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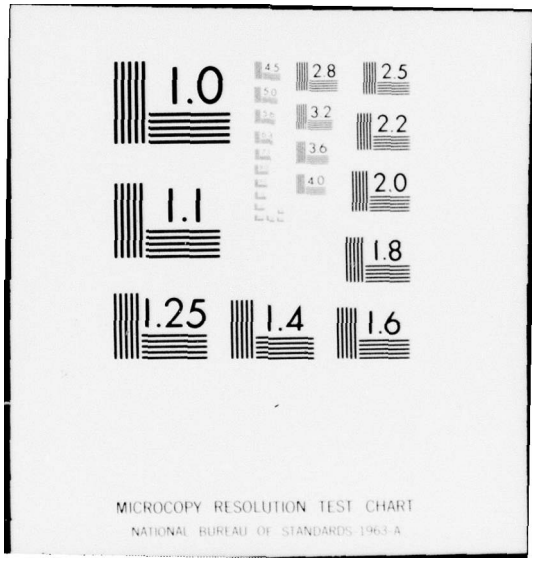
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SIGNAL DESIGN FOR EFFICIENT DETECTION IN RANDOMLY DISPERSIVE MEDIA

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ABSTRACT

The optimum structure of a signal to be transmitted over a randomly time-varying and frequency-selective medium is investigated. A model is developed that treats the medium as a randomly time-varying linear filter. By viewing the filter's transfer function as a homogeneous random field on the time-frequency plane, a second-order theory results that relates various second-order measures of the time and frequency structures of input and output processes.

A Neyman-Pearson detector is assumed, and a signal-design strategy, based on the asymptotic behavior of the false-dismissal probability when the detector is presented with a sequence of observations of the medium output, is developed. This approach leads to the strategy of maximizing the Kullback-Leibler information number. It is shown that this criterion minimizes the false-dismissal probability for any reasonable false-alarm probability when the medium satisfies Price's "low-energy coherence" condition.

The extrema of the Kullback-Leibler information number are investigated as a function of the noise-normalized eigenvalues of the output covariance kernel, and an energy-constrained maximum is found to occur when each degree of freedom possesses a signal-to-noise ratio in the vicinity of 2.

An optimum distribution of energy in the output time-frequency plane, which maximizes the Kullback-Leibler information number, is deduced. Also, a constraint is derived on the input ambiguity function that produces the best mean-square approximation to the optimum output energy distribution. A discussion on the general structure of input signals that satisfy this constraint in the high signal-to-noise ratio case is included. Finally, the signal-design criterion of maximizing the Kullback-Leibler information number is compared with the criterion of maximizing the divergence.

ACKNOWLEDGMENT

I wish to express my deep appreciation to Professor Thomas Kailath for his guidance during the course of this investigation. If this study achieves a coherent contribution to the problem of designing signals, it is surely because of our many fruitful discussions. Many of the conclusions presented in this study resulted from developing his suggestions or answering his questions. I also wish to thank Professor Norman Abramson, not only for his invaluable comments on the material but also for his constant inspiration throughout this study. In addition, I wish to thank Professor Allen Peterson for his careful reading of the dissertation and his constructive comments.

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CONTENTS

CONTENTS

TABLES

ILLUSTRATIONS

SIGNAL DESIGN FOR EFFICIENT DETECTION IN RANDOMLY DISPERSIVE MEDIA

I INTRODUCTION

This study presents a unified framework of concepts that relate to the reception of signals perturbed by a randomly time- and frequency-dispersive medium. In particular, the problem of designing efficient transmissions for the detection of such signals in the presence of noise is investigated within this framework. In practice, this signal-reception problem and the related signal-design problem arise in the detection of radar targets that are spread in both range and Doppler and arise in "on-off" signaling over randomly time-varying and frequency-selective radio channels.

We restrict our attention to the question of optimizing the detectability of a stochastic signal in the presence of noise. The stochastic signal of interest is, in general, a nonstationary random process that is the result of transmitting a known signal through a randomly dispersive medium with known spectral characteristics. Consideration is not given to signal-design aspects of the more general radar problem in which various characteristics of the target are estimated or the more general communications problem in which one of several signals is transmitted over the channel.

Following an introduction to the elements of the detection problem relevant to this study, in Chapter II, we devote Chapters III through VI to the development and discussion of a linear-system characterization of the randomly dispersive medium. Besides unifying various approaches to the problem of modeling dispersive media, this material discusses a second-order theory for randomly time-varying linear filters. Various measures of the second-order properties of the time and frequency structures of an arbitrary process are introduced, and the manner in which a

randomly dispersive medium transforms an input measure into an output measure is developed. The analysis presented in these chapters is germane to the synthesis problem treated in later chapters.

In Chapter VII, the detection problem is analyzed. Following the adoption of a Neyman-Pearson approach and an outline of a Karhunen-Loève exposition, the test statistic and various related quantities and useful facts are introduced.

In Chapter VIII, we present the concept of asymptotic relative efficiency of radar signals as a criterion for signal design. This criterion is based on the asymptotic behavior of the false-dismissal probability when the receiver is presented with a sequence of independent observations of the medium output. The Kullback-Leibler information number enters naturally into this material and serves as a useful measure of the efficiency per transmission of an input signal in reducing the false-dismissal probability.

Chapter IX is concerned with the investigation of the extrema of the Kullback-Leibler information number as a function of the noise-normalized eigenvalues of the output covariance kernel. This leads to a signal-design strategy of synthesizing input signals that provide degrees of freedom in the output signal with individual signal-to-noise ratio (SNR) in the vicinity of 2. The result is similar to a result obtained by Pierce [Ref. 1]. He concludes that a SNR in the vicinity of 3 for each degree of freedom minimizes an upper bound for the error probability in the symmetric binary-signaling case.

The Kullback-Leibler information number is investigated under Price's "low-energy coherence (LEC)" condition in Chapter X. We show that under LEC conditions maximizing the Kullback-Leibler information number is equivalent to Price's criterion of maximizing the receiver-output SNR. In addition, we demonstrate the asymptotic approach to normality of the Neyman-Pearson test statistic and conclude that both criteria yield optimum signal-design strategies when the noise-normalized eigenvalues of the output covariance kernel are suitably small.

In Chapter XI, we discuss the results of Chapter X in terms of the useful concept of the equivalent number of degrees of freedom of the output process.

Finally, in Chapter XII, an optimum shape for the distribution of energy in the output time-frequency plane is deduced. This shape maximizes the Kullback-Leibler information number by achieving the proper number of degrees of freedom for the available SNR. A constraint on the input ambiguity function is derived that ensures the best mean-square approximation to the optimum shape. We then treat the high-SNR case.

It is of interest to compare the Kullback-Leibler information number with other proposed measures of detection performance. Therefore, in Chapter XIII, maximizing the Kullback-Leibler information number and maximizing the divergence are discussed as signal-design criteria.

II DETECTION MODEL

In this chapter, we postulate a basic model that serves to formulate and analyze the detection problem treated in this study. The primary purpose of this model is to relate the properties of the received signal to the properties of the dispersive medium and the transmitted signal.

The spectra of signals transmitted in radar or radio communication applications are significantly nonzero only in a band of frequencies of width small compared to any frequency within the band. These signals, commonly referred to as narrow-band signals, admit a convenient complex envelope notation,

$$x(t) = \mathrm{Re}\{\mathbf{x}(t)e^{i2\pi f_0 t}\} \quad . \tag{1}$$

In this notation, $x(t)$ is the real narrow-band signal; f_0 is a frequency within the band; and $\mathbf{x}(t)$ is the complex envelope. The magnitude of $\mathbf{x}(t)$ is the conventional envelope of the narrow-band signal, while the phase of $\mathbf{x}(t)$ is the conventional phase measured with respect to the carrier phase, $2\pi f_0 t$. The complex envelope $\mathbf{x}(t)$ is a low-pass function and is readily visualized as the modulation of the carrier tone at frequency f_0. (See Appendix A.)

To characterize how the medium transforms an input signal into an output signal, the medium is viewed as a randomly time-varying linear filter. We account for time- and frequency-selective fading phenomena through the dependence of the filter's transfer function on time and frequency. Since we are concerned only with narrow-band signals, the medium can be modeled with an equivalent time-varying narrow-band filter that produces the same output as the medium for a class of properly restricted narrow-band inputs.

Because the medium is assumed to be linear, we can proceed formally in the usual manner and define system functions which yield the response of the medium to classes of elementary input signals. Owing to the random nature of the medium, however, the description of the system functions

obtained in this fashion necessarily requires concepts employed in the theory of random fields.

The fundamental system function associated with the medium is the time-variant transfer function $H(t, f)$. A simple experiment performed on the dispersive medium generates this important function:

If the input $x(t)$ is the CW tone

$$x(t) = \mathrm{Re}\{e^{i2\pi(f+f_0)t}\},$$

then the output of the medium in the absence of additive noise is given by

$$z(t) = \mathrm{Re}\{H(t, f)e^{i2\pi(f+f_0)t}\} \quad . \tag{2}$$

Thus $H(t, f)$ is the complex envelope of the medium response to a CW tone relative to the frequency of transmission. In this study, we assume that the time-variant transfer function is a complex random field defined on the time-frequency plane. That is, given n points $(t_1, f_1), \ldots, (t_n, f_n)$ in the time-frequency plane, the $2n$ random variables corresponding to the real and imaginary parts of $H(t_1, f_1), \ldots, H(t_n, f_n)$ will possess a joint probability distribution.

For fixed f, $H(t, f)$ defines a complex random narrow-band process in t that yields the amplitude and phase time variations of the medium response to a CW tone at frequency $f + f_0$. We can then view the transfer function $H(t, f)$ as a complex random surface yielding the amplitude and phase of the medium response as a function of time for CW tone inputs as a function of input frequency.

The system diagram of Fig. 1 depicts the model for the detection problem considered in this study. Upon observation of the process $\mathbf{y}(t)$ in the time interval $(0, T)$, the receiver must decide whether or not the

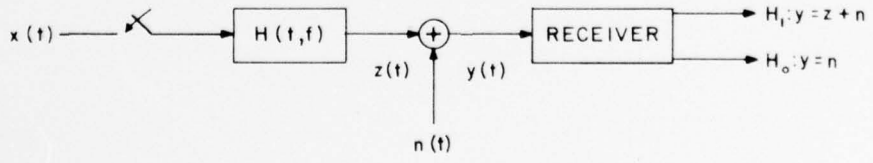

FIG. 1 SYSTEM DIAGRAM OF DETECTION MODEL

signal process $\mathbf{z}(t)$ has been added to the noise process $\mathbf{n}(t)$ (hypothesis H_1 or hypothesis H_0). When the signal process $\mathbf{z}(t)$ is present, it is the result of passing the signal $\mathbf{x}(t)$ through a randomly time-varying linear filter with transfer function $H(t, f)$.

III RANDOM FILTER MODEL

A. HOMOGENEOUS LINEAR FILTERS

This study treats an important and interesting class of randomly time-varying linear filters. We restrict our attention to random fields $H(t,f)$ that are homogeneous [Refs. 2 and 3]. The random field is homogeneous if

$$E[H^*(t,f)H(t+\alpha,f+\beta)] = R_H(\alpha,\beta) \tag{3a}$$

and

$$E[H(t,f)] = c \quad , \tag{3b}$$

where c is a constant complex quantity. (The symbol E indicates mathematical expectation.) Equation (3a) states that the random field $H(t,f)$ is wide-sense stationary in both the time direction and the frequency direction. The function $R_H(\alpha,\beta)$ is the time-frequency autocorrelation function and depends only on the time difference α and the frequency difference β when $H(t,f)$ is homogeneous.

Homogeneous random fields possess spectral representations in terms of orthogonal-increment random fields [Ref. 2]. The usual spectral representation [Refs. 2 and 4] of a wide-sense stationary random process indexed on a single real parameter is the one-dimensional version of this more general representation. Since $H(t,f)$ is homogeneous, it possesses a spectral representation of the form [Ref. 2]:

$$H(t,f) = \iint e^{i2\pi(\lambda t-\tau f)}dW(\lambda,\tau) \quad , \tag{4a}$$*

---

* Absence of limits on integrals indicates that the region of integration is from $-\infty$ to ∞.

where the $W(\lambda,\tau)$ field has orthogonal increments; that is

$$E[dW^*(\lambda_1,\tau_1)dW(\lambda_2,\tau_2)] = 0 \qquad \text{for} \qquad \lambda_1 \neq \lambda_2 \quad \text{or} \quad \tau_1 \neq \tau_2 \tag{4b}$$

and

$$E[|dW(\lambda,\tau)|^2] = dF_W(\lambda,\tau) \quad . \tag{4c}$$

Although Eq. (4a) is the result of purely mathematical considerations, it yields a meaningful physical interpretation. The integral in this equation is defined as an appropriate limit of summations of the form [Ref. 2]

$$\sum_k \sum_j \Delta W(\lambda_k,\tau_j) e^{i2\pi(\lambda_k t - \tau_j f)} \quad .$$

If $\mathrm{Re}\{e^{i2\pi(f+f_0)t}\}$ is the input to the filter, then, from Eq. (2), the output is an appropriate limit of summations of the form

$$\mathrm{Re}\{\sum_k \sum_j \Delta W(\lambda_k,\tau_j) e^{i2\pi(\lambda_k+f_0)\tau_j} e^{i2\pi(f+f_0+\lambda_k)(t-\tau_j)} \quad .$$

Let

$$\Delta U(\lambda_k,\tau_j) = \Delta W(\lambda_k,\tau_j) e^{i2\pi(\lambda_k+f_0)\tau_j} \quad ,$$

and consider one of the terms in the above summation,

$$\mathrm{Re}\{\Delta U(\lambda_k,\tau_j) e^{i2\pi(f+f_0+\lambda_k)(t-\tau_j)}\} \quad .$$

Observe that this term represents a contribution to the output due to the medium delaying the input by τ_j seconds, shifting the input frequency by λ_k cps, and modifying the input amplitude and phase according to the factor $\Delta U(\lambda_k,\tau_j)$. Thus the above summation decomposes the medium into a sum of incremental time-delay Doppler-shift channels, and $\Delta U(\lambda_k,\tau_j)$ is the complex envelope of the response of an incremental channel to a CW tone. Since $\Delta U(\lambda_k,\tau_j)$ does not depend on t or f, the incremental channels are time-flat (time-invariant) and frequency-flat (frequency-invariant). The incremental channels also possess the important orthogonality property

$$E[\Delta U^*(\lambda_1,\tau_1)\Delta U(\lambda_2,\tau_2)] = 0 \qquad \text{for} \qquad \lambda_1 \neq \lambda_2 \quad \text{or} \quad \tau_1 \neq \tau_2 \quad .$$

Orthogonality of the incremental channels is a necessary and sufficient condition for wide-sense stationarity of the random field $H(t,f)$ *in both* t and f.

It is interesting to note that expressing $H(t,f)$ in the summation form yields a tapped delay-line model equivalent to the filter model studied by Kailath [Ref. 5]. If we write

$$H(t,f) = \sum_k \sum_j \Delta W(\lambda_k, \tau_j) e^{i2\pi\lambda_k t} e^{-2\pi i \tau_j f}$$

and let

$$A_j(t) = \sum_k \Delta W(\lambda_k, \tau_j) e^{i2\pi\lambda_k t} ,$$

then $H(t,f)$ can be expressed

$$H(t,f) = \sum_j A_j(t) e^{-i2\pi\tau_j f} . \qquad (5)$$

A delay line possessing taps at the delays τ_j with a time-variable gain $A_j(t)$ at each tap realizes this transfer function. Hence the model for the dispersive medium treated in this study is essentially the Kailath model in which the tap gain functions are complex stationary random processes that are mutually orthogonal for all time shifts. Equation (5) illustrates the presence and origin of frequency-selective fading. At a particular instant of time, the transfer function varies with frequency; this frequency variation is caused by the time delays τ_j. If we write

$$B_k(f) = \sum_j \Delta W(\lambda_k, \tau_j) e^{-i2\pi\tau_j f} ,$$

then

$$H(t,f) = \sum_k B_k(f) e^{i2\pi\lambda_k t} . \qquad (6)$$

This transfer function can be realized by a sequence of time-varying elements in parallel [Ref. 6]. Each element is composed of a time-variant random filter $B_k(f)$ followed by a frequency translator which introduces the frequency translation λ_k. Equation (6) illustrates

the presence and origin of time-selective fading. At a particular frequency, the transfer function varies with time; this time variation is caused by the Doppler shifts λ_k.

B. MEDIUM SCATTERING FUNCTION

If $\mathrm{Re}\{e^{i2\pi(f+f_0)t}\}$ is the input to the medium, then the output of an incremental channel possessing Doppler shift λ_k and time delay τ_j has the form

$$\mathrm{Re}\{\Delta U(\lambda_k,\tau_j)e^{i2\pi(f+f_0+\lambda_k)(t-\tau_j)}\} \quad ,$$

which is merely a cosine function with random amplitude and phase. The ensemble average power of this cosine function can be expressed as

$$\frac{1}{2}E[|\Delta U(\lambda_k,\tau_j)|^2] = \frac{1}{2}\Delta F_W(\lambda_k,\tau_j) \doteq \frac{1}{2}S_v(\lambda_k,\tau_j)\Delta\lambda\Delta\tau \quad ,$$

where

$$F_W(\lambda,\tau) = \int_{-\infty}^{\lambda}\int_{-\infty}^{\tau} S_v(x,y)\,dx\,dy \quad .$$

Thus the surface $F_W(\lambda,\tau)$ [the real quantity appearing in Eq. (4c)] plots the average power scattered by the medium as a function of Doppler shift and time delay. The derivative $S_v(\lambda,\tau)$ is then the density of power scattered by the medium as a function of Doppler shift and time delay. For this reason, $S_v(\lambda,\tau)$ affords a particularly useful physical description of the medium and is called the medium scattering function [Refs. 7 and 8].

The spectral representation [Eq. (4a)] will be formally written in terms of a nonstationary white-noise random field $v(\lambda,\tau)$ rather than the orthogonal-increment random field $W(\lambda,\tau)$ [Refs. 9 and 10],

$$H(t,f) = \iint e^{i2\pi(\lambda t-\tau f)}v(\lambda,\tau)\,d\lambda\,d\tau \quad , \qquad (7a)$$

where

$$E[v^*(\lambda,\tau)v(\lambda+\xi,\tau+\nu)] = S_v(\lambda,\tau)\delta(\xi)\delta(\nu) \quad . \qquad (7b)$$

One can verify that Eqs. (7a) and (7b) imply a two-dimensional version of the Wiener-Khintchine theorem; that is, the time-frequency autocorrelation function and the scattering function constitute a Fourier transform pair,

$$R_H(\alpha,B) = \iint e^{i2\pi(\lambda\alpha-\tau\beta)} S_v(\lambda,\tau)\,d\lambda\,d\tau \tag{8a}$$

$$S_v(\lambda,\tau) = \iint e^{-i2\pi(\lambda\alpha-\tau\beta)} R_H(\alpha,\beta)\,d\alpha\,d\beta \quad . \tag{8b}$$

The above relationships [Eqs. (7) and (8)], which have been pointed out by Bello [Ref. 11], form the basis of a very fruitful and interesting approach to the problem of characterizing randomly time-varying media.

To satisfy the condition in Eq. (7b), we assume that the quadrature components of the random field $v(\lambda,\tau)$ form a mutually uncorrelated family of zero-mean random variables. This assumption implies that the constant in Eq. (3b) is zero. It is also assumed that for a given λ and τ the quadrature components of $v(\lambda,\tau)$ are identically distributed random variables. These assumptions (see Appendix A) lead to the following four-correlation functions for the quadrature components of $H(t,f)$:

$$E[H_R(t,f)H_R(t+\alpha,f+\beta)] = E[H_I(t,f)H_I(t+\alpha,f+\beta)] = \frac{1}{2}\,\mathrm{Re}\,\{R_H(\alpha,\beta)\} \tag{9a}$$

$$E[H_R(t,f)H_I(t+\alpha,f+\beta)] = -E[H_I(t,f)H_R(t+\alpha,f+\beta)] = \frac{1}{2}\,\mathrm{Im}\{R_H(\alpha,\beta)\} \quad , \tag{9b}$$

where

$$H(t,f) = H_R(t,f) + iH_I(t,f) \quad .$$

In modeling a scattering medium, it is reasonable to assume that the random field $H(t,f)$ is Gaussian. In this case, the above assumptions imply that the first-order probability density of the phase of $H(t,f)$ is uniform. This merely means that in the absence of prior information concerning the behavior of $H(t,f)$ one does not expect the phase of $H(t,f)$ for a scattering medium to favor a region of phase values at any particular point in the time-frequency plane. In other words, if we choose an arbitrary t and f and observe the phase of the response of a scattering medium to a CW tone

at frequency $f + f_0$ at time t, there is no physical reason *a priori* that the phase will assume some values at the expense of other values. The above assumptions also imply that the quadrature components of $H(t,f)$ possess zero mean, a fact that is intimately related to the uniform phase of $H(t,f)$.

We note that, when $H(t,f)$ is Gaussian and the conditions of Eqs. (9a) and (9b) are satisfied, the time-frequency autocorrelation function $R_H(\alpha,\beta)$ completely specifies the probability law of $H(t,f)$, and the quadrature components of $H(t,f)$ are homogeneous real random fields.

IV SYSTEM FUNCTIONS AND SYSTEM AUTOCORRELATION FUNCTIONS

A. SYSTEM FUNCTIONS

In this chapter, four equivalent representations for the output of a time-varying linear medium are deduced from the basic definition of $H(t,f)$ [Eq. (2)]. Each representation relates the input and output of the channel through one of four system functions. Each system function completely describes the channel in one of four possible rectangular coordinate systems. The four coordinate systems are formed by choosing Doppler shift or output time for the first coordinate and time delay or input frequency for the second coordinate.* For example, $H(t,f)$ is a system function where the first coordinate is output time and the second coordinate is input frequency. Furthermore, any system function can be obtained from any other system function through an appropriate Fourier transform relation. From a system point of view (the most convenient for our purposes), the spectral representation [Eq. (7a)] is a statement of transform relationships that exist between the four system functions. We begin by relating the input and output through the system function $H(t,f)$.

Let $\mathbf{X}(f)$ be the Fourier transform of the input complex envelope, then

$$x(t) = \mathrm{Re}\left\{\int \mathbf{X}(f)e^{i2\pi ft}df e^{i2\pi f_0 t}\right\}$$

$$= \int \mathrm{Re}\left\{\mathbf{X}(f)e^{i2\pi(f+f_0)t}\right\}df \quad .$$

The response of the medium to

$$\mathrm{Re}\left\{\mathbf{X}(f)e^{i2\pi(f+f_0)t}\right\}$$

---

* In addition to the four system functions discussed in this section, Bello [Ref. 11] considers four system functions that are obtained by choosing Doppler shift or input time for the first coordinate and time delay or output frequency for the second coordinate. In this manner, one obtains four pairs of system functions which Bello [Ref. 6] has defined to be dual operators. He also considers the four system functions obtained by choosing output time or output frequency for the first coordinate and input time or input frequency for the second coordinate.

is given by Eq. (2) as

$$\mathrm{Re}\left\{H(t,f)\mathbf{X}(f)e^{i2\pi(f+f_0)t}\right\} \quad .$$

Linearity of the medium implies that the medium response to $x(t)$ can be written

$$z(t) = \int \mathrm{Re}\left\{H(t,f)\mathbf{X}(f)e^{i2\pi(f+f_0)t}\right\} df$$

$$= \mathrm{Re}\left\{\int H(t,f)\mathbf{X}(f)e^{i2\pi ft}df e^{i2\pi f_0 t}\right\} \quad .$$

Thus we obtain the output complex envelope in terms of the Fourier transform of the input complex envelope through the relation

$$\mathbf{z}(t) = \int H(t,f)\mathbf{X}(f)e^{i2\pi ft}df \quad . \tag{10}$$

Equation (10) yields a Fourier representation of the output $\mathbf{z}(t)$ in terms of the input frequency spectrum. In this representation a component of the input at frequency f is weighted by the time-variant transfer function $H(t,f)$, which reflects the time-variable nature of the medium's frequency characteristics.

The spectral representation [Eq. (7a)] can be written

$$H(t,f) = \int h(t,\tau)e^{-i2\pi\tau f}dt \quad ,$$

where

$$h(t,\tau) = \int v(\lambda,\tau)e^{i2\pi\lambda t}d\lambda \quad .$$

The quantity $h(t,\tau)$ is a system function that relates the input and output in the following manner:

$$\mathbf{z}(t) = \int h(t,\tau)\mathbf{x}(t-\tau)d\tau \quad . \tag{11}$$

In Eq. (11), the function $h(t,\tau)$ is the channel time response at time t due to a unit impulse applied in the time domain at time $t-\tau$.

A third system function can be deduced by writing the spectral representation in the form

$$H(t,f) = \int V(\lambda,f)e^{i2\pi\lambda t}d\lambda \quad ,$$

where

$$V(\lambda,f) = \int v(\lambda,\tau)e^{-i2\pi\tau f}d\tau \quad .$$

The output $\mathbf{z}(t)$ can be expressed in terms of system function $V(\lambda,\tau)$ as

$$\mathbf{z}(t) = \iint V(u-f,f)\mathbf{X}(f)e^{i2\pi ut}df\,du \quad . \tag{12}$$

It follows from Eq. (12) that the Fourier transform of the output $\mathbf{z}(t)$ is given by

$$\mathbf{Z}(u) = \int V(u-f,f)\mathbf{X}(f)df \quad . \tag{13}$$*

In Eq. (13) the function $V(\lambda,f)$ is the medium frequency response at frequency $f+\lambda$ due to a unit impulse applied in the frequency domain at frequency f.

Finally, the output can be expressed in terms of the Doppler-delay random field $v(\lambda,\tau)$, which appears in the spectral representation [Eq. (7a)], as

$$\mathbf{z}(t) = \iint v(\lambda,\tau)\mathbf{x}(t-\tau)e^{i2\pi\lambda t}d\lambda\,d\tau \quad . \tag{14}$$

Equation (14) yields a representation of the output in terms of a system function that specifies the incremental time-delay and Doppler-shift characteristics of the medium. In this representation, the term

$$v(\lambda,\tau)\mathbf{x}(t-\tau)e^{i2\pi\lambda t}$$

corresponds to the medium delaying the input signal by τ seconds, shifting the input frequency spectrum by λ cps, and multiplying by the random complex gain $v(\lambda,\tau)$. Figure 2 summarizes the relationships between the four system functions.† In this figure, $(u \rightarrow v)$ symbolizes a Fourier transform from the u variable to the v variable.

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* The existence of $Z(f)$ is discussed in the following chapter.

† This diagrammatic representation is due to Bello [Ref. 11].

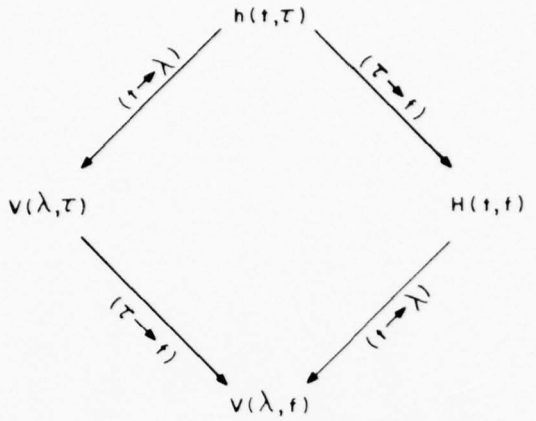

FIG. 2 SYSTEM FUNCTION RELATIONSHIPS

B. SYSTEM AUTOCORRELATION FUNCTIONS

Four system autocorrelation functions serve to characterize the second-order statistical properties of the system functions. We introduce these functions by simply computing the autocorrelation function associated with each system function:

$$E[H^*(t,f)H(t+\alpha,f+\beta)] = R_H(\alpha,\beta) \quad . \tag{15}$$

The function $R_H(\alpha,\beta)$ is the medium time-frequency autocorrelation function.

$$E[v^*(\lambda,\tau)v(\lambda+\nu,\tau+\xi)] = S_v(\lambda,\tau)\delta(\nu)\delta(\xi) \quad . \tag{16}$$

The function $S_v(\lambda,\tau)$ is the medium scattering function.

$$E[h^*(t,\tau)h(t+\alpha,\tau+\xi)] = R_h(\alpha,\tau)\delta(\xi) \quad . \tag{17}$$

The function $R_h(\alpha,\tau)$ is the delay-dependent time autocorrelation function of the medium.

$$E[V^*(\lambda,f)V(\lambda+\nu,f+\beta)] = R_V(\lambda,\beta)\delta(\nu) \quad . \tag{18}$$

The function $R_V(\lambda,\beta)$ is the Doppler-dependent frequency autocorrelation function of the medium.

The delta function $\delta(\xi)$ appears in Eqs. (16) and (17) because the time-variant transfer function $H(t,f)$ is wide-sense stationary in f. Similarly, the delta function $\delta(\nu)$ in Eqs. (16) and (18) is due to the wide-sense stationarity of $H(t,f)$ in t. The delta functions reflect the fact that the spectral components of a wide-sense stationary process possess the characteristics of nonstationary white noise [Eqs. (7a) and (7b)]. A corresponding set of Fourier transform relations exists between the system autocorrelation functions, similar to the relations that exist between the system functions. These relations are summarized in Fig. 3.

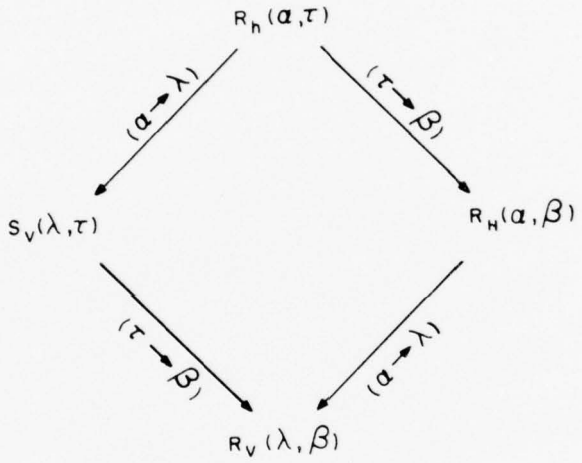

FIG. 3 SYSTEM AUTOCORRELATION FUNCTION RELATIONSHIPS

V GENERALIZED MEASURES OF THE SECOND-ORDER PROPERTIES OF AN ARBITRARY PROCESS

In many situations of interest to a communication engineer, the output of a dispersive medium is a nonstationary random process possessing finite energy. In this chapter, four useful functions are developed that serve to characterize the second-order properties of the time and frequency structures of signals of this type. These functions, although defined for nonstationary random processes, are meaningful when the process happens to be stationary or even deterministic. The usefulness of the measures developed in this section in analyzing a radar or communication system incorporating a scattering medium lies in the fact that the input and output measures are related simply and naturally through the system autocorrelation functions for a randomly time-varying linear filter. We shall find that in the second-order theory of randomly time-varying linear filters a mathematical structure exists that extends the very useful concepts that have been developed for deterministic time-invariant filters with stationary random inputs.

Let $\mathbf{z}(t)$ be a complex process, and define the time-autocorrelation function $R_{\mathbf{z}}(\alpha, t)$ as follows:

$$R_{\mathbf{z}}(\alpha, t) = E\left[\mathbf{z}^*\left(t - \frac{\alpha}{2}\right)\mathbf{z}\left(t + \frac{\alpha}{2}\right)\right] \quad . \tag{19}$$

Observe that $R_{\mathbf{z}}(\alpha, t)$ is Hermitian in α; that is,

$$R_{\mathbf{z}}^*(\alpha, t) = R_{\mathbf{z}}(-\alpha, t) \quad .$$

We shall call the Fourier transform of $R_{\mathbf{z}}(\alpha, t)$ with respect to α the energy density $S_{\mathbf{z}}(f, t)$,

$$S_{\mathbf{z}}(f, t) = \int R_{\mathbf{z}}(\alpha, t) e^{-i2\pi f\alpha} d\alpha \quad . \tag{20}$$

Since $R_{\mathbf{z}}(\alpha,t)$ is Hermitian in α, $S_{\mathbf{z}}(f,t)$ is a real function. [The energy density is nonnegative if and only if $R_{\mathbf{z}}(\alpha,t)$ is a positive definite function of α for all t.]

To analyze the frequency structure of the process $\mathbf{z}(t)$, it is convenient to deal with the Fourier transform

$$\mathbf{Z}(f) = \int \mathbf{z}(t)e^{-i2\pi ft}dt \quad . \tag{21}$$

If $\mathbf{z}(t)$ is a random process, then $\mathbf{Z}(f)$ is a random variable. It can be shown that the stochastic integral in Eq. (21) exists as a mean-square limit of Riemann sums if and only if

$$\int S_{\mathbf{z}}(f,t)dt < \infty \quad .$$

When $\mathbf{z}(t)$ is a stationary random process, this condition is not satisfied, since the frequency components of a stationary random process possess the characteristics of nonstationary white noise. If the process $\mathbf{z}(t)$ satisfies the condition

$$\int E[|\mathbf{z}(t)|]dt < \infty \quad ,$$

then Fubini's theorem implies that the integral defining $\mathbf{Z}(f)$ exists with probability one [Ref. 12].

The random process $\mathbf{Z}(f)$ possesses the frequency autocorrelation function $R_{\mathbf{Z}}(f,\beta)$, which is defined in a manner similar to the time autocorrelation function,

$$R_{\mathbf{Z}}(f,\beta) = E\left[\mathbf{Z}^*\left(f-\frac{\beta}{2}\right)\mathbf{Z}\left(f+\frac{\beta}{2}\right)\right] \quad . \tag{22}$$

The frequency autocorrelation function also possesses the Hermitian property

$$R^*_{\mathbf{Z}}(f,\beta) = R_{\mathbf{Z}}(f,-\beta) \quad .$$

Finally, we define the ambiguity function $\psi_{\mathbf{z}}(\alpha,\beta)$ as the Fourier transform of $R_{\mathbf{z}}(\alpha,t)$ with respect to t,

$$\psi_{\mathbf{z}}(\alpha,\beta) = \int R_{\mathbf{z}}(\alpha,t)e^{-i2\pi\beta t}dt \quad . \tag{23}$$

If $\mathbf{z}(t)$ is deterministic, then the above definition agrees with the usual definition of the ambiguity function [Refs. 13 and 14].

The Fourier transform relations between these second-order measures are displayed in Fig. 4, which is similar to the diagram used to display the relations between the system functions and the system autocorrelation functions.

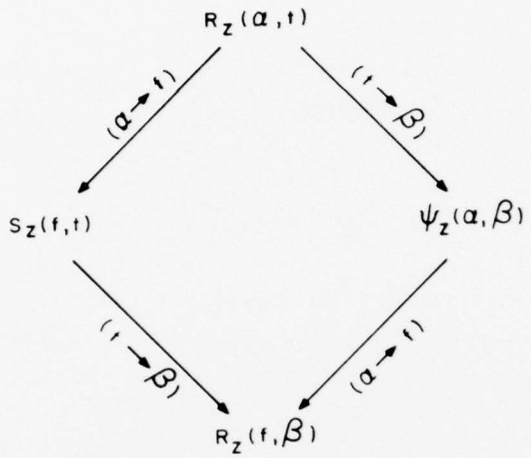

FIG. 4 TRANSFORM RELATIONSHIPS FOR THE MEASURES OF THE SECOND-ORDER PROPERTIES OF A PROCESS

The autocorrelation functions $R_z(\alpha, t)$ and $R_z(f, \beta)$ measure the correlation properties in the time and frequency domains, respectively, of the process $\mathbf{z}(t)$. Since these functions constitute a Fourier transform pair, the correlation properties in the time domain determine those in the frequency domain, and vice versa.

The ambiguity function $\psi_z(\alpha, \beta)$ has the interesting property of being the spectral decomposition of both the time variations of the time autocorrelation function and the frequency variations of the frequency autocorrelation function. For a fixed time difference α, $R_z(\alpha, t)$ is a function of time, and $\psi_z(\alpha, \beta)$, considered as a function of β, is a spectral decomposition of the time variations of $R_z(\alpha, t)$. Similarly, for a fixed frequency difference β, $R_z(f, \beta)$ is a function of frequency and $\psi_z(\alpha, \beta)$, considered as a function of α, is a spectral decomposition of the frequency variations of $R_z(f, \beta)$. It is also interesting to note that the dependence of the time autocorrelation function on the time

difference α is determined by the frequency variations of the frequency autocorrelation function, and the dependence of the frequency autocorrelation function on the frequency difference β is determined by the time variations of the time autocorrelation function.

The function $S_{\mathbf{z}}(f,t)$ possesses an energy density interpretation in the sense that the mean-square time envelope,

$$E[|\mathbf{z}(t)|^2] = R_{\mathbf{z}}(\alpha,t) = \int S_{\mathbf{z}}(f,t)df \quad ,$$

can be shown to be the average energy density in the time direction and the mean-square frequency envelope,

$$E[|\mathbf{Z}(f)|^2] = R_{\mathbf{Z}}(f,0) = \int S_{\mathbf{z}}(f,t)dt \quad ,$$

can be shown to be the average energy density in the frequency direction.

We demonstrate this property of $S_{\mathbf{z}}(\alpha,t)$ in the following considerations. Let $\mathbf{z}(t)$ be applied to a rectangular bandpass filter with height unity and cutoff points at the frequencies f_1 and f_2; then the mean value of the energy of the output $\mathbf{y}(t)$ can be written:

$$\begin{aligned} E\left[\int |\mathbf{y}(t)|^2 dt\right] &= E\left[\int |\mathbf{Y}(f)|^2 df\right] \\ &= E\left[\int_{f_1}^{f_2} |\mathbf{Z}(f)|^2 df\right] \\ &= \int_{f_1}^{f_2} R_{\mathbf{Z}}(f,0)\,df \\ &= \int_{f_1}^{f_2} \int S_{\mathbf{z}}(f,t)\,dt\,df \quad . \end{aligned}$$

Let $\mathbf{z}(t)$ be applied to a filter that multiplies the input function of time by unity if $t_1 \leq t \leq t_2$ and by zero for t otherwise; then the mean value of the energy of the output $\mathbf{y}(t)$ can be written:

$$E\left[\int |\mathbf{y}(t)|^2 dt\right] = E\left[\int_{t_1}^{t_2} |\mathbf{z}(t)|^2 dt\right]$$

$$= \int_{t_1}^{t_2} R_{\mathbf{z}}(0,t)\,dt$$

$$= \int_{t_1}^{t_2}\int S_{\mathbf{z}}(f,t)\,df\,dt$$

Thus, our interpretation of the function $S_{\mathbf{z}}(f,t)$ leads to a natural generalization of the spectrum concept employed in the theory of stationary random processes.

Moments of the energy density in the time and frequency directions are useful measures of the structure of the signal $\mathbf{z}(t)$. If we define the normalized time and frequency energy densities

$$p_T(t) = \frac{\int S_{\mathbf{z}}(f,t)\,df}{\iint S_{\mathbf{z}}(f,t)\,df\,dt} \tag{24a}$$

$$p_F(f) = \frac{\int S_{\mathbf{z}}(f,t)\,dt}{\iint S_{\mathbf{z}}(f,t)\,df\,dt} \tag{24b}$$

then the time and frequency moments can be defined

$$m_T^n(\mathbf{z}) = \int t^n p_T(t)\,dt = \frac{1}{(-2\pi i)^n}\,\frac{d^n}{d\beta^n}\,\frac{\psi_{\mathbf{z}}(0,\beta)}{\psi_{\mathbf{z}}(0,0)}\Bigg]_{\beta=0} \tag{25a}$$

$$m_F^n(\mathbf{z}) = \int f^n p_F(f)\,df = \frac{1}{(2\pi i)^n}\,\frac{d^n}{d\beta^n}\,\frac{\psi_{\mathbf{z}}(\alpha,0)}{\psi_{\mathbf{z}}(0,0)}\Bigg]_{\alpha=0} \tag{25b}$$

The mean signal time and the mean signal frequency of the process $\mathbf{z}(t)$ are given by the quantities $m_T^1(\mathbf{z})$ and $m_F^1(\mathbf{z})$, respectively. The quantity

$$\{m_T^2(\mathbf{z}) - [m_T^1(\mathbf{z})]^2\}^{1/2}$$

is a measure of the rms time dispersion of the process $\mathbf{z}(t)$, and the quantity

$$\{m_F^2(\mathbf{z}) - [m_F^1(\mathbf{z})]^2\}^{1/2}$$

is a measure of the rms frequency dispersion of the process $\mathbf{z}(t)$.

If $\mathbf{z}(t)$ is a wide-sense stationary random process, then the four signal measures can be written

$$R_{\mathbf{z}}(\alpha, t) = R_{\mathbf{z}}(\alpha) \tag{26a}$$

$$\psi_{\mathbf{z}}(\alpha, \beta) = R_{\mathbf{z}}(\alpha)\delta(\beta) \tag{26b}$$

$$R_{\mathbf{Z}}(f, \beta) = S_{\mathbf{z}}(f)\delta(\beta) \tag{26c}$$

$$S_{\mathbf{z}}(f, t) = S_{\mathbf{z}}(f) \quad , \tag{26d}$$

where $R_{\mathbf{z}}(\alpha)$ and $S_{\mathbf{z}}(f)$ are the usual autocorrelation function and power spectrum associated with a wide-sense stationary process. Note that Eq. (26c) states that the frequency components of a wide-sense stationary random process possess the characteristics of nonstationary white noise. Equation (26b) illustrates the tendency of the ambiguity function to display both the time and frequency correlation properties of the process $\mathbf{z}(t)$. When $\mathbf{z}(t)$ is wide-sense stationary, $\psi_{\mathbf{z}}(\alpha, \beta)$ is the product of a time autocorrelation function $R_{\mathbf{z}}(\alpha)$ and a frequency autocorrelation $\delta(\beta)$. If $\mathbf{z}(t)$ is wide-sense stationary white noise, then the ambiguity function becomes the product of delta functions, $\delta(\alpha)\delta(\beta)$, which indicates the independence of stationary white noise in both the time direction and the frequency direction. Thus, the width of the ambiguity function in the α direction is a measure of the time interval in which the process remains correlated, and the width of the ambiguity function in the β direction is a measure of the frequency interval in which the frequency components remain correlated.

VI INPUT-OUTPUT TRANSFORMATIONS OF SECOND-ORDER MEASURES FOR A RANDOMLY TIME-VARYING LINEAR FILTER

Let $\mathbf{x}(t)$ be the input complex envelope to a randomly time-varying linear filter with transfer function $H(t, f)$. We allow $\mathbf{x}(t)$ to be deterministic or random; if $\mathbf{x}(t)$ is random, then it is assumed to be independent of $H(t, f)$. When $\mathbf{z}(t)$ is the output complex envelope, the second-order measures associated with $\mathbf{z}(t)$ are related to the second-order measures associated with $\mathbf{x}(t)$ through the system autocorrelation functions. These relationships are of interest because they yield a simple means of determining the nature of the time and frequency dispersion suffered by signals transmitted over a dispersive medium.

The manner in which the medium transforms an input second-order measure into an output second-order measure is determined by the following fundamental relations:

$$R_{\mathbf{z}}(\alpha, t) = \int R_h(\alpha, \tau) R_{\mathbf{x}}(\alpha, t - \tau) d\tau \tag{27a}$$

$$\psi_{\mathbf{z}}(\alpha, \beta) = R_H(\alpha, \beta) \psi_{\mathbf{x}}(\alpha, \beta) \tag{27b}$$

$$R_{\mathbf{Z}}(f, \beta) = \int R_V(\lambda, \beta) R_{\mathbf{X}}(f - \lambda, \beta) d\lambda \tag{27c}$$

$$S_{\mathbf{z}}(f, t) = \iint S_v(\lambda, \tau) S_{\mathbf{x}}(f - \lambda, t - \tau) d\lambda d\tau \quad . \tag{27d}$$

Equation (27a) states that the output time autocorrelation function is the result of convolving the time variations of the input time autocorrelation function with the delay variations of the medium time autocorrelation function. Similarly, Eq. (27c) states that the output frequency autocorrelation function is the result of convolving the frequency variations of the input frequency autocorrelation function with the Doppler variations of the medium frequency autocorrelation function.

In Eq. (27b), the output ambiguity function is given as the product of the input ambiguity function and the medium time-frequency autocorrelation function. This relation yields an immediate description of the

dispersive effects of the medium on the second-order properties of input signals. For example, if the medium time-frequency autocorrelation function is essentially constant over the region in the $\alpha - \beta$ plane occupied by the input ambiguity function, then the medium is time-flat and frequency-flat relative to the input modulation. That is, the medium essentially preserves the time and frequency structures of the input signal. On the other hand, if the medium time-frequency autocorrelation function occupies a very narrow region in the $\alpha - \beta$ plane compared to the region occupied by the input ambiguity function, then the medium drastically alters the time and frequency structures of the input signal, to the extent that the structural characteristics of the output signal are determined by the medium rather than by the input signal. Thus, if it is desirable to maintain at the output of the medium the time and frequency structures possessed by the input signal, one should use input ambiguity functions that are narrow compared to the medium time-frequency autocorrelation function. However, if one wishes to measure the medium characteristics, it is desirable to use input ambiguity functions that are wide compared to the medium time-frequency autocorrelation. Similar conclusions can be drawn from Eq. (27d), which states that the output energy density is the result of convolving the input energy density with the medium scattering function. Observe that, if the input energy density occupies a very narrow region in the time-frequency plane relative to the region occupied by the medium scattering function in the Doppler-delay plane, then the output energy density is essentially determined by the medium scattering function. When the medium scattering function is very narrow compared to the input energy density, the output energy density is essentially determined by the input energy density.

A relationship that describes the output energy density in the time direction and a relationship that describes the output energy density in the frequency direction can be deduced from Eq. (27d). The output energy density in the time direction is given by the mean-square time envelope,

$$E[|\mathbf{z}(t)|^2] = \int S_z(f, t)\,dt \quad ,$$

which may be written

$$E[|\mathbf{z}(t)|^2] = \iiint S_v(\lambda, \tau) S_{\mathbf{x}}(f - \lambda, t - \tau)\,d\lambda\,d\tau\,df \quad . \qquad (28)$$

In Eq. (28) the integration with respect to f yields the term

$$\int S_{\mathbf{x}}(f-\lambda, t-\tau)\,df \;=\; \int S_{\mathbf{x}}(f, t-\tau)\,df \;=\; R_{\mathbf{x}}(0, t-\tau) \;=\; E[\,|\mathbf{x}(t-\tau)|^2\,] \quad .$$

The function

$$\mathrm{T}(\tau) \;=\; \int S_v(\lambda, \tau)\,d\lambda \tag{29}$$

will be called the medium delay profile, since it yields the density of power scattered by the medium as a function of time delay. We perform the integration with respect to f and λ in Eq. (28) and obtain

$$E[\,|\mathbf{z}(t)|^2\,] \;=\; \int \mathrm{T}(\tau) E[\,|\mathbf{x}(t-\tau)|^2\,]\,d\tau \quad . \tag{30}$$

Equation (30) states that the output mean-square time envelope is obtained by convolving the input mean-square time envelope with the medium delay profile. A similar expression for the output energy density in the frequency direction can be obtained from the mean-square frequency envelope,

$$E[\,|\mathbf{Z}(f)|^2\,] \;=\; \int S_{\mathbf{z}}(f, t)\,dt \quad .$$

The function

$$\Lambda(\lambda) \;=\; \int S_v(\lambda, \tau)\,d\tau \tag{31}$$

will be called the medium Doppler profile, since it yields the density of power scattered by the medium as a function of Doppler shift. One can write the output mean-square frequency envelope as follows:

$$E[\,|\mathbf{X}(f)|^2\,] \;=\; \int \Lambda(\lambda) E[\,|\mathbf{X}(f-\lambda)|^2\,]\,d\lambda \quad . \tag{32}$$

Equation (32) states that the output mean-square frequency envelope is the result of convolving the input mean-square frequency envelope with the medium Doppler profile.

The Doppler profile is the power spectrum of the wide-sense stationary time variations of the time-variant transfer function,

$$E[H^*(t, f) H(t+\alpha, f)] \;=\; R_H(\alpha, 0)$$

$$\Lambda(\lambda) \;=\; \int R_H(\alpha, 0)\, e^{-i2\pi\lambda\alpha}\,d\alpha \quad .$$

Hence the power spectrum of the time variations of the medium response to a CW tone at a fixed frequency is given by the Doppler profile.

The delay profile is the power spectrum of the wide-sense stationary frequency variations of the transfer function,

$$E[H^*(t,f)H(t,f+\beta)] = R_H(0,\beta)$$

$$\mathrm{T}(\tau) = \int R_H(0,\beta)\, e^{-i2\pi\tau\beta}\, d\beta \quad .$$

Hence the delay profile yields the power spectrum of the variations of the response of the medium at a fixed time as the input frequency of the CW tone is varied.

By employing the following normalized delay and Doppler profiles,

$$q_{\mathrm{T}}(\tau) = \frac{\mathrm{T}(\tau)}{\int \mathrm{T}(\tau)\, d\tau} \tag{33a}$$

$$q_{\Lambda}(\lambda) = \frac{\Lambda(\lambda)}{\int \Lambda(\lambda)\, d\lambda} \quad , \tag{33b}$$

we obtain moments of the medium scattering function

$$m_{\mathrm{T}}^n(H) = \int \tau^n q_{\mathrm{T}}(\tau)\, d\tau = \frac{1}{(-2\pi i)^n} \frac{d^n}{d\beta^n} \frac{R_H(0,\beta)}{R_H(0,0)}\Bigg]_{\beta=0} \tag{34a}$$

$$m_{\Lambda}^n(H) = \int \lambda^n q_{\Lambda}(\lambda)\, d\lambda = \frac{1}{(2\pi i)^n} \frac{d^n}{d\alpha^n} \frac{R_H(\alpha,0)}{R_H(0,0)}\Bigg]_{\alpha=0} \quad . \tag{34b}$$

Mean-square time and frequency dispersions of the output process $\mathbf{z}(t)$ can be computed by employing Eqs. (25a) and (25b):

$$m_T^2(\mathbf{z}) - [m_T^1(\mathbf{z})]^2 = m_{\mathrm{T}}^2(H) - [m_{\mathrm{T}}^1(H)]^2 + m_T^2(\mathbf{x}) - [m_T^1(\mathbf{x})]^2 \tag{35a}$$

$$m_F^2(\mathbf{z}) - [m_F^1(\mathbf{z})]^2 = m_{\Lambda}^2(H) - [m_{\Lambda}^1(H)]^2 + m_F^2(\mathbf{x}) - [m_F^1(\mathbf{x})]^2 \quad . \tag{36b}$$

These equations state that the output mean-square time (frequency) dispersion is the sum of the input mean-square time (frequency) dispersion and the mean-square time-delay (Doppler) spread of the medium.

Figure 5 summarizes the relationships existing among the input second-order measures, the system autocorrelation functions, and the output second-order measures.

In this figure, input measures appear on the inner ring, medium measures on the middle ring, and output measures on the outer ring. The symbols *, **, and ·, which denote single convolution, double convolution, and multiplication, respectively, describe how quantities on the outer ring are obtained from the corresponding quantities on the middle and inner rings.

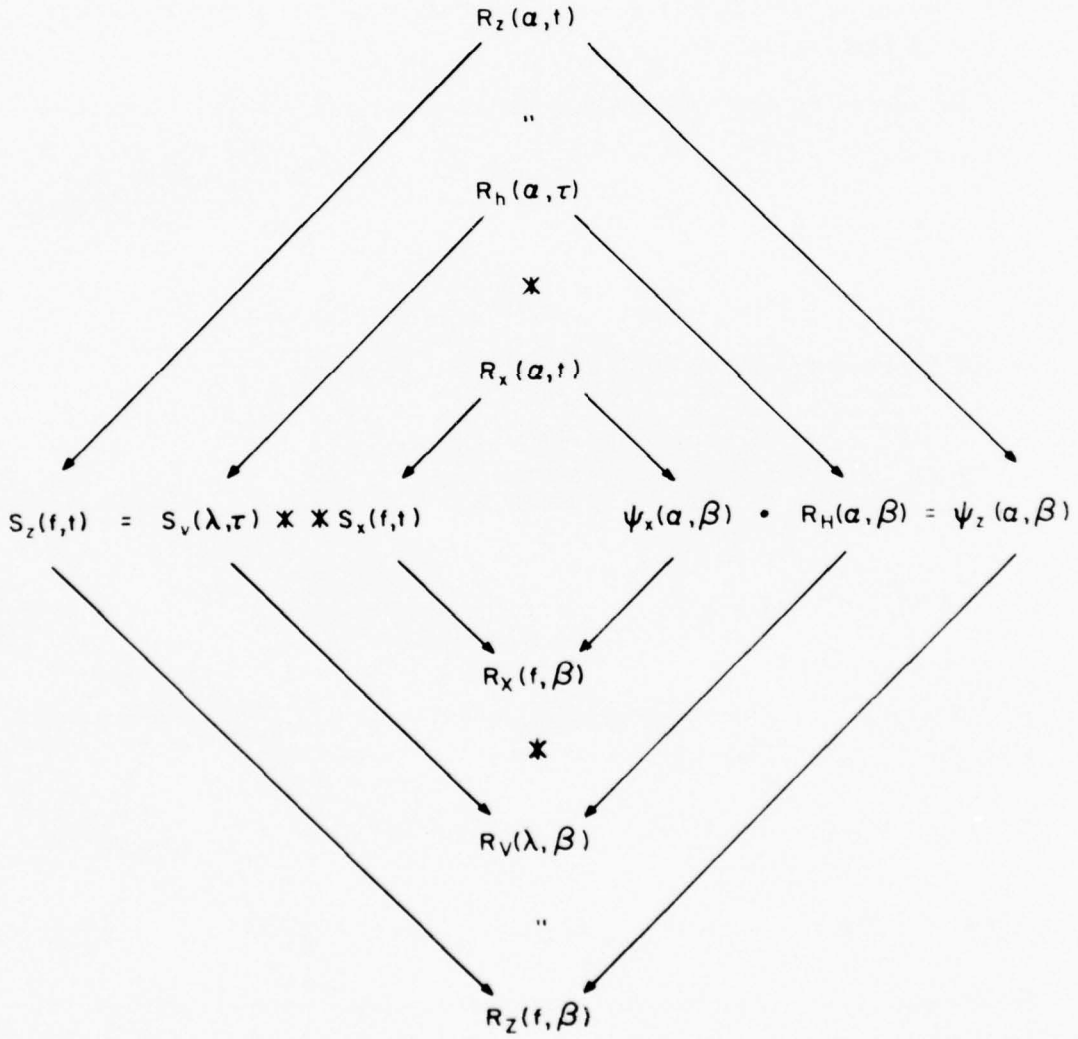

FIG. 5 SUMMARY OF INPUT, MEDIUM, AND OUTPUT SECOND-ORDER RELATIONSHIPS

VII ANALYSIS OF DETECTION PROBLEM

A. DISCUSSION

On the basis of observing the process $\mathbf{y}(t)$ in the time interval $(0,T)$, the receiver must choose between the hypotheses:

$$H_0: \quad \mathbf{y}(t) = \mathbf{n}(t), \qquad \text{noise alone}$$

$$H_1: \quad \mathbf{y}(t) = \mathbf{n}(t) + \mathbf{z}(t), \qquad \text{signal plus noise.}$$

In other words, the task of the receiver is to decide whether the probability measure governing the observed process $\mathbf{y}(t)$ is the noise probability measure or the signal-plus-noise probability measure. The following assumptions define the two probability measures:

(1) The medium transfer function is a complex, homogeneous, Gaussian random field with zero mean and time-frequency autocorrelation function $R_H(\alpha,\beta)$.

(2) The complex envelope of the transmitted signal $\mathbf{x}(t)$ is a known function of time.

(3) The noise complex envelope $\mathbf{n}(t)$ is a complex, stationary, Gaussian random process with zero mean and a white-noise spectrum of intensity N_0 and is independent of the random field $H(t,f)$. (See Appendix A.)

These assumptions imply that when $\mathbf{z}(t)$ is present $\mathbf{y}(t)$ is, in general, a complex, nonstationary, Gaussian random process with zero mean and covariance function that depends on the transmitted signal and the time-frequency autocorrelation function $R_H(\alpha,\beta)$. When $\mathbf{z}(t)$ is not present, the third assumption implies that, in the interval $(0,T)$, $\mathbf{y}(t)$ is equal to complex, stationary, Gaussian noise with zero mean and a white-noise spectrum of intensity N_0.

Since the probability measures associated with the signal-plus-noise and noise hypotheses are completely determined by their respective covariance functions, it can be seen that the receiver's task of signal detection is in essence one of determining which covariance function governs the process $\mathbf{y}(t)$. In view of the dependence of the signal-plus-noise covariance

function on an interaction between the transmitted signal and the dispersive characteristics of the medium, it is possible that a judicious choice of input signal will produce a signal-plas-noise covariance function that in some sense yields the most efficient alternative to the noise covariance function. Cast in more physical terms, this statement raises the questions of the existence of a preferred energy distribution in the time-frequency plane at the output of the medium and—if it exists—how it can be achieved through proper design of the transmitted signal.

We assume that the receiver is an optimum Neyman-Pearson detector, that is, that the detection procedure employed by the receiver yields the smallest false-dismissal probability for a given false-alarm probability [Ref. 15]. By the fundamental lemma of Neyman and Pearson [Ref. 16], the receiver can realize this optimum detection procedure by computing any monotone increasing function of the likelihood ratio. This results in a real-valued random variable, called the test statistic, defined on the space of all possible observations $\{\mathbf{y}(t)\}$. Hence the receiver processes a particular observation $\mathbf{y}(t)$ and summarizes this observation with a real number that is sufficient to effect the optimum detection procedure. For a given false-alarm probability α, a threshold L is determined that is exceeded by the test statistic with probability equal to α when noise alone is present. The receiver then decides that the signal $\mathbf{z}(t)$ is present when the test statistic is greater than L and is not present when the test statistic is less than L. This procedure yields the smallest obtainable false-dismissal probability among all the detection procedures that have false-alarm probability equal to α.

B. DERIVATION OF THE TEST STATISTIC

In order to calculate the likelihood ratio, we proceed in the usual manner and treat the Karhunen-Loève expansion of the process $\mathbf{y}(t)$ in the interval $(0,T)$. Since the book by Helstrom [Ref. 17] contains an excellent Karhunen-Loève exposition for the detection problem considered in this study, it is sufficient to present only an outline of the main results of Helstrom's treatment and to briefly discuss and clarify points that are pertinent to the present discussion. In this manner, we maintain continuity, introduce notation, and point out various important facts and relations used in later chapters.

Let the covariance functions of the various zero-mean, complex Gaussian processes involved in the detection problem be:

$$K_{\mathbf{z}}(t,s) = E[\mathbf{z}^*(t)\mathbf{z}(s)]$$

$$K_{\mathbf{n}}(t,s) = E[\mathbf{n}^*(t)\mathbf{n}(s)]$$

$$K_{\mathbf{y}}(t,s) = E[\mathbf{y}^*(t)\mathbf{y}(s)] = K_{\mathbf{z}}(t,s) + K_{\mathbf{n}}(t,s) \quad .$$

The Karhunen-Loève expansion of the process $\mathbf{y}(t)$ in the interval $(0,T)$ generates the process $\hat{\mathbf{y}}(t)$ [the difference between $\hat{\mathbf{y}}(t)$ and $\mathbf{y}(t)$ is discussed in Appendix B], which possesses the orthonormal series expansion

$$\hat{\mathbf{y}}(t) = \sum_n \mathbf{y}_n \mathbf{f}_n(t) \quad , \tag{36a}$$

where

$$\mathbf{y}_n = \int_0^T \mathbf{f}_n^*(t)\mathbf{y}(t)dt \quad . \tag{36b}$$

If the functions $\{\mathbf{f}_n(t)\}$ are the eigenfunctions of the integral equation

$$\int_0^T \mathbf{f}_n(t)K_{\mathbf{z}}(t,s)dt = \lambda_n \mathbf{f}_n(s) \quad , \quad s\in(0,T) \quad , \tag{37a}$$

then

$$\int_0^T \mathbf{f}_n^*(t)\mathbf{f}_m(t)dt = \delta_{nm} \tag{37b}$$

and

$$E[\mathbf{y}_n^*\mathbf{y}_m] = (\lambda_n + N_0)\delta_{nm} \quad , \tag{37c}$$

where δ_{nm} is the Kronecker delta function. Since the covariance function $K_{\mathbf{z}}(t,s)$ is Hermitian and nonnegative definite, the eigenvalues $\{\lambda_n\}$ are real and nonnegative and can be arranged in descending order,

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0 \quad .$$

The assumption that led to the relations involving the real and imaginary parts of the medium time-frequency autocorrelation function

[Eqs. (9a) and (9b)] lead to similar relations for the real and imaginary parts of the output covariance function $K_{\mathbf{z}}(t,s)$. Let

$$\mathbf{z}(t) = z_R(t) + iz_I(t) \quad .$$

If the Doppler-delay random field $v(\lambda,\tau)$ satisfies the orthogonality condition of Eq. (7b) by possessing quadrature components that form a mutually independent family of zero-mean random variables and if both quadrature components at a given λ and τ are identically distributed, then (see Appendix A)

$$K_{z_R z_R}(t,s) = K_{z_I z_I}(t,s) = \frac{1}{2}\,\mathrm{Re}\{K_{\mathbf{z}}(t,s)\} \tag{38a}$$

$$K_{z_R z_I}(t,s) = -K_{z_I z_R}(t,s) = \frac{1}{2}\,\mathrm{Im}\{K_{\mathbf{z}}(t,s)\} \quad . \tag{38b}$$

The covariance and cross-covariance functions of the quadrature components of the complex envelope of a real narrow-band *stationary* process must also satisfy the conditions of Eqs. (38a) and (38b)(Appendix A). Since the noise is stationary, $\mathbf{y}(t)$ is the sum of two independent processes satisfying these conditions, which implies that, if

$$\mathbf{y}(t) = y_R(t) + iy_I(t) \quad ,$$

then

$$K_{y_R y_R}(t,s) = K_{y_I y_I}(t,s) = \frac{1}{2}\,\mathrm{Re}\{K_{\mathbf{y}}(t,s)\} \tag{39a}$$

$$K_{y_R y_I}(t,s) = -K_{y_I y_R}(t,s) = \frac{1}{2}\,\mathrm{Im}\{K_{\mathbf{y}}(t,s)\} \quad . \tag{39b}$$

One can verify that the conditions in Eqs. (39a) and (39b) cause the expectation $E[\mathbf{y}(t)\mathbf{y}(s)]$ to be zero for all t and s. Since

$$E[\mathbf{y}_n\mathbf{y}_m] = \int_0^T \int_0^T \mathbf{f}_n^*(t)\mathbf{f}_m^*(s)E[\mathbf{y}(t)\mathbf{y}(s)]\,dt\,ds \quad ,$$

the expectation $E[\mathbf{y}_n\mathbf{y}_m]$ is identically zero for all n and m. The two relations

$$E[\mathbf{y}_n^*\mathbf{y}_m] = (\lambda_n + N_0)\delta_{nm}$$

and

$$E[\mathbf{y}_n \mathbf{y}_m] = 0$$

imply the following conditions on the covariances and cross covariances of the quadrature components of the $\mathbf{y}_n$:

$$E[y_{nR} y_{mR}] = E[y_{nI} y_{mI}] = \frac{1}{2}(\lambda_n + N_0)\delta_{nm} \tag{40a}$$

$$E[y_{nR} y_{mI}] = E[y_{nI} y_{mR}] = 0, \qquad \text{all } m \text{ and } n \tag{40b}$$

where

$$\mathbf{y}_n = y_{nR} + i y_{nI} \quad .$$

The above relations [Eqs. (40a) and (40b)] apply for the signal-plus-noise hypothesis; the corresponding relations for the noise hypothesis are obtained by setting λ_n to zero.

We can now conclude that the quadrature components of the $\mathbf{y}_n$ form a mutually independent family of zero-mean Gaussian random variables with variance $(\lambda_n + N_0)/2$ under hypothesis H_1 and variance $N_0/2$ under hypothesis H_0.

The likelihood ratio $l(\mathbf{y})$ can be formally obtained by forming the limiting ratio of the probability measures on a sequence of the $\mathbf{y}_n$ under the two hypotheses,

$$l(\mathbf{y}) = \lim_{n \to \infty} \frac{f(\mathbf{y}_1, \ldots, \mathbf{y}_n | H_1)}{f(\mathbf{y}_1, \ldots, \mathbf{y}_n | H_0)}$$

$$l(\mathbf{y}) = \prod_{m=1}^{\infty} \frac{1}{1 + \frac{\lambda_m}{N_0}} \exp\left\{ \sum_{n=1}^{\infty} \frac{\lambda_n |\mathbf{y}_n|^2}{N_0(\lambda_n + N_0)} \right\} \quad . \tag{41}$$

Equation (41) implies that a suitable test statistic has the form

$$S = \sum_{n=1}^{\infty} \frac{\lambda_n |\mathbf{y}_n|^2}{\lambda_n + N_0} \quad . \tag{42}$$

If $g(s,t)$ is the solution of the integral equation

$$g(r,t) + \frac{1}{N_0}\int_0^T K_{\mathbf{z}}(r,s)g(s,t)ds = K_{\mathbf{z}}(r,t) \quad , \tag{43}$$

then $g(r,t)$ possesses the expansion

$$g(r,t) = \sum_{n=1}^{\infty} \frac{N_0\lambda_n}{\lambda_n + N_0}\,\mathbf{f}_n^*(r)\mathbf{f}_n(t) \tag{44}$$

and the summation for S in Eq. (42) can be obtained from a term-by-term integration of the expression [Ref. 17]

$$S = \int_0^T\int_0^T \mathbf{y}(s)g(s,t)\mathbf{y}^*(t)dsdt \quad . \tag{45}$$

The optimum receiver realizes the quadratic-form processing indicated in Eq. (45). Various optimum receiver structures have been deduced by Price [Ref. 18], Middleton [Ref. 19], and Kailath (Refs. 20, 21, and 22].

C. CHARACTERISTIC FUNCTION OF THE TEST STATISTIC

If we set

$$\frac{\lambda_n|\mathbf{y}_n|^2}{\lambda_n + N_0} = w_n$$

in Eq. (42), then

$$S = \sum_n w_n \quad , \tag{46}$$

and the characteristic function of S under the hypothesis $H_j, j = 0,1$, is given by

$$\Phi_S(u|H_j) = \prod_n \Phi_{w_n}(u|H_j) = \prod_n E[e^{\imath u w_n}|H_j] \quad . \tag{47}$$

One can easily determine that

$$E[e^{\imath u w_n}|H_1] = \frac{1}{1 - \imath\lambda_n u} \tag{48a}$$

and

$$E[e^{iuw_n}|H_0] = \frac{1}{1 - i\dfrac{N_0}{\lambda_n + N_0}\lambda_n u} \quad ; \tag{48b}$$

hence,

$$\Phi_S(u|H_1) = \prod_n \frac{1}{1 - i\lambda_n u} \quad , \tag{49a}$$

and

$$\Phi_S(u|H_0) = \prod_n \frac{1}{1 - i\dfrac{N_0}{\lambda_n + N_0}\lambda_n u} \quad . \tag{49b}$$

We obtain the corresponding probability densities by the method of residues,

$$f_S(s|H_1) = \sum_n \exp\left(-\frac{s}{\lambda_n}\right) \prod_{k \neq n} \left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1} \quad , \tag{50a}$$

$$f_S(s|H_0) = \sum_n \exp\left\{-\frac{\lambda_n + N_0}{N_0\lambda_n} s\right\} \prod_{k \neq n} \left(1 - \frac{\lambda_k + N_0}{\lambda_n + N_0}\frac{\lambda_n}{\lambda_k}\right)^{-1} \quad . \tag{50b}$$†

By using Eqs. (50a) and (50b), an interesting relation between the two probability densities can be verified. This relation involves the Fredholm determinant [Ref. 24],

$$D(u) = \prod_{n=1}^{\infty} (1 + \lambda_n u) \quad , \tag{51}$$

and assumes the following simple form:

$$f_S(s|H_0) = D\left(\frac{1}{N_0}\right) e^{-s/N_0} f_S(s|H_1) \quad . \tag{52}$$

† Grenander, *et al.* [Ref. 23] discuss a numerical technique for approximating these probability laws.

Important properties of the statistic S are deduced from the logarithm of the characteristic function.

If the Nth moment $E[S^N|H_j]$ exists, then $\log \phi_S(u|H_j)$ can be expanded in terms of the first N derivatives [Ref. 25],

$$\log \phi_S(u|H_j) = \sum_{n=1}^{N} C_n[S|H_j] \frac{(iu)^n}{n!} + R_N(u) \quad , \tag{53}$$

where

$$\lim_{|u| \to 0} |u|^n R_n(u) = 0 \quad ,$$

and the cumulants $C_n[S|H_j]$ are defined by

$$C_n[S|H_j] = \frac{1}{i^n} \frac{d^n}{du^n} \log \phi_S(u|H_j) \quad . \tag{54}$$

For the statistic S, the cumulants assume a particularly symmetric form,

$$C_n[S|H_1] = \sum_{m=1}^{\infty} \lambda_m^n \tag{55}$$

$$C_n[S|H_0] = \sum_{m=1}^{\infty} \left(\frac{N_0 \lambda_m}{N_0 + \lambda_m} \right)^n \quad . \tag{56}$$

If the covariance function $K_z(s,t)$ is continuous, then by Mercer's theorem [Ref. 26],

$$K_z(s,t) = \sum_n \lambda_n \mathbf{f}_n^*(t) \mathbf{f}_n(s) \quad . \tag{57}$$

This series expansion for the kernel $K_z(s,t)$ implies that the cumulants $C_n[S|H_1]$ result from the n-fold integration

$$\sum_m \lambda_m^n = \int_0^T \cdots \int_0^T K_z(t_n, t_1) \prod_{j=1}^{n-1} K_z(t_j, t_{j+1}) dt_1 \cdots dt_n \quad . \tag{58}$$

Similarly, the series expansion of Eq. (44) for the kernel $g(s,t)$ implies that the cumulants $C_n[S|H_0]$ result from the n-fold integration

$$\sum_m \left(\frac{N_0 \lambda_m}{N_0 + \lambda_m} \right)^n = \int_0^T \cdots \int_0^T g(t_n, t_1) \prod_{j=1}^{n-1} g(t_j, t_{j+1}) dt_1 \cdots dt_n \quad . \tag{59}$$

The mean and variance of S are obtained from the first two cumulants,

$$E[S|H_1] = \sum_n \lambda_n = \int_0^T K_z(t,t) dt \tag{60a}$$

$$\mathrm{Var}[S|H_1] = \sum_n \lambda_n^2 = \int_0^T \int_0^T |K_z(s,t)|^2 ds dt \tag{60b}$$

$$E[S|H_0] = \sum_n \frac{N_0 \lambda_n}{N_0 + \lambda_n} = \int_0^T g(t,t) dt \tag{61a}$$

$$\mathrm{Var}[S|H_0] = \sum_n \left(\frac{N_0 \lambda_n}{N_0 + \lambda_n} \right)^2 = \int_0^T \int_0^T |g(s,t)|^2 ds dt \quad . \tag{61b}$$

The statistic S is a random variable that can be obtained as the limit of a sequence of random variables, W_1, W_2, ...,

$$S = \lim_{N \to \infty} W_N \quad , \tag{62a}$$

where

$$W_N = \sum_{n=1}^{N} w_n \quad . \tag{62b}$$

The W_N will converge in mean square [Ref. 27] (and hence in probability) to the random variable S, if

$$\lim_{n \to \infty} E[(S - W_n)^2] = 0 \quad ,$$

where

$$E[(S - W_N)^2] = \sum_{n=N+1}^{\infty} E[w_n^2] \quad .$$

Since

$$E[w_n^2 | H_1] = 2\lambda_n^2$$

and

$$E[w_n^2 | H_0] = 2\left(\frac{N_0\lambda_n}{N_0 + \lambda_n}\right)^2 ,$$

mean-square convergence of the W_N to S under both hypotheses follows as a result of the condition

$$\sum_n \lambda_n^2 < \infty .$$

However, the above condition is implied by the continuity of $K_z(s,t)$ on the square $s\in[0,T]$, $t\in[0,T]$, since $|K_z(s,t)|$ must then be bounded on the square and

$$\sum_n \lambda_n^2 = \int_0^T\int_0^T |K_z(s,t)|^2 ds dt < \infty .$$

In Ref. 28, Loève proves that convergence in probability is equivalent to the stronger mode of convergence, convergence with probability one, when the random variables w_n are independent. Hence, the W_N converge to S with probability one when the kernel $K_z(s,t)$ is continuous.

VIII ASYMPTOTIC RELATIVE EFFICIENCY OF RADAR SIGNALS

The receiver treated in this study is a Neyman-Pearson detector. For a given false-alarm probability α, the receiver achieves the lowest possible false-dismissal probability by comparing the statistic S with a proper threshold. Since the probability law of the random variable S under either hypothesis depends on the eigenvalues of the covariance kernel $K_z(t,s)$, which, in turn, is determined by an interaction between the input signal and the dispersive medium, the resulting false-dismissal probability will, in general, depend on the structure of the input signal.

In this chapter, we introduce the problem of designing an input signal that optimizes the efficiency of the Neyman-Pearson detection procedure employed by the receiver.

The signal-design criterion discussed in this study is based on the asymptotic behavior of the false-dismissal probability when the decision-making element of the Neyman-Pearson receiver is presented with a sequence of independent observations of the statistic S. All members of the sequence are either samples from the noise hypothesis H_0 or samples from the signal-plus-noise hypothesis H_1. The posing of the signal-design aspects of the present problem in the realm of large sample theory is justifiable not only because of important radar applications in which the transmitted signal is repeated several times and communication applications involving the reception of several diversity transmissions but also because an important underlying principle of signal design can be evolved with tractable mathematics and well-known results. Although the "single-sample" receiver differs markedly from the "multiple-sample" receiver, we shall show that the large sample theory yields an important result that is interpretable as a measure of the efficiency per observation presented to the receiver. This measure serves as a reasonable suboptimum criterion for comparing different transmissions in lieu of the presently intractable minimum error-probability criterion for the single-sample receiver.

Consider the following sequence of tests that can be performed by the Neyman-Pearson detector. For each $n = 1, 2, \ldots$, the detector must choose between H_1 or H_0 on the basis of n independent observations, $S_1, S_2, \ldots, S_n$, of the statistic S. We conclude from Eq. (52) that the likelihood ratio for the nth test can be written

$$\prod_{j=1}^{n} \frac{f_S(S_j|H_1)}{f_S(S_j|H_0)} = \frac{1}{D^n\left(\frac{1}{N_0}\right)} \exp\left(\frac{1}{N_0} \sum_{j=1}^{n} S_j\right) . \tag{63}$$

Therefore,

$$\delta_n = \frac{1}{n} \sum_{j=1}^{n} S_j$$

is a sufficient statistic. Let $L_1, L_2, \ldots,$ be a sequence of thresholds satisfying

$$P[\delta_n \geq L_n|H_0] = \alpha .$$

In this manner, we generate a sequence of tests possessing false-alarm probability equal to α. The false-dismissal probability of the nth test will be denoted by β_n:

$$\beta_n = P[\delta_n < L_n|H_1] .$$

Various investigators have studied the asymptotic behavior of β_n [Refs. 29, 30, and 31]. Kullback [Ref. 31] presents a simple derivation for the following theorem which Chernoff [Ref. 29] attributes to unpublished work of C. Stein.

Theorem:

For any value of α, $0 < \alpha < 1$,

$$\lim_{n\to\infty} (\beta_n)^{1/n} = e^{-I(H_0:H_1)} . \tag{64}$$

where $I(H_0:H_1)$ is the Kullback-Leibler information number,

$$I(H_0:H_1) = -\int f_S(s|H_0) \log \frac{f_S(s|H_1)}{f_S(s|H_0)}\, ds . \tag{65}$$

Thus the number $I(H_0:H_1)$ measures the exponential rate at which the false-dismissal probability approaches zero with an increasing number of observations. When n is large, maximizing $I(H_0:H_1)$ corresponds to minimizing the false-dismissal probability. For this reason, we investigate the properties of a transmission $\mathbf{x}(t)$ that maximizes the associated Kullback-Leibler information number $I(H_0:H_1;\mathbf{x})$.

The above result can also be interpreted in terms of the asymptotic relative efficiency of two competitive choices, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, for the transmitted signal. For each input signal, we construct a sequence of tests possessing false-alarm probability α. There results a sequence of false-dismissal probabilities $\{\beta_{n_1}\}$ for input signal $x_1(t)$ and a sequence of false-dismissal probabilities $\{\beta_{n_2}\}$ for input signal $\mathbf{x}_2(t)$. If two sequences of integers, $\{n_1\}$ and $\{n_2\}$, can be found so that $\beta_{n_1} = \beta_{n_2}$, then it follows from Eq. (64) that

$$\lim_{\substack{n_1\to\infty \\ n_2\to\infty}} \frac{n_1}{n_2} = \frac{I(H_0:H_1;\mathbf{x}_2)}{I(H_0:H_1;\mathbf{x}_1)} \quad . \tag{66}$$

Thus the right-hand side of Eq. (66) is the limiting ratio of the number of times each signal must be transmitted to yield the same false-dismissal probability for a given false-alarm probability. If $I(H_0:H_1;\mathbf{x}_1) > I(H_0:H_1;\mathbf{x}_2)$, then, in this sense, the signal $\mathbf{x}_1(t)$ is more efficient in reducing the false-dismissal probability per transmission than the signal $\mathbf{x}_2(t)$.

Kullback [Ref. 31] bases his proof of Eq. (64) on the easily established inequality

$$I(H_0:H_1) \geq \frac{1}{n}\left(\alpha \log \frac{\alpha}{1-\beta_n} + (1-\alpha)\log\frac{1-\alpha}{\beta_n}\right) \quad . \tag{67}$$

He observes that, since the right-hand side of this inequality monotonically increases with decreasing β_n, for $\alpha \leq 1 - \beta_n \leq 1$, one obtains a lower bound to the minimum possible β_n, say β_n^*, for a given $I(H_0:H_1)$ and a fixed value of α, $0 < \alpha < 1$. We note that, for the single-sample receiver, maximizing $I(H_0:H_1)$ corresponds to minimizing the lower-bound β_1^* for a given α.

IX MAXIMIZING THE KULLBACK-LEIBLER INFORMATION NUMBER

A. PRELIMINARY CONSIDERATIONS

By using the relation between the two probability laws in Eq. (52) and the definition of $I(H_0:H_1)$ in Eq. (65), the Kullback-Leibler information number can be expressed in terms of the eigenvalues of the kernel $K_z(t,s)$:

$$I(H_0:H_1) = \log D\left(\frac{1}{N_0}\right) - \frac{1}{N_0} E[S|H_0] \tag{68a}$$

$$I(H_0:H_1) = \sum_n \log\left(1 + \frac{\lambda_n}{N_0}\right) - \frac{1}{N_0} \sum_n \frac{N_0\lambda_n}{N_0 + \lambda_n} \quad . \tag{68b}$$

Denote by γ_n the nth eigenvalue normalized to the noise intensity,

$$\gamma_n = \frac{\lambda_n}{N_0} \quad ;$$

then

$$I(H_0:H_1) = \sum_n \log\,(1 + \gamma_n) - \sum_n \frac{\gamma_n}{1 + \gamma_n} \quad . \tag{69}$$

We shall also refer to the noise-normalized eigenvalue γ_n as the SNR of the nth degree of freedom of the process $\mathbf{y}(t)$.

In this chapter, the fact that the eigenvalues are constrained by the dispersive properties of the medium is ignored, and the problem of maximizing $I(H_0:H_1)$ under the simple constraint

$$\sum_n \gamma_n \leq \frac{E_0}{N_0} \tag{70}$$

is investigated. The results of the present chapter are then interpreted in a later chapter to develop useful signal-design criteria when the eigenvalues are constrained by a dispersive medium. Since

$$\sum_n \lambda_n = \int_0^T K_{\mathbf{z}}(t,t)\,dt = E\left[\int_0^T |\mathbf{z}(t)|^2\,dt\right] ,$$

the inequality (70) bounds the average signal energy-to-noise intensity ratio at the output of the medium. This is equivalent to bounding the energy of the transmitted signal.

We are concerned with the behavior of $I(H_0:H_1)$ over all Cauchy sequences $\{\gamma_n\}_{n=1}^{\infty}$ of nonnegative and nonincreasing terms satisfying

$$\sum_n \gamma_n \leq \frac{E_0}{N_0} .$$

Our task is to find a sequence $\delta \in \Gamma$,

$$\Gamma = \left[\{\gamma_n\}_{n=1}^{\infty} , \ 0 \leq \gamma_{n+1} \leq \gamma_n , \ \sum_n \gamma_n \leq \frac{E_0}{N_0}\right] ,$$

such that $I(H_0:H_1)$ is maximized. Since

$$0 \leq \log (1 + \gamma_n) - \frac{\gamma_n}{1 + \gamma_n} \leq \gamma_n ,$$

convergence of the sum

$$\sum_n \gamma_n$$

implies convergence of the sum

$$\sum_n \log (1 + \gamma_n) - \frac{\gamma_n}{1 + \gamma_n} .$$

For any sequence $\gamma \in \Gamma$, let γ^m denote the finite sequence obtained from the first m coordinates of γ,

$$\gamma^m = \left\langle \gamma_1^m, \gamma_2^m, \ \ldots, \ \gamma_m^m \right\rangle .$$

Similarly, let ψ_m denote the finite sum

$$\psi_m(\gamma^m) = \sum_{n=1}^{m} \log\,(1 + \gamma_n^m) - \frac{\gamma_n^m}{1 + \gamma_n^m} \quad , \tag{71}$$

then $I(H_0:H_1)$ evaluated at γ is defined as the limit

$$I(H_0:H_1) = \lim_{m \to \infty} \psi_m(\gamma^m) \quad .$$

It is clear that the finite sum ψ_m will possess a constrained relative extremum at a point where all γ_n^m are equal. This follows since both ψ_m and the constraint

$$\sum_{n=1}^{m} \gamma_n^m \leq \frac{E_0}{N_0}$$

are symmetric in the γ_n^m.

In the remaining sections of this chapter, we prove that $I(H_0:H_1)$ achieves an absolute maximum at a sequence in Γ possessing a finite number of nonzero equal coordinates with coordinate values in the vicinity of 2. The proof of this statement is based on finding a sequence $\delta \in \Gamma$ that satisfies the following condition:

For a given available SNR, E_0/N_0, there exists an integer N such that

$$\psi_m(\delta^m) \geq \psi_m(\gamma^m)$$

for all $m > N$ and all $\gamma \in \Gamma$.

Since $I(H_0:H_1)$ is defined as the limit of the ψ_m, the above condition implies the existence of an absolute maximum:

Assume there is a $\zeta \in \Gamma$ such that

$$\lim_{m \to \infty} \psi_m(\zeta^m) > \lim_{m \to \infty} \psi_m(\delta^m) \quad ,$$

then it follows from elementary properties of the limit that there is an integer M such that

$$\psi_m(\zeta^m) > \psi_m(\delta^m)$$

for all $m > M$, which contradicts the assumed property of the sequence δ.

The existence of a preferred sequence of normalized eigenvalues implies the existence of a preferred distribution of available SNR among the degrees of freedom at the output of the medium. This is not surprising. It is intuitively clear that all distributions are not equivalent, as it is clear that the performance of a communication system depends on the strength and number of independent diversity paths when the total SNR is fixed.

In addition to being physically reasonable, our results are in accord with the findings of other investigators who have been concerned with problems related to the one treated in this study. We have mentioned the work of Pierce [Ref. 1] in which he concludes that a SNR in the vicinity of 3 for each degree of freedom minimizes an upper bound for the error probability in the symmetric binary-signaling case. In Ref. 32 Wainstein and Zubakov pose the following question: How many transmissions (denote by N) of a given signal are required to achieve a specified α and β with the minimum expenditure of total SNR, if each transmission is independently perturbed by a time-invariant, zero-mean, complex Gaussian gain? By appealing to the central limit theorem (which requires very large N and moderate α and β), they find that the optimum value of N, in general, depends on both α and β; however, for $\alpha = \beta$, the optimum value of N yields a SNR of exactly 2 for each transmission. In Ref. 33 Price investigates the error-probability behavior of an "on-off" communication system transmitting a constant-envelope pulse over a purely time-varying channel with exponential correlation function. For a given available SNR and fixed time constant in the exponential correlation function, his results indicate the existence of an optimum pulse duration. Since the eigenvalues depend on the product of the channel time constant and the pulse duration, his work implies the existence of an optimum set of normalized eigenvalues.

B. FIRST MAXIMIZATION PROBLEM

Let m and k be fixed integers, with $k \leq m$, and set $j = m - k$. Denote by the ψ_k and ψ_j the restrictions of the ψ_n to the corresponding k- and j-dimensional spaces. In this section, we treat the problem of maximizing the sum

$$\psi_k(\gamma^k) + \psi_j(\gamma^j) \quad ,$$

subject to the constraints

(i) $0 \leq \gamma_n^k \leq 1 \quad , \quad n = 1, 2, \ldots, k$

(ii) $\gamma_n^j > 1 \quad , \quad n = 1, 2, \ldots, j \quad , \quad \sum_n \gamma_n^j \leq c_0 \quad .$

The function ψ_k is the contribution to ψ_m due to apportioning part of the available SNR to degrees of freedom possessing individual SNR less than or equal to unity. Similarly, ψ_j is the contribution due to degrees of freedom with individual SNR greater than unity.

Problems involving finding the absolute extrema of functions are considerably simplified if the functions happen to be convex or concave in the region of interest. Although the decomposition of ψ_m into the two functions ψ_k and ψ_j may seem artificial, we demonstrate that this decomposition leads to a simplification in that ψ_k is convex and ψ_j is concave, subject to the constraints (i) and (ii). The convexity or concavity of these functions follows as an immediate consequence of the following lemma [Ref. 34].

Lemma:

Let r be any positive integer and O_r an open convex set in $\mathcal{E}_r$. The function ψ_r is convex (concave) on O_r if the quadratic form,

$$Q(\gamma^r, h^r) = \sum_n \sum_m h_n^r h_m^r \frac{\partial^2}{\partial \gamma_n^r \partial \gamma_m^r} \psi_r(\gamma^r) \geq 0 \; [Q(\gamma^r, h^r) \leq 0] \quad ,$$

for all $\gamma^r \in O_r$ and all $h^r \in \mathcal{E}_r$. Strict convexity or strict concavity results if the inequality is strict for every $h^r \neq 0$.

Since

$$Q(\gamma^r, h^r) = \sum_{n=1}^{r} (h_n^r)^2 \frac{1 - \gamma_n^r}{(1 + \gamma_n^r)^3} \quad , \tag{72}$$

the function ψ_k is strictly convex in any open convex set satisfying the constraint (i), and the function ψ_j is strictly concave in any open convex set satisfying the constraint (ii).

The fact that the first partial derivatives

$$\frac{\partial}{\partial \gamma_n^k} \psi_k(\gamma^k) = \frac{\gamma_n^k}{(1 + \gamma_n^k)^2} \quad , \quad n = 1, 2, \ldots, k \quad ,$$

are nonnegative implies that ψ_k achieves its maximum value, subject to the constraint (i), at the point whose coordinates are all unity.

We now investigate the behavior of ψ_j in the region

$$P_j = \{\gamma^j : \gamma_n^j > 1, \; n = 1, 2, \ldots, j\}$$

and on the hyperplane $H_j(c)$, defined by the equation

$$\sum_{n=1}^{j} \gamma_n^j = c \quad , \quad j < c \leq c_0 \quad .$$

If ψ_j has a constrained relative *extremum in* the hyperplane $H_j(c)$ at some point γ_0^j, then the Lagrange multiplier rule ensures the existence of a constant σ such that γ_0^j is a critical point of the function

$$\phi_j(\gamma^j) = \psi_j(\gamma^j) + \sigma\left(\sum_{n=1}^{j} \gamma_n^j - c\right) \quad .$$

We observe that all second partial derivatives of ϕ_j are identical to the second partial derivatives of ψ_j. Hence, the lemma implies that ϕ_j is strictly concave in the region P_j for all choices of the constants σ and c. Concavity of ϕ_j ensures that this function possesses at most one critical point in P_j, which (if it exists) must be an absolute maximum of the function in this region. Since ϕ_j is a symmetric function of the γ_n^j, $n = 1, 2, \ldots j$ (the value of ϕ_j is invariant under all permutations of the coordinates of a particular γ^j), this critical point can occur only at a point where all the γ_n^j are equal. This statement follows from the observation that a critical point at a point possessing at least two unequal coordinates implies the existence of another critical point obtained by permuting the two

unequal coordinates (because of the symmetry of ϕ_j), which contradicts the strict concavity of ϕ_j. *Furthermore, since* σ is so chosen that the critical point is in the hyperplane $H_j(c)$, this critical point must be the point

$$\gamma_n^j = \frac{c}{j} \quad , \quad n = 1, 2, \ldots, j \quad .$$

It remains to demonstrate that a σ can be so chosen that the point

$$\gamma_n^j = \frac{c}{j} \quad , \quad n = 1, 2, \ldots, j$$

is indeed a critical point. A critical point occurs when the j first partial derivatives,

$$\frac{\partial}{\partial \gamma_n^j} \phi_j(\gamma^j) = \frac{\gamma_n^j}{(1 + \gamma_n^j)^2} + \sigma \quad ,$$

are zero. The choice

$$\sigma = - \frac{jc}{(j + c)^2}$$

is sufficient to ensure the occurrence of a critical point when all coordinates are equal to c/j. Hence, the function ψ_j attains a constrained absolute maximum in the hyperplane $H_j(c)$ at this point.

We conclude that a point $\gamma^m \in \Gamma^m$ that maximizes the sum $\psi_k(\gamma^k) + \psi_j(\gamma^k)$ under the constraints (i) and (ii) possesses k coordinates equal to unity and j coordinates equal to c/j. *The optimum* value of c is determined to be the maximum value c_0, because the partial derivative,

$$\frac{\partial}{\partial c} \psi_j(\gamma^j) \Big]_{\substack{\gamma_n^j = \frac{c}{j} \\ \text{all } n}} = \frac{cj}{(j + c)^2} \quad ,$$

is positive for $0 < c \leq c_0$.

C. SECOND MAXIMIZATION PROBLEM

In this section, we investigate the problem of determining a dimension r such that ψ_r is maximum under the conditions that all coordinates of γ^r are equal and $\sum_{n=1}^{r} \gamma_n^r = c_0$. In other words, the optimum division of an available SNR among degrees of freedom possessing equal SNR will be determined.

For a given r,

$$\gamma_n^r = c_0/r \quad , \quad n = 1, 2, \ldots, r \quad ,$$

and

$$\psi_r(\gamma^r) = r\left\{\log\left(1 + \frac{c_0}{r}\right) - \frac{c_0/r}{1 + c_0/r}\right\} \quad .$$

Let

$$\frac{c_0}{r} = x \quad ;$$

then

$$\psi_r(\gamma^r)\Big]_{\substack{\gamma_n^r = x \\ \text{all } n}} = c_0 e(x) \quad , \tag{73}$$

where

$$e(x) = \frac{\log(1 + x)}{x} - \frac{1}{1 + x} \quad . \tag{74}$$

The function $e(x)$ measures the efficiency of distributing the available SNR, c_0, among degrees of freedom with SNR equal to x. A plot of the negative of the function e appears in Fig. 6. One can verify that e achieves its maximum value at the point $x = 2.16$ (accurate to two decimal places). Observe that the maximum of e is rather broad and the value 2.16 is not very critical. The function e varies by no more than 1 percent for $1.9 \leq x \leq 2.5$. However, e decreases very rapidly for x decreasing below 1.5 and decreases monotonically from 90 percent of the maximum for x increasing above 4.7.

For a given c_0, the optimum dimension can be found by choosing an integer r^* such that the value of the function plotted in Fig. 6

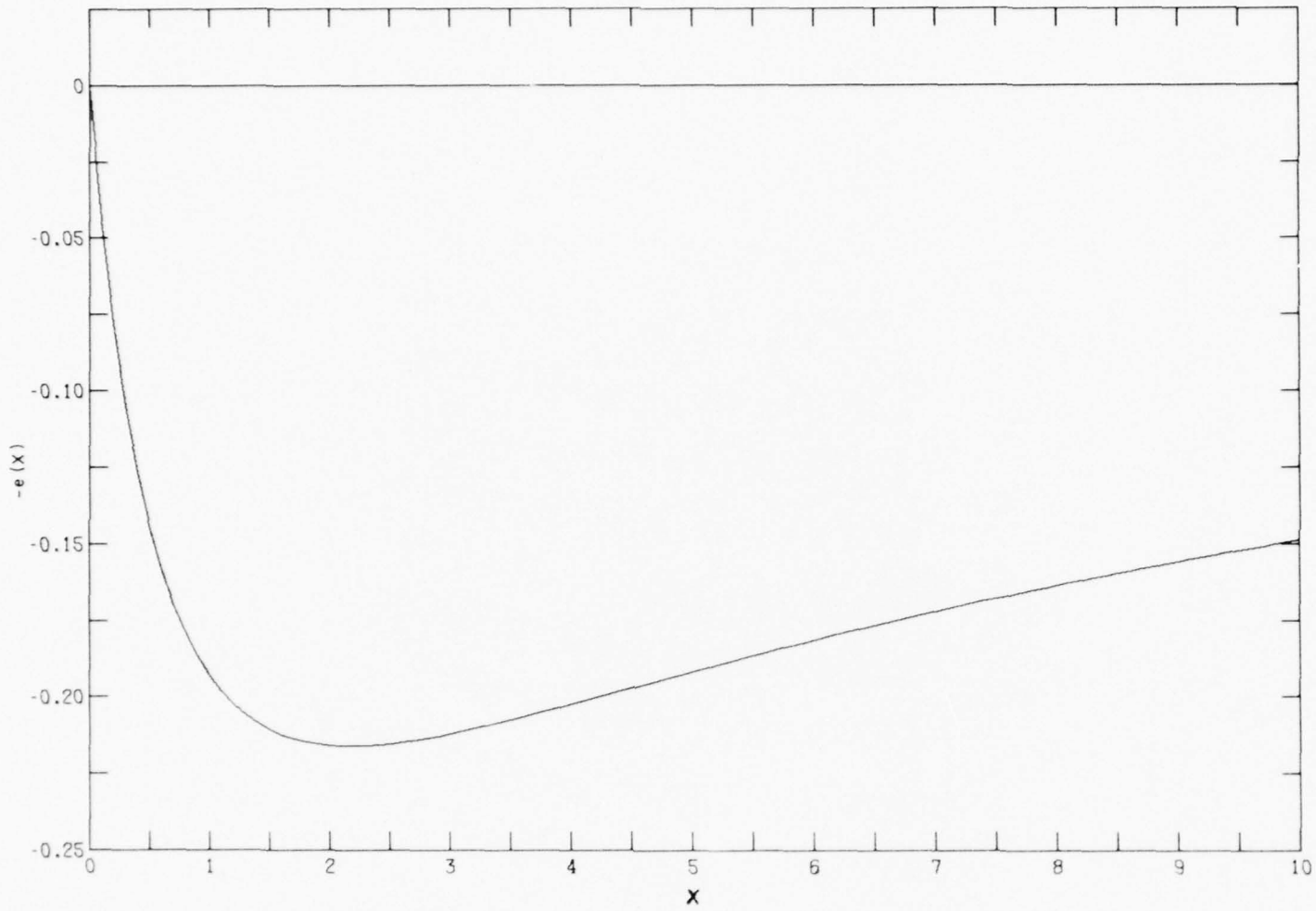

FIG. 6 NEGATIVE OF THE EFFICIENCY FUNCTION

at the point $x^* = c_0/r^*$ is the *minimum* obtainable value for all possible choices of $x = c_0/r$. In general, this procedure will yield a SNR in the vicinity of 2 for each coordinate. In view of the relatively broad minimum of this function, an excellent first approximation to the optimum dimension can be determined by simply referring to the curve.

In the first maximization problem, we have found that the function ψ_j achieves its maximum in the region defined by constraint (ii) at a point where each coordinate is equal to c_0/j. The above result demonstrates the existence of an optimum value for the dimension j that results in coordinate values in the vicinity of 2.

Furthermore, it is evident from the above arguments that the k coordinates equal to unity in the solution to the first maximization problem are in fact wasteful and their combined SNR is more efficiently employed if coordinate values in the vicinity of 2 are chosen.

We now combine the results of the first and second maximization problems and obtain the following maximization principle.

Maximization principle:

The Kullback-Leibler information number is maximized subject to the constraint on the available SNR,

$$\sum_n \gamma_n \leq E_0/N_0 \quad ,$$

by choosing m degrees of freedom possessing SNR E_0/mN_0 (that is, $\lambda_n = E_0/m$ for $n = 1, 2, \ldots m$) in the vicinity of 2, such that

$$e\left(\frac{E_0}{mN_0}\right) \geq e\left(\frac{E_0}{rN_0}\right) \quad ,$$

where

$$e(x) = \frac{\log\ (1 + x)}{x} - \frac{1}{1 + x} \quad ,$$

for any integer r.

X THE KULLBACK-LEIBLER INFORMATION NUMBER UNDER THE CONDITION OF "LOW-ENERGY COHERENCE"

A. DISCUSSION

Important practical problems exist in which the noise-normalized eigenvalues of the kernel $K_z(t,s)$ are necessarily small, owing to a severe scattering of transmitted energy in the dispersive medium and a practical limitation on the peak power transmitted. This situation can occur, for example, in the detection of radar-astronomical targets. Price has studied this problem in detail and refers to the case in which the largest normalized eigenvalue γ_1 is much less than unity as the "low-energy-coherence (LEC)" condition [Ref. 35]. The important constraint to be considered in problems of this nature is the maximum achievable SNR of each individual degree of freedom rather than the total SNR which, in practice, can be made large by obtaining many degrees of freedom with transmissions possessing large durations or bandwidths.

Price has shown that under the condition of LEC the quadratic-form processor of the Neyman-Pearson receiver [Eq. (45)] approaches a SNR-maximizing quadratic-form processor [Ref. 35]. The latter suboptimum processor computes a statistic,

$$M = \int_0^T \int_0^T \mathbf{y}^*(t) m(t,s) \mathbf{y}(s) dt ds \quad , \tag{75}$$

such that the receiver-output SNR,

$$\mathcal{R}(M) = \frac{\{E[M|H_1] - E[M|H_0]\}^2}{\mathrm{Var}[M|H_0]} \tag{76}$$

is maximized, subject to an energy constraint on the transmitted signal, over all possible choices of the kernel $m(t,s)$. Although the statistic M is of the same form as the statistic S, the kernel $m(t,s)$ is much simpler to compute than the kernel $g(t,s)$. It is easily verified that $\mathcal{R}(M)$ is maximized when $m(t,s)$ is equal to the kernel $K_z(t,s)$ [Ref. 35].

Price [Ref. 8] also investigated the problem of finding a transmission (of fixed energy) that maximizes the receiver-output SNR, $\mathcal{R}(S)$ under LEC conditions for the Neyman-Pearson detector. He concluded that $\mathcal{R}(S)$ is maximized by maximizing the quantity

$$\iint |R_H(\alpha,\beta)|^2 |\psi_{\mathbf{z}}(\alpha,\beta)|^2 d\alpha d\beta \quad ,$$

subject to an energy constraint on $\mathbf{x}(t)$.

In view of the equivalence of SNR-maximizing processing and Neyman-Pearson processing for LEC conditions demonstrated by Price (with the resulting conclusion that increasing $\mathcal{R}$ improves the error performance of the detection system), it is clear that maximizing the above quantity is an optimum signal-design strategy for LEC conditions.

Since the criterion of maximizing the Kullback-Leibler information number is based on large sample theory, we are not ensured that this criterion will, in general, yield signal-design procedures that minimize the false-dismissal probability at a given false-alarm probability for the single-sample receiver. However, in this chapter we demonstrate that maximizing the number $I(H_0:H_1)$ under LEC conditions is equivalent to maximizing the ratio $\mathcal{R}(S)$ and hence is identical with Price's criterion for suitably small normalized eigenvalues.

In addition, it is demonstrated that the key phenomenon in the LEC case is the approach of the noise-normalized statistic S/N_0 to a normal random variable. It will be shown that, as S/N_0 approaches normality, maximizing $I(H_0:H_1)$ minimizes the false-dismissal probability for any false-alarm probability, under the reasonable restriction that the false-alarm probability is not chosen so small that a false-dismissal probability less than one-half cannot be achieved.

B. EQUIVALENCE OF THE KULLBACK-LEIBLER INFORMATION NUMBER AND THE RECEIVER-OUTPUT SNR FOR LEC CONDITIONS

For $\gamma_n < 1$, the terms involved in the expression for $I(H_0:H_1)$ appearing in Eq. (69) possess the expansions

$$\log\,(1 + \gamma_n) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (\gamma_n)^k \quad ,$$

$$\frac{\gamma_n}{1 + \gamma_n} = \sum_{k=1}^{\infty} (-1)^{k-1} (\gamma_n)^k \quad ,$$

which result in the following expression for $I(H_0:H_1)$:

$$I(H_0:H_1) = \sum_{k=2}^{\infty} (-1)^k \left(1 - \frac{1}{k}\right) \sum_n (\gamma_n)^k \quad . \tag{77}$$

Low-energy coherence implies that γ_1 is much less than unity,

$$\begin{aligned}
&\text{LEC} \Longrightarrow \gamma_1 << 1 \Longrightarrow \\
&\gamma_n << 1, \quad \text{all } n \Longrightarrow \\
&\gamma_n^3 << \gamma_n^2, \quad \text{all } n \Longrightarrow \\
&\sum_n \gamma_n^3 << \sum_n \gamma_n^2 \Longrightarrow \frac{\sum_n \gamma_n^3}{\sum_n \gamma_n^2} << 1 \quad .
\end{aligned}$$

From Eq. (77), we obtain the limiting value of $I(H_0:H_1)$,

$$\lim_{\frac{\sum_n \gamma_n^3}{\sum_n \gamma_n^2} \to 0} I(H_0:H_1) = \frac{1}{2} \sum_n \gamma_n^2 \quad . \tag{78}$$

Therefore, under LEC conditions, the Kullback-Leibler information number is dominated by the sum of the squares of the normalized eigenvalues. The maximization of $I(H_0:H_1)$, subject to an energy constraint, is achieved by maximizing the variance of S under hypothesis H_1 [Eq. (60b)]

$$\sum_n \lambda_n^2 = \int_0^T \int_0^T |K_z(s,t)|^2 ds dt \quad , \tag{79a}$$

subject to the constraint on the mean of S under hypothesis H_1 [Eq. (60a)]

$$\sum_n \lambda_n = \int_0^T K_z(t,t) dt \leq E_0 \quad . \tag{79b}$$

For the Neyman-Pearson detector, the ratio $R(S)$ assumes the form

$$R(S) = \frac{\left(\sum_n \lambda_n - \sum_n \frac{N_0 \lambda_n}{N_0 + \lambda_n}\right)^2}{\sum_n \left(\frac{N_0 \lambda_n}{N_0 + \lambda_n}\right)^2} = \frac{\left(\sum_n \frac{\gamma_n^2}{1 + \gamma_n}\right)^2}{\sum_n \left(\frac{\gamma_n}{1 + \gamma_n}\right)^2} \quad .$$

If $\gamma_n \ll 1$, $R(S)$ approaches $\sum_n \gamma_n^2$, which demonstrates the equivalence of $I(H_0:H_1)$ and $R(S)$ when the normalized eigenvalues are suitably small.

To relate the integration in Eq. (79a) to the input signal, it is more convenient to deal with the autocorrelation function $R_z(\alpha, t)$ [Eq. (19)],

$$R_z(\alpha, t) = K_z\left(t - \frac{\alpha}{2}, \; t + \frac{\alpha}{2}\right) \quad . \tag{80}$$

Hereafter, we assume that the medium has finite memory L, that is,

$$h(t, \tau) = 0 \quad \text{for } \tau > L \quad ,$$

and that the transmitted signal $\mathbf{x}(t)$ is zero outside a finite time interval of duration T_x. Then, if the duration of the observation interval T is equal to $L + T_x$, the integral in Eq. (79a) can be written as an integration of the squared magnitude of $R_z(\alpha, t)$ over the entire $\alpha - t$ plane,

$$\sum_n \lambda_n^2 = \iint |R_z(\alpha, t)|^2 d\alpha dt \quad . \tag{81}$$

The integral in Eq. (81) can be expressed in terms of the four second-order measures of the process $\mathbf{z}(t)$ (Fig. 4) by applying the Parseval relation,

$$\sum_n \lambda_n^2 = \iint |R_z(\alpha, t)|^2 d\alpha dt = \iint |\psi_z(\alpha, \beta)|^2 d\alpha d\beta$$

$$= \iint |R_Z(f, \beta)|^2 df d\beta = \iint |S_z(f, t)|^2 df dt \quad . \tag{82}$$

Since

$$\psi_z(\alpha, \beta) = R_H(\alpha, \beta) \psi_x(\alpha, \beta) \quad , \tag{27b}$$

the quantity $\sum_n \lambda_n^2$ is conveniently related to the input signal as follows:

$$\sum_n \lambda_n^2 = \iint |R_H(\alpha,\beta)|^2 |\psi_{\mathbf{x}}(\alpha,\beta)|^2 d\alpha d\beta \quad . \tag{83}$$

A similar argument applied to the sum of the eigenvalues yields

$$\sum_n \lambda_n = \int_0^T K_{\mathbf{z}}(t,t)dt = \int R_{\mathbf{z}}(0,t)dt = \psi_{\mathbf{z}}(0,0) = R_H(0,0)\psi_{\mathbf{x}}(0,0) \quad .$$

The energy constraint, $\sum_n \lambda_n \leq E_0$, bounds the maximum input energy $\psi_{\mathbf{x}}(0,0)$ since $R_H(0,0)$ is a constant.

Rewriting Eq. (83),

$$\sum_n \lambda_n^2 = R_H^2(0,0)\psi_{\mathbf{x}}^2(0,0) \iint \left|\frac{R_H(\alpha,\beta)}{R_H(0,0)}\right|^2 \left|\frac{\psi_x(\alpha,\beta)}{\psi_x(0,0)}\right|^2 d\alpha d\beta \quad ,$$

we conclude that $\sum_n \lambda_n^2$ is maximized (subject to the constraint $\sum_n \lambda_n \leq E_0$) by choosing a signal $\mathbf{x}(t)$ that maximizes

$$\frac{\sum_n \lambda_n^2}{\psi_{\mathbf{z}}^2(0,0)} = \iint |R_H^0(\alpha,\beta)|^2 |\psi_{\mathbf{x}}^0(\alpha,\beta)|^2 d\alpha d\beta \quad , \tag{84}$$

where

$$R_H^0(0,0) = 1 \quad \text{and} \quad \psi_{\mathbf{x}}^0(0,0) = 1 \quad ,$$

and scaling this signal so that its energy is maximum.

C. APPROACH OF THE NOISE-NORMALIZED STATISTIC TO NORMALITY

In this section we briefly investigate the limiting probability law of the noise-normalized statistic S/N_0 under LEC conditions. From Eq. (42) the noise-normalized statistic can be written as the sum

$$\frac{S}{N_0} = \sum_n \frac{\lambda_n |\mathbf{y}_n|^2}{N_0(\lambda_n + N_0)} \quad .$$

Let

$$v_n = \frac{\lambda_n |\mathbf{y}_n|^2}{N_0(\lambda_n + N_0)} \tag{85a}$$

and

$$V_N = \sum_{n=1}^{N} v_n \quad ; \tag{85b}$$

then, under hypothesis H_1, the v_n are exponentially distributed with characteristic function $(1 - i\gamma_n u)^{-1}$ and $P[\lim_{N\to\infty} V_N = S/N_0] = 1$ for $K_z(s,t)$ continuous.

The independent random variables $\{v_n\}$ obey the central limit theorem if Lypunov's condition [Ref. 36] is satisfied:

$$\lim_{N\to\infty} \frac{1}{[\mathrm{Var}(V_N)]^{\frac{2+\delta}{2}}} \sum_{n=1}^{N} E[|v_n - E(v_n)|^{2+\delta}] = 0 \quad , \tag{86}$$

for some $\delta > 0$. Let $\delta = 1$, then

$$E[|v_n - E(v_n)|^{2+\delta}] = E[|v_n - \gamma_n|^3] \quad ,$$

$$E[|v_n - \gamma_n|^3] \le \{E[(v_n - \gamma_n)^2]\}^{1/2}\{E[(v_n - \gamma_n)^4]\}^{1/2} = 3\gamma_n^3$$

and

$$\{\mathrm{Var}[V_N]\}^{\frac{2+\delta}{2}} = \left(\sum_{n=1}^{N} \gamma_n^2\right)^{3/2} \quad .$$

Therefore, the sum of positive terms on the left-hand side of Eq. (86) is bounded above by the quantity

$$\frac{3}{\left(\sum_{n=1}^{N} \gamma_n^2\right)^{1/2}} \frac{\sum_{n=1}^{N} \gamma_n^3}{\sum_{n=1}^{N} \gamma_n^2} \quad .$$

Hence S/N_0 approaches a normal random variable with mean $\sum_n \gamma_n$ and variance $\sum_n \gamma_n^2 > 0$ if

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \gamma_n^3}{\sum_{n=1}^{N} \gamma_n^2} = 0 \quad .$$

The LEC condition implies the inequality

$$\frac{\sum_n \gamma_n^3}{\sum_n \gamma_n^2} \ll 1 \quad .$$

The above ratio can not be zero unless $\sum_n \gamma_n^2$ diverges, which implies that $\sum_n \gamma_n$ also diverges; this can occur only if the average energy at the output of the medium is infinite.

Although the γ_n are all less than unity, the two summations $\sum_n \gamma_n$ and $\sum_n \gamma_n^2$ will, in general, be quite large for practical systems operating under LEC conditions. The first summation is the available SNR at the input to the receiver and the second summation is Price's receiver-output SNR. If the system performs its task of detection satisfactorily, then both input and output SNR must be appropriately large. However, for suitably small γ_n, the ratio of the sum $\sum_n \gamma_n^3$ to the sum $\sum_n \gamma_n^2$ can be made vanishingly small. As this ratio approaches zero, the limiting false-dismissal probability becomes

$$\lim_{\frac{\sum_n \gamma_n^3}{\sum_n \gamma_n^2} \to 0} P[S/N_0 < L | H_1] = \Phi\left[\frac{L - \sum_n \gamma_n}{(\sum_n \gamma_n^2)^{1/2}}\right] \quad , \qquad (87)$$

where Φ is the unit normal probability distribution. If the threshold L is less than $\sum_n \gamma_n$, the probability of false dismissal is less than one-half and is minimized subject to the constraint $\sum_n \gamma_n \leq E_0/N_0$ by maximizing $\sum_n \lambda_n^2$ for $\sum_n \lambda_n = E_0$.

This conclusion is based on simple arguments that are true in the limit. Consequently, we do not readily obtain a quantitative measure of the loss in system performance when the limiting case is approximately achieved and the above signal-design strategy is employed. However, Price's thorough analysis of the LEC case [Ref. 35] deals with this question. Besides obtaining quantitative measures of system loss for the SNR-maximizing receiver (as opposed to the Neyman-Pearson receiver), he calculated bounds for the loss incurred by employing the signal-design strategy of maximizing $\sum_n \gamma_n^2$ (as opposed to the minimum error-probability criterion). Price concludes that these system losses are small when LEC conditions prevail.

XI EQUIVALENT NUMBER OF DEGREES OF FREEDOM

In Eq. (46) the statistic S has been expressed as a sum of exponentially distributed random variables,

$$S = \sum_n w_n \quad .$$

The Karhunen-Loève expansion decomposes the output of the medium into independent degrees of freedom $\{\mathbf{z}_n\}$, and w_n is the contribution of the nth degree of freedom to the statistic S.

The ratio

$$\frac{E^2[S|H_1]}{Var[S|H_1]}$$

is a useful measure of the statistical stability of S. If there are only N degrees of freedom with eigenvalues all equal to λ, then the above stability ratio becomes

$$\frac{E^2[S|H_1]}{Var[S|H_1]} = \frac{\left(\sum_{n=1}^{n} \lambda_n\right)^2}{\sum_{n=1}^{n} \lambda_n^2} = \frac{N^2\lambda^2}{N\lambda^2} = N \quad .$$

For the infinite-dimensional case, it is convenient to define a quantity π_0 as the equivalent number of degrees of freedom,

$$\pi_0 = \frac{E^2[S|H_1]}{Var[S|H_1]} = \frac{\left[\int R_{\mathbf{z}}(0,t)dt\right]^2}{\iint |R_{\mathbf{z}}(\alpha,t)|^2 d\alpha dt} = \frac{\psi_{\mathbf{z}}^2(0,0)}{\iint |\psi_{\mathbf{z}}(\alpha,\beta)|^2 d\alpha d\beta} \quad ,$$

$$\pi_0 = \frac{1}{\iint |\psi_{\mathbf{z}}^0(\alpha,\beta)|^2 d\alpha d\beta} \quad . \tag{88}$$

Expressed in terms of the input-signal and medium properties, the equivalent number of degrees of freedom of the output process becomes

$$\pi_0 = \frac{1}{\iint |R_H^0(\alpha,\beta)|^2 |\psi_{\mathbf{z}}^0(\alpha,\beta)|^2 d\alpha d\beta} \quad . \tag{89}$$

If, in the infinite-dimensional case, most of the energy of the process $\mathbf{z}(t)$ is contained in a finite set of degrees of freedom

$$\left(\sum_{n=1}^{\infty} \lambda_n \doteq \sum_{n=1}^{N} \lambda_n \right)$$

with approximately equal eigenvalues, then π_0 is a measure of the number of degrees of freedom in this dominating set.

It is interesting to observe that the quantity π_0 in Eq. (89) is identical to the inverse of the limiting form of the energy-normalized Kullback-Leibler information number [Eq. (84)] for LEC conditions. Therefore, maximizing $I(H_0:H_1)$ for the LEC case is achieved by minimizing the equivalent number of degrees of freedom.

This result supports intuitive reasoning based on physical considerations. The LEC case is characterized by a lack of sufficient available SNR to distribute among the many degrees of freedom introduced by the medium. The receiver must then detect the presence of a large number of very noisy degrees of freedom. We improve detection capability by choosing an input signal that minimizes the number of degrees of freedom and hence produces degrees of freedom with higher SNR than would otherwise be obtained.

The criterion of minimizing the equivalent number of degrees of freedom in the LEC situation also yields intuitively satisfying results for the general structure of the transmitted signal. We first observe that this criterion dictates the choice of an input signal with ambiguity function possessing a squared envelope $|\psi_{\mathbf{x}}^0(\alpha,\beta)|^2$ which resembles the squared envelope of the time-frequency autocorrelation function $|R_H^0(\alpha,\beta)|^2$. In this manner, the common volume shared by these two functions is maximized; the Schwartz inequality bounds the maximum achievable common volume,

$$\iint |R_H^0(\alpha,\beta)|^2 |\psi_{\mathbf{x}}^0(\alpha,\beta)|^2 d\alpha d\beta \leq \left(\iint |R_H^0(\alpha,\beta)|^4 d\alpha d\beta \right)^{1/2} \left(\iint |\psi_{\mathbf{x}}^0(\alpha,\beta)|^4 d\alpha d\beta \right)^{1/2} .$$

If $|R_H(\alpha,\beta)|^2$ is the squared envelope of the ambiguity function of a deterministic signal, then π_0 is minimized if and only if

$$|\psi_x^0(\alpha,\beta)|^2 = |R_H^0(\alpha,\beta)|^2 \quad .$$

The above statements imply that the shape of the input energy density $S_x(f,t)$ should resemble the medium scattering function $S_v(\lambda,\tau)$. If $S_v(\lambda,\tau)$ is wide in λ and narrow in τ (large Doppler spread and small time-delay spread), we should transmit a signal that is wide in frequency and narrow in time duration, for example, a narrow pulse. Sending a narrow pulse through a medium with a large Doppler spread and a small time-delay spread corresponds to transmitting a signal with an energy distribution in the time-frequency plane that is least likely to be severely diffused by the dispersive medium. In this manner, we minimize the variability (number of degrees of freedom) of the output process $\mathbf{z}(t)$.

On the other hand, if a long-duration signal is transmitted, the rapid time variations of the medium (implied by the large Doppler spread) are given full opportunity to severely distort the transmitted signal. This implies an excessive generation of degrees of freedom for LEC conditions as opposed to the short-duration transmission.

It is interesting to compute π_0 when the medium is time- and frequency-invariant, that is, when

$$R_H^0(\alpha,\beta) = 1 \qquad \text{for all } \alpha \text{ and } \beta.$$

This is the time-frequency autocorrelation function for the time- and frequency-flat medium which merely multiplies the input signal by a random complex gain. It is evident that the output of such a medium contains only one degree of freedom; hence

$$(\pi_0)^{-1} = \iint |\psi_x^0(\alpha,\beta)|^2 d\alpha d\beta = 1 \quad . \tag{91}$$

Equation (91) is a well-known relation which is usually derived from the basic definition of $\psi_x(\alpha,\beta)$ given in Eq. (23) [Ref. 13].

XII SIGNAL DESIGN FOR RANDOMLY DISPERSIVE MEDIA

A. DISCUSSION

The general problem of choosing an input signal that maximizes the Kullback-Leibler information number when the eigenvalues are constrained by an arbitrarily dispersive medium is difficult and at present is unsolved. The solution to this problem involves maximizing the quantity

$$I(H_0:H_1) = \int_0^{1/N_0}\int_0^T g(t,t,x)\,dt\,dx - \frac{1}{N_0}\int_0^T g\left(t,t,\frac{1}{N_0}\right)dt \quad , \tag{92}$$

where $g(r,t,x)$ is the resolvent kernel for the integral equation

$$g(r,t,x) + x\int_0^T K_z(r,s)g(s,t,x)\,ds = K_z(r,t) \quad . \tag{93}$$

However, by using the results obtained for the simple energy-constrained case (finite number of nonzero eigenvalues), it is possible to deduce an optimum shape for the output energy density $S_z(f,t)$. When the medium is dispersive, the assumption of a finite number of nonzero eigenvalues implies, in general, that the output process is limited in both time and frequency. Hence, the resulting nonrealizable optimum output energy density based on a finite number of eigenvalues must be approximated by a realizable energy density.

We have shown that $I(H_0:H_1)$ is maximized when the available SNR is distributed equally among the eigenfunctions so that each degree of freedom possesses a SNR in the vicinity of 2. In this chapter we demonstrate that to obtain a group of equal eigenvalues it is sufficient that the energy density $S_z(f,t)$ be constant over a region in the t - f plane.

It is convenient to deal with a time- and frequency-shifted process $\mathbf{z}(t)$ with mean signal time and mean signal frequency equal to zero,

$$\iint t S_z(f,t)\,df\,dt = 0 \tag{94a}$$

$$\iint f S_{\mathbf{z}}(f,t)\,dt\,df = 0 \quad . \tag{94b}$$

B. AN OPTIMUM SHAPE FOR THE OUTPUT ENERGY DENSITY

Since it is assumed that a finite duration input produces a finite duration output, $\mathbf{z}(t)$ is zero for t outside the finite interval $(-T/2, T/2)$. Therefore, the eigenfunctions $\{\mathbf{f}_n(t)\}$ can be defined to be zero outside the interval $(-T/2, T/2)$, and the integration can be performed over all t in the integral equation defining the eigenfunctions and the eigenvalues [Eq. (37a)],

$$\int \mathbf{f}_n(t) K_{\mathbf{z}}(t,s)\,dt = \lambda_n \mathbf{f}_n(s) \quad .$$

In this section, we are concerned with the covariance kernel $K_{\mathbf{Z}}(f,\gamma)$ of the frequency process $\mathbf{Z}(f)$, where

$$\mathbf{Z}(f) = \int \mathbf{z}(t) e^{-i2\pi f t}\,dt \quad , \tag{95a}$$

and

$$K_{\mathbf{Z}}(f,\gamma) = E[\mathbf{Z}^*(f)\mathbf{Z}(\gamma)] \quad . \tag{95b}$$

The kernel $K_{\mathbf{Z}}(f,\gamma)$ is the double Fourier transform of the kernel $K_{\mathbf{z}}(t,s)$,

$$K_{\mathbf{Z}}(f,\gamma) = \iint K_{\mathbf{z}}(t,s) e^{i2\pi(ft-\gamma s)}\,dt\,ds \quad . \tag{96}$$

Let $\mathbf{F}_n(f)$ be the Fourier transform of the nth eigenfunction,

$$\mathbf{F}_n(f) = \int \mathbf{f}_n(t) e^{-i2\pi f t}\,dt \quad ;$$

then the functions $\{\mathbf{F}_n(f)\}$ constitute an orthonormal set and satisfy the following integral equation:

$$\int K_{\mathbf{Z}}(f,\gamma)\mathbf{F}_n(f)\,df = \lambda_n \mathbf{F}_n(\gamma) \quad . \tag{97}$$

Therefore, the kernel $K_{\mathbf{Z}}(f,\gamma)$ possesses eigenfunctions $\{\mathbf{F}_n(f)\}$ and eigenvalues $\{\lambda_n\}$.

In Ref. 37, Zakai proves that if $\mathbf{z}(t)$ is time-limited and if the function $K_{\mathbf{Z}}(f,f)$ is continuous and satisfies

$$\int_{-\infty}^{+\infty} \frac{K_{\mathbf{Z}}(f,f)}{1+f^2}\, df < \infty \quad ,$$

then the sample representation,

$$\mathbf{Z}(f) = \sum_k \mathbf{Z}\left(\frac{k}{T}\right) \operatorname{sinc} \pi T\left(f - \frac{k}{T}\right) \quad , \tag{98}$$

of the nonstationary random process $\mathbf{Z}(f)$ converges with probability one to $\mathbf{Z}(f)$. Since

$$\int_{-\infty}^{+\infty} \frac{K_{\mathbf{Z}}(f,f)}{1+f^2}\, df < \int_{-\infty}^{+\infty} K_{\mathbf{Z}}(f,f)\, df = E\left[\int_{-\infty}^{+\infty} |\mathbf{Z}(f)|^2 df\right] = E\left[\int_{-\infty}^{+\infty} |\mathbf{z}(t)|^2 dt\right] \quad ,$$

the process $\mathbf{Z}(f)$ possesses a sample representation with probability one if the average energy at the output of the medium is finite when the function $K_{\mathbf{Z}}(f,f)$ is continuous. Equation (98) implies that the kernel $K_{\mathbf{Z}}(f,\gamma)$ can be expanded

$$K_{\mathbf{Z}}(f,\gamma) = \sum_k \sum_j K_{\mathbf{Z}}\left(\frac{k}{T}, \frac{j}{T}\right) \operatorname{sinc} \pi T\left(f - \frac{k}{T}\right) \operatorname{sinc} \pi T\left(\gamma - \frac{j}{T}\right) \quad . \tag{99}$$

We now assume that $S_{\mathbf{z}}(f,t)$ is constant in the t-direction for t in the interval $(-T/2, T/2)$:

$$S_{\mathbf{z}}(f,t) = \begin{cases} P(f) & \text{for} \quad -\dfrac{T}{2} \le t \le \dfrac{T}{2} \\ 0 & \text{for} \quad t \text{ otherwise} \end{cases} \quad . \tag{100}$$

This assumption implies that the autocorrelation function,

$$R_{\mathbf{Z}}(f,\beta) = E\left[\mathbf{Z}^*\left(f - \frac{\beta}{2}\right)\mathbf{Z}\left(f + \frac{\beta}{2}\right)\right] \quad ,$$

has the form

$$R_{\mathbf{Z}}(f,\beta) = \int S_{\mathbf{z}}(f,t)e^{-i2\pi\beta t}dt = TP(f)\ \mathrm{sinc}\ \pi\beta T \quad . \tag{101}$$

It follows that the kernel $K_{\mathbf{Z}}(f,\gamma)$ is given by

$$K_{\mathbf{Z}}(f,\gamma) = TP\left(\frac{f+\gamma}{2}\right)\ \mathrm{sinc}\ \pi(f-\gamma)T \quad , \tag{102a}$$

which yields

$$K_{\mathbf{Z}}(k/T, j/T) = \begin{cases} E\left[\left|\mathbf{Z}\left(\frac{k}{T}\right)\right|^2\right] & , \quad \text{for} \quad k = j \\ 0 & , \quad \text{for} \quad k \neq j \end{cases} \quad . \tag{102b}$$

The expansion of Eq. (99) can now be rewritten,

$$K_{\mathbf{Z}}(f,\gamma) = \sum_n \frac{E\left[\left|\mathbf{Z}\left(\frac{n}{T}\right)\right|^2\right]}{T}\sqrt{T}\ \mathrm{sinc}\ \pi T\left(f-\frac{n}{T}\right)\sqrt{T}\ \mathrm{sinc}\ \pi t\left(\gamma-\frac{n}{T}\right) \quad . \tag{103}$$

Equation (103) can be recognized as Mercer's theorem [Ref. 26] where the eigenfunctions and eigenvalues of the kernel $K_{\mathbf{Z}}(f,\gamma)$ are given by

$$\mathbf{F}_n(f) = \sqrt{T}\ \mathrm{sinc}\ \pi T\left(f-\frac{n}{T}\right) \tag{104a}$$

and

$$\lambda_n = \frac{E\left[\left|\mathbf{Z}\left(\frac{n}{T}\right)\right|^2\right]}{T} = P\left(\frac{n}{T}\right) \quad . \tag{104b}$$

It is now possible to deduce a (nonrealizable) shape $\hat{S}_{\mathbf{z}}(f,t)$ for the output energy density that is optimum in the sense that it produces the maximum number of degrees of freedom with SNR in the vicinity of 2 and hence maximizes the Kullback-Leibler information number. For convenience, it is assumed that the available SNR, $\psi_{\mathbf{z}}(0,0)/N_0$, is an even multiple of 2.16, the optimum SNR of an individual degree of freedom. (In general the SNR per degree of freedom must be determined by applying the maximization principle.) If

$$\hat{S}_z(f,t) = 2.16N_0 \quad \text{for} \quad \begin{cases} -\dfrac{\psi_z(0,0)}{4.32N_0T} \le f \le \dfrac{\psi_z(0,0)}{4.32N_0T} \\ -\dfrac{T}{2} \le t \le \dfrac{T}{2} \end{cases} \quad , \tag{106a}$$

$$= 0 \qquad \text{for} \qquad (f,t) \quad \text{otherwise} \quad ,$$

then Eq. (104b) implies

$$\frac{\lambda_n}{N_0} = \frac{E\left[\left|\mathbf{Z}\left(\frac{n}{T}\right)\right|^2\right]}{TN_0} = \frac{\int \hat{S}_z\left(\frac{n}{T},t\right)dt}{TN_0} = 2.16 \quad \text{for} \quad |n| \le \frac{\psi_z(0,0)}{4.32N_0}$$

$$= 0 \qquad \text{for} \qquad |n| > \frac{\psi_z(0,0)}{4.32N_0} \quad . \tag{106b}$$

The "pillbox" shape $\hat{S}_z(f,t)$ achieves the proper SNR per degree of freedom. Furthermore, any time- and frequency-limited shape that achieves the same number of degrees of freedom as $\hat{S}_z(f,t)$ must possess a time-bandwidth product equal to or greater than the time-bandwidth product of $\hat{S}_z(f,t)$.

C. MEAN-SQUARE APPROXIMATION OF THE OPTIMUM ENERGY DENSITY

Since $\hat{S}_z(f,t)$ is limited in both time and frequency, it can not be realized as a true energy density at the output of the medium. Moreover, the particular dispersive properties of the medium constrain the allowable shape that the output energy density can assume. However, if this particular pillbox shape can be closely approximated with a realizable energy density, then we are ensured of the existence of a dominating set of degrees of freedom with the proper SNR. To this end, we seek an input signal $\mathbf{x}(t)$ such that the associated output energy density $S_z(f,t)$ minimizes the (normalized) mean-square error,

$$\epsilon(\mathbf{x}) = \frac{\iint [S_z(f,t) - \hat{S}_z(f,t)]^2 df dt}{\iint \hat{S}_z^2(f,t) df dt} \quad . \tag{107}$$

The broad minimum of the negative of the efficiency function plotted in Fig. 6 indicates that the optimum SNR of 2.16 per degree of freedom is not critical in the equal-eigenvalue case. Similarly, the performance of the system will not be highly sensitive to small differences between the eigenvalues as long as their magnitudes are in the vicinity of 2. By minimizing the error $\epsilon(\mathbf{x})$, we are in effect generating a preferred eigenvalue distribution based on approximating the nonrealizable optimum shape $\hat{S}_z(f,t)$. We remark that if $\mathbf{z}(t)$ is not time-limited, the minimization of the error $\epsilon(\mathbf{x})$ in Eq. (107) is still meaningful. That is, the optimality of the shape $\hat{S}_z(f,t)$ (ignoring physical realizability) depends only on the fact that we assumed zero-mean Gaussian probability laws for the output and white-noise processes.

The mean-square error can be expressed directly in terms of the input ambiguity function and the medium time-frequency autocorrelation function. Let

$$W = \frac{\psi_z(0,0)}{2.16N_0T} \quad ;$$

then

$$\epsilon(\mathbf{x}) = 1 - \frac{\psi_z(0,0)}{2.16N_0} \iint R_H^{0*}(\alpha,\beta)\,\psi_x^{0*}(\alpha,\beta)\left\{2\,\mathrm{sinc}\,\pi W\alpha\,\mathrm{sinc}\,\pi T\beta - R_H^0(\alpha,\beta)\psi_x^0(\alpha,\beta)\right\}d\alpha d\beta \quad . \tag{108}$$

Therefore, the best mean-square approximation to the pillbox $\hat{S}_z(f,t)$ is obtained by choosing an input signal $\mathbf{x}(t)$ with ambiguity function $\psi_x(\alpha,\beta)$ that maximizes the quantity

$$\mathcal{J}(\mathbf{x}) = \iint R_H^{0*}(\alpha,\beta)\psi_x^{*0}(\alpha,\beta)\{2\,\mathrm{sinc}\,\pi W\alpha\,\mathrm{sinc}\,\pi T\beta - R_H^0(\alpha,\beta)\psi_x^0(\alpha,\beta)\}d\alpha d\beta \quad . \tag{109}$$

The problem of obtaining the ambiguity function that maximizes $\mathcal{J}(\mathbf{x})$ is difficult and has not been solved. However, the quantity $\mathcal{J}(\mathbf{x})$ can be used to choose an input signal from a finite class of competitive transmissions or it can be used to optimize a parameter of an otherwise specified signal. In addition, we shall arrive at an important implication concerning the structure of efficient transmissions by deriving an upper bound for $\mathcal{J}(\mathbf{x})$.

The quantity $\mathcal{J}(\mathbf{x})$ can also be expressed in terms of the output energy density,

$$\mathcal{J}(\mathbf{x}) = \frac{\frac{2}{TW}\int_{-T/2}^{T/2}\int_{-W/2}^{W/2} S_z(f,t)\,df\,dt}{\iint S_z(f,t)\,df\,dt} - \frac{\iint S_z^2(f,t)\,df\,dt}{\left[\iint S_z(f,t)\,df\,dt\right]^2} \quad . \tag{110}$$

By using the Schwartz inequality

$$\int_{-T/2}^{T/2}\int_{-W/2}^{W/2} S_z(f,t)\,df\,dt \le (TW)^{1/2}\left[\iint S_z^2(f,t)\,df\,dt\right]^{1/2} \quad , \tag{111}$$

and noting that the equivalent number of degrees of freedom π_0 can be written

$$\pi_0 = \frac{1}{\iint |\psi_z^0(\alpha,\beta)|^2\,d\alpha\,d\beta} = \frac{\left[\iint S_z(f,t)\,df\,dt\right]^2}{\iint S_z^2(f,t)\,df\,dt} \quad , \tag{112}$$

the following upper bound on $\mathcal{J}(\mathbf{x})$ is obtained:

$$\mathcal{J}(\mathbf{x}) \le \frac{2\pi_0^{1/2} - (TW)^{1/2}}{(TW)^{1/2}\,\pi_0} \quad . \tag{113}$$

Observe that the upper bound is maximum when

$$\pi_0 = TW = \frac{\psi_z(0,0)}{2.16N_0} \quad , \tag{114}$$

and the maximum value of this bound is $2.16N_0/\psi_z(0,0)$. This choice of π_0 corresponds to choosing the (mean)^2-to-variance ratio of the statistic S (under hypothesis H_1) equal to the (mean)^2-to-variance ratio of a statistic derived from a finite number of degrees of freedom with individual SNR equal to 2.16.

An interesting and useful conclusion follows as a result of the upper bound in the inequality (113). For the normalized mean-square error to be less than unity [$\mathcal{S}(\mathbf{x})$ positive], it is necessary that

$$\pi_0 > \frac{TW}{4} = \frac{\psi_z(0,0)}{8.64N_0} \quad , \tag{115a}$$

or

$$(\pi_0)^{-1} = \iint |R_H^0(\alpha,\beta)|^2 |\psi_x^0(\alpha,\beta)|^2 d\alpha d\beta < \frac{8.64N_0}{\psi_z(0,0)} \quad . \tag{115b}$$

The inequality (115a) places a lower bound on the equivalent number of degrees of freedom required to effectively approximate the pillbox energy density $\hat{S}_z(f,t)$. If this inequality is not satisfied, then the L_2 norm of the error, $S_z(f,t) - \hat{S}_z(f,t)$, is greater than the L_2 norm of the function we are attempting to approximate.

D. HIGH-SNR CASE

If the available SNR, $\psi_z(0,0)/N_0$, is large and the medium is not highly dispersive, then it is likely that we will have

$$\iint |R_H^0(\alpha,\beta)|^2 d\alpha d\beta >> \frac{8.64N_0}{\psi_z(0,0)} \quad . \tag{116}$$

We define an available SNR to be high (relative to the dispersive state of the medium) if the inequality (116) is satisfied. For example, if

$$\frac{S_v(\lambda,\tau)}{\iint S_v(\lambda,\tau)d\lambda d\tau} = \frac{1}{LB} \quad \text{for} \quad \begin{cases} -\frac{L}{2} \le \tau \le \frac{L}{2} \\ -\frac{B}{2} \le \lambda \le \frac{B}{2} \end{cases} ,$$

$$= 0 \quad \text{for} \quad (\lambda,\tau) \quad \text{otherwise} \quad ,$$

and $LB \ll 1$ (an underspread medium), then

$$\iint |R_H^0(\alpha,\beta)|^2 d\alpha d\beta = \frac{1}{LB} \gg 1 \quad .$$

If $\psi_z(0,0)/8.64N_0 > 1$, then the inequality of (116) is satisfied.

In the high-SNR case, it is important to choose a signal $\mathbf{x}(t)$ with ambiguity function $\psi_{\mathbf{x}}^0(\alpha,\beta)$ that counteracts $R_H^0(\alpha,\beta)$ in the integral

$$\int\int |R_H^0(\alpha,\beta)|^2 |\psi_{\mathbf{x}}^0(\alpha,\beta)|^2 d\alpha d\beta$$

and causes the result of the integration to be small enough to satisfy the inequality (115b). Both $|R_H^0(\alpha,\beta)|^2$ and $|\psi_{\mathbf{x}}^0(\alpha,\beta)|^2$ achieve a maximum value of unity (at the origin) so that the above integral is always less than the integral on the left-hand side of the inequality (116). In contrast to the LEC case in which the above integral is maximized in order to minimize the number of degrees of freedom, we must, in the present situation, generate degrees of freedom by making this integral appropriately small. In general, this can be accomplished most efficiently by choosing an input signal with ambiguity function that tends to minimize the common volume shared by both $|R_H^0(\alpha,\beta)|^2$ and $|\psi_{\mathbf{x}}^0(\alpha,\beta)|^2$. (Of course, one must not make the integral too small and thus generate an excessive number of degrees of freedom with too small a SNR.)

An attempt to obtain a small common volume shared by the two functions $|R_H^0(\alpha,\beta)|^2$ and $|\psi_{\mathbf{x}}^0(\alpha,\beta)|^2$ has the effect of concentrating transmitted signal energy in the region of the time-frequency plane where the scattering introduced by the medium is most severe; hence, the above signal-design strategy results in transmitted signals possessing time-frequency structures that emphasize the susceptibility of the signal to the dispersive effects of the medium. In this manner, one obtains an output that is sufficiently random in the sense that it is associated with an adequately large number of degrees of freedom for the available SNR.

For example, if $R_H(\alpha\ \beta)$ is narrow in α and wide in β, then one should choose an input signal with an ambiguity function that is wide in α and narrow in β. This dictates transmitting a narrow-band,

long-duration signal (for example, a long pulse) for detecting the presence of a target possessing a small time-delay spread and a large Doppler spread. The resulting output will be a rapidly time-varying process of long duration. Extending this argument to other possibilities yields Table 1, which tabulates the desired relative magnitudes of signal bandwidth and duration for various combinations of Doppler and time-delay spreads for the high-SNR case.

Table 1

SIGNAL PARAMETERS FOR VARIOUS CHANNEL CONDITIONS—HIGH-SNR CASE

MEDIUM PARAMETERS		SIGNAL PARAMETERS		REPRESENTATIVE EXAMPLES
Doppler Spread	Time-Delay Spread	Bandwidth	Duration	
Large	Small	Small	Large	Long Pulse
Small	Large	Large	Small	Short Pulse
Small	Small	Large	Large	Constant-Envelope Spread Spectrum

The derivatives of $S_z(f,t)$ at the origin possess the form

$$\left[\frac{\partial^n}{\partial f^n}\frac{\partial^m}{\partial t^m}S_z(f,t)\right]_{(f,t)=(0,0)} = (-1)^n(i2\pi)^{n+m}\iint \alpha^n\beta^m R_H(\alpha,\beta)\psi_x(\alpha,\beta)\,d\alpha\,d\beta \quad .$$

The choice of an input signal that achieves a small common volume for the functions $|R_H^0(\alpha,\beta)|^2$ and $|\psi_x^0(\alpha,\beta)|^2$ will also tend to make the magnitude of the product, $R_H(\alpha,\beta)\psi_x(\alpha,\beta)$, small in the α-β plane. From the above equation, it can be seen that this has a flattening effect on the output energy density, since the derivatives at the origin will tend to be small.

XIII THE KULLBACK-LEIBLER INFORMATION NUMBER VERSUS DIVERGENCE AS A CRITERION FOR SIGNAL DESIGN

Grettenberg [Ref. 38] has suggested the signal-design criterion of maximizing the divergence between the signal-plus-noise and noise hypotheses. Since maximizing either the Kullback-Leibler information number or the divergence is a suboptimum strategy when the medium is randomly time-varying and frequency-selective and the receiver is a Neyman-Pearson detector, it is of interest to discuss the differences between the two criteria.

The divergence is defined as the sum of two Kullback-Leibler information numbers,

$$J(H_1,H_0) = I(H_1:H_0) + I(H_0:H_1) \tag{117a}$$

$$J(H_1,H_0) = \int \{f_S(s|H_1) - f_S(s|H_0)\} \log \frac{f_S(S|H_1)}{f_S(S|H_0)}\, ds \tag{117b}$$

$$J(H_1,H_0) = \sum_n \gamma_n - \sum_n \frac{\gamma_n}{1+\gamma_n} \quad . \tag{117c}$$

For LEC conditions ($\gamma_1 \ll 1$), the divergence is dominated by the sum of the squares of the noise-normalized eigenvalues,

$$J(H_1,H_0) = \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} (-1)^k \gamma_n^k \quad ,$$

$$\lim_{\frac{\sum_n \gamma_n^3}{\sum_n \gamma_n^2} \to 0} J(H_1,H_0) = \sum_n \gamma_n^2 \quad . \tag{118}$$

Therefore, for suitably small normalized eigenvalues, maximizing the divergence is equivalent to maximizing the Kullback-Leibler information number $I(H_0:H_1)$ [Eq. (78)]. Both criteria, in view of the approach to

normality of the statistic S/N_0, yield an optimum signal-design strategy for LEC conditions.

We can conclude, from the lemma on page 46, that for any finite dimension m, the divergence,

$$J(H_1,H_0) = \sum_{n=1}^{m} \gamma_n - \sum_{n=1}^{m} \frac{\gamma_n}{1+\gamma_n} ,$$

is convex in any open convex set in the region

$$\Gamma^m = \left\{ \gamma : \gamma_n \geq 0,\ n = 1,2,\ldots,m,\ \sum_{n=1}^{m} \gamma_n \leq \frac{E_0}{N_0} \right\}$$

and hence attains its maximum in the $(m-1)$-dimensional boundary of this region. Reiterating this argument until the dimension is unity, we conclude that the divergence is maximized in the region Γ^m by choosing one degree of freedom possessing the total available SNR.

The divergence criterion of choosing one degree of freedom independent of the available SNR is in marked contrast to the criterion of dividing the available SNR equally among degrees of freedom with individual SNR in the vicinity of 2. Since the divergence is a convex function of the γ_n, this criterion will tend to minimize the number of degrees of freedom at the output of a dispersive medium. When the available SNR is large and the medium is not highly dispersive (high-SNR case discussed in the last chapter), maximizing $I(H_0:H_1)$ dictates a signal-design strategy that is in opposition to the strategy dictated by maximizing $J(H_1,H_0)$. The first strategy directs one to generate degrees of freedom at the output; the second strategy directs one to limit the number of degrees of freedom at the output.

To assess the results obtainable by using the two criteria and to shed light on the above difficulty, we have computed various typical receiver operating points for the Neyman-Pearson detector. Table 2 tabulates the false-dismissal probabilities, $\beta[I(H_0:H_1)]$ and $\beta[J(H_1,H_0)]$, obtained, respectively, by maximizing $I(H_0:H_1)$ and $J(H_1,H_0)$, for the same value of false-alarm probability α. The various error probabilities were computed from the gamma probability law; the number of

Table 2

COMPARISON OF VARIOUS RECEIVER OPERATING POINTS ACHIEVED BY THE KULLBACK-LEIBLER INFORMATION-NUMBER AND DIVERGENCE CRITERIA

SNR = $10 \log_{10} \Psi_z(0,0)/N_0$	π_0	α	$\beta[I(H_0:H_1)]$	$\beta[J(H_1,H_0)]$
10 db	5	1.31×10^{-1}	1.09×10^{-1}	1.78×10^{-1}
13 db	10	7×10^{-2}	3.18×10^{-2}	1.19×10^{-1}
16 db	20	6×10^{-3}	9.29×10^{-3}	1.17×10^{-1}
17.8 db	30	2.18×10^{-3}	1.13×10^{-3}	9.6×10^{-2}
19 db	40	2.28×10^{-4}	3.57×10^{-4}	9.85×10^{-2}
20 db	50	4.65×10^{-5}	7.36×10^{-5}	8.75×10^{-2}

degrees of freedom used to compute α and $\beta[I(H_0:H_1)]$ (denoted by π_0) was determined by dividing the available SNR by the near-optimum value of 2. Similarly, the error probabilities, α and $\beta[J(H_1,H_0)]$, were computed by using one degree of freedom with the total available SNR.

At moderate values of SNR, the computed points in Table 2 display a striking difference in the false-dismissal probabilities, $\beta[I(H_0:H_1)]$ and $\beta[J(H_1,H_0)]$. If the available SNR is 20 db and a false-alarm probability of 4.65×10^{-5} is chosen, then $\beta[J(H_1,H_0)]$ is more than 10^3 times greater than $\beta[I(H_0:H_1)]$.

The improvement in error-probability performance gained by dividing the SNR among independent degrees of freedom is not surprising. By receiving many degrees of freedom (with the proper SNR), we obtain a diversity improvement affording a protection against the time- and frequency-selective effects of the medium that is not obtained by receiving a signal with only one degree of freedom. If all the received signal energy resides in one degree of freedom, it is more probable for this one degree of freedom to be in a time- or frequency-selective null (that is, the gain of the medium is low for the particular degree of freedom) than it is for many independent degrees of freedom to simultaneously be in a time- or frequency-selective null.

XIV CONCLUSIONS

In attacking the problem of designing efficient signals for transmission over randomly dispersive media, we have proceeded along two fairly distinct lines of analysis: the mathematical characterization of the medium and the analysis of the detection problem.

By viewing the medium transfer function as a homogeneous random field on the time-frequency plane, we have introduced a unified system theory of random filters which has provided the necessary framework in which to cast the signal-design problem. In addition, it has been demonstrated throughout this study that this model affords a comprehensive and superior physical viewpoint for the communication engineer's task of treating systems that incorporate dispersive media. The various second-order measures of the time and frequency structures of a process and their resulting transformations through a dispersive medium provide not only a fundamental and complete second-order theory for the statistical description of such signals but also a useful physical interpretation of the dispersive effects of the medium on input signals.

The analysis of the detection problem has ultimately led to the choice of maximizing the Kullback-Leibler information number as a signal-design strategy. The number $I(H_0:H_1;\mathbf{x})$ (which may be regarded as an asymmetric distance between the signal-plus-noise and noise probability laws) is a measure of the ability of the Neyman-Pearson detector to discriminate against H_1 when H_0 is true and the transmitted signal is $\mathbf{x}$. However, $I(H_0:H_1;\mathbf{x})$ also measures the exponential rate at which the false-dismissal probability approaches zero for an increasing sequence of observations when the false-alarm probability is fixed. In this sense, $I(H_0:H_1;\mathbf{x})$ may be regarded as a measure of the ability of the detector to discriminate against H_0 when H_1 is true, and, indeed, it is this property of $I(H_0:H_1;\mathbf{x})$ that has been exploited in the present study. It is clear that the Kullback-Leibler information numbers should not be rigidly interpreted as only unidirectional measures of the ability of the detector to discriminate one hypothesis

from the other, since the performance of the detector under one hypothesis is not independent of its performance under the other. For example, maximizing a measure of the detector's capability to discriminate against H_1 when H_0 is true may imply, for a fixed false-alarm probability, that a lower threshold can be employed, which, in turn, may imply a lower false-dismissal probability.

We have shown that the criterion of maximizing $I(H_0:H_1;\mathbf{x})$ yields an optimum signal-design strategy (the false-dismissal probability is minimized at any reasonable false-alarm probability) for suitably small noise-normalized eigenvalues at the output of the medium. Additionally, it has been shown that this criterion, in the general case, dictates the very reasonable signal-design strategy of transmitting a signal that produces an output possessing degrees of freedom with individual SNR in the vicinity of 2. In other words, it is desirable to apportion the available SNR equally among degrees of freedom with the proper individual SNR and hence to obtain a diversity protection against the time- and frequency-selective fading phenomena of the medium.

In contrast to the divergence criterion which tends to minimize the degrees of freedom at the output of the medium at all SNR, the Kullback-Leibler information-number criterion tends to limit or generate degrees of freedom at the output as a function of the available SNR and the dispersive state of the medium. Geometrically, this behavior is explained by the convexity of the divergence as a function of the noise-normalized eigenvalues, whereas the Kullback-Leibler information number is convex or concave in a region where the noise-normalized eigenvalues are all less than or greater than one, respectively.

On the one hand, if the available SNR and the dispersive state of the medium are such that the SNR of an individual degree of freedom is necessarily small, then maximizing $I(H_0:H_1;\mathbf{x})$ tends to limit the region in the time-frequency plane occupied by the output signal energy. This strategy directly combats the medium dispersion. On the other hand, if the available SNR and the dispersive state of the medium are such that the SNR of an individual degree of freedom tends to be large, then maximizing $I(H_0:H_1;\mathbf{x})$ tends to enlarge the region

in the time-frequency plane occupied by the output signal energy. This strategy uses the medium dispersion to create the desired number of degrees of freedom.

The results of the medium-characterization and detection-problem analyses have been combined to yield synthesis constraints on the input ambiguity function. One constraint amounts to designing an input signal that minimizes the number of degrees of freedom at the output of the medium when the SNR of each degree of freedom is small. This is achieved by maximizing the common volume shared by the squared envelopes of the medium time-frequency autocorrelation function and the input ambiguity function or, equivalently, choosing an input energy density resembling the medium scattering function. The other constraint ensures the best mean-square approximation to an energy density that is nonzero and constant only over a rectangular region in the output time-frequency plane. We have shown that the latter nonrealizable energy density provides the maximum number of degrees of freedom with SNR in the vicinity of 2 and hence maximizes the Kullback-Leibler information number.

For the high-SNR case, a lower bound has been deduced for the equivalent number of degrees of freedom required by the output process to effectively approximate the optimum energy density. We have concluded that, if the SNR is large and the medium is not highly dispersive, then it is desirable to transmit signals that emphasize the susceptibility of the transmission to the dispersive effects of the medium. This strategy amounts to using the dispersion introduced by the medium to achieve a desirable diversity reception. The required degree of diversity is achieved by making the common volume shared by the squared envelopes of the medium time-frequency autocorrelation function and the input ambiguity function suitably small or, equivalently, by choosing an input energy density sufficiently different from the medium scattering function.

APPENDIX A

COMPLEX ENVELOPES

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COMPLEX ENVELOPES

This appendix is concerned with defining the complex envelopes of the various signals that enter in the system analysis.

1. INPUT COMPLEX ENVELOPES

In communication and radar applications, the transmitted signal is the result of amplitude and phase modulating a CW tone. That is, in any particular application, an amplitude modulation $A(t)$ and a phase modulation $\phi(t)$ are basic system inputs that define the transmitted signal

$$x(t) = A(t) \cos 2\pi[f_0 t + \phi(t)] \quad . \tag{A.1}$$

The input complex envelope is a complex function of time with amplitude $A(t)$ and phase $\phi(t)$

$$\mathbf{x}(t) = A(t)e^{i2\pi\phi(t)} \quad . \tag{A.2}$$

Equation (A.1) can be rewritten in terms of the input complex envelope

$$x(t) = \text{Re}\{\mathbf{x}(t)e^{i2\pi f_0 t}\} \quad . \tag{A.3}$$

In general, the complex envelope $\mathbf{x}(t)$ can be quite arbitrary. The amplitude and phase functions need not be related, or, equivalently, the quadrature components of $\mathbf{x}(t)$ can be chosen independently. For example, practical applications exist in which the quadrature components of $\mathbf{x}(t)$ may be derived from two independent binary information sources, or one of the quadrature components may serve as a pilot tone to be used in the demodulation process at the receiver, while the other quadrature component is information modulated.

It is assumed that the demodulation process employed by the receiver is the inverse of the modulation process employed by the

transmitter. That is, if the receiver were directly connected to the transmitter, then, in the absence of additive noise, the receiver would exactly reproduce the input amplitude and phase modulation functions. It has been assumed that $x(t)$ is a narrow-band signal; practical narrow-band systems closely approximate the above idealized assumptions.

2. OUTPUT COMPLEX ENVELOPES

The system functions associated with the randomly dispersive medium explicitly define the output complex envelope as a transformation of the input complex envelope. For example, we can write the output complex envelope in the form

$$\mathbf{z}(t) = \iint v(\lambda,\tau)\mathbf{x}(t-\tau)e^{i2\pi\lambda t}d\lambda d\tau \quad . \tag{A.4}$$

When the transfer function $H(t,f)$ is homogeneous, we have found that the system function $v(\lambda,\tau)$ in the above equation must, in general, be a nonstationary, complex, white-noise random field,

$$E[v^*(\lambda_1,\tau_1)v(\lambda_2,\tau_2)] = S_v(\lambda_1,\tau_1)\delta(\lambda_1-\lambda_2)\delta(\tau_1-\tau_2) \quad . \tag{A.5}$$

To satisfy the condition in Eq. (A.5), it has been assumed that the quadrature components of the random field $v(\lambda,\tau)$ form a mutually uncorrelated family of zero-mean random variables, and additionally, it has been assumed that for a given λ and τ the quadrature components of $v(\lambda,\tau)$ are identically distributed random variables. These assumptions lead to the following conditions on the quadrature components of the random field $v(\lambda,\tau)$:

$$E[v_R(\lambda_1,\tau_1)v_R(\lambda_2,\tau_2)] = E[v_I(\lambda_1,\tau_1)v_I(\lambda_2,\tau_2)] = \frac{1}{2}S_v(\lambda_1,\tau_1)\delta(\lambda_1-\lambda_2)\delta(\tau_1-\tau_2) \tag{A.6a}$$

$$E[v_R(\lambda_1,\tau_1)v_I(\lambda_2,\tau_2)] = E[v_I(\lambda_1,\tau_1)v_R(\lambda_2,\tau_2)] = 0 \quad \text{for all } \lambda_1, \tau_1, \lambda_2, \text{ and } \tau_2. \tag{A.6b}$$

The physical significance of these assumptions merits some discussion. We have interpreted the Doppler-delay random field $v(\lambda,\tau)$ in terms of a decomposition of the medium into incremental Doppler-shift time-delay channels. In this decomposition, the complex number

$v(\lambda,\tau)d\lambda d\tau$ corresponds to the random amplitude and phase introduced by an incremental channel. The orthogonality relation of Eq. (A.5) is equivalent to the following relations:

$$E[v_R(\lambda_1,\tau_1)v_R(\lambda_2,\tau_2)] + E[v_I(\lambda_1,\tau_1)v_I(\lambda_2,\tau_2)] = S_v(\lambda_1,\tau_1)\delta(\lambda_1-\lambda_2)\delta(\tau_1-\tau_2) \tag{A.7a}$$

$$E[v_R(\lambda_1,\tau_1)v_I(\lambda_2,\tau_2)] - E[v_I(\lambda_1,\tau_1)v_R(\lambda_2,\tau_2)] = 0 \quad . \tag{A.7b}$$

It is reasonable to assume that the quadrature components $v_R(\lambda,\tau)$ and $v_I(\lambda,\tau)$ are identically distributed random fields. This assumption and Eq. (A.7a) lead to the condition of Eq. (A.6a). It has been assumed that $v_R(\lambda,\tau)$ and $v_I(\lambda,\tau)$ posses zero mean. Since our assumptions so far have implied that $v_R(\lambda,\tau)$ and $v_I(\lambda,\tau)$ are individually uncorrelated in the $\lambda-\tau$ plane, a reasonable interpretation of Eq. (A.7b) leads us to assume that the random fields $v_R(\lambda,\tau)$ and $v_I(\lambda,\tau)$ possess zero cross correlation for all shifts in both arguments. In other words, we assume that the orthogonality condition of Eq. (A.5) is obtained by a completely uncorrelated scattering mechanism. This assumption implies Eq. (A.6b). For the zero-mean Gaussian medium, the relations of Eq. (A.6) simply imply that the incremental channels are statistically independent and that each incremental channel introduces a uniformly distributed random phase and a Rayleigh-distributed random gain.

Equations (A.6a) and (A.6b) imply that

$$E[v(\lambda_1,\tau_1)v(\lambda_2,\tau_2)] = 0 \tag{A.8}$$

for all λ_1, τ_1, λ_2, and τ_2. We can deduce the following relation from Eqs. (A.4) and (A.8),

$$E[\mathbf{z}(t)\mathbf{z}(s)] = 0 \quad , \quad \text{for all } t \text{ and } s. \tag{A.9}$$

Finally, Eq. (A.9) implies that

$$E[z_R(t)z_R(s)] = E[z_I(t)z_I(s)] \tag{A.10a}$$

$$E[z_R(t)z_I(s)] = -E[z_I(t)z_R(s)] \quad . \tag{A.10b}$$

3. COMPLEX ENVELOPES OF STATIONARY RANDOM PROCESSES

If $n(t)$ is a real stationary random process with spectral density (one-sided) $S_n(f)$, then $n(t)$ has the spectral representation [Ref. 4]:

$$n(t) = \int_0^\infty \cos 2\pi\lambda t\, du(\lambda) + \int_0^\infty \sin 2\pi\lambda t\, dv(\lambda) \quad , \tag{A.11}$$

where $u(\lambda)$ and $v(\lambda)$ are real orthogonal-increment random processes with

$$E[\{du(\lambda)\}^2] = E[\{dv(\lambda)\}^2] = S_n(\lambda)d\lambda$$

and

$$E[du(\lambda)dv(\lambda)] = 0 \quad .$$

The complex envelope of $n(t)$ relative to the frequency f_0 is developed by shifting the process $n(t)$ by f_0 cps in the frequency domain. In Eq. (A.11), let $\lambda = f_0 + f$ and define:

$$u_0^+(f) = \begin{cases} u(f_0 + f) + u(f_0 - f) & 0 \le f \le f_0 \\ u(f + f_0) & f_0 < f \end{cases}$$

$$u_0^-(f) = \begin{cases} u(f_0 + f) - u(-f + f_0) & 0 \le f \le f_0 \\ u(f_0 + f) & f_0 < f \end{cases}$$

$$v_0^+(f) = \begin{cases} v(f_0 + f) + v(f_0 - f) & 0 \le f \le f_0 \\ v(f_0 + f) & f_0 < f \end{cases}$$

$$v_0^-(f) = \begin{cases} v(f_0 + f) - v(f_0 - f) & 0 \le f \le f_0 \\ v(f_0 + f) & f_0 < f \end{cases} .$$

The above four processes possess orthogonal increments, and the increments of the u processes are orthogonal to the increments of the v processes. The spectral representation in Eq. (A.11) can be written in the form

$$n(t) = x(t)\cos 2\pi f_0 t - y(t)\sin 2\pi f_0 t \quad , \tag{A.12}$$

where $x(t)$ and $y(t)$ are real stationary random processes with spectral representations

$$x(t) = \int_0^\infty \cos 2\pi f t \, du_0^+(f) + \int_0^\infty \sin 2\pi f t \, dv_0^-(f) \tag{A.13a}$$

$$y(t) = \int_0^\infty \sin 2\pi f t \, du_0^-(f) - \int_0^\infty \cos 2\pi f t \, dv_0^+(f) \quad . \tag{A.13b}$$

The following relations can be verified by direct computation:

$$R_x(\tau) = E[x(t)x(t+\tau)] = \int_0^\infty S_x(f)\cos 2\pi f \tau \, d\tau \tag{A.14a}$$

$$R_y(\tau) = E[y(t)y(t+\tau)] = R_x(\tau) \tag{A.14b}$$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = -R_{yx}(\tau) = \int_0^\infty S_{xy}(f)\sin 2\pi f \tau \, d\tau \quad , \tag{A.14c}$$

where

$$S_x(f) = \begin{cases} S_n(f_0+f) + S_n(f_0-f) & 0 \le f \le f_0 \\ S_n(f_0+f) & f_0 < f \end{cases} \tag{A.15a}$$

and

$$S_{xy}(f) = \begin{cases} S_n(f_0+f) - S_n(f_0-f) & 0 \le f \le f_0 \\ S_n(f_0+f) & f_0 < f \end{cases} \quad . \tag{A.15b}$$

If $n(t)$ is white noise with spectral intensity N_0, then

$$S_x(f) = \begin{cases} 2N_0 & 0 \le f \le f_0 \\ N_0 & f_0 < f \end{cases} \tag{A.16a}$$

and

$$S_{xy}(f) = \begin{cases} 0 & 0 \leq f \leq f_0 \\ N_0 & f_0 < f \end{cases} . \qquad \text{(A.16b)}$$

In the analysis of narrow-band systems, in which the spectra of the quadrature components of the received signals are significantly non-zero only in a range of frequencies much less than the center frequency f_0, $S_x(f)$ is essentially flat and $S_{xy}(f)$ is essentially zero relative to the frequency range of interest. We then obtain the approximations:

$$R_x(\tau) \doteq 2N_0\delta(\tau) \qquad \text{(A.17a)}$$

and

$$R_{xy}(\tau) \doteq 0 . \qquad \text{(A.17b)}$$

APPENDIX B

SIGNIFICANCE OF THE ERROR BETWEEN THE OBSERVED PROCESS AND ITS KARHUNEN-LOÈVE EXPANSION

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SIGNIFICANCE OF THE ERROR BETWEEN THE OBSERVED PROCESS AND ITS KARHUNEN-LOÈVE EXPANSION

The statistic S is based on the process $\hat{\mathbf{y}}(t)$ instead of the observed process $\mathbf{y}(t)$. Price [Ref. 35] points out that, in general, the two process are not equivalent, because of the difficulty in obtaining a complete set of orthonormal functions that satisfactorily represent a process containing white noise. However, Price offers a plausible argument for ignoring the difference between the two processes.

In this appendix we extend one of Price's arguments [Ref. 35] and prove that the quadrature components of the error process $\boldsymbol{\epsilon}(t)$ defined by the relation

$$\mathbf{y}(t) = \hat{\mathbf{y}}(t) + \boldsymbol{\epsilon}(t) \tag{B.1}$$

are mutually independent of the quadrature components of the process $\hat{\mathbf{y}}(t)$ for all time shifts under both hypotheses. We first prove that

$$E[\hat{\mathbf{y}}^*(s)\boldsymbol{\epsilon}(t)] = 0 \quad , \quad s,t, \in (0,T) \quad , \tag{B.2}$$

$$E[\hat{\mathbf{y}}^*(s)\epsilon(t)] = \sum_n E[\mathbf{y}_n^*\boldsymbol{\epsilon}(t)]\mathbf{f}_n^*(s) \tag{B-3}$$

$$\begin{aligned} E[\mathbf{y}_n^*\epsilon(t)] &= E\left[\{\mathbf{y}(t) - \hat{\mathbf{y}}(t)\}\int_0^T \mathbf{y}^*(s)\mathbf{f}_n(s)ds\right] \\ &= \int_0^T K_{\mathbf{y}}(s,t)\mathbf{f}_n(s)ds - \sum_m \left[\int_0^T\int_0^T \mathbf{f}_m^*(r)K_{\mathbf{y}}(s,r)\mathbf{f}_n(s)drds\right]\mathbf{f}_m(t) \\ &= \int_0^T [K_{\mathbf{z}}(s,t) + N_0\delta(t-s)]\mathbf{f}_n(s)ds \\ &\quad - \sum_m \left\{\int_0^T\int_0^T \mathbf{f}_m^*(r)[K_{\mathbf{z}}(s,r) + N_0\delta(t-s)]\mathbf{f}_n(s)drds\right\}\mathbf{f}_m(t) \\ &= (\lambda_n + N_0)\mathbf{f}_n(t) - \sum_m \left\{\int_0^T \mathbf{f}_m^*(r)\mathbf{f}_n(r)[\lambda_n + N_0]dr\right\}\mathbf{f}_m(t) \\ &= (\lambda_n + N_0)\mathbf{f}_n(t) - (\lambda_n + N_0)\mathbf{f}_n(t) = 0 \quad . \end{aligned}$$

Observe that the above arguments apply to the noise hypothesis by setting the λ_n to zero. We conclude from Eq. (B.3) that

$$E[\hat{\mathbf{y}}^*(s)\boldsymbol{\epsilon}(t)] = 0 \quad , \quad s,t \in (0,T) \quad ,$$

under both hypotheses.

The expectation

$$E[\hat{\mathbf{y}}(s)\epsilon(t)]$$

is also equal to zero for $s,t \in (0,T)$:

$$E[\hat{\mathbf{y}}(s)\boldsymbol{\epsilon}(t)] = \sum_n E[\mathbf{y}_n\boldsymbol{\epsilon}(t)]\mathbf{f}_n(s) \quad . \tag{B.4}$$

$$E[\mathbf{y}_n\boldsymbol{\epsilon}(t)] = E\left[\{\mathbf{y}(t) - \hat{\mathbf{y}}(t)\}\int_0^T \mathbf{y}(s)\mathbf{f}_n^*(s)ds\right]$$

$$= \int_0^T E[\mathbf{y}(t)\mathbf{y}(s)]\mathbf{f}_n^*(s)ds - \sum_m \left\{\int_0^T\int_0^T \mathbf{f}_m^*(r)E[\mathbf{y}(r)\mathbf{y}(s)]\mathbf{f}_n^*(s)drds\right\}\mathbf{f}_m(t)$$

$$= 0 \quad ,$$

since

$$E[\mathbf{y}(t)\mathbf{y}(s)] = E[\mathbf{y}(r)\mathbf{y}(s)] = 0$$

under either hypothesis. Hence, from Eq. (B.4) we conclude

$$E[\hat{\mathbf{y}}(s)\boldsymbol{\epsilon}(t)] = 0 \quad , \quad s,t \in (0,T) \quad . \tag{B.5}$$

The relation

$$E[\hat{\mathbf{y}}^*(s)\boldsymbol{\epsilon}(t)] = 0$$

implies the following conditions on the quadrature components:

$$E[\hat{y}_R(s)\epsilon_R(t)] + E[\hat{y}_I(s)\epsilon_I(t)] = 0 \tag{B.6}$$

$$-E[\hat{y}_I(s)\epsilon_R(t)] + E[\hat{y}_R(s)\epsilon_I(t)] = 0 \quad . \tag{B.7}$$

Similarly, the relation

$$E[\hat{\mathbf{y}}(s)\boldsymbol{\epsilon}(t)] = 0$$

implies the additional conditions on the quadrature components:

$$E[\hat{y}_R(s)\epsilon_R(t)] - E[\hat{y}_I(s)\epsilon_I(t)] = 0 \qquad \text{(B.8)}$$

$$E[\hat{y}_I(s)\epsilon_R(t)] + E[\hat{y}_R(s)\epsilon_I(t)] = 0 \quad . \qquad \text{(B.9)}$$

Solving Eqs. (B.6) through (B.9) yields zero for all four expectations. Since the random variables possess zero-mean Gaussian distributions, the quadrature components $\epsilon_R(t)$ and $\epsilon_I(t)$ are mutually independent of the quadrature components $\hat{y}_R(s)$ and $\hat{y}_I(s)$.

Equations (36a) and (36b) imply that the process $\hat{\mathbf{y}}(t)$ can be written

$$\hat{\mathbf{y}}(t) = \sum_n \left[\int_0^T \mathbf{z}(s)\mathbf{f}_n^*(s)ds\right]\mathbf{f}_n(t) + \sum_n \left[\int_0^T \mathbf{n}(s)\mathbf{f}_n^*(s)ds\right]\mathbf{f}_n(t) \quad . \qquad \text{(B.10)}$$

For dispersive media that possess a continuous covariance kernel $K_z(t,s)$ (a reasonable assumption for physical media), the orthonormal functions $\{\mathbf{f}_n(t)\}$ form a complete set for representing the output process $\mathbf{z}(t)$,

$$\mathbf{z}(t) = \sum_n \left[\int_0^T \mathbf{z}(s)\mathbf{f}_n^*(s)ds\right]\mathbf{f}_n(t) \quad . \qquad \text{(B.11)}$$

Equations (B.1), (B.10), and (B.11) imply that the error $\boldsymbol{\epsilon}(t)$ is given by

$$\boldsymbol{\epsilon}(t) = \mathbf{n}(t) - \sum_n \left[\int_0^T \mathbf{n}(s)\mathbf{f}_n^*(s)ds\right]\mathbf{f}_n(t) \quad . \qquad \text{(B.12)}$$

Therefore, under either hypothesis, the error $\boldsymbol{\epsilon}(t)$ depends only on the noise $\mathbf{n}(t)$ and contains no information concerning the presence or absence of the signal $\mathbf{z}(t)$[Ref. 35]. In this case, it is clear that the independence of the processes $\boldsymbol{\epsilon}(t)$ and $\mathbf{y}(t)$ follows as a result of a noise component of the process $\mathbf{n}(t)$ that is in a space orthogonal to the space spanned by the eigenfunctions $\{\mathbf{f}_n(t)\}$ [a consequence of Eq. (B.12)]. We can argue (heuristically) that, since the signal component is zero in the direction of this orthogonal space, there is no need to consider observations from this space. (This is easily verified in the finite dimensional case.) In other words, if it is known that a particular coordinate of the observation contains noise alone that is independent of the noise on all the other coordinates, we gain nothing

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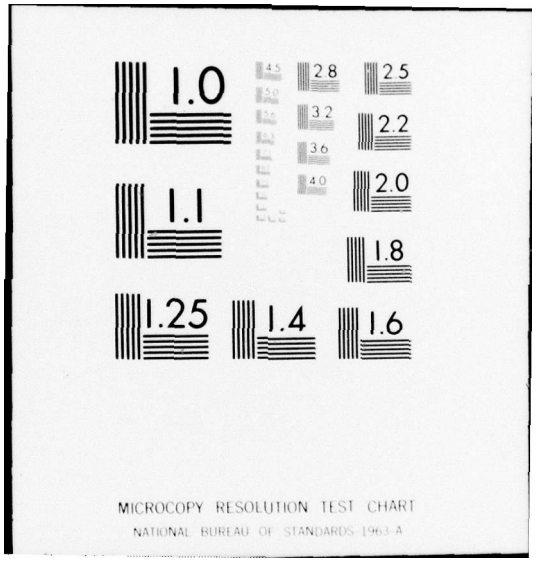
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by obtaining this coordinate. No information is obtained concerning the presence or absence of the signal. No information is obtained concerning the noise on coordinates that may contain signal. More precisely, it is necessary only to consider the projection of the noise $\mathbf{n}(t)$ on the space spanned by the signal process for the detection problem treated in this study.

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