now on. On the other hand, we will for the moment consider a special case where the spectral band of interest, obviously containing the signal[6], is such that the statistical properties of the noise (spectral density and cross-correlation spectrum) can be considered as being independent of frequency. Although, strictly speaking, such a hypothesis may be made in a band that is arbitrarily large, it is clear that, in practice, the best chance of realizing the hypothesis lies in the use of narrow bands. Thus, it (the hypothesis) is very desirable in the practical cases of radar or sonar signals, which are in principle narrow band - that is, of a bandwidth that is small compared with the center frequency.

Let us assume that we are dealing with signals lying in a narrow band centered at ν_0. That is, to a first approximation, only the statistical properties of noise at

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[6]Or, more precisely, the quasi-total of the signal energy, since a signal of finite duration has no spectral limits.

the frequency ν_0 will enter into the problem; in other words, we are interested only in the values $C_{jk}(\nu_0)$. The matched filtering is thus composed of the filters $p_k(\nu_0)$ distributed over the N inputs (Fig. V-1), then the summation of these N filter outputs, followed by the filter $S^*(\nu)$. This last filtering, narrow band by hypothesis, assures to some extent the exclusiveness of the frequency ν_0 between the input and output of the system. The filters $p_k(\nu_0)$ must be interpreted in the following way: sinusoidal signal, with frequency ν_0, unit amplitude, and zero phase, denoted by $\alpha(t)$, applied at input E_k, comes out with amplitude $|p_k(\nu_0)|$ and phase Arg $p_k(\nu_0)$. Likewise, the proper gain $G_p(\nu_0)$ of the system represents the amplitude and phase of the response at P for the same unitary signal $\alpha(t)$ applied at the N inputs. Describing all the transformations by the modulus and phase obtained under the above condition, all equations previously written and, in particular, the matrix equations remain valid when the functions of ν are replaced by their values at $\nu = \nu_0$. For example, the unitary signal $\alpha(t)$ applied at the N inputs may be described by the same column matrix α (all elements = 1) that previously represented the spectrum of $\delta(t)$. We may write then

$$G_p(\nu_0) = \tilde{\alpha}p \ ,$$

where p is the column matrix of the $p_k(\nu_0)$. We will thus be able to examine what the preceding formalism becomes in terms of the narrow-band approximation and to deduce from it, should the occasion arise, its special properties.

At this point, we have no need to isolate the narrow band by means of filters since $S^*(\nu)$ itself takes that responsibility. On the other hand, the "filters" $p_k(\nu_0)$ may always be realized since they consist only of a "phase shifter" and an "amplifier", that is, very simple electronic circuits (provided, however, that $|p_k(\nu_0)|$ is bounded).

Nevertheless, if we had to construct the $C_{jk}(\nu_0)$ themselves, we would need to filter, or rather prefilter, the N inputs with narrow-band filters ϕ centered at ν_0, which are identical and have a gain of unity in the narrow band and zero gain elsewhere. If the $B_j(t)$ were put into these filters ϕ, the outputs would be narrow-band noise $b_j(t)$. We have seen in Paragraph III-2 that the complex number $C_{jk}(\nu_0)$ is the "spectral line" in the crosscorrelation of $b_j(t)$ with $b_k(t)$. In fact, the narrower the filters ϕ, the closer the crosscorrelation function is to a sinusoidal function of τ with frequency ν_0. This function has a phase and an amplitude specified exactly by $C_{jk}(\nu_0)$(see Eq. III-11). Thus, technically, it is the crosscorrelation of $b_j(t)$ and $b_k(t)$ which $C_{jk}(\nu_0)$ will represent, and this same complex number defines the crosscorrelation function and the "crosscorrelation spectrum."

Since this crosscorrelation may be constructed only with nonlinear operations (products) produced by interactions between spectral bands, preliminary isolation (of bands) by prefiltering is indispensable. We will return to this point when it is required to consider the $C_{jk}(\nu_0)$ in order to construct the $p_k(\nu_0)$ (see Chapter IX).

VI-2. Normalized Proper Filtering - Technical Advantages

Multiple filtering (in the sense which we have outlined) which defines proper filtering of the system (see Fig. V-1), is given on the basis of Eq. (V-3):

$$p_k(\nu_0) = \sum_j M_{jk}(\nu_0) \Big/ \Delta(\nu_0) \quad . \qquad \text{(VI-1)}$$

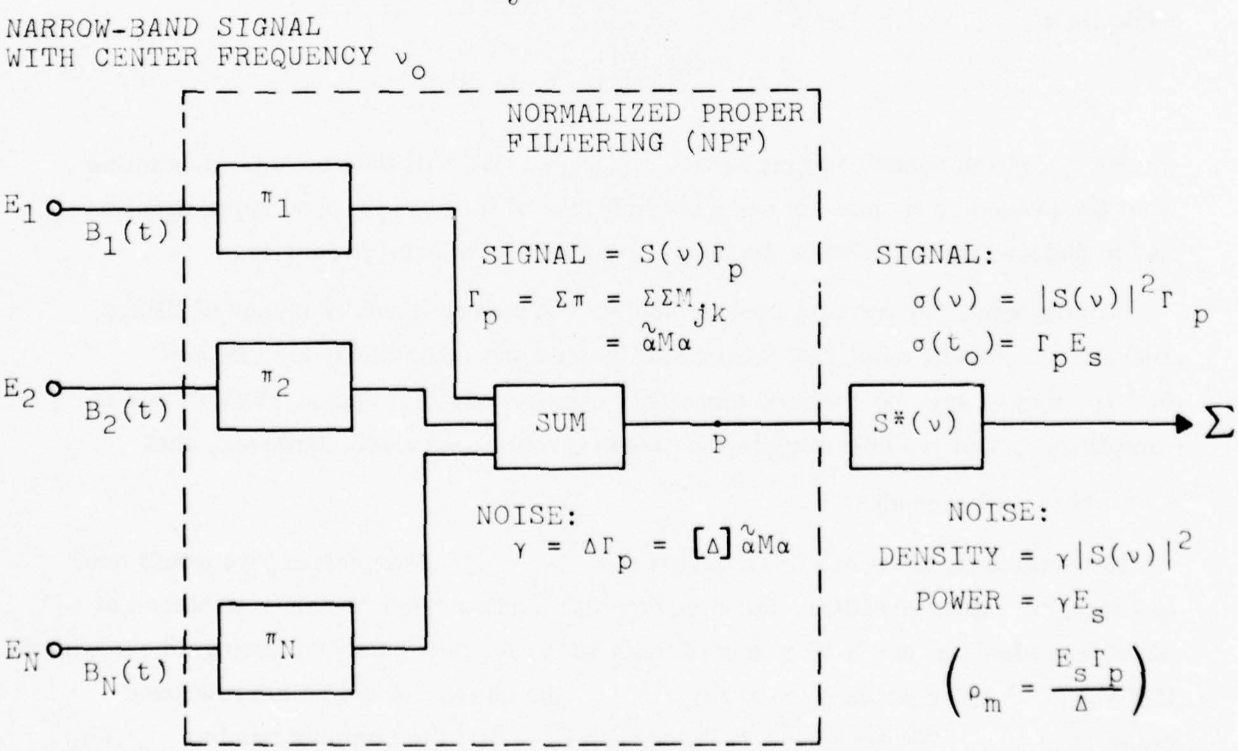

Fig. VI-1. Normalized Matched Filtering

The proper gain of the system (see Eq. (V-6)) becomes

$$G_p(\nu_0) = \sum_j \sum_k M_{jk}(\nu_0) \Big/ \Delta(\nu_0) \quad . \tag{VI-2}$$

We have seen (Eqs. (V-8) and (V-28)) that $G_p(\nu_0)$ is real and non-negative, its numerator and its denominator each being real and non-negative. The optimum signal-to-noise ratio is (Eq. (V-7.a))

$$\rho_M = G_p(\nu_0) \, . \, E_s \ , \tag{VI-3}$$

where E_s denotes the signal energy at an arbitrary input. We know that the filters in a matched filtering process are definable except for a constant real factor. This property applies here also. Thus, the filters p_k (the notation ν_0 will be understood) of Eq. (VI-1) all contain a common factor $\frac{1}{\Delta}$. This factor depends upon the properties of the noise. However, in all cases where Δ is not zero, the p_k may be replaced by the filters

$$\pi_k = \sum_j M_{jk} \ . \tag{VI-4}$$

Henceforth, we will distinguish between simple proper filtering (SPF), defined by Eqs. (VI-1) and (VI-2), and normalized proper filtering (NPF), defined by Eq. (VI-4), to which corresponds a normalized proper gain,

$$\Gamma_p = \sum_j \sum_k M_{jk} = \sum_k \pi_k \ , \tag{VI-5}$$

which is real and non-negative.

It is understood that these limitations apply only in the case of the "narrow-band" approximation. The performance of the NPF remains the same as that of the SPF and, following Eq. (VI-3), depends in particular upon Δ:

$$\rho_M = \frac{\Gamma_p}{\Delta} E_s = G_p \; E_s \quad . \tag{VI-6}$$

Now we must determine the way in which the NPF, as defined by Eq. (VI-4), behaves when $\Delta = 0$. This will be the objective of Paragraph VI-5.

It may be noted that the NPF presents an interesting technological advantage; although ρ_m always depends upon Δ, the filters (π_k) themselves do not depend upon it. Thus (see Chapter IX), Δ is a polynomial of order N in terms of the C_{jk} whereas the π_k are only of order $N-1$. Computations which involve taking the product of complex numbers and which lead to the "fabrication" of the π_k have a lower degree of difficulty than those required for the computation of Δ.

VI-3. Invariance of the NPF in the Presence of the Same Noise Superimposed on Each of the $B_j(t)$

A property common to both SPF and NPF is the fact that the phase of the signals (narrow band) at the output of the summation of the N channels is unchanged. This is tied in with the fact that Γ_p and G_p are real and non-negative.

A second special property of NPF is connected with the structure of the π_k. Equation (VI-4) shows that π_k is the sum of the cofactors of a given column, i.e., a determinant of order N formed from Δ by replacing the elements of the column by ones.

It is easily verified that if the same constant a is added to each of the elements of such a determinant other than those of the k^{th} column, the value of the determinant, and hence that of π_k, is not changed. The physical interpretation of this property is as follows.

An NPF is unchanged in the presence of a stationary supplementary noise that is identical at all inputs, not correlated with the $B_j(t)$, and superimposed upon them. Let us assume, in fact, that the same noise $n(t)$ is added to all the $B_j(t)$. The crosscorrelation functions $C_{jk}(\tau)$ become

$$N_{jk}(\tau) = E\left\{\left[B_j(t) + n(t)\right] \left[B_k(t+\tau) + n(t+\tau)\right]\right\}, \qquad \text{(VI-7)}$$

which is, because of the lack of correlation between $n(t)$ and the other noise,

$$N_{jk}(\tau) = C_{jk}(\tau) + C_n(\tau),$$

where $C_n(\tau)$ is the autocorrelation of $n(t)$.

Hence, the Fourier transform is

$$N_{jk}(\nu) = C_{jk}(\nu) + C_n(\nu) , \qquad \text{(VI-8)}$$

which is, for a narrow band,

$$N_{jk}(\nu_0) = C_{jk}(\nu_0) + C_n(\nu_0) . \qquad \text{(VI-9)}$$

Thus, all the $C_{jk}(\nu_0)$ are increased by the same amount $a = C_n(\nu_0)$, which is, moreover, the spectral density of $n(t)$ at $\nu = \nu_0$. This modification leaves the value of π_k unchanged; hence the NPF is unchanged, as we have already seen. It is evident, however, that the performance (ratio ρ_m) is affected. Equation (VI-6) shows, in fact, that if Γ_p is unchanged (Eq. (VI-5)), Δ itself is altered; it may be shown that it increases, becoming

$$\Delta + a\,\Gamma_p$$

and that, consequently, as would be expected, the signal-to-noise ratio decreases. It may be said that everything capable of constructing the π_k from the C_{jk} will be, a priori, unaffected by a noise $n(t)$ at the inputs.

An analogous reasoning allows us, with the same result, to replace the noise $n(t)$ by a sinusoidal signal of "infinite" duration and frequency lying in the narrow-band of interest. We concede that under quasi-stationary conditions (see Paragraph IX-2) with a long, real signal, it may be assumed that the π_k are not altered by the presence of the signal at the inputs.

VI-4. Matrix Notation for NPF

The matrix notation for SPF and NPF is derived from the equations of Paragraph V-3 and from the definition of the column matrix π for NPF.

$$\pi = [\Delta]\, p . \qquad \text{(VI-10)}$$

For SPF the notation is that of Paragraph V-3 , and only the interpretation of the functions of ν which appear in it is changed, following the convention of Paragraph VI-1. For NPF, using Eq. (V-3) , the following may be derived:

$$\pi = [\Delta]\, C^{-1} \alpha . \qquad \text{(VI-11)}$$

Let M be the adjoint matrix of C (the matrix of cofactors) defined by

$$\tilde{M} = [\Delta]\; C^{-1} \; ; \qquad \text{(VI-12)}$$

then,

$$\pi = \tilde{M}\alpha \; . \qquad \text{(VI-13)}$$

The gain of the NPF may be written (see Paragraph V1-1)

$$\Gamma_p = \tilde{\alpha}\pi \; , \qquad \text{(VI-14)}$$

that is,

$$\Gamma_p = \tilde{\alpha}\,\tilde{M}\alpha = [\Delta]\;\tilde{\alpha}\,C^{-1}\alpha = [\Delta]\;\tilde{\alpha}\,p \; . \qquad \text{(VI-15)}$$

These equations must be compared with Eq. (VI-5) in order to be defined. The spectral density at the output of the NPF is

$$\gamma = \pi^{+} C \pi = [\Delta^2]\;\tilde{\alpha}\,C^{-1}\alpha = [\Delta]\;\tilde{\alpha}\,M\,\alpha , \qquad \text{(VI-16)}$$

or

$$\gamma = \Delta\,\Gamma_p \; . \qquad \text{(VI-16a)}$$

Having replaced an SPF by an NPF, let us complete the system by adding a filter $S^*(\nu)$ in series with the summation of the N channels. If the narrow-band signal $S(\nu)$ is applied at the input, the spectrum of the signal after summation of the N channels is $\Gamma_p\; S(\nu)$. At the output $\sum$ (see Fig. VI-1) it is $\Gamma_p|S(\nu)|^2$. Its value (maximum) at time $t_0 = 0$ is $\Gamma_p\;E_s$. The noise density, which has a value γ at point P, becomes equal to $\gamma|S(\nu)|^2$ at $\sum$, and its power becomes γE_s.

Figure VI-1 summarizes the properties of an NPF followed by a filter $S^*(\nu)$ (normalized matched filtering).

VI-5. Limiting Behavior of the NPF-Eliminable Noise and Cutoff

The NPF has been defined, starting with the SPF, for $\Delta \neq 0$. Let us assume it to be realized in terms of the C_{jk} in accordance with Eq. (VI-4) . The values of π_k and of Γ_p are well defined and remain defined if Δ goes to zero[7]. Then what is the performance of the NPF under these circumstances? We have already stressed the characteristics of the "physical limitations" of the hypothesis $\Delta = 0$. We will treat it here, however, in a formal way, in order to justify the use of the NPF. In fact, note that, in this case, the filters p_k (Eq. (VI-1) are not realizable since they are not defined. This is then a particularly regrettable case of failure of the SPF (see the end of Paragraph IV-6) since it corresponds to an "infinite" signal-to-noise ratio (Eq. (VI-6)). On the other hand, the NPF is realizable and can yield this "infinite" S/N ratio provided that $\Gamma_p \neq 0$.

Such performance may be obtained - π_k remaining bounded and hence the output signal remaining bounded only by nulling the noise (while the SPF produces an infinite signal since the p_k are unbounded). This is clear from Eq. (VI-16) where γ is nulled when $\Delta = 0$, the matrix M and the C_{jk} themselves remaining bounded. Thus, the performance of the NPF is better than that of the SPF in the case where $\Delta = 0$; it is defined and hence it provides the limiting performance of matched filtering by nulling the noise at its output.

This performance exists, in fact, only if Γ_p itself does not simultaneously go to zero, that is, if the system does not "cut off", nulling both the noise and the signal. This would be the case in particular if the matrix C were of rank $\leq N-2$. All the cofactors M_{jk} would then go to zero, as would Γ_p.

Thus, the "limiting performance" of an NPF system is possible only if the matrix C is of rank $N-1$.

If the matrix C is of rank $N-1$, it may be shown that Γ_p can be zero only if $\Delta = 0$.

An NPF, then, has two possible modes of behavior:

a. The case of eliminable interference characterized by $\Delta = 0$ and $\Gamma_p \neq 0$.. The noise is nulled and the signal is not.

b. The case of cutoff, characterized by $\Gamma_p = 0$ (hence $\Delta = 0$). The multiple filter lets nothing pass, by letting its gain go to zero. It may be shown that

[7]The word "becomes" contradicts "stationary." Let us assume a very slow change in the noise, and hence of the C_{jk}, allowing time for adjustments in the value of the π_k, and concede that in this case a situation exists where $\Delta = 0$.

such a case occurs only by simultaneous nulling of all the π_k, that is, simultaneous cutoff on all channels. Returning to the case where the matrix C is of order $N-2$, it is obviously still a case of cutoff.

VI-6. **Limiting Case of a Zero Noise at One Input**

The case in which one of the noise inputs is found to be zero is a special case of systems with eliminable interference. It is obviously a limiting case and merely a formal one, since in such a case it is evident that one need only use that single input at which the noise is zero. If, however, one of the noises "becomes" zero (see note of Paragraph (VI-5)), let us verify that the NPF immediately makes use of that fact. In fact, if $B_k(t)$ is the noise that goes to zero, all C_{jk} go to zero for all j. The matrix C, provided with a column and a line of zeros, is reduced to rank $N-1$. The representation of the π_j in the form of determinants taken from Δ (by replacing the j^{th} column by ones) shows that all of them go to zero except π_k, which becomes equal to M_{kk} (real, non-negative) and also, in this case, equal to Γ_p. Thus, the only channel that remains open is the one in which the noise is zero, which obviously results in zero output noise. Note, in passing, that the occurrence of two input noises $B_j(t)$, being zero,is a case of cutoff, all the π_j then being zero. The matrix C, moreover, is of rank $N-2$. Although this is of a particularly formal case, we see here an example where the NPF fails because of an excess, so to speak, of good noise behavior. It is clear that we may get an output, in the case of cutoff, only by trying systems having fewer than N inputs. (See Paragraph VII-7 on the optimal use of an antenna.)

VI-7. Influence of Independent Noises Superimposed on the N Inputs.

We have just considered some limiting cases of "eliminable interference" and "cutoff," all of which involve the reduction of the rank of the matrix C. Now, let us demonstrate that the presence of unavoidable thermal noise at the inputs is sufficient, in practice, to exclude these limiting cases.

Thermal noise appears as noise inputs $n_j(t)$ produced in the input amplifiers or in the receiving elements themselves, superimposed upon the $B_j(t)$, which may be considered "external" interference noise.

The noises $n_j(t)$ are characterized by

a. non-zero densities,

b. no mutual correlation, and

c. no correlation between themselves and the $B_j(t)$.

In this case, the matrix C' of such a system is

$$C'_{jk} = C_{jk}, \qquad \text{for } j \neq k ,$$

$$C'_{jj} = C_{jj} + d_j ;$$

that is,

$$C' = C + d , \tag{VI-17}$$

calling d the diagonal matrix made up of the spectral densities of the noises $n_j(t)$. This matrix d is

a. of rank N since no d_j is zero, and

b. Hermitian since it is diagonal and real.

Let us assume that the rank of C' is less than N ; this means that there exists a non-zero column matrix (all u_j are non-zero) such that

$$C'u = 0 ,$$

which results in

$$u^+C'u = 0 ; \tag{VI-18}$$

taking into account Eq. (VI-17) , Eq. (VI-18) may be written

$$u^+Cu + u^+du = 0 . \tag{VI-19}$$

The two bilinear terms of (VI-19), being non-negative, must each go to zero separately. The column matrix u would then have to satisfy the following:

$$u^+du = \sum_j |u_j|^2 d_j = 0 ,$$

which is impossible since all d_j are $\neq 0$ and all the u_j cannot go to zero.

Thus the matrix C' cannot be of a rank less than N. It is clear that this property is not limited to the narrow-band case.

More generally, it may be said that, if some portion of the interference noise uncorrelated throughout the N inputs, the limiting cases will not occur.

CHAPTER VII

TWO-INPUT CASE - COHERENT NOISE

Summary

The case of two inputs is treated here in a detailed manner, including some numerical calculations. The effectiveness of an NPF with two inputs is compared with the "direct sum." The theoretical gain procured from the NPF is calculated as a function of two parameters:

a. the ratio of the two input powers, and

b. the complex correlation coefficient Θ (amplitude and phase of the normalized correlation function).

This gain is represented by a diagram which gives the curves for level as a function of the position of Θ in the complex plane. Naturally, it is always greater than 1 and is "infinite" in the case of eliminable noise.

Two limiting cases are settled, one by an equality of performance, the other, the case of cutoff, by an exceptional inferiority of the NPF compared with the direct sum. The latter case yields the same result as a single input.

"Coherent noise" is eliminable from the inputs, and the mechanism for its elimination is considered.

VII-1. General Equations

Let us apply the preceding results to a system with two inputs. The four matrix elements C_{jk} are C_{11} and C_{22}, spectral densities of the two noise inputs at $\nu = \nu_0$;

and $C_{12} = C_{21}^*$, which represents the crosscorrelation of the two noises at $\nu = \nu_0$.

Let us recall the relation

$$|C_{12}|^2 \leq C_{11} \; C_{22},$$

which expresses the fact that Δ of matrix C is non-negative. NPF is defined by (VI-4).

$$\pi_1 = C_{22} - C_{12} \qquad \text{(VII-1)}$$

$$\pi_2 = C_{11} - C_{21} \; .$$

Note that if the spectral densities are equal the two filtering processes are complex conjugates. Let us not consider here the case where one of noises is zero, a case which comes under general consideration in Paragraph VI-7. We have

$$\Gamma_p = \pi_1 + \pi_2 = C_{11} + C_{22} - C_{12} - C_{21} \,. \qquad \text{(VII-2)}$$

Note that Γ_p is nothing more than the spectral density (at $\nu = \nu_0$), of the difference between the two noises (see Fig. IX-6). In fact, the crosscorrelation of that difference is, by definition,

$$E \left\{ \left[B_1(t) - B_2(t) \right] \left[B_1(t+\tau) - B_2\,(t+\tau) \right] \right\} ,$$

or

$$C_{11}\,(\tau) - C_{12}(\tau) - C_{21}(\tau) + C_{22}(\tau) \quad ,$$

whose Fourier transform, according to (VII-2), is equal to $\Gamma_p(\nu)$.

VII-2. Eliminable Interference and Cutoff.

Let

$$\left.\begin{aligned} C_{11} &= q\sqrt{C_{11}\;C_{22}} \,, \\ C_{22} &= \frac{1}{q}\sqrt{C_{11}\;C_{22}} \,, \qquad \text{real, positive } q = \sqrt{\frac{C_{11}}{C_{22}}} \\ C_{12} &= \Theta\sqrt{C_{11}\;C_{22}} \,, \qquad |\theta| \leq 1 \text{ (Eq. (III-13)} \end{aligned}\right\} \text{(VII-3)}$$

Thus Θ is a complex number that represents the amplitude and phase of the normalized correlation function for two noise inputs (or degree of complex correlation).

The matrix C and the preceding quantities may be written

$$\sqrt{C_{11}\;C_{22}} \begin{bmatrix} q & \Theta \\ \Theta^* & \frac{1}{q} \end{bmatrix} \qquad \text{(VII-4)}$$

$$\left.\begin{aligned}
\Delta &= C_{11}\, C_{22}\, (1-|\theta|^2) \\
\Gamma_p &= 2\sqrt{C_{11}\, C_{22}}\left[\frac{1}{2}\left(q+\frac{1}{q}\right) = R\,[\theta]\right] \\
\pi_{11} &= \sqrt{C_{11}\, C_{22}}\left(\frac{1}{q}-\theta\right) \\
\pi_{22} &= \sqrt{C_{11}\, C_{22}}\,(q-\theta^{*}) .
\end{aligned}\right\} \qquad \text{(VII-5)}$$

Setting aside the case where one of the two densities is zero, systems with eliminable interference are defined by

$$\left.\begin{aligned}
&|\theta| = 1 \\
&\frac{1}{2}\left(q+\frac{1}{q}\right) - R[\theta] \neq 0 ,
\end{aligned}\right\} \qquad \text{(VII-6)}$$

where R designates the real part.

We always have $\frac{1}{2}\left(q+\frac{1}{q}\right) \geq 1$ for any q. (VII-7)

Since $|\theta| = 1$ in order that the inequality (VII-6) be realized, it is necessary that $\theta \neq 1$, that is, that θ have a non-zero argument.

In more technical terms, let us say that, if two noises are "totally correlated" $(|\theta|=1)$ without their "average phases" being equal $(\theta \neq 1)$, we have a system with eliminable interference.

Thus, conditions for cutoff of the system require, according to the preceding remarks, that $\theta = 1$ and as a result

$$\frac{1}{2}\left(q+\frac{1}{q}\right) = 1 \quad ;$$

hence, $q = 1$.

Summarizing, we have

$$\left.\begin{aligned}
\theta &= 1 \\
q &= 1 .
\end{aligned}\right\} \qquad \text{(VII-8)}$$

Thus, the case of cutoff may occur only for noises having equal spectral densities at ν_0 $(q=1)$.

Moreover, the condition $\theta = 1$ itself reduces Eq. (VII-5) to

$$\Delta = 0, \text{ or rather } C_{12} = C_{21} = \sqrt{C_{11}\, C_{22}}$$

$$\begin{aligned}
\Gamma_p &= \left(\sqrt{C_{11}} - \sqrt{C_{22}} \right)^2 \\
\pi_1 &= \sqrt{C_{22}} \left(\sqrt{C_{22}} - \sqrt{C_{11}} \right) \\
\pi_2 &= \sqrt{C_{11}} \left(\sqrt{C_{11}} - \sqrt{C_{22}} \right) .
\end{aligned} \qquad \text{(VII-9)}$$

If, besides, the two densities are equal, the system cuts off by simultaneously nulling $\pi_1, \pi_2, \ldots\ldots$ and Γ_p, as shown by Paragraph VI-6.

Thus, in order for a two-input NPF to cut off, it is required that

a. the spectral densities (and consequently, the powers) be equal and

b. the "average phases" be equal, that is, that

$$C_{12} = C_{21} = \sqrt{C_{11}\, C_{22}} = C_{11} = C_{22} . \qquad \text{(VII-10)}$$

Let us return to the general case. The signal-to-noise ratio is (see Eq. (VI-6))

$$\rho_m = \frac{\Gamma_p}{\Delta} E_s = \frac{C_{11}+C_{22}-C_{12}-C_{21}}{C_{11}\, C_{22}-|C_{12}|^2} \cdot E_s . \qquad \text{(VII-11)}$$

It is "infinite" for systems with eliminable interference and indeterminate in the case of cutoff.

Finally, if the two noises are uncorrelated, $C_{12} = C_{21} = 0$ and we have

$$\begin{aligned}
\pi_1 &= C_{22} \\
\pi_2 &= C_{11} \\
\Gamma_p &= C_{11} + C_{22} \\
\rho_m &= \frac{C_{11}+C_{22}}{C_{11}\, C_{22}} .
\end{aligned} \qquad \text{(VII-12)}$$

VII-3. Advantages of Matched Filtering Over the Direct Sum

We may represent the "benefits" derived from matched filtering in the following way:

The two inputs E_1 and E_2 may be considered to be receiving elements of an antenna. Assume that the signal, identical at the two elements, is carried by a plane wave whose wave front is parallel to the straight line $E_1 E_2$. The process of "direction summation" is most commonly used to enhance the signal-to-noise ratio, which means that proper antenna filtering is replaced by a simple sum (Fig. VII-1). The signal, upon summation, has for an amplitude spectrum $2\ S(\nu)$. Subsequently received by the filter $S^*(\nu)$, it yields an output signal $2|S(\nu)|^2$ whose value at time $t_0 = 0$ is

$$2 \int |S(\nu)|^2 \, d\nu = 2\ E_s$$

and whose instantaneous power at that time is $4E_s^2$.

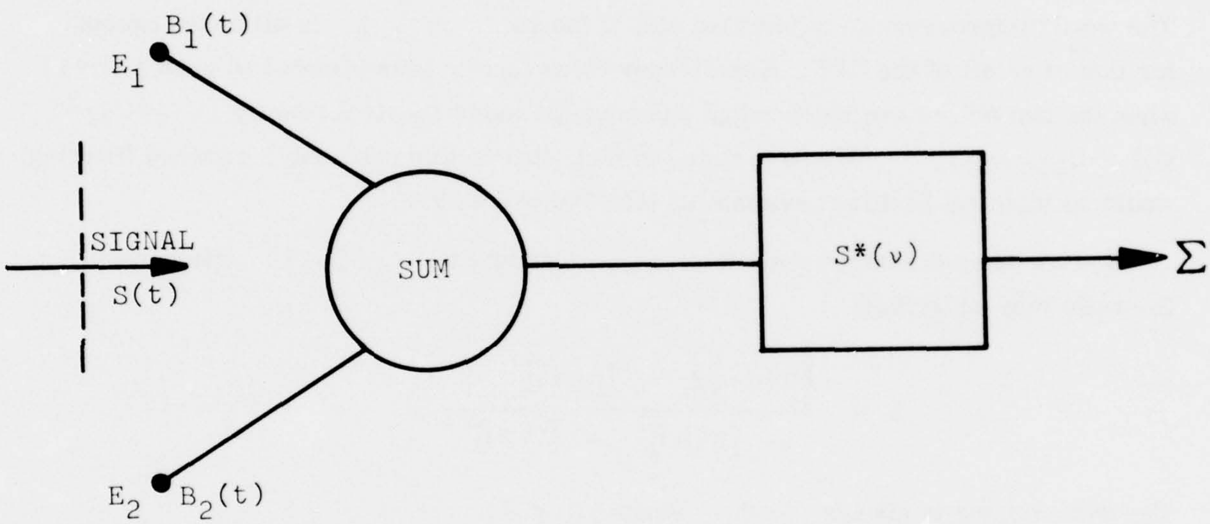

Fig. VII-1. Direct Summation

The spectral density of the noise at the point of summation is obviously

$$d_\Sigma = C_{11} + C_{22} + C_{12} + C_{21}, \qquad \text{(VII-13)}$$

and the noise power at the output of the filter $S^*(\nu)$ is

$$d_\Sigma \int |S(\nu)|^2 \, d\nu = d_\Sigma\ E_s . \qquad \text{(VII-14)}$$

The signal-to-noise ratio of the process is therefore

$$\rho_\Sigma = \frac{4}{C_{11}+C_{22}+C_{12}+C_{21}} \cdot E_s \quad . \tag{VII-15}$$

The improvement given by matched filtering (or NPF) over the "direct sum" is expressed by

$$b = \frac{\rho_m}{\rho_\Sigma} = \frac{(C_{11}+C_{22}-C_{12}-C_{21})\ (C_{11}+C_{22}+C_{12}+C_{21})}{4\quad C_{11}C_{22}-|C_{12}|^2} \quad . \tag{VII-16}$$

The word "improvement" is justified only if the ratio $b \geq 1$ in all cases except for that of cutoff of the NPF. It is already clear that no improvement is gained $(b=1)$ when the two noises are independent and have the same spectral density $(C_{12}=C_{21}=0$ and $C_{11}=C_{22})$. We have seen, in fact, that in this case only, matched filtering would be identical to direct summation (see Paragraph V-2).

Let us make it obvious that $b \geq 1$. Taking Eq. (VII-3) into account, the ratio may be written

$$b = \frac{\left[\frac{1}{2}(q+\frac{1}{q})\right]^2 - \left[R(\Theta)\right]^2}{1-\left[R(\Theta)\right]^2 - \left[I(\Theta)\right]^2} \quad . \tag{VII-17}$$

The denominator is always ≥ 0, since $|\Theta| \leq 1$.

Since $\frac{1}{2}\ (q+\frac{1}{q}) \geq 1$ for any q, (VII-7)

we may let

$$\left[\frac{1}{2}\ (q+\frac{1}{q})\right]^2 = m^2 + 1 \tag{VII-18}$$

with m real and positive.

Hence,

$$b = \frac{m^2+1 - \left[R(\Theta)\right]^2}{1-\left[R(\Theta)\right]^2 - \left[I(\Theta)\right]^2} = 1 + \frac{m^2 + \left[I(\Theta)\right]^2}{1-\left[R(\Theta)\right]^2 - \left[I(\Theta)\right]^2} \quad , \tag{VII-19}$$

which is a form from which it is obvious that $b \geq 1$. Uncorrelated noises $(\theta=0)$ with the same spectral density $(m=0)$ correspond to the case $b = 1$ (zero improvement).

The case $|\theta| = 1$, which makes the denominator of b equal to zero and gives an "infinite" improvement, corresponds to the case of eliminable interference, as we have alreay seen.

The improvement b remains undetermined in two cases.

a. The first case:

$$m = 0, \ \theta = 1, \tag{VII-20}$$

which corresponds to the case of cutoff for an NPF. The signal-to-noise ratio for direct summation, given by Eq. (VII-15), may be written

$$\rho_\Sigma = \frac{4E_s}{2\sqrt{C_{11}\ C_{22}}} \cdot \frac{1}{\frac{1}{2}\left(q+\frac{1}{q}\right) + R(\theta)}$$

$$= \frac{2E_s}{\sqrt{C_{11}\ C_{22}}} \cdot \frac{1}{1+m^2+R(\theta)} \tag{VII-21}$$

and is not zero in the case (VII-20). Thus, the indeterminacy is resolved as a very striking inferiority of the NPF compared with direct summation. It may be said that, with noise inputs having the same spectral densities and "in phase" $(\theta=0)$, the NPF cuts off while direct summation produces a signal-to-noise ratio equal to E_S/C_{11}, which is identical to that produced with a single input and a filter $S^*(\nu)$. Thus, the direct sum does not contribute anything itself, but at least it does not "spoil" the result. Such a case is found, for example, when the noise carried by a plane wave comes from the same direction as the signal and hence is identical at the two inputs. In this case alone, and provided that such a noise input is indeed the only interference – otherwise the NPF adjusts itself for the other noise inputs, according to Paragraph VI-3 – the NPF cuts off. In practice, the presence of independent thermal noise at the two inputs, combined with some of the considerations of Paragraph VI-3, prevents cutoff, even if the condition $\theta = 1$ does not represent in itself the limit of a limiting condition $(|\theta| = 1)$.

b. The second, equally idealized, case where b is undetermined corresponds to

$$m = 0$$

$$\Theta = -1 \quad . \qquad \text{(VII-22)}$$

Here, the NPF behaves as in the case of eliminable interference and gives an "infinite" signal-to-noise ratio. It is found, however, that the direct sum does the same thing since the two noise inputs are in phase opposition. Note that the two filters π_1 and π_2 are now real and identical (see Eq. (VII-5) with $q = 1$ and $\Theta = 1$) and, because of this fact, produce the sum precisely, except for a multiplicative factor. Thus, the indeterminacy is resolved as an equality of performance.

VII-4. A System of Graphs Representing the Gain of the NPF

Let us set aside the two extreme cases examined above. The improvement b, always greater than unity, may be represented for each value of m, by a surface (Eq. (VII-19)) above the complex plane for Θ. This surface is entirely contained in the vertical cylinder of radius 1 and is tangent to it at infinity (the case of eliminable interference). Furthermore, this surface is symmetric with respect to the two planes $I(\Theta) = 0$ and $\text{Arg } \Theta = \frac{\pi}{2}$, and it is sufficient to represent it within a 90 degree dihedron (in a single quadrant). It may be represented there by level curves. Figure VII-2 shows, in four successive quarter planes, the level curves of the surfaces for four values of m, that is, four values of the ratio of the spectral densities C_{11}/C_{22}. These four values are

$$C_{11}/C_{22} = 1 \qquad 10 \log C_{11}/C_{22} = 0 \text{ db}$$

$$C_{11}/C_{22} = 1.4 \qquad 10 \log C_{11}/C_{22} = 1.5 \text{ db}$$

$$C_{11}/C_{22} = 2 \qquad 10 \log C_{11}/C_{22} = 3 \text{ db}$$

$$C_{11}/C_{22} = 2.9 \qquad 10 \log C_{11}/C_{22} = 4.5 \text{ db} \quad .$$

The greater the ratio C_{11}/C_{22}, the higher the "base point" of the surface (corresponding to $\theta = 0$, $b_0 = 1 + m^2$) and the more rapidly the surface rises, as $|\theta|$ increase, toward the "infinite" values which correspond to $|\theta| = 1$. The level curves are ellipses whose major axes lie along the real axis. These ellipses tend toward circles when b becomes very large. We may say, then, that regardless of the value of C_{11}/C_{22} the more "advantageous" of two values of θ having the same modulus is the least real one.

The critical zone corresponding to $m=0$ and $\theta = 1$ has been crosshatched in Fig. VII-2 .

VII-5. Coherent Interference (Jamming) - Mechanism for its Elimination

We will call interference noise coherent if it is carried by a plane wave coming from a direction other than that of the signal. This is the important and practical case of a jammer or of a localized distant source. Two noise inputs $B_1(t)$ and $B_2(t)$ differ from each other only by a time delay.

We will generalize the definition of coherent interference a little and assume that

$$B_2(t) = a\,B_1(t-u) , \tag{VII-23}$$

where a is a real multiplicative factor and u is the time lag. It is then simple to verify that

$$C_{12}(\tau) = a\,C_{11}(\tau-u) , \tag{VII-24}$$

and as a result,

$$C_{12}(\nu) = a\,C_{11}(\nu)\,e^{-2\pi i\nu u} . \tag{VII-25}$$

Since we have, furthermore,

$$C_{22}(\nu) = a^2\,C_{11}(\nu) , \tag{VII-26}$$

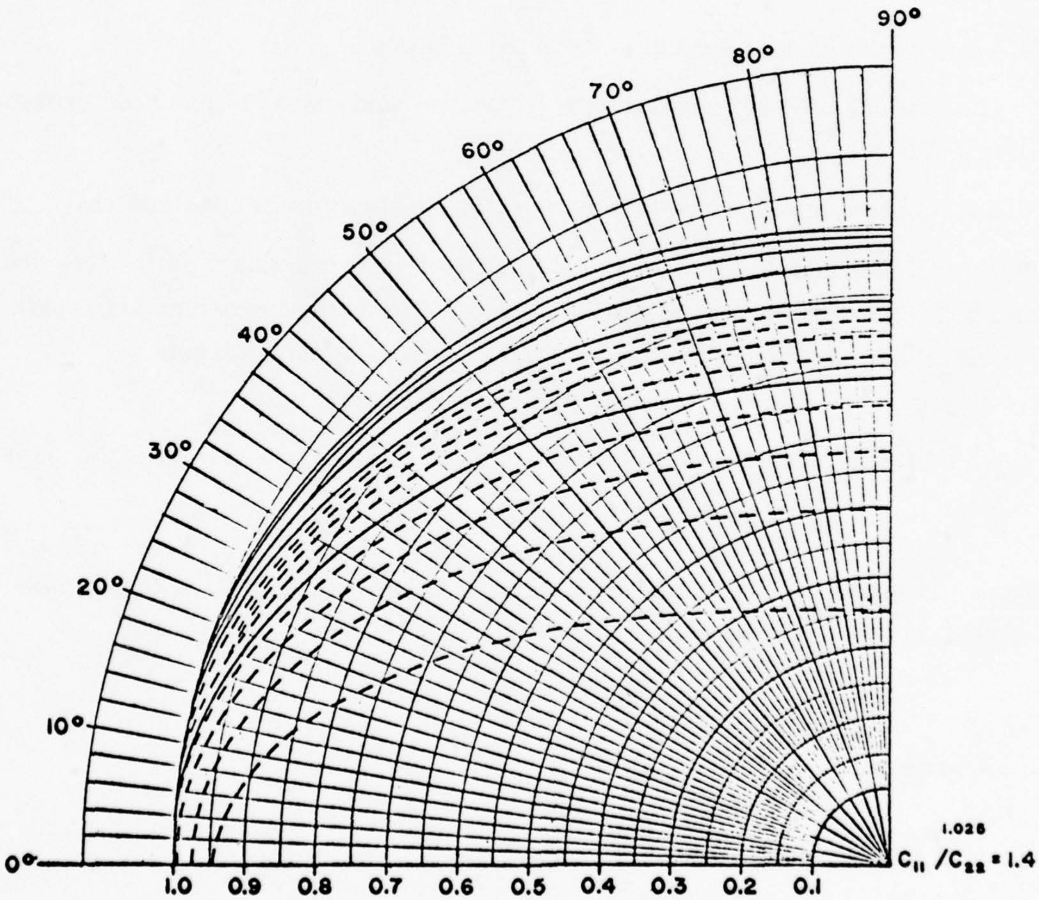

Fig. VII-2A. Graphical Representation of NPF Gain

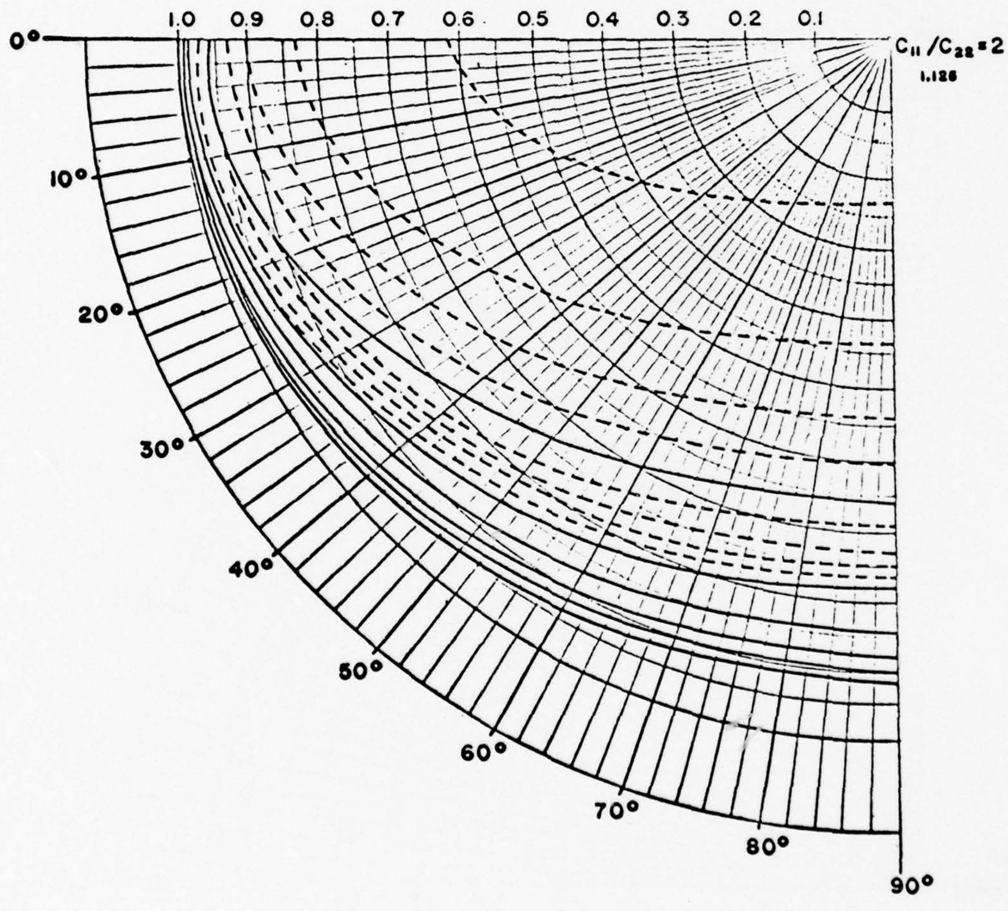

Fig. VII-2B. Graphical Representation of NPF Gain

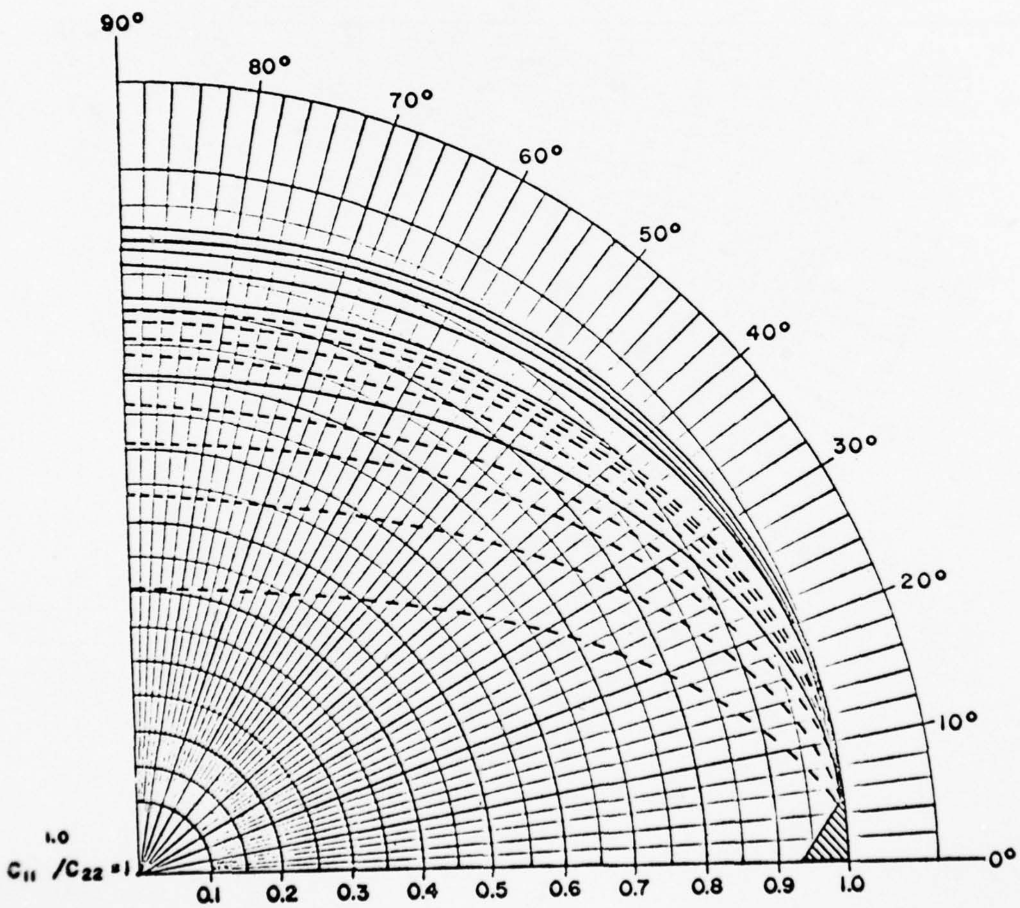

Fig. VII-2C. Graphical Representation of NPF Gain

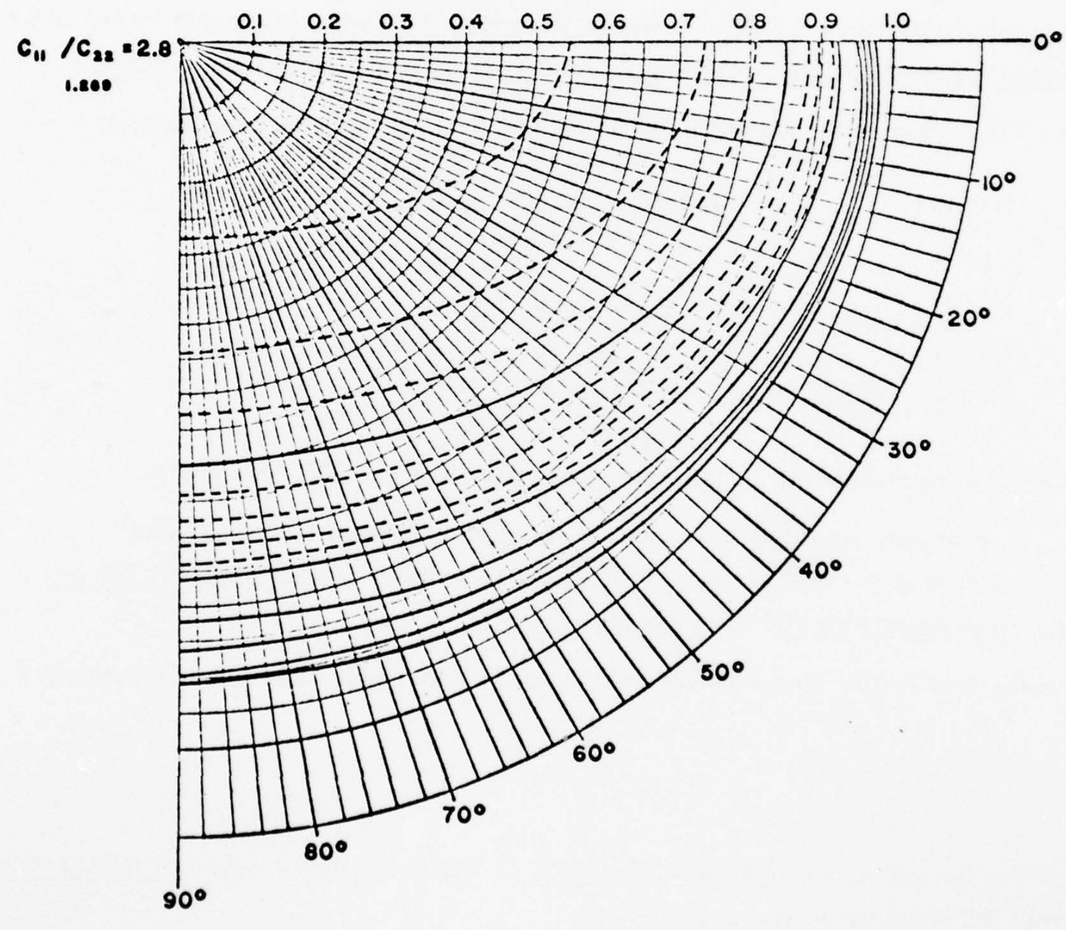

Fig. VII-2D. Graphical Representation of NPF Gain we may write, for frequency $\nu = \nu_0$,

$$C_{12}(\nu_0) = e^{-2\pi i\nu_0 u} \sqrt{C_{11}(\nu_0) \,.\, C_{22}(\nu_0)} \; . \qquad \text{(VII-27)}$$

The number Θ defined by Eq. (VII-3) is

$$\Theta = e^{-2\pi i\nu_0 u} \; . \qquad \text{(VII-28)}$$

Thus, we have the case $|\Theta| = 1$ which gives a system with eliminable interference. Hence, coherent interference is eliminable interference in a system with two inputs.

Figure VII-3 illustrates how the noise is eliminated. In fact, the two filtering processes π_1 and π_2 are, except for the factor $\sqrt{C_{11}\ C_{22}}$ (see Eq. (VII-5)),

$$\left.\begin{aligned} \pi_1 &= a - e^{-2\pi i\nu_0 u} \\ \pi_2 &= \frac{1}{a} + e^{+2\pi i\nu_0 u} \; . \end{aligned}\right\} \qquad \text{(VII-29)}$$

Each may be considered to be the sum of two filters in parallel. The term $e^{-2\pi i\nu_0 u}$ represents a delay u, and its conjugate represents an "advance" . In the general sum of the four terms, noise inputs cancel each other in pairs, but two opposing terms come from different inputs.

Likewise, it is easily seen how the case of cutoff may occur, when simultaneously

$$a = 1$$

$$u = 0 \quad ,$$

that is, when the two noise inputs are identical, or rather when the coherent interference comes from the same direction as the signal.

The four terms of the noise destructively interfere in pairs on the same input. Thus a two-input system may eliminate coherent interference provided that the interference does not come from the same direction as the signal. It is clear, in fact, that in this latter case, no spatial discrimination between signal and noise being possible, one may do no better than when using a single input; direct summation itself results in no improvement (Paragraph VII-3a).

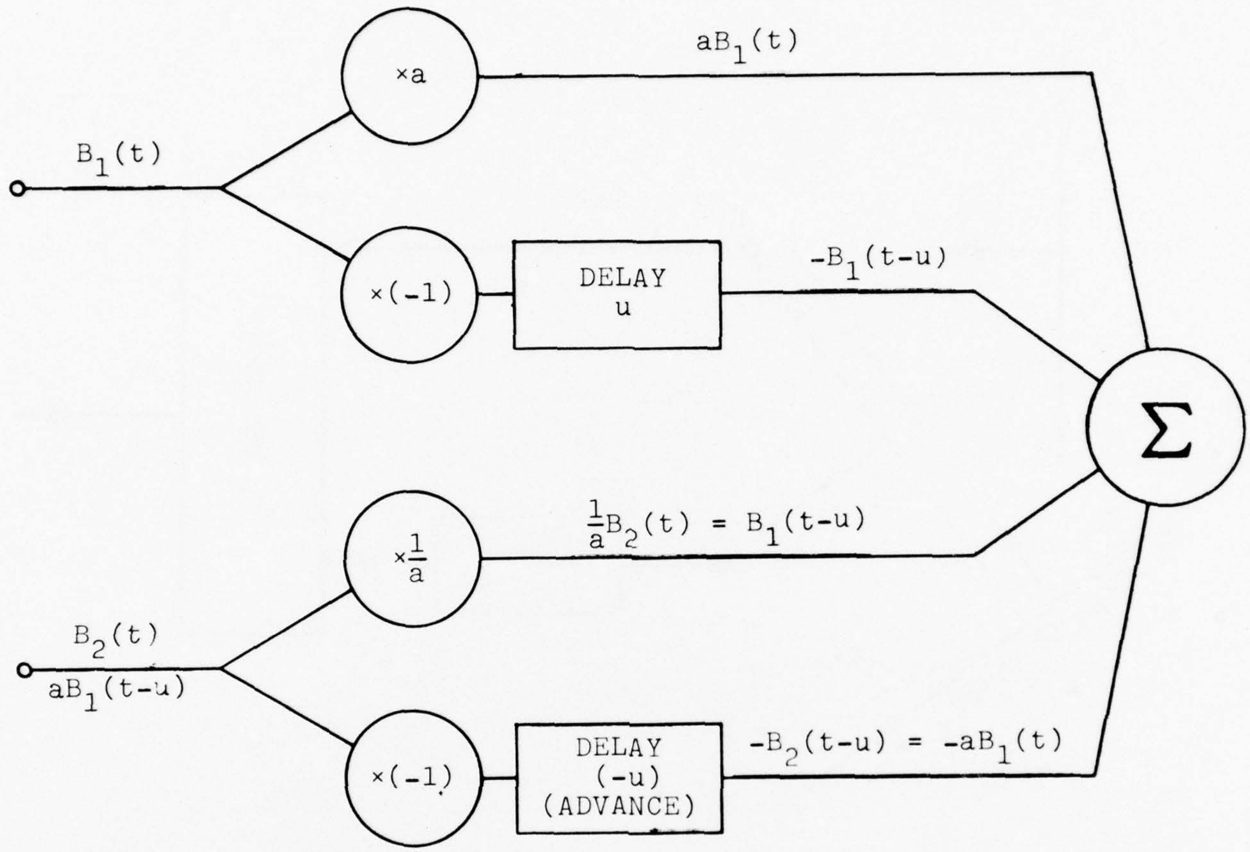

Fig. VII-3

VII-6. Return to a Remark Made in Chapter I - The Special Case of Coherent Interference

The reasoning illustrated by Fig. VII-3 does not necessarily use the "narrow-band" hypothesis; in fact, coherent interference is interference that is eliminable over a broad band.

Figure VII-4 shows a simpler method of effecting this cancellation in the case where $a = 1$. By conveniently delaying the noise at one input and taking a difference, the noise is effectively cancelled without cancelling the signal, if the signal comes from another direction. This simple case was brought up in Chapter I to suggest that direct summation is not always the best method.

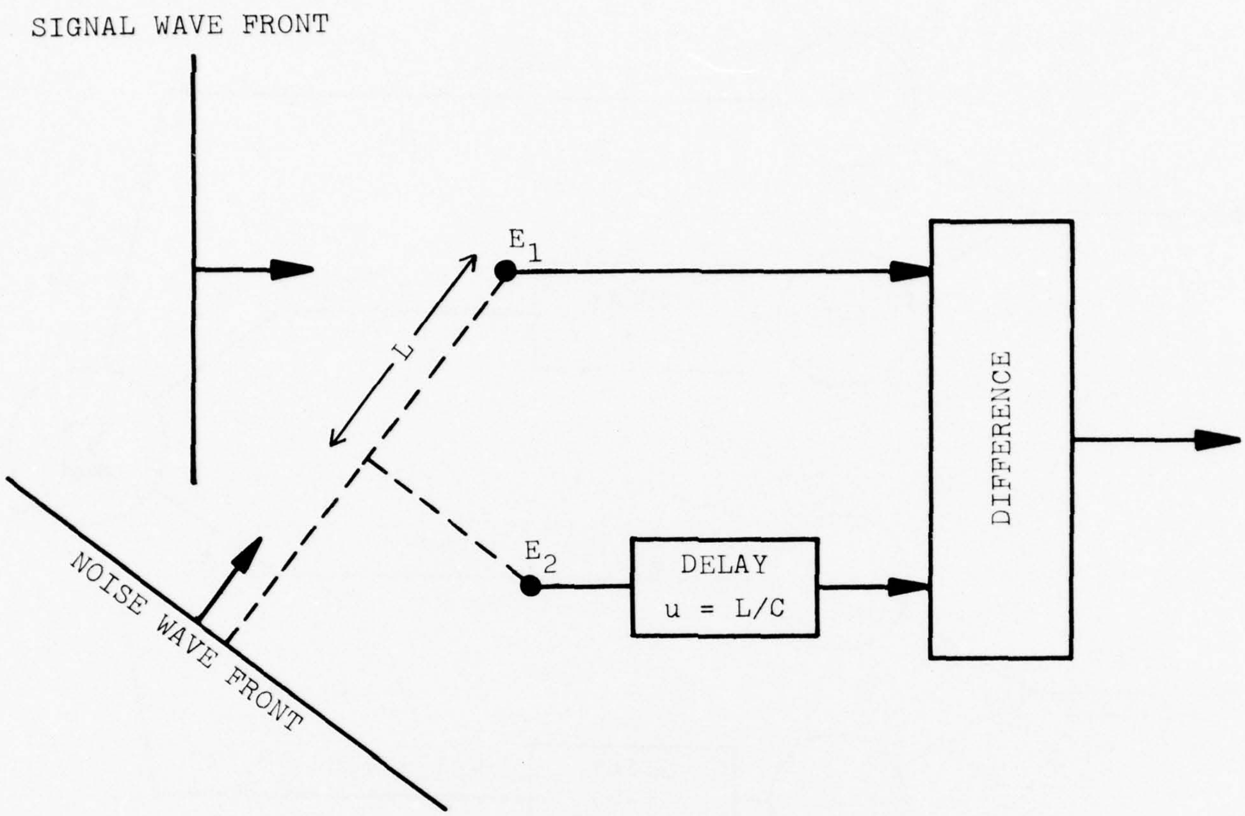

Fig. VII-4

VII-7. Optimum Use of an Antenna

A few of the preceding remarks may be very quickly extended to the case of 3 inputs, permitting their generalization to N inputs. It is easy to verify, for example, that "coherent interference," which is eliminable interference with two inputs, results in cutoff with three or more inputs.[8]

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[8]Each of the N noise inputs are derived from a reference noise by (Eq. (VII-23))

$$b_j(t) = a_j \, b_0 \, (t-u_j) \ ,$$

or $$C_{jk}(\nu) = a_j \, a_k \, C_0(\nu) \, e^{-2\pi i\nu(u_j-u_k)} \ ,$$

where $C_0(\nu)$ is the spectral density of $b_0(t)$. After putting them in convenient form, it may be proven that the determinant of matrix C is reduced to a determinant of order N, all of whose elements are equal to 1. Thus the rank of the system is 1 in all cases.

To eliminate a jammer, for example, assuming that it is the only interference, a system of only two inputs would be required.

Thus with different N's, various possibilities are offered for eliminating interference, but these possibilities depend upon the value of N.

On the other hand, the number of possibilities is greater for larger N (number of solutions to $\Delta = 0$). It is indicated that, in order to derive the most advantage from an antenna with N elements, several systems should be used:

a. a system with two elements which are used directly,

b. a system of 3 elements which may include the two preceding elements, and

c. then, a system with 4, 5 ... N elements,

the last making use of the entire antenna. We have at our disposal, then, all possibilities offered by the antenna from the point of view of "eliminable interference."

For example, since coherent interference will be eliminable on a system with two elements, it doesn't matter that it causes cutoff on the others. Other noise configurations will be eliminable with 3 elements, etc.

CHAPTER VIII

MATCHED FILTERING AND DIRECTIVITY

Summary

After a review of the definition and physical meaning of antenna directivity, as well as the definition of omnidirectional noise, this chapter establishes the fact that optimization of directivity is only a special case of matched filtering applied to special crosscorrelation properties created at the antenna by omnidirectional noise.

Thus, theories and processes tending to optimize directivity optimize signal detection only to the degree that the more or less implicit hypothesis of omnidirectional noise is valid.

A few special properties of plane or linear antennas are reviewed.

VIII-1. Limits of the Validity of the Directivity Concept

Current usage in technical literature assumes that the way to favor detection of a signal carried by a plane wave is to obtain, by means of an antenna, "the greatest directivity" in the direction of that wave. We have just seen, however, that optimization of detection depends upon the statistical relationships of the interference noise inputs to the antenna elements. If we claim to optimize detection by optimizing "directivity," we become involved with a more or less implicit hypothesis concerning the nature of the statistical relationships. We will see, in fact, that this hypothesis is that of "omnidirectional noise," which corresponds to an exact definition (as does coherent interference) and, consequently, to a well-defined correlation between two elements. Likewise, we will see that, as would be expected, to optimize directivity is nothing more than performing matched filtering in the presence of omnidirectional noise, that is, treating a special case of the spatial structure of the noise. Thus, from this chapter we may derive ways to modify some of the hasty conclusions regarding the general character of this directivity concept, to which a considerable theoretical and technical effort has been attached. To assume that a noise is omnidirectional when we know nothing about it is a simple hypothesis but it is not an optimum procedure.

VIII-2. Definition and Physical Meaning of Directivity

First, let us recall the definition of directivity. The "complex directivity" is a function of frequency and of two spatial parameters defining the direction of a plane wave. Let us combine these two spatial parameters into the symbol ω. The complex function $D(\omega, \nu)$ gives the phase and amplitude of the voltage observed at the antenna output in the presence of a plane wave having direction ω and frequency ν. The output is defined as the sum of the voltages coming from the antenna elements through, if need be, a linear filter in each channel (multiple filtering).

The quantity

$$f(\nu) = \frac{4\pi |D(\omega_0, \nu)|^2}{\int |D(\omega, \nu)|^2 \, d\omega} \qquad \text{(VIII-1)}$$

is called the directivity factor relative to the direction ω_0 (or, in shortened, but less correct form, "directivity"), where $d\omega$ represents an elementary solid angle about direction ω. To optimize the directivity is, in fact, to optimize $f(\nu)$.

The physical meaning of this quantity is obvious. The square of the magnitude of $D(\omega, \nu)$ has the meaning of power per unit solid angle. The denominator of $f(\nu)$ is thus the total power of independent contributions, each assigned a direction ω. We may still say that it is the power at the output of an antenna when the antenna is in a field of plane waves coming from all directions, which are independent of direction (even infinitesimally separated) and have the same spectral density[9]. This, then, is the precise definition of omnidirectional interference.

The numerator of $f(\nu)$ may be said to be the power received by the antenna in omnidirectional noise if the antenna possesses the same response for all directions that it has for the direction ω_0. Consequently, to say that "the directivity is high" (or $f(\nu)$ is high) is the same as saying that the antenna is particularly suited

[9] That is, "perfectly diffuse radiation."

to "avoid" plane waves coming from directions other than ω_0, thereby achieving a spatial selection; this has the form of a linear filtering process with a gain that is a function of direction. We may also say that the numerator is, except for some factor, the power produced at the output by a plane wave coming from direction ω_0. Thus, $f(\nu)$ also represents the signal-to-noise ratio (ratio of mean powers) at the output for a signal plane wave from direction ω_0 and an omnidirectional interference noise field.

The definition of $f(\nu)$, then, is strictly related to that of omnidirectional noise, and the use of the concept of "directivity factor" implies the omnidirectional noise hypothesis. The value of $f(\nu)$ depends upon antenna geometry, which is assumed here to be defined by the N elements and upon the filtering, if required, in each channel. It is the optimization of $f(\nu)$ by means of this multiple filtering which we will discuss here.

VIII-3. Expression for the "Directivity Factor" as a Function of the Gains of the Multiple Filters

Now, let us consider a collection of N antenna elements $E_1, E_2, \ldots E_N$ (Fig. VIII-1), all identical as far as electromagnetic or electroacoustic receiving elements are concerned, and a phase reference point $\underline{0}$.

For a plane wave propagating in direction ω with frequency ν and unit amplitude, the voltage received by element E_j is represented in terms of amplitude and phase by

$$e^{2\pi i \frac{\nu}{c} (\vec{OE}_j \cdot \vec{\omega})},$$

where $\vec{\omega}$ is the unit vector in direction ω. The scalar quantity $\vec{OE}_j \cdot \vec{\omega}$ represents the path difference of the plane wave between O and E_j; when divided by c (propagation velocity) it is the "time lag" or "advance" of E_j with respect to O.

Let us begin by making "identical signals" by applying to each input a filtering process which compensates for the path differences among the different inputs for a plane wave coming from direction ω_0. Hence, we are led to apply to E_j a "filtering process"

$$e^{-2\pi i \frac{\nu}{c} (\vec{OE}_j \cdot \vec{\omega}_o)} ,$$

which, over the frequencies ν , is merely an ideal algebraic "time lag"

$$\frac{\vec{OE}_j \cdot \vec{\omega}}{C} .$$

Next let us apply to each input a supplementary filtering $q_j(\nu)$, which is unspecified for the time being and which will be varied in order to optimize the directivity factor $f(\nu)$.

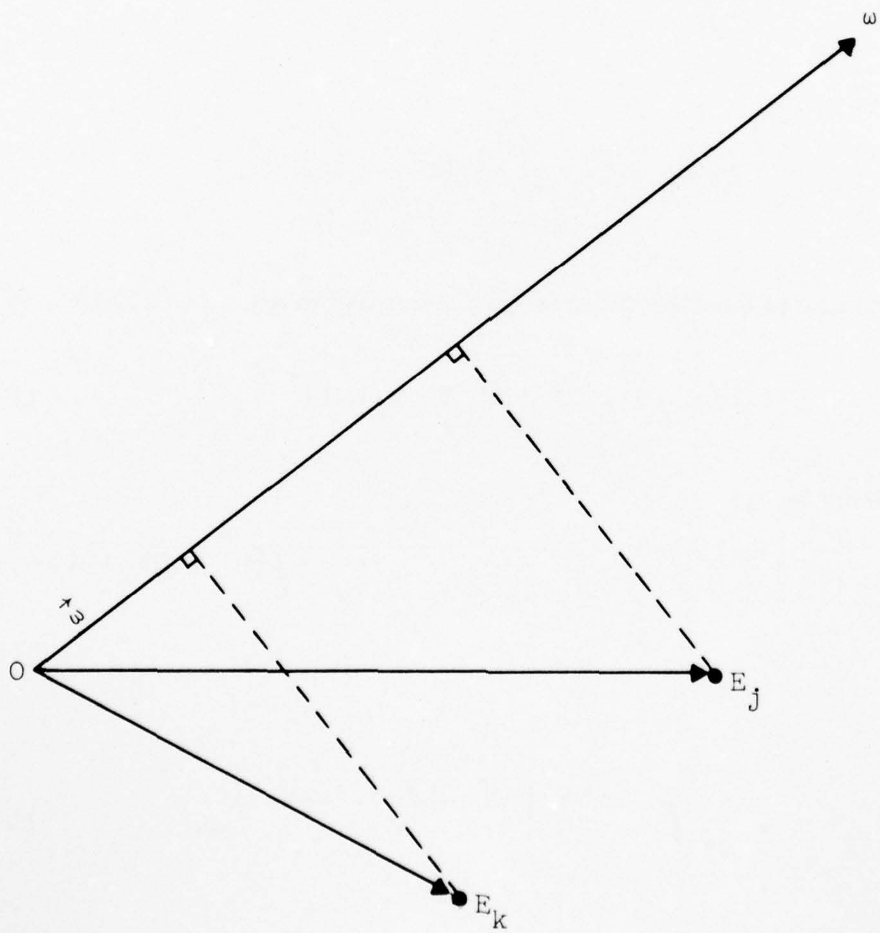

Fig. VIII-1

The voltage at the output of the filter $q_j(\nu)$ in channel E_j for a plane wave coming from direction ω is represented by

$$q_j(\nu)\ e^{2\pi i \frac{\nu}{c} [\vec{OE}_j \ . \ (\vec{\omega}-\vec{\omega}_0)]} \quad .$$

Thus, the previously defined "complex directivity function" $D(\omega,\nu)$ is (see reference (13))

$$D(\omega,\nu) = \sum_j q_j(\nu)\ e^{2\pi i \frac{\nu}{c} [\vec{OE}_j \ . \ (\vec{\omega}-\vec{\omega}_0)]} \quad , \qquad \text{(VIII-2)}$$

and we have

$$D(\omega_0,\nu) = \sum_j q_j(\nu) \quad . \qquad \text{(VIII-3)}$$

The numerator of the directivity factor, according to Eq. (VIII-1), is

$$4\pi |D(\omega \ ,\nu)|^2 = 4\pi | \sum_j q_j(\nu)|^2 \ . \qquad \text{(VIII-4)}$$

The denominator is

$$I = \int |D(\omega,\nu)|^2\ d\omega - \sum_j \sum_k q_j(\nu)\ q_k^*(\nu) \int e^{2\ i \frac{\nu}{c} [(\vec{OE}_j - \vec{OE}_k) . (\vec{\omega}-\vec{\omega}_0)]}\ d\omega . \qquad \text{(VII-5)}$$

Let

$$V_{jk}(\nu) = \frac{1}{4\pi} \int e^{2\pi i \frac{\nu}{c} [(\vec{OE}_j - \vec{OE}_k) . (\vec{\omega}-\vec{\omega}_0)]}\ d\omega \qquad \text{(VIII-6)}$$

with the result

$$I = 4\pi \sum_j \sum_k q_j(\nu)\, q_k^*(\nu)\, V_{jk}(\nu) \quad . \qquad \text{(VIII-7)}$$

Note that

$$\left.\begin{aligned} V_{jk}(-\nu) &= V_{jk}^*(\nu) \\ V_{kj}(\nu) &= V_{jk}^*(\nu) \\ |V_{jk}(\nu)| &\le 1 \\ V_{jj}(\nu) &= 1 \end{aligned}\right\} \quad . \qquad \text{(VIII-8)}$$

On the other hand, I is real and non-negative for all q_j. The expression for the directivity factor is

$$f(\nu) = \frac{\left| \sum_j q_j(\nu) \right|^2}{\sum_j \sum_k q_j(\nu)\, q_k^*(\nu)\, V_{jk}(\nu)} \quad , \qquad \text{(VIII-9)}$$

and the problem consists of finding the $q_j(\nu)$ that maximize $f(\nu)$.

VIII-4. Optimization of the Directivity Factor

The solution has an obvious relationship to the reasoning of Chapter III and is purposely presented here in an analogous and brief manner. We will temporarily omit the frequency notation (ν), which will remain understood.

Equation (VIII-9) defines $f(\nu)$ for the q_j, which are themselves defined except for a complex factor. This factor may be chosen such that the numerator of $f(\nu)$ is normalized; that is,

$$\sum_j q_j = 1 \quad , \qquad \text{(VIII-10)}$$

which is the condition under which the q_j will be constrained. Consider complex numbers r_j for which

$$\sum_j r_j = 0 \quad . \tag{VIII-11}$$

All numbers $x_j = q_j + \alpha r_j$ with real arbitrary α will satisfy the normalization condition for the numerator

$$\sum_j x_j = 1 \quad .$$

Let h_j, in particular, be a solution sought among the q_j constrained by (VIII-10). This solution must minimize the value of $I/_{4\pi}$:

$$\frac{I}{4\pi} = \sum_j \sum_k q_j q_k^* V_{jk} \quad . \tag{VIII-7}$$

Thus, in particular, the value of I for $q_j = h_j$ is less than any value of I for $q_j = h_j + \alpha r_j$ for any r_j constrained by (VIII-11) . We may write the inequality

$$I_h \leq I_h + \alpha r \quad , \tag{VIII-12}$$

which may be expressed as

$$\sum_j \sum_k h_j h_k^* V_{jk} \leq \sum_j \sum_k (h_j + \alpha r_j)(h_k^* + \alpha r_k^*) V_{jk} \quad , \tag{VIII-13}$$

or

$$\alpha^2 \sum_j \sum_k r_j r_k^* V_{jk} + \alpha \sum_j \sum_k (r_j h_k^* + h_y r_k^*) V_{jk} \geq 0 \quad . \tag{VIII-14}$$

The coefficient of α^2 is real and non-negative, since it is a particular value of $I/_{4\pi}$ for $q_j = r_j$. In order that the inequality (VIII-14) be true for any α , we must not allow the sign of the term in α to dominate for small values of $|\alpha|$. The coefficient of α, which is divided into two complex conjugate terms, must therefore go to zero. Setting one of the terms equal to zero we get the condition

$$\sum_j \sum_k r_j h_k^* V_{jk} = 0 \qquad \text{(VIII-15)}$$

for all r_j constrained by (VIII-11). Equation (VIII-15) may be written

$$\sum_j r_j \left[\sum_k h_k^* V_{jk} \right] = 0 \quad . \qquad \text{(VIII-16)}$$

Comparing this with Eq. (VIII-11), it is clear that all the coefficients of the r_j on (VIII-16) must be equal to the same number μ. The solution, then, is given by

$$\sum_k h_k^* V_{jk} = \mu, \; j = 1,2,\ldots.N. \qquad \text{(VIII-17)}$$

The value of $I/_{4\pi}$, minimized by the solution h, is (see Paragraph VIII-7)

$$\frac{I_m}{4\pi} = \sum_j \sum_k h_j h_k^* V_{jk} = \sum_j h_j \left[\sum_k h_k^* V_{jk} \right] , \qquad \text{(VIII-18)}$$

$$= \mu \sum_j h_j = \mu \quad ,$$

taking (VIII-17) into account and the fact that h_j is constrained, as are all the q_j (Eq. (VIII-10)).

The directivity factor f, made a maximum by the solution h, is

$$f_m = \frac{4\pi}{I_m} \quad . \qquad \text{(VIII-19)}$$

The solution (VIII-17) written in matrix form is

$$Vh^* = \left[\frac{I_m}{4} \right] \alpha \quad , \qquad \text{(VIII-20)}$$

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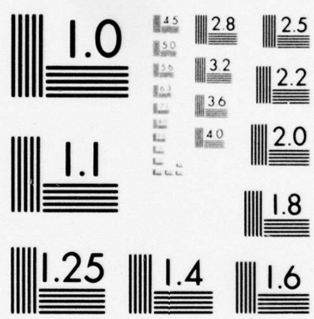

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where V is the matrix whose elements are the V_{jk} (Hermitian with respect to the indices), h is the column matrix of the h_j, and α is the column matrix all of whose elements are ones.

VIII-5. Correlation Matrix for Omnidirectional Noise

We have just calculated the "filtering" that optimizes the directivity of an ensemble of N antenna elements. The proper filtering corresponding to omnidirectional noise will now be calculated, in order to verify the fact that it is identical to the filtering process h defined by (VIII-20). The similarity between Eqs. (VIII-20) and (III-45) (or rather (V-3), since it is a case of proper filtering) clearly shows that we need only prove that the matrix V^* is the matrix of crosscorrelation spectra $(C_{jk}(\nu))$ in the case of omnidirectional noise after the signal is made identical at the N inputs.

Thus, the problem is reduced to

a. evaluating the crosscorrelation between two inputs E_j and E_k,
b. taking into account the alteration of that crosscorrelation by making the signals at E_j and E_k identical, and
c. confirming that the result is V_{jk}.

The first step is a classical calculation which already has been done for some special cases by B. Picinbono[10] and is reproduced here using a general notation compatible with the rest of the paper.

Let us call $b_0{}^{\omega}(t)$ the elementary noise received at point 0 from direction ω (see Fig. VIII-1). The noise received at E_j from the same direction is

$$b_j{}^{\omega}(t) = b_0{}^{\omega}\left(t + \frac{\vec{OE_j} \cdot \vec{\omega}}{C}\right) . \qquad \text{(VIII-21)}$$

The corresponding noise at E_k is $b_k{}^{\omega}(t) = b_0{}^{\omega}\left(t + \frac{\vec{OE_k} \cdot \vec{\omega}}{C}\right)$. (VIII-22)

[10]Not published.

The elementary crosscorrelation (for direction ω) is then

$$m_{jk}^{\omega}(\tau) = E\left\{b_j^{\omega}(t)\cdot b_k^{\omega}(t+\tau)\right\} = m_0^{\omega}\left(\tau + \frac{(\vec{OE}_k - \vec{OE}_j)\cdot\vec{\omega}}{C}\right), \quad \text{(VIII-23)}$$

designating by $m_0^{\omega}(\tau)$ the elementary autocorrelation at 0. Thus, we have, taking Fourier transforms,

$$m_{jk}^{\omega}(\nu) = m_0^{\omega}(\nu)\; e^{+2\pi i\frac{\nu}{c}\left[(\vec{OE}_k - \vec{OE}_j)\cdot\vec{\omega}\right]}, \quad \text{(VIII-24)}$$

where $m_0^{\omega}(\nu)$ designates the spectral density in direction ω. Then, by definition, omnidirectional noise is composed of uncorrelated contributions having the same spectral density in all directions.

Thus,

a. $m_0^{\omega}(\nu)$ is, in fact, independent of ω, and may be written $m_0(\nu)$, and
b. the elementary crosscorrelations and their transforms add simply with integration with respect to ω, giving the total crosscorrelation. Thus the total crosscorrelation spectrum at inputs E_j and E_k is

$$Y_{jk}(\nu) = \int m_{jk}^{\omega}(\nu)\, d\omega = m_0(\nu)\int e^{2\pi i\frac{\nu}{c}\left[(\vec{OE}_k - \vec{OE}_j\cdot\vec{\omega})\right]}\, d\omega. \quad \text{(VIII-25)}$$

The scalar product of (VII-25) only involves the distance E_jE_k. The classical result, according to which $y_{jk}(\nu)$ is <u>real and even in ν</u>, is easily justified on the basis of this equation. The result is that $y_{jk}(\tau)$ is <u>even in τ,</u> for any pair of points. This result is physically obvious, the data being symmetrical with respect to the median plane of E_jE_k. Applying a delay $(+\tau)$ at E_j or $(-\tau)$ at E_k gives the same result. Furthermore, the crosscorrelation does not depend upon the distance E_jE_k.

VIII-6. Matched Filtering in Omnidirectional Noise-Similarity to Optimization of Directivity Factor

The second step will take into account the fact that input E_j has been affected by a "delay" $e^{-2\pi i\frac{\nu}{c}(\vec{OE}_j \cdot \vec{\omega})}$ and E_k by a "delay" $e^{-2\pi i\frac{\nu}{c}(\vec{OE}_k \cdot \vec{\omega})}$.

Transposing Eq. (III-10), where the two preceding filtering processes play the roles of $R_j(\nu)$ and $R_k(\nu)$, and $y_{jk}(\nu)$ plays the part of $C_{jk}(\nu)$, the new crosscorrelation spectrum, after these "delays" make the signals identical, is

$$\gamma_{jk}(\nu) = y_{jk}(\nu)\ e^{2\pi i\frac{\nu}{c}(\vec{OE}_j - \vec{OE}_k)\cdot\vec{\omega}_0} . \tag{VIII-26}$$

As a result (see (VIII-25)),

$$\gamma_{jk}(\nu) = m_0(\nu) \int e^{-2\pi i\frac{\nu}{c}\ [(\vec{OE}_j - \vec{OE}_k)\cdot(\vec{\omega} - \vec{\omega}_0)]}\, d\omega . \tag{VIII-27}$$

Let us compare the preceding equation to Eq. (VIII-6). Except for a real factor $4\pi\ m_0(\nu)$, the $\gamma_{jk}(\nu)$ are identical to the $V^*_{jk}(\nu)$. These $\gamma_{jk}(\nu)$ are the crosscorrelation spectra of the noise inputs for which we propose to find the proper filtering. As we know, the solution, except for a real factor, is

$$\gamma p = \alpha , \tag{VIII-29}$$

where γ is the matrix of the $\gamma_{jk}(\nu)$ and p is the column matrix of the proper filters. It may be written in the form

$$Vp^* = \alpha , \tag{VIII-30}$$

which is equivalent to the solution (VIII-20).

In conclusion, proper matched filtering in omnidirectional noise is that filtering which optimizes directivity.

The spectral density at an arbitrary input is $4\pi m_0(\nu)$. It remains the same after the input signals are made identical.

Thus, the quantity

$$\frac{\gamma_{jk}(\nu)}{4\pi m_0(\nu)} = V_{jk}^{*}(\nu)$$

represents the normalized crosscorrelation of the noise inputs at E_j and E_k after the signals have been made identical.

Whenever the vectors $\vec{OE}_j$ are perpendicular to the direction ω_0, the crosscorrelation $\gamma_{jk}(\nu)$ is, except for a multiplicative factor, reduced to $Y_{jk}(\nu)$ (Eq. (VIII-26)), that is, a real quantity that is even in ν but not necessarily non-negative. The signals are then identical without the need of introducing any delays, since all the inputs are in the same plane normal to ω_0. In this case, the matrices γ and $\underline{V}$ are real and symmetric (hence, always Hermitian). The matched filter p is real. Hence, in omnidirectional noise and for inputs situated in the same plane normal to the reference direction, matched filtering is real.

VIII-7. Application to Two and Three Inputs

Let us apply the preceding results to a system having two elements. We will assume that they are arranged along a line parallel to the plane wave front, such that the signal is the same at the two inputs. Let us use the narrow-band approximation. In omnidirectional noise, the power of the interference noise is the same at the two inputs. Furthermore, the crosscorrelation spectrum is real. The two filters π_1 and π_2 (see Eq. (VII-5)) are real and identical. They may be replaced by the direct summation, which means that there is no efficient matched filter for two inputs in omnidirectional noise. This is physically obvious from the symmetry of the parts played by the two inputs. Note that we have here a very exceptional case, where the direct sum is a part of the matched filtering although the noise inputs are not independent (see Paragraph V-2); but this is true only for two inputs in omnidirectional noise. With two antenna elements, it may be only a question of their separation distance for improvement of the signal-to-noise ratio. It is well known that they must be placed with a separation such that the crosscorrelation of the two noise inputs, which is a function of this separation, is minimized.

If a is this separation, calculation of $V_{jk}(\nu)$ from Eq. (VIII-6) gives

(a) $$\frac{\sin\frac{2\pi\nu a}{c}}{\frac{2\pi\nu a}{c}}$$

in the case of two point receivers;

(b) in the case of two parallel line elements of undefined length, we have a planar situation where the corresponding expression is

$$J_0\left(\frac{2\pi\nu a}{c}\right) .$$

In each case there exists an optimum separation distance between the two elements (for a given frequency).

Let us call $F(a)$ one of the two preceding functions and consider the case of three equidistant elements lined up in the order 1, 2, 3.

The matrix of the V_{jk} is:

$$\begin{bmatrix} 1 & F(a) & F(2a) \\ F(a) & 1 & F(a) \\ F(2a) & F(a) & 1 \end{bmatrix} .$$

Consequently, the filters π_1, π_2, , and π_3 are

$$\pi_1 = \pi_3 = [1-F(a)] \quad [1-F(2a)]$$

$$\pi_2 = [1-F(2a)] \quad [1+F(2a)-2F(a)] .$$

Finally, the weighting (real filtering) to apply to the two end elements (1 and 3) with reference to the central element is

$$\frac{1-F(a)}{1+F(2a)-2F(a)} \cdot$$

It is simple to generalize this method to an arbitrary number of elements with arbitrary spacing.

CHAPTER IX

GENERATION OF PROPER FILTERING - AUTOADAPTIVE SYSTEMS - PRACTICAL STATIONARITY CONDITIONS

Summary

In this chapter, we will deal with the technology, or at least the principles of the technology, of matched filtering. It is a question of constructing the complex quantities representing the N filters, and then making use of them to do the filtering.

Furthermore, we desire to ensure that the filters and filtering processes will evolve in a continuous manner, adapting to slow variations in the statistical relationships of the noise inputs. The expression "slow variations" is defined by the assumption of a "practical duration of stationarity" which is large compared to the time constants used and the duration of the signal. The essential steps are:

a. formulation of the elements of the correlations matrix C_{jk} in the form of electrical voltages,

b. formulation of the product of several elements, of cofactors M_{jk} and of the "filters" π_k themselves, and

c. a method of "filtering" each channel, starting with the complex number representing its filter.

Finally, the block diagrams for the case of three inputs and for two inputs are presented. These diagrams may be used to verify the permanence of the filtering process in the presence of noise, superimposed on the inputs, which is not correlated with the initial noise inputs.

XI-1. Representation of a Crosscorrelation (Narrow Band) of Two Slowly Varying Voltages

For the moment, we will treat an aspect of matched filtering that is close to the technology without going into the details of the technology. That aspect is the effective realization of matched filtering in the special case of the narrow-band approximation.

A filter π of an NPF, for example, is made up of a combination of C_{jk} (a homogeneous polynomial of degree $N-1$). Hence, first of all, these C_{jk} must be "fabricated." We know that $C_{jk}(\nu_0)$ represents the crosscorrelation of two narrow-band noises $b_j(t)$ and $b_k(t)$ resulting from the respective filtering $B_j(t)$ and $B_k(t)$ by filters of bandwidth $\emptyset$ centered at the frequency ν_0 (and at $-\nu_0$, considering only the case of filters with real unit impulse responses (see Paragraph III-2)).

We will assume that the spectral band $\emptyset$ contains all the signal, while remaining narrow enough such that $C_{jk}(\nu)$ remains practically equal to $C_{jk}(\nu_0)$. Under these conditions the crosscorrelation function of $b_j(t)$ and $b_k(t)$ is

$$\gamma_{jk}(\tau) = 2\emptyset|C_{jk}(\nu_0)| \text{ Cos } \left[2\pi\nu_0\tau + \arg C_{jk}(\nu_0)\right] \quad . \tag{IX-1}$$

The value of $\gamma_{jk}(\tau)$ for $\tau = 0$ is

$$\gamma_{jk}(\tau)_{\tau=0} = 2\emptyset R\left[C_{jk}(\nu_0)\right] \quad . \tag{IX-2}$$

Hence, $\gamma_{jk}(\tau)_{\tau=0}$ is the mathematical expectation of the product of the two noises and, by virtue of ergodicity, the time average of this product. Thus, the quantity $R\left[C_{jk}(\nu_0)\right]$ may be realized, except for the factor $2\ \emptyset$, in the following way (Fig. IX-1):

a. by means of a multiplier, produce, in the form of an electrical voltage, the product of the noises $b_j(t)$ and $b_k(t)$ received at inputs E_j and E_k after prefiltering $\emptyset$;

b. take the time average of the multiplier output voltage by means of a sufficiently narrow low-pass filter (an integrator with an adequate time constant), delivering a DC component plus a fluctuating component which is small for a large time constant; and

c. the output voltage of the low pass filter represents, except for the residual fluctuation, the time average of the product of the two noise inputs, or $R\left[C_{jk}(\nu_0)\right]$.

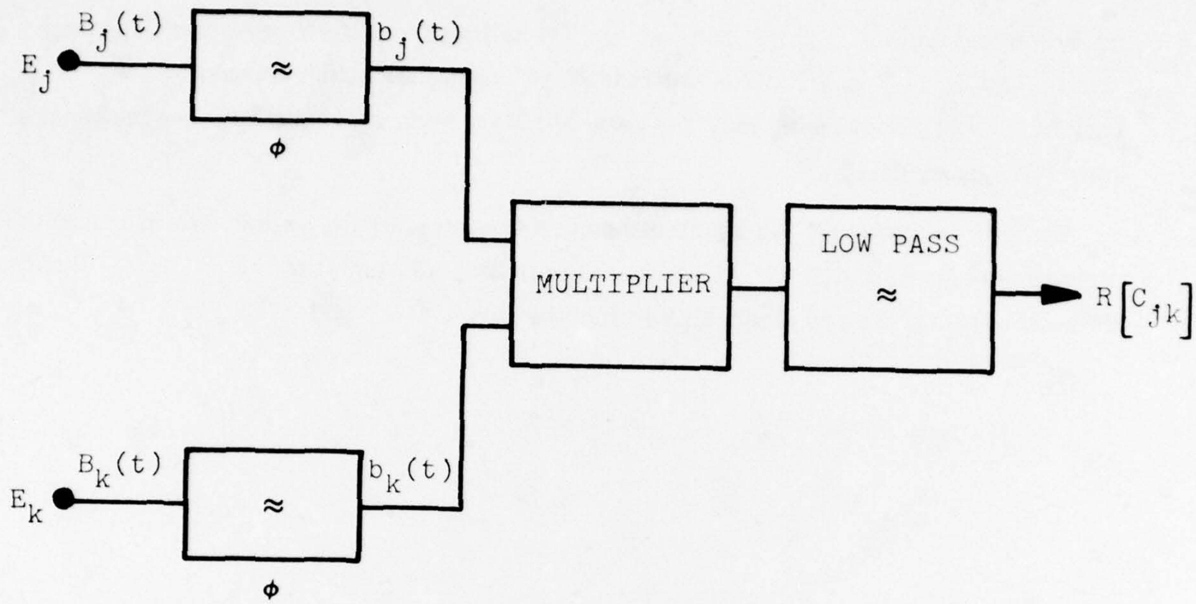


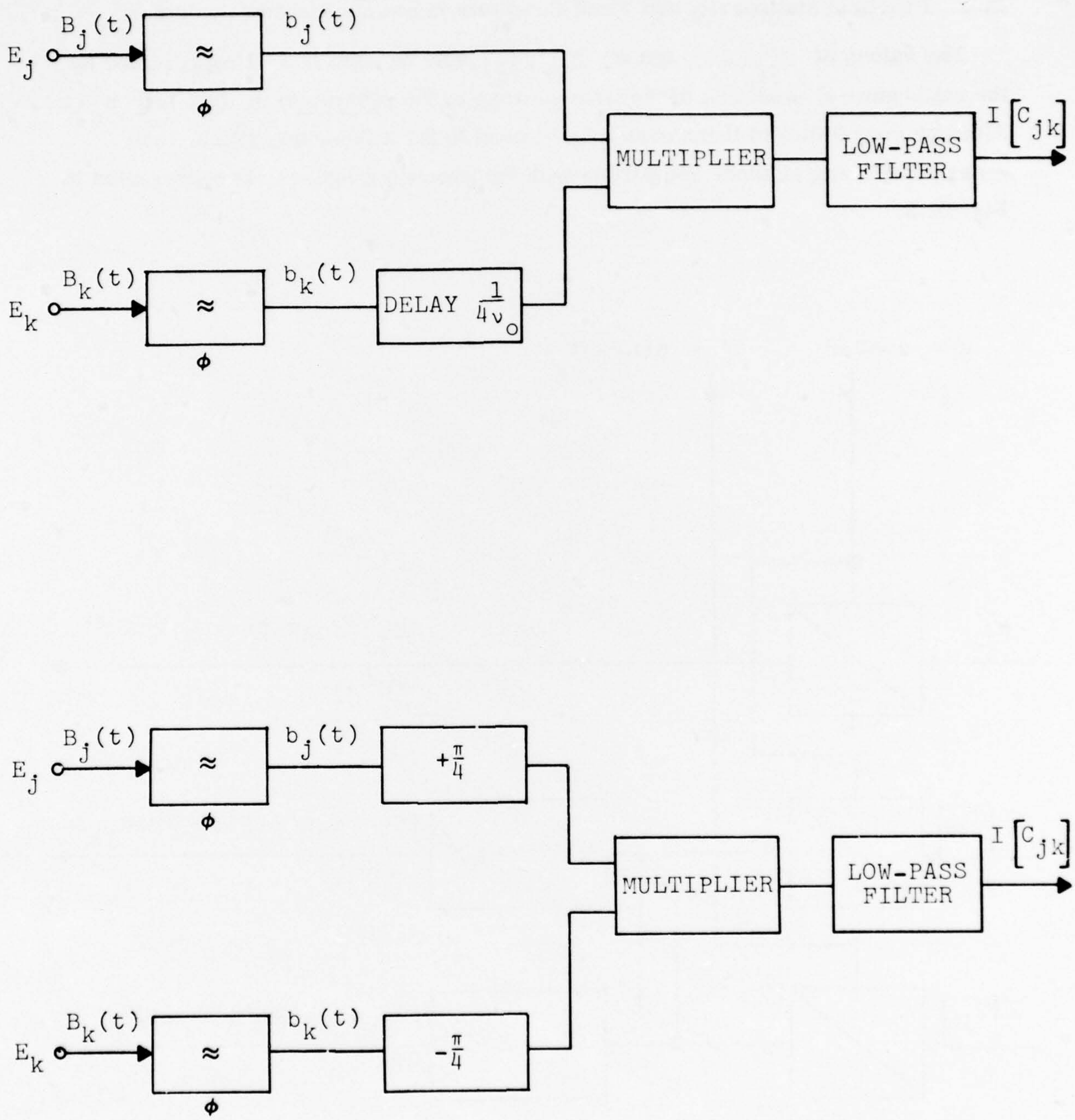
Fig. IX-1

If we were also able to produce a voltage representing $I\left[C_{jk}(\nu_0)\right]$, we could represent the complex number $C_{jk}(\nu_0)$ by two electrical voltages. In order to do this, let us take the two noise inputs $b_j(t)$ and $b_k(t)$ and let $\tau = \frac{1}{4\nu_0}$ in Eq. (IX-1). It is clear that

$$\gamma_{jk}\left(-\frac{1}{4\nu_0}\right) = 2\phi I\left[C_{jk}(\nu_0)\right] \quad . \qquad \text{(IX-3)}$$

Thus, $I\left[C_{jk}(\nu_0)\right]$ is, except for the factor $2\,\emptyset$, the mathematical expectation of the time average of a voltage obtained by taking the product of $b_j(t)$ with a noise voltage $b_k(t)$, itself obtained by delaying $b_k(t)$ by $\frac{1}{4\nu_0}$.

Since we are dealing with a narrow band, this time delay is equivalent to a phase lag of $\pi/2$. This amounts to the same thing as realizing an advance of $\pi/4$ in the phase of $b_j(t)$ and a lag of $\pi/4$ in the phase of $b_k(t)$. The two processes of generating $I\left[C_{jk}(\nu_0)\right]$ illustrated in Fig. IX-2 are equivalent. The output voltage, except for some fluctuation, represents the value of $I\left[C_{jk}(\nu_0)\right]$.



E_j
B_j(t)
≈
φ
b_j(t)
E_k
B_k(t)
≈
φ
b_k(t)
DELAY 1/4ν_o
MULTIPLIER
LOW-PASS FILTER
I [C_jk]
E_j
B_j(t)
≈
φ
b_j(t)
+π/4
E_k
B_k(t)
≈
φ
b_k(t)
-π/4
MULTIPLIER
LOW-PASS FILTER
I [C_jk]

Fig. IX-2

IX-2. Practical Stationarity and Time Constants in an Autoadaptive System

The values of $C_{jj}(\nu_0)$ and of $C_{kk}(\nu_0)$ may be equally well represented by the mathematical expection of the time average of the squares of $b_j(t)$ and $b_k(t)$. Although very different techniques may be used to form these quantities, their generation, using methods compatible with the preceding figures, is represented in Fig. IX-3.

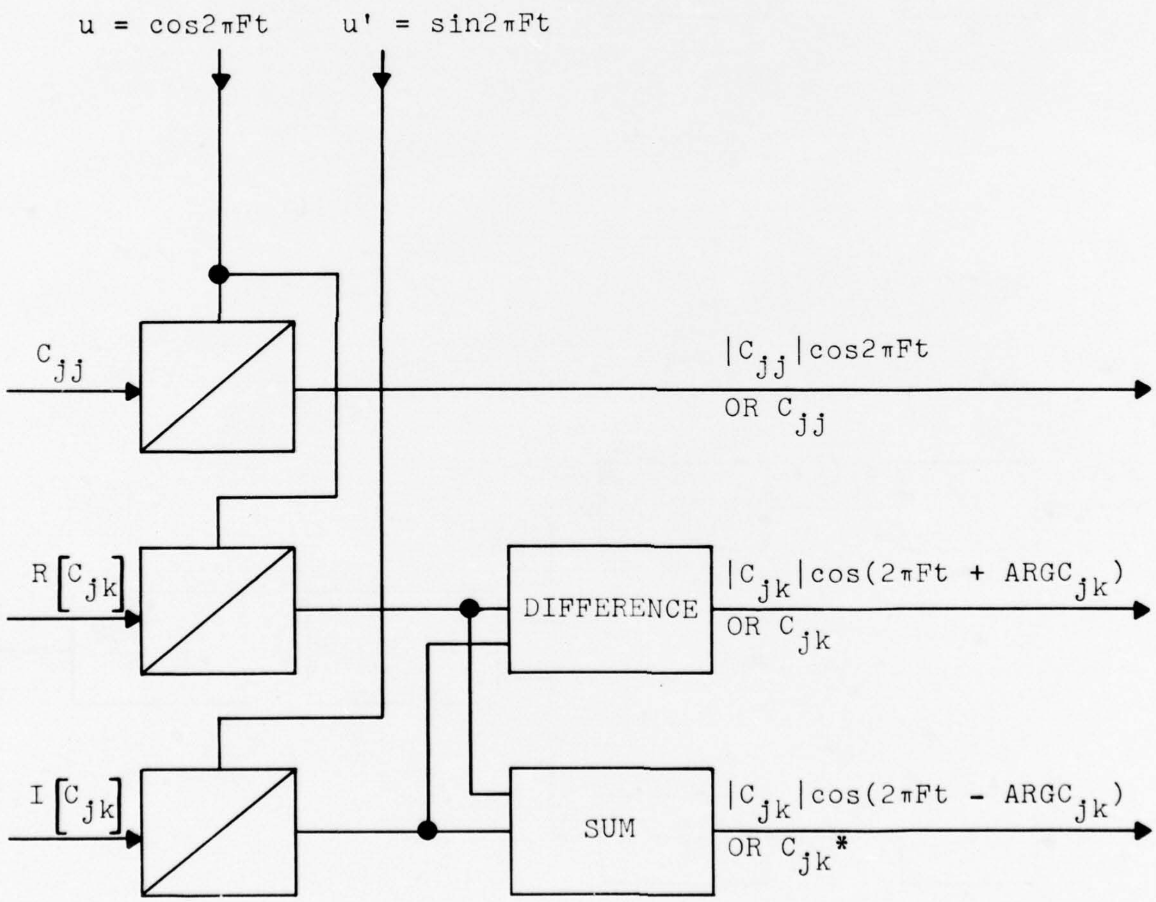


Fig. IX-3. Representation of the Complex Numbers C_{jk} by AC Voltages with Frequency F

Finally, for two inputs, we have formed four voltages which represent

$C_{jj}(\nu_0), C_{kk}(\nu_0)$, R $\left[C_{jk}(\nu_0)\right]$, and I $\left[C_{jk}(\nu_0)\right]$. From now on we shall omit the indication of frequency ν_0.

With all the possible C_{jk} it is possible to construct the filters π_k. This consists of obtaining complex numbers (in the form of voltages) equal to the π_k. Next, properly defined filtering consists of applying the filters to the noise inputs $b_k(t)$ and taking the sum of the filter outputs. Later we will see how to realize the π_k and the filtering process itself. But, for the present, it is interesting to note that if the noise inputs are strictly stationary, the C_{jk} may theoretically be obtained in the form of strictly continuous voltages with the aid of infinitely narrow low-pass filters. In order to avoid confusion, from now on we will call these filtering processes integrations.

In practice, the C_{jk} are obtained to within only a fluctuation whose amplitude is dependent upon the integration time constant. On the other hand, since the noise inputs vary slowly with time (limited stationarity), the C_{jk} obtained from the above methods are going to evolve slowly. The filters π_k derived from them will also evolve, achieving an autoadaptation of the system that follows the slow variations of the statistical relationships of the $B_j(t)$.

Consequently, if an "average stationarity time" θ_s can be defined (an essentially practical concept), it will be useless to integrate the outputs of multipliers whose time constants θ are greater than this average time θ_s. On the other hand, if the time constant θ is not greater than the inverse of the filter bandwidth $\emptyset$, it is ineffectual in practice.

Thus we have imposed the following conditions.

a. The signal is narrow band, lying within the band ϕ. Thus, its time duration T is greater than $1/\phi$.[11]

[11]The margin between T and $1/\phi$ allows the possibility of several signal types in the case of doppler effect, for example. In particular, in the case of long signals with a "single frequency," the output of the proper filtering may be handled by "band division," which corresponds to matched filtering for each possible value of doppler shift (see reference (23)).

b. The band $\emptyset$, although larger than that of the signal, is narrow enough for the narrow-band approximation to be valid, that is, such that $C_{jk}(\nu)$ are equal to $C_{jk}(\nu_0)$ within the band.

c. The noise may be considered as stationary for duration θ_s, which is very large compared with $1/\emptyset$ and T. The integration time constant is both large compared with $1/\emptyset$ and T and small compared with θ_s.

$$1/\emptyset < T << \theta << \theta_s \tag{IX-4}$$

Thus, the C_{jk} are defined for durations on the order of θ_s. They are considered as being perfectly constant over the duration θ, which is the duration over which they are computed by an integration of products.

d. Since θ is large compared with $1/\emptyset$, the multiplier - integrator system may be considered to be a correlator "with strong integration" (reference 7, Chapter V). The power associated with the fluctuation of the output is proportional to $1/\emptyset\theta$, that is, to a very small quantity. This fluctuation is considered to be negligible.

In order to establish the ideas more solidly, taking inspiration from a practical case, let us assume that $\emptyset$ is on the order of 100 cps at a center frequency of 10 kc, which is certainly a narrow band. The duration of signals is on the order of a second, and their bandwidth is a few cycles per second. The interference noise is stable - let us say "practically stationary" - for durations of from 30 to 60 seconds. We may envisage integration time constants on the order of 5 to 10 seconds.

IX-3. Representation of a Crosscorrelation by an AC Voltage

Having thus defined the practical conditions of stationarity which allow the use of the preceding theoretical developments, let us return to the C_{jk} represented by constant, or more precisely, slowly varying voltages. We will indicate briefly, in the rest of this chapter, how the principles of matched filtering with N inputs may be implemented.

We will see later that in order to apply filtering πk to noise $b_k(t)$ it is particularly convenient to express the complex number π_k in the form of an AC voltage of fixed frequency, whose amplitude and phase are equal to the modulus and argument, respectively, of π_k. Moreover, it is quite natural to represent any complex numbers by AC voltages. Thus, although we will "fabricate" the C_{jk}

in the form of DC voltages, we are going to convert them to AC voltages with an arbitrary frequency F.

Let us start with a voltage source of frequency F, which will serve as a phase reference at that frequency. Its amplitude is normalized to the value 1,

or

$$u = \cos 2\pi F t \quad .$$

Take the product of u with the DC voltage representing C_{jj} (a simple modulator is adequate). We then obtain the voltage $C_{jj} \cos 2\pi F t$ with zero phase, which will henceforth represent the complex number C_{jj} - here it is real and non-negative. Similarly, we will have the voltage $C_{kk} \cos 2\pi F t$.

Take the product of u with the DC voltage representing $R\left[C_{jk}\right]$. We obtain

(see Fig. IX-3) $$V_1 = R\left[C_{jk}\right] \cos 2\pi F t. \qquad \text{(IX-5)}$$

From voltage u , derive a voltage u' of the same (normalized) amplitude but given a phase lag of $\pi/2$:

$$u' = \sin 2\pi F t \; .$$

Take the product of u' with the DC voltage which represents $I\left[C_{jk}\right]$. We have

$$V_2 = I\left[C_{jk}\right] \sin 2\pi F t \quad . \qquad \text{(IX-6)}$$

We find that the difference between the voltages, $V_1 - V_2$, represents in amplitude (modulus) and in phase (argument) the complex number C_{jk}. The sum V_1 and V_2 represents C^*_{jk} . Hereafter, the C_{jk} and C^*_{jk} are represented by voltages with frequency F and appropriate amplitude and phase.

IX-4. Principles of Producing a Product of Complex Numbers in the Form of an AC Voltage

The following operation consists of realizing products of the C_{jk}. The products are also complex numbers representable by AC voltages. Let a_1, for example, be represented by

$$|a_1|\cos\left[2\pi F t + (\arg a_1)\right]$$

and a_2 by

$$|a_2|\cos[2\pi Ft + (\arg a_2)] \quad .$$

Using a multiplier, take the product of the two voltages. Two spectral components are obtained, one at frequency $2F$,

$$W = |a_1|\cdot|a_2|\cos\left\{4\pi Ft + (\arg a_1) + (\arg a_2)\right\} \quad , \tag{IX-7}$$

and the other at frequency zero. We may eliminate the second component by a high-pass filter or, even better, isolate the first by a band-pass filter wide enough to let the slow variations in $|a_1|\cdot|a_2|$ and in $(\arg a_1 + \arg a_2)$ pass through.

Now the voltage W represents the product of two complex numbers a_1 and a_2. Thus the product of two complex numbers C_{jk} is represented by an AC voltage at frequency $2F$. The phase reference voltage for these products is obviously the voltage with frequency $2F$, obtained by squaring voltage u.

We have at our disposal, at present, monomials of degree $N-1$ that enter into the composition of a π_k. We may omit, in this process, the intermediate computation of cofactors M_{jk}, which are sums and differences of the above monomials (π_k being itself the sum of the cofactors of the k^{th} column of Δ). The phase reference at frequency $(N-1)\ F = f$ is handled in the same way.

IX-5. Narrow-band Linear Filtering Using Multiplication By a Voltage Representing The Filter

The final step consists of "filtering" the k^{th} channel with π_k, that is, changing the amplitude and phase of the voltage at frequency ν_0 which appears at each channel

a. by multiplying its amplitude by $/\pi_k/$, and

b. by increasing its phase by π_k.

Thus, π_k is represented by the "filtering voltage" ϕ:

$$\phi = |\pi_k|\cos(2\pi ft + \arg \pi_k) \quad , \tag{IX-8}$$

letting $f = (N-1)\ F$.

The input voltage (to be filtered) is

$$\chi = \alpha\cos(2\pi\nu_o t + \lambda), \tag{IX-9}$$

and we wish to obtain

$$y = \alpha|\pi_k|\cos(2\pi\nu_o t + \lambda + \arg \pi_k). \tag{IX-10}$$

Take the product of the voltage $\emptyset$ and the voltage χ and filter the component at frequency $(\nu_0 + f)$.

$$z = |\pi_k|\cos(2\pi(f + \nu_o)t + \lambda + \arg \pi_k). \tag{IX-11}$$

The above voltage has the amplitude and phase of y, but it is not at frequency ν_0. In certain cases, this fact may not be troublesome. Thus, the output of the matched filter is obtained in the form of a narrow-band voltage with center frequency $f + \nu_0$. If, however, we have to recover the frequency ν_0, it may be done with the aid of the "phase reference" voltage at frequency f.

$$r = \cos 2\pi f t \ . \tag{IX-12}$$

Taking the product of z and r and filtering for the component at ν_0 (demodulation with r), the voltage y itself may be obtained (except for an unimportant factor of 1/2).

The only thing remaining is to take the sum of the outputs of all the channels in order to obtain normalized proper filtering. Thus we have given a very brief glance at realization of an NPF. Numerous variations are possible in the technology itself, but the basic circuit remains as a <u>multiplier of two narrow-band voltages</u>. Simple modulators may be used to realize certain of the multiplications (when one of the factors has a normalized amplitude). In the majority of cases, we need either an analog multiplier or a digital multiplier of quantized voltages accurate enough to minimize inaccuracy in the average value of the products (for example, geometric quantization, see reference (24)).

In order to realize **a** CSPF, we have seen that it is necessary to <u>divide</u> the result of the preceding NPF by $\Gamma_p = \Sigma\pi_k$. To form Γ_p presents little difficulty, and it will be obtained on the form of a voltage having a frequency f and zero phase (Γ_p is real and non-negative).

Of itself, the amplitude of this voltage is of interest to us and will be determined by means of an envelope detector in the form of a constant or slowly varying voltage, always of the same sign. Subsequently, the output from the NPF must be divided by this voltage.

This operation of division by a voltage with a constant sign is technologically possible, although a little more difficult than multiplication. Thus the CSPF is realized.

IX-6. Flow Charts for Two-and Three-Input Systems

Figure IX-4 shows the block diagram for an NPF in the case of 3 inputs, or more precisely, one of the possible block diagrams. The complexity of this equipment rests primarily with the large number of analog multipliers that it requires. Within the present state of the art, it appears difficult to handle more than 4 or 5 inputs. For this reason, it is natural to turn toward less perfect but more easily realized solutions, such as that of connecting the CSPF's together, the principles of which are discussed in Paragraph VI-9 and Paragraph VI-10.

The theory of a system with two inputs is treated in Chapter VII. Figure IX-5 represents a possible realization of the corresponding circuit. This figure is similar to Fig. IX-4. Six analog multipliers are required to realize a 2-input NPF, and a supplementary divider for an CSPF (the squares may be realized in a number of ways). The other elements (modulators, pass-band filters, integrators) are classical.

Figure IX-6 shows a simplified circuit useful only in the case of two inputs having the same spectral density. As we know, the filters π_1 and π_2 are complex conjugates.[12] Equation VII-5 applies to this case, which is characterized by

$$C_{11} = C_{22} = C, \text{ from which } q = 1 \text{ and } C_{12} = C\theta \qquad \text{(IX-13)}$$

and, in particular,

---

[12] Recall that π_1 and π_2 are the system functions (complex gains) of the filters for the frequency ν_0.

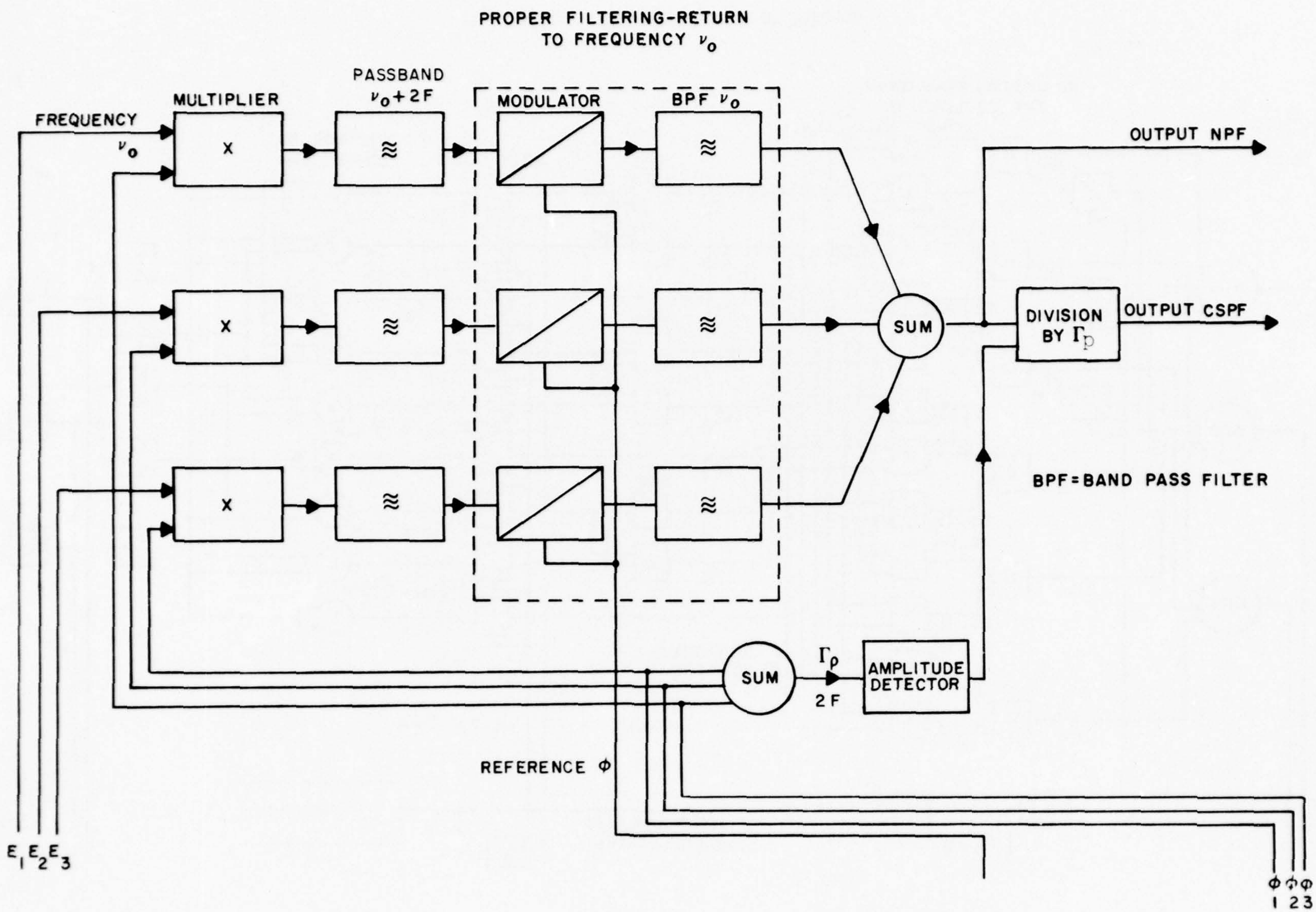


Fig. IX-4A. Narrow-Band Matched Filtering with Three Inputs

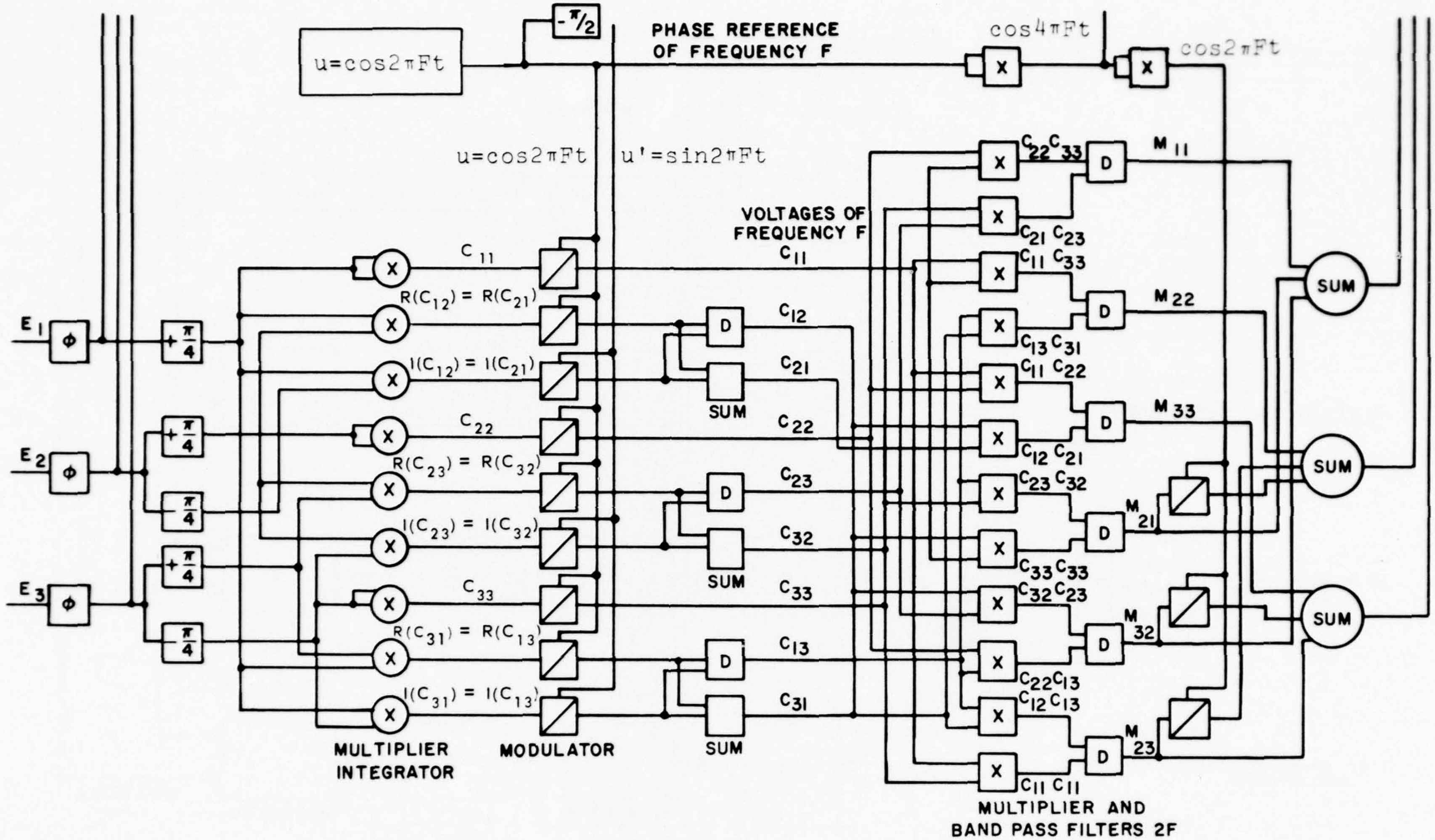


Fig. IX-4B. Block Diagram of Filters

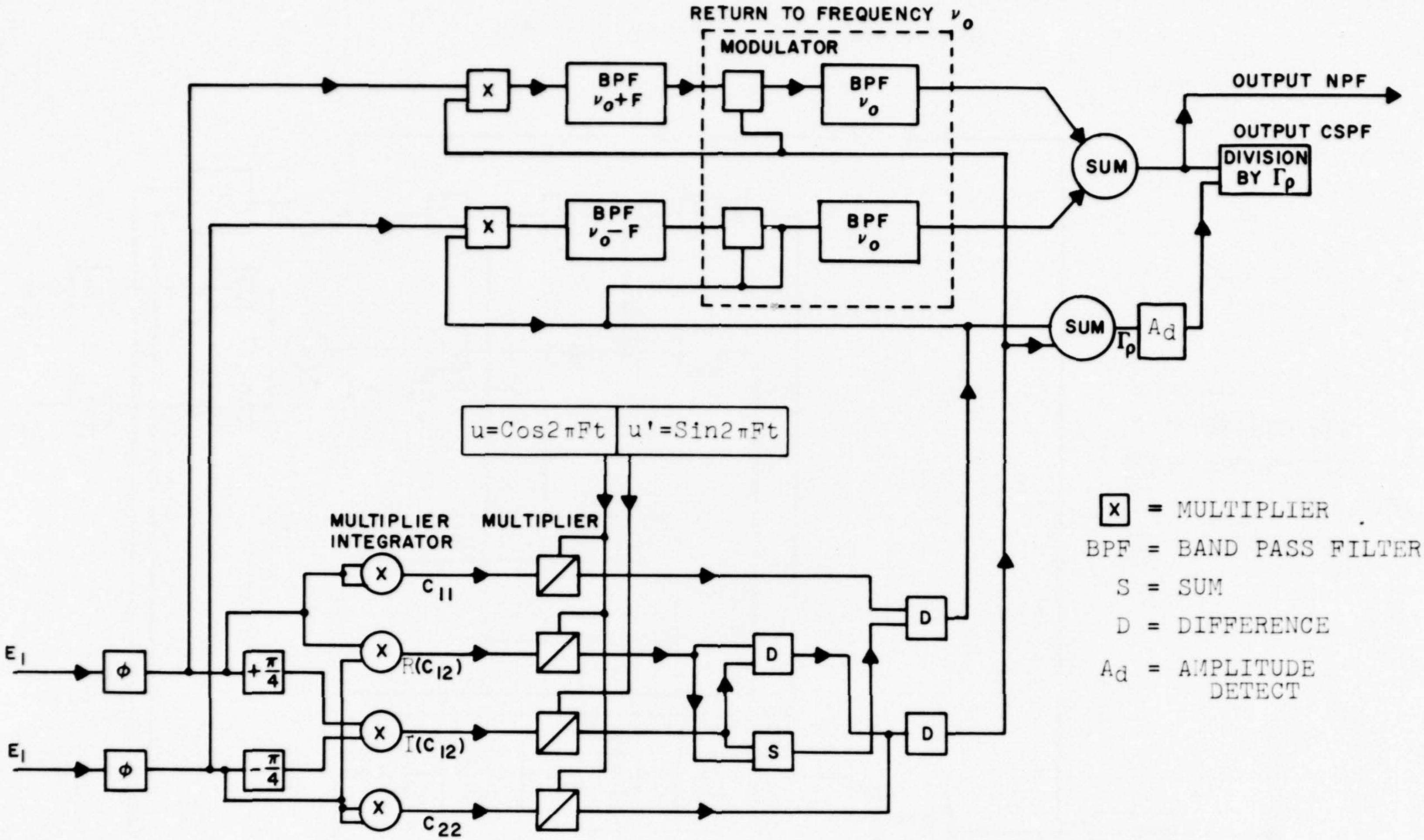


Fig. IX-5. Two-Input System – General Case

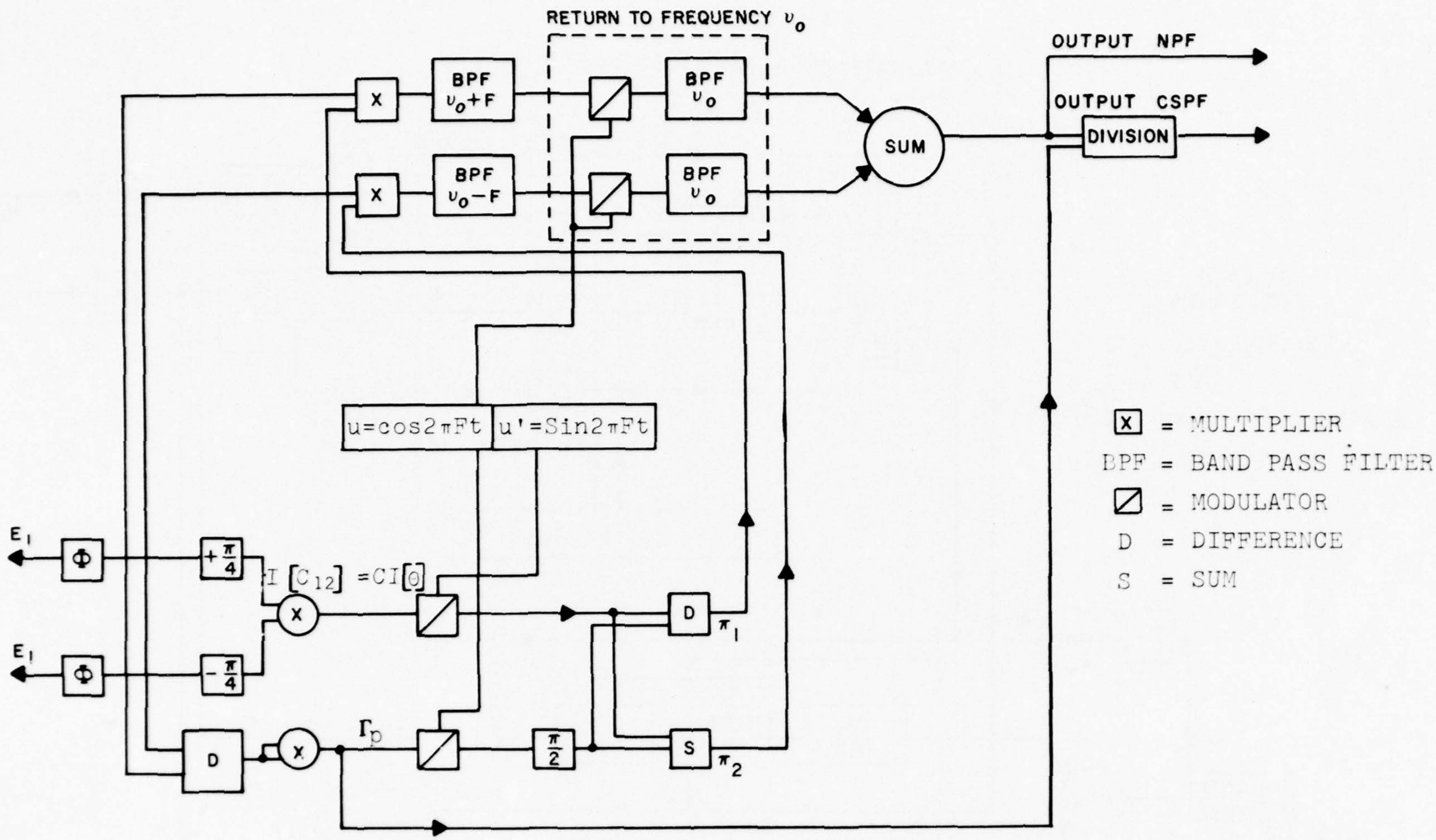


Fig. IX-6. Two-Input – Equal Power Case

$$\Gamma_p = 2\ C\ (1-R(\theta))$$

$$\pi_1 = C\ (1-\theta) = \frac{1}{2}\ \Gamma_p\ -i\ CI(\theta) \qquad \text{(IX-14)}$$

$$\pi_2 = C\ (1-\theta^*) = \frac{1}{2}\ \Gamma_p\ +i\ CI(\theta)\ ,$$

and it may be seen that the real part of both π_1 and π_2 is nothing more than $\Gamma_{p/2}$, that is, half the spectral density of the difference. This justifies the structure of the circuit of Fig. IX-6, which requires only 4 analog multipliers.

IX-7. Verification of one Property of the NPF in the Preceding Diagrams

It has been demonstrated in Paragraph VI-3 that an additional noise input $n(t)$, uncorrelated with $B_1(t)$ and $B_2(t)$ and superimposed upon them changes the behavior of neither the NPF nor the CSPF. This property may be verified in the sample circuits of Figs. IX-4, IX-5, and IX-6. Its verification is particularly simple in Fig. IX-6. First of all, it is obvious that everything that is common to the two inputs - the noise $n(t)$ or the contingent signal - disappears in the difference of the two and, because of this difference, does not influence the process of synthesizing π_1 and π_2.

It may be proven that the noise $n(t)$ disappears in the multiplication-integration of two inputs which differ in phase by $\pi/2$. In fact, the two narrow-band noises $b_1(t)$ and $b_2(t)$ may be written, making more evident their low frequency components (whose bandwidth is on the order of ϕ),

$$\begin{aligned} b_1(t) &= M_1(t)\ \cos\ 2\pi\ \nu_0\ t + N_1(t)\ \sin\ 2\pi\ \nu_0\ t\ , \\ b_2(t) &= M_2(t)\ \cos\ 2\pi\ \nu_0\ t + N_2(t)\ \sin\ 2\pi\ \nu_0\ t\ , \end{aligned} \qquad \text{(IX-15)}$$

and the noise $n(t)$ may be written

$$n(t) = p(t)\ \cos\ 2\pi\ \nu_0\ t + q(t)\ \sin\ 2\pi\ \nu_0\ t\ ,$$

where neither $p(t)$ nor $q(t)$ is correlated with $M_1(t)$, $M_2(t)$, $N_1(t)$, or $N_2(t)$. Furthermore, the mathematical expectations of all the low frequency components are zero (reference (7), page 46 , Eq. 3-6-31).

Following the procedure of the diagram in Fig. IX-6, we see that the multiplier forms the product of

$$\left[M_1(t) + p(t)\right] \cdot \left[-\sin 2\pi \nu_0 t\right] + \left[N_1(t) + q(t)\right] \cdot \left[\cos 2\pi \nu_0 t\right]$$

and

$$\left[M_2(t) + p(t)\right] \cdot \left[\cos 2\pi \nu_0 t\right] + \left[N_2(t) + q(t)\right] \cdot \left[\sin 2\pi \nu_0 t\right]$$

(IX-16)

Integration done by a low-pass filter eliminates components of frequency $2\nu_0$. As a strong integrator ($\theta >> 1/_{\emptyset}$, . see Eq.(IX-4)), it forms the time average - or *the mathematical expectation* - of the low frequency terms in the product, that is, of

$$V(t) = [N_1(t) + q(t)] [M_2(t) + p(t)] - [M_1(t) + p(t)] [N_2(t) + q(t)]$$

In $V(t)$, the term $p(t) \cdot q(t)$ is eliminated (its time average is zero, two components of the same stationary noise being uncorrelated at a given instant of time ((7),Eq. 3-6-35.).

The time averages of all products such as $q(t)\ M_2(t)$ or $p(t)\ N_1(t)$ are zero, since they are equal (with no correlation) to the product of the averages of each factor.

Summarizing, in the time average of $V(t)$, at the output of the integrator, only quantities depending upon $b_1(t)$ and $b_2(t)$ remain. Thus, the noise $n(t)$ is eliminated in the process of synthesizing π_1 and π_2; and the predicted property is therefore confirmed. The preceding demonstration still applies if $n(t)$ is replaced by a sinusoidal function of frequency ν_0. It is enough to assume in this case, that

$$q(t) = 0$$

$$p(t) = p_0 \;.$$

The voltage $V(t)$ becomes $V_0(t)$:

$$V_0(t) = N_1(t)\left[M_2(t) + p_0\right] - \left[M_1(t) + p_0\right] N_2(t) \;,$$

and p_0 is eliminated from the time average of $V_0(t)$ because the averages of $N_1(t)$ and $N_2(t)$ are zero.

If a sinusoidal function of unlimited duration - having the characteristic of stationarity - does not change the value of π_1 and π_2, the same will be true, certainly, for the narrow-band signal $S(t)$ (narrower than ϕ). In fact, this signal is a sinusoidal function with frequency ν_o and slowly varying amplitude. Furthermore, its duration is small compared with the integration time constant. For these two reasons, the signal, identical at the two inputs, does not depend upon the multiplier-integrator output, nor, consequently, upon the values of π_1 and π_2. Similar verification of this property may be made using Figs. IX-5 and IX-4.

The $I\left[C_{jk}\right]$ are not altered by the noise $n(t)$ or by a sinusoidal time function of frequency ν_o. The $R\left[C_{jk}\right]$ are all increased by the same quantity, which subsequently vanishes when the difference is taken.

CONCLUSIONS

The general area covered by this study is an investigation of the best way of detecting the presence – and only the presence – of weak signals in noise interference and, consequently, of improving electromagnetic and acoustic detection over a wide range of processes.

In this area are included two methods of approach to this problem. The first is statistical detection theory, which makes use of the concept of estimation. The other is the examination of the optimization of certain judiciously chosen criteria (signal-to-noise ratio).

Although historically unrelated, the answers obtained coincide for conditions usually found in practice.

The Matched Filter Theorem is one of those optimizations and concerns a particular signal in a stationary noise field. It forms the point of departure for this study.

We have tried to modify this theorem – which handles only a single signal and a single noise – to include the multiple aspects of detection by an antenna with several receiving elements. Then we have examined most of the forms taken by the solution for different statistical relationships presented by the noise inputs. We wished to stress the very general character of this treatment by showing how it includes and extends beyond the concept of directivity, which is generally used for the same purpose. Finally, we have pursued the work to the point of block diagrams for the realization of such systems, in the particular case of narrow-band signals, and from the point of view of an automatic adaptation, with gradual changes in a quasi-stationary noisefield.

REFERENCES

(1) A. BLANC-LAPIERRE
R. FORTET
J. KAMPE DE FERIET

"Théorie des Fonctions Aléatoires"
(Masson & Cie, Paris 1953

(2) D. MIDDLETON

An introduction to statistical communication theory"
(McGraw-Hill Book Company, Inc., N.Y. 1960)

(3) W. B. DAVENPORT
W. L. ROOT

"An introduction to the theory of random signals and noise"
(McGraw-Hill Book Company, Inc., N.Y.,1958)

(4) L. A. WAINSTEIN
V. D. ZUBAKOV

"Extraction of signals from noise" (translated from the Russian, by R. A. SILVERMAN)
(Prentice-Hall Inc., Englewood Cliffs, New Jersey,1962)

(5) B. M. DWORK

"Detection of a pulse superimposed on fluctuation noise"
(Proc. IRE, July 1950, pp. 771-774)

(6) A. ANGOT

"Complèments de Mathématiques à l'usage des Ingénieurs de l'Electronique et des Télécommunications"
(Editions de la Revue d'Optique, Paris,1952)

(7) A. BLANC-LAPIERRE
B. PICINBONO

"Propriétés Statistiques du bruit de Fond"
(Masson & Cie, Paris,1961)

(8) A. BLANC-LAPIERRE
(avec la collaboration de G. BONNET, P. FAURE, H. MERMOZ, B. PICINBONO)

"Modèles statistiques pour l'étude des phénomènes de fluctuations"
(Masson & Cie, Paris,1963)

(9) P. M. WOODWARD

"Probability and information theory with application to Radar"
(Pergamon Press, London,1957)

(10) D. O. NORTH

"Analysis of the factors which determine signal/noise.
Discrimination in Radar"
(Report PTR-6c, RCA Laboratories, June 1943)

(11) J. VAN VLECK & D. MIDDLETON

"A theorical comparison on the visual, aural and meter reception of pulsed signals in the presence of noise"
(Journal of Applied Physics, Vo. 17, pp. 940-971, Novembre 1946)

(12) L. A. ZADEH & J. R. RAGAZZINI

"Optimum filters for the detection of signals in noise"
(P.T.R.E., Vol. 40, Number 10 pp. 1223-1231, Octobre 1952)

(13) B. PICINBONO

Thèse présentée à la Faculté des Sciences d'Alger
"Sur certains problèmes concernant la détection des signaux faibles"

(14) LICHNEROWICZ

"Algèbre et Analyse Linéaire"
(Masson & Cie, Paris,1956)

(15) G. H. BROWN

"Pattern synthesis -- Simplified methods of array design to obtain a desired directive pattern"
(R.C.A. Review, Sept. 1959,Vol. 20, pp. 398-412)

(16) P. M. WOODWARD

"A method of calculating the field over a plane aperture required to produce a given polar diagram"
"Journal I.E.E. 1947, 93 part III,page 1554)

(17) S. A. SCHELKUNOFF

"A mathematical theory of linear array"
(Bell System Technical Journal, 1943, 22, page 80)

(18) Allen S. DUNBAR

"On the theory of antenna beam shaping"
(Journal of Applied Physics Vol. 23, n° 8, Août 1952, page 847)

(19) S. DRABOWITCH

"Quelques applications de la théorie du signal à la technique des antennes"
(Revue Technique C.F.T.H., n° 33, page 7, 1960)

(20) V. G. WELSBY & D. G. TUCKER

"Multiplicative receiving arrays"
(The Journal of British IRE, Vol. 19, n° 6, Juin 1959, page 369)

(21) Van der MAAS & H. GRUENBERG

"Note on a simplified calculation for Dolph. Tchebycheff arrays"
(Journal of Applied Physics, 1956, 27, n° 8, page 962)

(22) M. BOUIX

"Etude à l'infini du rayonnement électromagnétique harmonique. Dispositifs rayonnants à l'émission et à la réception"
(Annales des Télécommunications, Sept. - Oct. 1963)

(23) P.Y. ARQUES

"Détection des signaux faibles" - Performance de la division de bande partielle dans la détection des signaux à bande étroite"
(Annales des Télécommunications, tome 19, n°s 3-4, 1964)

(24) G. BONNET

(C.R. Académie des Sciences, 255, pp 1064-1066, 1962)

(25) E. M. GLASER

"Signal detection by adaptive filters"
(IRE Transactions, I.T. 7, n° 2, page 87, 1961)

(26) H. L. GROGINSKI

"On the design of adaptive systems"
(IRE Nat. Con. Rec. Part. 4, page 160, 1958)

(27) D. GABOR
W. L. P. WILBY
R. WOODCOCK

"An universal non-linear filter, predictor and simulator which optimizes itself by a learning process"
(Congrès Inst. Elect. Engrs, Paper n° 3270, Juil. 60)

(28) D. MIDDLETON

"An introduction to statistical communication theory"
(McGraw-Hill Book Company, 1960)

(29) D. MIDDLETON
Van METER

"Detection and extraction of signals in noise from the point of view of statistical decision theory"
(J. Soc. Ind. Appl. Math. 3, 192, 1955)

(30) "Modern statistical approach to reception in Communication theory"
(IRE Trans. Of Inform. Theory, PG IT-4, Sept. 1954)

(31) P. L. STOCKLIN

Conferences on Signal Theory
(unpublished at this time)

(32) B. PICINBONO
J. de SUSO BARBA

"Sur une généralisation du filtrage optimum en détection-intégration"
(Institut d'Etudes Nucléaires d'Alger
Note présentée à l'Académie des Sciences)

MATCHED FILTERING & OPTIMAL USE OF AN ANTENNA (MERMOZ)

USC PUB. NO. 745, 29 JUNE 1966.

TYPES OF PROCESSORS

I. OPTIMIZATION CRITERIA ARE BASED ON THE RECONSTRUCTION OF THE MOST PROBABLE FORM OF THE SIGNAL, ie, THE BEST "GUESS" AT THE ORIGINAL FORM AND CONTENT OF THE TRANSMITTED SIGNAL.

II. OPTIMIZATION CRITERIA ARE BASED ON THE DETECTION OF THE PRESENCE OF A SIGNAL OF A GIVEN FORM. IT IS NOT REQUIRED THAT THE SIGNAL BE ACCURATELY REPRODUCED.

ANTENNA GAIN

ASSUMPTIONS ON NOISE FIELD: I. STATISTICAL INDEPENDENCE AMONG THE NOISE OUTPUTS FROM N ELEMENTS.

II. OMNIDIRECTIONAL NOISE.

I. => OUTPUT SIGNAL POWER INCREASES AS N^2 WHILE THAT OF NOISE INCREASES AS N. THUS GAIN $= \frac{N^2}{N} = N$.

II. => GAIN = DIRECTIVITY.

III. COHERENT INTERFERENCE

III. => OPTIMUM GAIN BY USING 2 PHONES WITH APPROPRIATE TIME DELAY.

IV. RANDOM NOISE STATISTICS (TIME DEPENDENT)

IV. => OPTIMUM GAIN BY ADAPTIVE PROCESSOR.

SIGNAL-TO-NOISE RATIO: $\rho = \dfrac{\text{INSTANTANEOUS POWER OF THE SIGNAL AT ARBITRARY TIME } t.}{\text{AVERAGE POWER OF NOISE.}}$

NOTATION: $S_1(t), \ldots, S_N(t)$ WHERE $S_i(t)$ IS THE SIGNAL AT THE i^{th} PHONE.

$B_1(t), \ldots, B_N(t)$ " $B_i(t)$ IS THE NOISE " " " " .

$C_{jk}(\tau) = E\{B_j(t)\, B_k(t+\tau)\}$ ARE CROSSCORRELATIONS OF NOISE FIELD.

$R_1(t), \ldots, R_N(t)$ ARE N FILTERS WHICH MAXIMIZE ρ WHEN THEIR N OUTPUTS ARE SUMMED.

$C_{jk}(\nu)$, THE FOURIER TRANSFORM OF $C_{jk}(\tau)$, IS THE CROSSCORRELATION SPECTRA.

NOTE: $C_{jk}(\tau) = C_{kj}(-\tau)$ $\qquad C_{jk}(\nu) = C^*_{jk}(-\nu) = C^*_{kj}(\nu)$

$[C_{jk}(\tau)]^2 \leq C_{jj}(0) \cdot C_{kk}(0)$, for all τ.

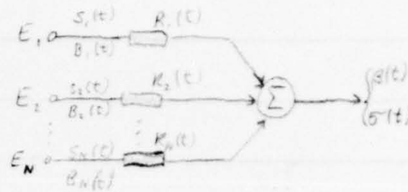
DEFINE $b_i(\tau)$ TO BE THE OUTPUT OF $R_i(\tau)$ APPLIED TO $B_i(\tau)$. THEN

$\gamma_{jj}(\nu) = C_{jj}(\nu)\, |R_j(\nu)|^2$ AND $\gamma_{jk}(\tau) = E\{b_j(t) \cdot b_k(t+\tau)\}$

$\gamma_{jk}(\tau) = \int R^*_j(\nu) \cdot R_k(\nu) \cdot C_{jk}(\nu)\, e^{-2\pi i \nu \tau}\, d\nu.$

$\gamma_{jk}(\nu) = R^*_j(\nu) \cdot R_k(\nu) \cdot C_{jk}(\nu)$

$|C_{jk}(\nu)|^2 \leq C_{jj}(\nu) \cdot C_{kk}(\nu)$, for all ν, or $\begin{vmatrix} C_{jj} & C_{jk} \\ C_{kj} & C_{kk} \end{vmatrix} \geq 0$. all ν.



SIGNAL AT THE OUTPUT OF FILTER $R_j(t)$

IS $\int R_j(\theta)\, S_j(t-\theta)\, d\theta$

AND AT THE OUTPUT OF THE SUMMER

IS $\sigma(t) = \sum_j \int R_j(\theta)\, S_j(t-\theta)\, d\theta$

FOR NOISE — $\int R_j(u)\, B_j(t-u)\, d\theta$ & $\beta(t) = \sum_j \int R_j(u)\, B_j(t-u)\, du$.

THE AVERAGE NOISE POWER IS $P_\beta = E\{\beta^2(t)\} = E\{[\sum_j \int R_j(u)\, B_j(t-u)\, du]^2\}$

$$= \sum_j \sum_k \iint R_j(u)\, R_k(v)\, C_{jk}(u-v)\, du\, dv \geq 0$$

TO MAXIMUMIZE $\rho = \dfrac{[\sigma(t_0)]^2}{P_\beta}$.

(I) $\sigma(t_0) = \sum_j \int R_j(\theta)\, S_j(t-\theta)\, d\theta = k$. then minimize P_β.

Consider impulse responses $r_1(t), r_2(t) \ldots r_N(t)$ $\ni \sum_j \int r_j(\theta)\, S_j(t_0-\theta)\, d\theta = 0$.

Then all FILTERING OF THE FORM $R_j(t) + \alpha r_j(t)$ satisfies (I)

Let $h_j(t)$ be a filter process which minimizes P_β

i.e. $R_j(t) = h_j(t) \Rightarrow [\sigma(t_0)]^2 = k^2$ & P_β IS A MINIMUM.

Thus $\sum_j \sum_k \iint h_j(u) h_k(v)\, C_{jk}(u-v)\, du\, dv \leq \sum_j \sum_k \iint [h_j(u) + \alpha r_j(u)][h_k(v) + \alpha r_k(v)]\, C_{jk}(u-v)\, du\, dv$

for all α & r_j

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29 June 1966. i-vii + 116 p., figs.

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1. Filters (Electromagnetic wave)
2. Signals—Processing
3. Signal-to-noise ratio
4. Antennas

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III. Title

been translated to serve as a ready reference work. The study itself lies in the area of research that is concerned with the improvement of long-range-detection techniques. Subjects covered include (1) a formal solution to matched filtering with N inputs, (2) the case of identical signals—proper filtering—orthogonal images of a system with N noise inputs, (3) narrow-band approximation—variation of proper filtering, (4) case of two inputs—coherent noise, (5) matched filtering and directivity, and (6) generation of proper filtering—auto-adaptive systems—practical stationarity conditions.

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