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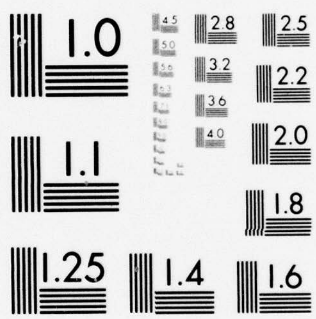

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Theoretical Explanation of Spectral Slopes in Stratospheric Turbulence Data and Implications for Vertical Transport

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18 October 1976

AERONOMY DIVISION PROJECT 6687
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This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

John Howard

Chief Scientist

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This paper is motivated by ~~our~~ the requirement to improve understanding of vertical motion of pollutants in the stratosphere. One method to estimate vertical transport due to the effects of turbulence is by means of the "effective diffusivity" coefficient. To calculate this parameter, it is often necessary to know the value of ε, the rate of turbulent dissipation. This parameter, ε, is also important to know in the context of the global numerical stratospheric simulation models now being created for environmental assessment purposes. One of the most often used methods to measure ε employs turbulence spectra.

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20. Abstract (Continued)

These spectra as seen in the HICAT data, however, have anomalous properties which must be understood before we can intelligently use them outside the isotropic range for the purpose of measuring ε. The HICAT power spectra of stratospheric turbulence had -5/3 slopes on log-log graphs that extended up to wavelengths as long as 40,000 ft. Standard turbulence theory of the inertial subrange would account for this observation only to wavelengths of about 150 ft. The purpose of this report is to extend existing theory to explain this phenomenon. The results should apply to all turbulence generated by the Kelvin-Helmholtz instability in a stratified fluid far from a boundary. They also may explain certain well known spectral nonlinearities, but the most important practical outcome is that a spectral estimate of turbulent dissipation rate must be based on itotropic range wave numbers to avoid error despite the -5/3 slope.

Preface

The author wishes to express thanks to Dr. R. E. Good for his interest in this problem and the many encouraging and stimulating remarks he made during its progress. In addition, the author is grateful to Dr. A. F. Quesada, Dr. C. Yang, and S. P. Zimmerman for several helpful suggestions.

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Illustrations

Theoretical Explanation of Spectral Slopes in Stratospheric Turbulence Data and Implications for Vertical Transport

I. INTRODUCTION

Stratospheric turbulence, like tropospheric CAT, takes the form of layers which are broad in the horizontal direction (for example, 25 km) but thin in the vertical direction (typically between 100 m to 1000 m depending on how measured (Rosenberg and Dewan,[1] Anderson,[2] and Barat.[3] The power spectra of the turbulent velocity fluctuations within such layers have slopes (on log-log plots) which are roughly in the vicinity of -5/3 to wavelengths as long as 10,000 ft, or possibly as long as 40,000 ft. (See Figure 1.) More specifically, these slopes fall in the range of -1.5 to -1.7 for horizontal and -1.25 to -1.4 for vertical fluctuations (HICAT data*). While one would expect slopes of -5/3 for spectra in the inertial sub-range, one would not ordinarily expect to find them at wavelengths far exceeding the scale for isotropy. Isotropy is found up to 100 to 150 ft in the case of stratospheric turbulence (HICAT data*). The purpose of this paper is to explain the presence of this anomalous -5/3 range of spectra in data from stratospheric

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(Received for publication 30 September 1976)

*References to HICAT data are found in Crooks, et al, ref. 45.)

1. Rosenberg, N.W., and Dewan, E.M. (1975) Stratospheric Turbulence and Vertical Effective Diffusion Coefficients, AFCRL-TR-75-0519.
2. Anderson, A.D. (1957) Free-Air turbulence, J. Meteorol. 14:477-494.
3. Barat, J. (1975) Étude expérimentale de la structure du champ de turbulence dans la moyenne stratosphere, C.R. Acad. Sc. Paris 280(Serial B):691-693.

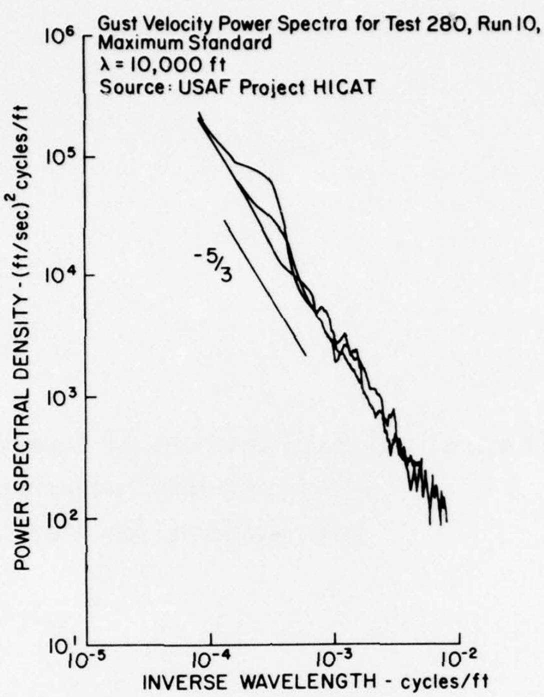

Figure 1. Typical HICAT Turbulence Spectrum Showing -5/3 Slope Beyond the Isotropic Range

turbulence. As mentioned in the abstract, this is of use in the general vertical transport problem.

It is generally agreed that CAT, and presumably most stratospheric turbulence, is generated via the Kelvin-Helmholtz (K-H) instability mechanism (Rosenberg and Dewan,[1] and Dutton[4, 5]). When the shear across a stratified layer is sufficient to cause the Richardson number to go below 0.25, an unstable wave presumably builds up and eventually breaks down in such a way as to generate a layer of turbulence. Excellent experimental information and photographs will be

4. Dutton, J.A. (1971) CAT aviation and atmospheric science, Rev. Geophys. 9:613-657.

5. Dutton, J.A. (1973) Recent perspectives on turbulence in the free atmosphere, in N.K. Vinnichenko, N.Z. Pinus, S.M. Shmeter, and G.N. Shur, Turbulence in the Free Atmosphere, Consultants Bureau, New York.

found in the work of Thorpe,[6] Woods,[7,8] Battan,[9] and Ludlam.[10] The classic theoretical paper is that of Rosenhead.[11]

The K-H mechanism is responsible for turbulence to be found below the surface of the ocean, and it has been said that the resulting thin layers of turbulence which are thus generated intermittently in space and time, are the primary cause of what vertical transport there is in the interior of the world ocean (Woods and Wiley[12]). It is interesting to note that, in this connection, some stratospheric and other physicists have become aware that there is a useful and close analogy between the dynamics of the ocean and the stratosphere; for example, see Thorpe,[13] Rosenberg and Dewan,[1] Woods and Wiley,[12] and Stewart and Bolgiano.[14]

An anomalous -5/3 slope (that is, one found outside the inertial range) was not seen for the first time in the HICAT spectral data by any means, but rather it has been repeatedly seen in stratified media. For example, Pond et al[15] reported the results of an experiment where the theoretical lower limit of the inertial range was $k = 0.1\ cm^{-1}$ (k is the wavenumber) whereas, in contrast, the $k^{-5/3}$ spectrum provided a reasonable fit to $k = 0.005\ cm^{-1}$! In another case, Stewart[16] made the following statement which is perhaps the most precise description of the problem at hand: "It is well known that -5/3 spectra appear very frequently when one observes along a horizontal path in a stratified fluid. These spectra are observed even when it is quite apparent that the turbulence observed is very far from isotropic. ··· This point needs more carefull consideration ···· than it has yet received."

6. Thorpe, S.A. (1973b) Experiments on stability and turbulence in a stratified shear flow, J. Fluid Mech. 61(Part 4):731-751.

7. Woods, J.D. (1969) On Richardson's number as a criterion for laminar-turbulent-laminar transition in the ocean and atmosphere, Radio Sci. 12:1289-1298.

8. Woods, J.D. (1968) Wave-induced shear instability in the summer thermodine, J. Fluid. Mech. 32:791-800.

9. Battan, L.J. (1973) Radar Observation of the Atmosphere, University of Chicago Press.

10. Ludlam, F.H. (1967) Characteristics of billow clouds and their relation to clear air turbulence, Quart. J. Roy. Met. Soc. 93:419-435.

11. Rosenhead, L. (1931) The formation of vortices from a surface of discontinuity, Proc. Roy. Soc. 134:170-192.

12. Woods, J.D., and Wiley, R.L. (1972) Billow turbulence and ocean microstructure, Deep Sea Research and Oceanic Abst. 19:87-121.

13. Thorpe, S.A. (1973a) CAT in the Laboratory, Weather.

14. Stewart, R.W. and Bolgiano, R. (1969) Comments in: Clear air turbulence and its detection, edited by Y.H. Pao and A. Goldberg, Plenum Press, p. 515-520.

15. Pond, S., Steward, R.W., and Burling, R.W. (1963) Turbulence spectra in the wind over waves, J. Atm. Sci. 20:319-321.

16. Stewart, R.W. (1969) Turbulence and waves in a stratified atmosphere, Radio Sci. 4:1269-1278.

In another case, Pao[17] described this phenomenon in a stratified turbulent flow as seen in a towing tank as follows: "However, the turbulence is too weak to have a distinct $f^{-5/3}$ inertial range. This 'anomalous -5/3 subrange,' we believe, is due to the presence of the internal wave spectral peak, lifting up the spectral curve as a result. ...However, our results (Figure 3) indicate that the presence of the $f^{-5/3}$ frequency range does not warrant the existence of an inertial subrange in a stably stratified fluid." Figure 3 was a spectrum with -5/3 slope on the log-log plot showing the anomalous -5/3 subrange.

Finally, in this connection, Dutton[18] wrote the following: "One of the important properties of atmospheric turbulence is that the -5/3 inertial subrange does appear to exist, but for reasons unknown the -5/3 relation holds at much smaller wave numbers than would be expected from the theory."

From the previous information, it is clear that the anomalous -5/3 effect in stratified fluids is well known and that a theoretical explanation for it would be of value at this time. (One further mention of this subject will be found in Ellison.[19])

In order to gain some appreciation of how far the HICAT spectra differ from the usual predictions of the theory for the inertial subrange, we now make a comparison. In the case of shear generated turbulence in the absence of buoyancy, the maximum wavelength for isotropy is derived by comparing the mean rate of strain to the rate of strain of the turbulence. The latter is, from a similitude argument, given by $(k^3 \phi(k))^{1/2}/2\pi$ where $\phi(k)$ is the three-dimensional scalar wavenumber spectrum of the velocity fluctuations. Letting μ be the scale of the mean velocity, ℓ the outer scale, S the mean strain rate ($S \sim \mu/\ell$), ε the rate of dissipation, s the turbulent strain rate and using $\phi = 1.5\, \varepsilon^{2/3} k^{-5/3}$, $\varepsilon \sim \mu^3/\ell$, and making $s/S = 10$, Tennekes and Lumley[20] showed that isotropy should not extend down much further than the value of k where

$$k\ell = 350 \quad .$$

This implies that the maximum wavelength for isotropy would not be much greater than about 2 percent of the outer length.

17. Pao, Y.H. (1973) Measurements of internal waves and turbulence in two-dimensional stratified shear flows, Bound-Layer Meteorol. 5:177.

18. Dutton, J.A. (1970) Effects of turbulence on aeronautical systems, Progress in Aeronautical Sciences, 11, edited by D. Küchemann et al, Pergamon Press, Oxford.

19. Ellison, T.H. (1956) Atmospheric Turbulence, Surveys in Mechanics (p. 425), edited by T.H. Batchelor and R.M. Davies, Cambridge University Press.

20. Tennekes, H., and Lumley, J.L. (1972) A First Course in Turbulence, MIT Press, Cambridge, Massachusetts.

When the drain of kinetic turbulent energy due to work done against the stable buoyancy forces is taken into account, Lumley[21] has shown that the inertial range will not extend to lower wavenumbers than k_b (inverse "buoyancy length") given by

$$k_b \sim \sqrt{\frac{N_1^3}{\varepsilon}} \tag{1}$$

where

$$N_1^2 = \frac{g}{\Theta}\,\frac{d\Theta}{dz} \tag{2}$$

and where Θ is the average potential temperature, g the acceleration of gravity, and N_1 is the buoyancy frequency.

In the HICAT data the measured thickness of turbulent layers ranged from 500 ft to 7,000 ft with most of them under 3000 ft or about 1 km. Taking 1 km as the outer length, the first criterion above implies that 20 m would be the approximate maximum wavelength for isotropy. The bouyancy length criterion, according to Zimmerman and Loving,[22] Monograph I, CIAP gives such values as 15 m, 51 m, and 40 m. As already indicated, the data showed that isotropy in the HICAT measurements extended to wavelengths between 30 m and 45 m. Thus, by all criteria the inertial range should not extend much beyond about 50 m or so. But, as already mentioned, the -5/3 spectrum (in the approximate sense) extends to several thousands of feet if not tens of thousands of feet in wavelength. It is precisely this discrepancy which we now set out to explain.

An important observation, made by Gifford,[23] is that one should expect that spatial aliasing effects will significantly extend the -5/3 range of wavenumbers to smaller values in the one-dimensional spectra as compared to the three-dimensional spectrum. In the example studied by Gifford, he showed that the -5/3 range of the one-dimensional spectrum extended to a value of k about five times smaller than the smallest -5/3 range value of k of the three-dimensional spectrum. Thus, a three-dimensional spectrum with inertial range extending to 50 m could appear out to 250 m in the one-dimensional spectrum due to spatial aliasing effects. But as we have seen, we must explain -5/3 ranges out to wavelengths of 10,000 to 40,000 ft (say, beyond at least 5 km). This could only happen if the anomalous

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21. Lumley, J. L. (1964a) The spectrum of nearly inertial turbulence in a stably stratified fluid, J. Atm. Sci. 21:99-102.

22. Zimmerman, S. P., and Loving, N. (1975) The Natural Stratosphere of 1974, CIAP Monograph 1, Final Report, DOT, CIAP., DOT-TST-75-51.

23. Gifford, F. (1959) The interpretation of meteorological spectra and correlations, J. Meteorol. 16:344-346.

-5/3 range in the three-dimensional spectrum extended at least as far as the "outer length" of the 1 km layer thickness. The large 40,000 ft range would be compatible with an outer length of the largest layer thickness seen in HICAT. In the following, we therefore set out to explain -5/3 ranges all the way to outer length.

An important consequence of the anomalous -5/3 range is that it can lead to erroneous estimates of ε when the method of Stewart and Grant[24] is employed. In other words, as Pao[17] has written, it "...should be considered as a caution flag to those who derive turbulent energy dissipation rate ε from spectra measured with relatively slow-response instruments...". This is because the experimenter could mistake a -5/3 range for a true inertial subrange, and the latter might begin at wavenumbers higher than the resolution of his instruments.*

The plan of this paper is briefly as follows. First we will examine the equations of motion as well as the spectral equations for turbulent motion and temperature fluctuations. The physics of these equations and their individual terms will be discussed, and a spectral theory of K-H turbulence will be developed which predicts a -5/3 range extending from the Kolmogorov microscale to the integral or outer scale. A simple physical picture will then be given of the "dual cascade" responsible for the phenomenon.

2. SPECTRAL EQUATIONS OF TURBULENCE AND TEMPERATURE FLUCTUATIONS IN STRATIFIED SHEAR FLOW

In this section, we shall follow the treatment given by Lin et al.[25] While more suited to our purposes because of its generality, the latter treatment is essentially along the same lines as that given in such standard works as Hinze,[26] and Lumley and Panofsky.[27] First, we consider the Navier-Stokes equation. We assume constant viscosity, incompressibility, and validity of the Boussinesq

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*A typical relation between turbulent diffusivity, K_e, and the dissipation rate, ε, is $K_e = (\text{const.})\,\varepsilon^{1/3} k_1^{-4/3}$ where $N \equiv$ buoyancy frequency, and $k_1 \equiv (\text{const.})\,\varepsilon^{-1/2} N^{3/2}$. (See Zimmerman and Loving.[22])

24. Stewart, R.W., and Grant, H.L. (1962) Determination of the rate of dissipation of turbulent energy near the sea surface in the presence of waves, J. Geophys. Res. 67:3177-3180.

25. Lin, J.T., Panchev, S., and Cermak, J. (1969a) Turbulence spectra in the buoyancy subrange of thermally stratified shear flows, Project Themis, Technical Rept. No. 1, College of Engineering, Colorado State University.

26. Hinze, J.O. (1959) Turbulence, McGraw-Hill, New York.

27. Lumley, J.L., and Panofsky, H.A. (1964) The Structure of Atmospheric Turbulence, Interscience Publishers, New York.

approximation. Letting P be pressure, μ the viscosity, ρ_o the mean density, ρ the density fluctuation, and U_i the i^{th} component of the velocity we can write

$$\rho_o \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} - \rho g_i + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial U_i}{\partial x_j} \right) \tag{3}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \quad . \tag{4}$$

Next we must have the equation for the temperature field which, in turn, controls the density. This is simply the convective diffusion equation for heat flow

$$\rho_o C_p \left(\frac{\partial T}{\partial t} + U_i \frac{\partial T}{\partial x_i} \right) = \mu_T \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i} \right) \tag{5}$$

where C_p is the specific heat capacity at constant pressure assumed constant, μ_T is the thermal conductivity also assumed constant, and T is the temperature.

The equations for the two point correlations of the velocity and temperature fluctuations are obtained in the usual manner, using Reynold's decomposition

$$U_i = \overline{U}_i + u_i \quad , \quad P = \overline{P} + p_i$$

$$\rho = \overline{\rho} + \rho_1 \quad , \quad T = \overline{T} + \theta$$

where the overbar denotes time average. Defining $\nu \equiv \mu/\overline{\rho}$, $\nu_T = \mu_T/(\rho_o C_p)$, using $\theta/\overline{T} = -\rho_1/\overline{\rho}$, assuming stationarity of mean flow, and performing the usual manipulations as well as the tensor contraction, we obtain

$$\overset{\text{I}}{\frac{\partial}{\partial t}\overline{(u_i u_i')}} + \Bigg(\overset{\text{II}}{\{\delta_{1i}\overline{u_i' u_3} + \delta_{1i}\overline{u_i u_3'}\}} + \overset{\text{III}}{r_3 \frac{\partial}{\partial r_1}\overline{(u_i u_i')}} \Bigg) \frac{d\overline{U}_1}{dx_3}$$

$$+ \overset{\text{IV}}{\frac{\partial}{\partial r_j}(\overline{u_i' u_j' u_i} - \overline{u_i u_j u_i'})} = \overset{\text{V}}{\frac{g}{\overline{T}}(\overline{u_i' \theta} + \overline{u_i \theta'})\,\delta_{i3}} + \overset{\text{VI}}{2\nu \frac{\partial^2 \overline{(u_i u_i')}}{\partial r_j \partial r_j}} \tag{6}$$

$$\overset{\text{I}}{\frac{\partial \overline{(\theta'\theta)}}{\partial t}} + \overset{\text{II}}{(\overline{\theta' u_3} + \overline{\theta u_3'}) \frac{d\overline{T}}{dx_3}} + \overset{\text{III}}{r_3 \frac{\partial}{\partial r_1} \overline{(\theta'\theta)} \frac{d\overline{U}}{\partial x_3}}$$

$$+ \overset{\text{IV}}{\frac{\partial}{\partial r_j} (\overline{\theta\theta' u_j'} - \overline{\theta\theta' u_j})} = \overset{\text{V}}{2\nu_T \frac{\partial^2 \overline{(\theta'\theta)}}{\partial r_j \partial r_j}} \qquad (7)$$

where primes refer to displaced points $\vec{x}' = \vec{x} + \vec{r}$ and $r_j = x_j' - x_j$, local homogeneity was assumed, the approximation $\overline{U}_1' - \overline{U}_1 = r_3 \cdot (d\overline{U}_1/dx_3)$ was used, and $\overline{U}_1$ and $\overline{T}$ were assumed to be functions of only the vertical coordinate, x_3. The reader should consult the references for further details.

We now physically identify terms, and since they all involve two point correlations, it will be more convenient to consider the zero lag case.

I = rate of change per unit mass of the kinetic energy fluctuations with respect to time.

II = rate of production of velocity fluctuations due to the vertical transport (u_3) of horizontal fluctuations, u_1, in the presence of vertical mean shear, $d\overline{U}_1/dx_3$.

III = (as will be discussed below) the effects of the distortion and rotation of turbulent motion due to mean shear.

IV = nonlinear convective effects involving vortex stretching of smaller eddies by larger eddies.

Both III and IV are divergences, and energy is neither created nor destroyed by them.

V = rate of work done against stable buoyancy forces which results in the increase of the potential energy of the temperature (or density) fluctuation field.

VI = rate of turbulent energy dissipation into heat by viscosity.

In the case of Eq. 7 we have:

I = rate of change of mean square temperature fluctuations with respect to time.

II = rate of production of temperature fluctuations due to vertical transport (u_3) of temperature fluctuations (θ) in the presence of vertical mean gradient $d\overline{T}/dx_3$.

III = the distortion and rotation of temperature fluctuations due to mean shear which is analogous to term III in the equation for kinetic energy above.

IV = nonlinear convective effects involving vortex stretching as in the kinetic energy equation.

V = rate of temperature fluctuation dissipation by molecular diffusion effects.

The spectral equations are obtained in the usual manner by Fourier transformation of the two point correlations. We next consider the three dimensional scalar wave number spectra obtained by averaging over spherical shells in k space. Equations 6 and 7 are thus transformed into

$$\underbrace{\frac{\partial\phi(k,t)}{\partial t}}_{\mathrm{I}} + \underbrace{\phi_{uw}(k,t)\frac{d\overline{U}_1}{dx_3}}_{\mathrm{II}} - \underbrace{\left(k_1\frac{\partial E_{i,i}}{\partial k_3}\right)_{SP.AV.}\cdot\frac{d\overline{U}_1}{dx_3}}_{\mathrm{III}} = \underbrace{F(k,t)}_{\mathrm{IV}} + \underbrace{\frac{g}{\overline{T}}\phi_{wT}(k,t)}_{\mathrm{V}} - \underbrace{2\nu k^2\phi(k,t)}_{\mathrm{VI}} \tag{8}$$

and

$$\underbrace{\frac{\partial\phi_{TT}(k,t)}{\partial t}}_{\mathrm{I}} + \underbrace{\phi_{wT}(k,t)\frac{d\overline{T}}{dx_3^*}}_{\mathrm{II}} - \underbrace{\left(k_1\frac{\partial E_{T\,i,i}}{\partial k_3}\right)_{SP.AV.}\cdot\frac{d\overline{U}_1}{dx_3}}_{\mathrm{III}} = \underbrace{F_{TT}(k,t)}_{\mathrm{IV}} - \underbrace{2\nu_T k^2\phi_{TT}(k,t)}_{\mathrm{V}} \tag{9}$$

where $x_3^* \equiv x_3/2$, and where $E_{i,i}$ and $E_{T\,i,i}$ are the diagonal elements of the non-averaged three-dimensional spectra.

The numbering of these spherically averaged spectral terms corresponds to the previous numbering of the correlation terms. We now give the physical interpretations. For Eq. (8) we have:

I = the rate of change with respect to time of $\phi(k,t)$, the kinetic energy spectral density at k.

II = rate of "turbulent production" at k. ϕ_{uw} is the real part of the cross spectrum between the streamwise and vertical components of the velocity fluctuations.

III = transfer in k-space of kinetic energy due to the distortion effects of mean shear. It is from lower to higher values of k in the anisotropic region. The reader may consult discussions in Hinze[26] and Lumley and Panofsky[27] for further information.

IV = transfer in k-space of kinetic energy due to the inertial cascade from low values of k where production is important to high values of k, where molecular dissipation occurs. It represents the net input or output (that is, the "build-up") of energy at k and when F = 0, the cascade is steady in k-space since $F \equiv -\partial\varepsilon(k)/\partial k$.

V = rate of turbulent buoyancy "production," or, as in the present stable case, buoyant "dissipation." It refers to kinetic energy loss at k by conversion into buoyant potential energy. ϕ_{wT} is the real part of the cross-spectrum taken between fluctuations of temperature and vertical velocity.

VI = viscous dissipation rate at k considered negligible for all but the highest values of k near the Kolmogorov microscale.

The terms for Eq. (9) are:

I = rate of change with respect to time of $\phi_{TT}(k,t)$, the temperature fluctuation spectral density at k.

II = rate of production of temperature fluctuations at k. Comparison with term V of Eq. (8) shows that multiplication of the present term by $(g/\overline{T})/\overline{T}'$ (where $T' \equiv d\overline{T}/dx^*$) gives term V for potential energy. In fact, if Eq. (9) is multiplied by this "conversion factor," our temperature fluctuation equation becomes the equation for the potential energy of the fluctuations.

III = transfer in k-space, due to mean shear. Lin et al[25] appear to be the first to have incorporated this term in the equation for temperature fluctuations.

IV = the inertial cascade term for the temperature fluctuations.

V = molecular heat dissipation rate at k which is assumed negligible for small k.

We now turn to a more detailed physical description of the terms of these equations in the context of K-H turbulence.

3. INTERPRETATION OF THE TURBULENCE PRODUCTION TERMS

3.1 Kinetic Energy (KE) Production Terms

Insight into the production terms II and V of Eq. (6) in the context of K-H turbulence, is provided by the discussions of Ludlam[10] and Businger[28, 29] who derived the critical Richardson's numbers on the basis of a simple physical model for the potential and kinetic energy budget. In the following, we will see why the production of turbulence from mean shear II and the buoyancy dissipation (or negative buoyant "production"), V, are approximately equal but opposite in sign for Kelvin-Helmholtz turbulence. Later on, this crucial observation will be used to explain the -5/3 spectrum.

The following argument leads to Richardson number, $R_i = 0.25$ as a necessary condition for turbulent breakdown to occur when there is a shear across a stable layer (cf. also Miles[30]). The argument is based on the fact that the available kinetic energy from the shear flow must exceed the potential energy of buoyancy work (involved in the vertical transport of parcels of fluid across the layer in the mixing process) if turbulence is to take place.

Consider the idealized case of a horizontal layer across which is a linear, stable temperature gradient and a linear shear of the horizontal velocity. If parcels of fluid of unit mass at the top and bottom of the layer are adiabatically interchanged, one must do work against the buoyancy force, $g\Delta\theta/\Theta_o$ where $\Delta\theta$ is the change of potential temperature across the layer and Θ_o is the average potential temperature.* $\Delta\theta = (d\Theta/dx_3)\Delta x_3$ where Δx_3 is layer thickness. The work done to move one parcel across the layer is

$$\frac{g}{\Theta_o}\int_o^{\Delta x_3} \frac{d\Theta}{dx_3}\,\Delta x'_3\, d\Delta x'_3 = \frac{g}{\Theta_o}\frac{d\Theta}{dx_3}\frac{(\Delta x_3)^2}{2} . \tag{10}$$

*Previously we have assumed incompressibility, however here we are considering the more general case treated by Businger.[28]

28. Businger, J.A. (1969a) Note on the critical Richardson number(s), Quart. J. Roy. Meteorol. Soc. 95:653-654.

29. Businger, J.A. (1969b) On the energy supply of clear air turbulence, in Clear Air Turbulence and its Detection, edited by Y.H. Pao and A. Goldburg, Plenum Press.

30. Miles, J.W. (1963) On the stability of heterogeneous shear flows, Part II, J. Fluid Mech. 16:209-227.

The potential energy, (PE), generated by an interchange of two parcels originally located on opposite sides of the layer is therefore

$$PE = \frac{g}{\Theta_o} \frac{d\Theta}{dx_3} (\Delta x_3)^2 \quad . \tag{11}$$

Next consider the kinetic energy, (KE), available from parcel exchange across the layer, due to the shear. Letting $\overline{U}_1$ be the mean horizontal velocity and $\Delta\overline{U}_1$ be the difference in horizontal velocity across the layer, we find the following available KE due to the averaging out of the parcel moments after interchange:

$$\begin{aligned} KE &= \frac{1}{2}\left[\overline{U}_1^2 + (\overline{U}_1 + \Delta\overline{U}_1)^2 - 2\left(\overline{U}_1 + \frac{1}{2}\Delta\overline{U}_1\right)^2\right] \\ &= \frac{1}{4}\left(\frac{d\overline{U}_1}{dx_3}\right)^2 (\Delta x_3)^2 \quad . \end{aligned} \tag{12}$$

Equating KE = PE we get:

$$R_i = \text{(gradient Richardson number)} = \frac{g}{\Theta_o} \frac{(\partial\Theta/\partial x_3)}{(\partial\overline{U}_1/\partial x_3)^2} = 0.25 \quad . \tag{13}$$

We now relate this to Eq. (6). Letting x' = x there, we see that term II is proportional to $[\overline{u_1 u_3}\, d\overline{U}_1/dx_3]$ while term V is proportional to $[\overline{u_3\theta}\, g/\overline{T}]$. The ratio of these terms is:

$$R_{if} = \text{(flux Richardson number)} = \frac{\overline{u_3\theta}\, g}{\overline{T}(\overline{u_1 u_3})(d\overline{U}/dx_3)} \quad . \tag{14}$$

Letting K_m be the eddy viscosity and K_h the eddy conductivity defined by

$$-\overline{u_1 u_3} \equiv K_m\, d\overline{U}_1/dx_3$$

and

$$-\overline{\theta u_3} \equiv K_h\, d\overline{T}/dx_3 \tag{15}$$

we obtain

$$R_{if} \cong R_i \quad \text{when} \quad K_m \approx K_h \quad .$$

In the case where intimate mixing can occur, KE = PE implies $R_i = 1$ as shown by Businger[29] with arguments similar to the above. This in turn implies that turbulence will begin to decay when $R_i = 1$ due to a lack of net energy input. In the case of K-H turbulence, the value of R_i must be roughly in the range of 0.25 to 1.0 for the "active life" that is, the energy "fed" period of the turbulence layer. It should be now clear that terms II and V of Eq. (6) physically relate to KE and PE in the presence of the velocity shear and stable potential gradient.

Next we consider the corresponding terms II and V of the spectral equation, Eq. (6), and we see that they are the spectral form of the KE and PE at a given k. Their ratio is the spectral flux Richardson number.

Also, note that the "parcel interchange" picture of Businger is physically related to the terms $\overline{\theta u_3}$ and $\overline{u_1 u_3}$. These relate to the PE and KE of the turbulence due to u_3, that is, vertical transport. Unlike Businger's case, incompressibility is assumed in our treatment; thus, potential temperature is replaced by ordinary temperature.

Another point is that the production of turbulent KE can also be regarded from a point of view that does not involve parcel exchange but rather an energy input from mean shear to turbulence by means of vortex stretching. As Tennekes and Lumley[20] (P41) showed, a vertical shear tends to "feed" vortices by this process if the latter are aligned at 45° from the horizontal.

3.2 The Potential Energy (PE) Production Terms

As mentioned previously, Eqs. (7) and (9) for temperature fluctuations can be regarded as the equations for potential energy. For example, when Eq. (9) is multiplied by $(g/\overline{T})/\overline{T}'$ it becomes the "PE spectral equation." There is but one "production term," namely II. It represents the rate of increase of the squared temperature fluctuations, θ^2, in the presence of a mean vertical temperature gradient though the action of vertical transport by velocity fluctuations. For example, an increase of θ takes place when a fluid parcel is transported down the gradient.

This mechanism of production is entirely analogous to the momentum case discussed previously. Also notice that the buoyancy sink term, V of the KE spectrum equation, Eq. (8), corresponds to the buoyancy source term, II of the PE spectrum equation, Eq. (9). This latter point explains a certain peculiarity pointed out in discussions of the so called "buoyancy subrange" (see for example Lumley,[21] and Bolgiano,[31, 32]) namely that whenever the KE spectrum is made to

31. Bogliano, R. (1959) Turbulent spectra in stably stratified atmosphere, J. Geophys. Res. 64:2226.

32. Bolgiano, R. (1962) Structure of turbulence in stratified media, J. Geophys. Res. 67:3015-3023.

be steeper than -5/3 by a predominant buoyancy dissipation effect, the PE spectrum is at the same time made less steep than -5/3. These slopes bear a reciprocal relation in general, and the above description enables one to see the physical reason for this effect.

4. INTERPRETATION OF THE CASCADE TERMS

As we have seen, the nonlinear convective terms of the equations of motion lead to the divergence of triple correlation terms, IV, in Eqs. (6) and (7). These give rise to F and F_{TT}, the cascade terms in Eqs. (8) and (9). In an inertial range, these terms equal zero, implying a steady flow down the scale cascade in k-space. The -5/3 slope depends on a steady, conservative flow (without sources sinks, "build-ups," or "decays" along the way). We now review the physics of the cascade processes in the PE and KE spectra.

4.1 Inertial (KE) Cascade

Since the inertial cascade plays a key role in this paper, it will be useful to examine it closely. First we will consider the famous treatment by Onsager[33] which, to date, seems to be the most informative one available. Next we shall discuss the vortex stretching mechanism responsible for it, and finally the Heisenberg concept will be mentioned since it gives an interesting physical picture and will be used later.

Onsager[33] proceeded by directly Fourier transforming the Navier-Stokes equations. The nonlinear convective term depends upon the difference of wavenumbers, $(\pm \vec{k}' \pm \vec{k})$. Assuming that k, as well as differences of k, are of the order of (1/L) where "L" is the size of the largest eddies, he found that k' would be at most 2/L, where k is regarded as the driving mode and k' the mode receiving the energy. Applying this reasoning to subsequent cascade steps results in a geometric progression. For this reason, the wavenumber doubles at each step; hence, Δk between steps $\approx$ k. The amount of energy transferred at each step is therefore of the order $k\phi(k)$. Let Q(k) represent the flux of energy past wavenumber k in k-space. Then, in the continuous limit,

$$Q(k) \approx \frac{k\,\phi(k)}{\tau(k)} \tag{16}$$

33. Onsager, L. (1949) Statistical hydrodynamics, Nuovo Cimento 6:(Ser. 9, Supp. 2):279-287.

where $\tau(k)$ is a characteristic time for a step at wavenumber k. On apparently dimensional grounds, Onsager chose

$$\tau = 1/\sqrt{k^3\phi(k)} \quad . \tag{17}$$

Alternatively, Corrsin[34] deduced this result physically from

$$\tau = \frac{\text{kinetic energy per step}}{\text{energy transfer rate}} \quad . \tag{18}$$

He obtained a denominator by analogy with $\overline{u_i u_k}(\partial \overline{U}_i/\partial x_k)$:

$$\frac{(\text{spectral velocity})^3}{(\text{spectral length})} = \frac{(k\phi)^{3/2}}{(1/k)} = \sqrt{k^5\phi^3} \quad . \tag{19}$$

Hence

$$\tau(k) = \frac{k\phi}{\sqrt{k^5\phi^3}} = \frac{1}{\sqrt{k^3\phi}} \quad \text{QED.} \tag{20}$$

Using this in Eq. (16)

$$Q(k) \approx \phi(k)^{3/2}\, k^{5/2} \quad . \tag{21}$$

Assuming that the cascade in k space is divergenceless and conservative,

$$\frac{\partial Q}{\partial k} \equiv -F = 0 \quad . \tag{22}$$

Setting:

$$Q(k) = \varepsilon \quad , \tag{23}$$

$$\phi(k) \approx \varepsilon^{2/3}\, k^{-5/3} \quad . \tag{24}$$

This of course agrees with the usual result obtained by a similitude argument.

The doubling at each step accelerates the cascade and therefore causes the amount of energy at k to decrease with increasing k in a way which is analogous to the manner that a convectively accelerated stream of incompressible fluid will become more narrow as it speeds up.

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34. Corrsin, S. (1958) Local Isotropy in Turbulent Shear Flow, NACA RM 58-B11.

Physically, the cascade comes about by the vortex stretching action of the larger eddies on the smaller ones. Mathematically, this is represented by term IV in Eq. (6). It can be shown, however, that

$$\frac{\partial}{\partial x_j}\left(\frac{1}{2}\,\overline{u_i u_i u_j}\right) = \overline{u_i u_j s_{ij}} \tag{25}$$

where

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad . \tag{26}$$

But $(-\overline{u_i u_j s_{ij}})$ is the amount of energy per unit mass and per unit time gained by a disturbance with velocity components $u_i u_j$ in strain rate s_{ij} (see Tennekes and Lumley[20] p. 257). If, for example, there is compression in the y direction and stretch in the x direction (Figure 2) the energy exchange rate is equal to

$$Q = s(u_2^2 - u_1^2) \quad . \tag{27}$$

Since ω_1 is increased and ω_2 decreased, u_2^2 increases while u_1^2 decreases. Hence, $(u_2^2 - u_1^2)$ starts at zero and becomes positive. The net effect is to increase the energies of eddies by this deformation. The cascade is therefore visualized as the transfer of energy by the distortion effects of large eddies upon the smaller eddies. The interested reader should consult the excellent discussion of Tennekes and Lumley[20] for further quantitative information about this process. They show for example that this leads to a derivation of the -5/3 slope. Since $v^2 \sim k\phi$ dimensionally,

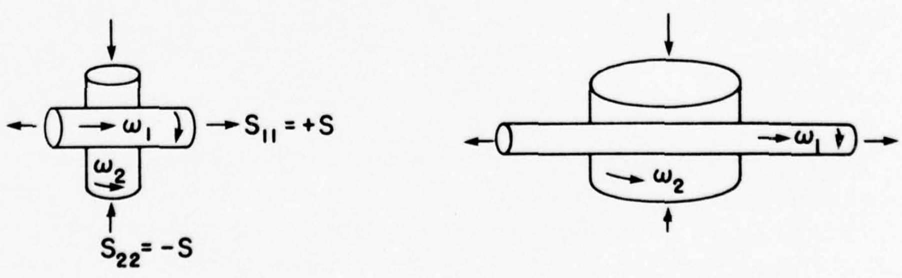

Figure 2. Illustration of Vortex-Stretching Mechanism of the Turbulence Cascade

$$Q(k) \sim k\phi(k)\, s(k) \quad , \tag{28}$$

and, also dimensionally

$$s(k) \sim (k^3\phi)^{1/2} \quad . \tag{29}$$

Equations (23), (28), and (29) thus lead directly to

$$\phi(k) \sim \varepsilon^{2/3}\, k^{-5/3} \quad . \quad \text{[see Eq. (24)]}$$

The computationally useful concept of Heisenberg is that the energy transfer across k can be regarded as a turbulent viscosity effect (see Hinze[26]):

$$\int_o^k F\, dk = -2\,\nu(k) \int_o^k k^2\, \phi(k)\, dk \tag{30}$$

where $\nu(k)$ is the effective turbulent viscosity defined (from dimensions)

$$\nu(k) = \alpha \int_k^\infty \sqrt{\frac{\phi(k)}{k^3}}\, dk \equiv \alpha K(k) \tag{31}$$

where $\alpha \approx 1$.

Various generalizations of this approach have been advanced and we shall later make use of a very general one due to Panchev.[35]

4.2 Temperature Fluctuation (PE) Cascade

The mechanism of the previous inertial cascade represents a small version of the "turbulent production process" due to average shear. In a similar manner, the temperature cascade is a small version of the production process due to mean temperature gradient.

As above, the cascade term IV in Eq. (7) is the divergence of a triple correlation. Considering IV as a one point correlation, we follow the treatment of Tennekes and Lumley. Briefly, IV can be represented as a miniaturized form of

35. Panchev, S. (1971) Random Functions and Turbulence, Pergamon Press, New York.

$$(\overline{\theta u_3})\left(\frac{\partial \overline{T}}{\partial x_3}\right) = \text{(temperature variance production rate)} \quad . \qquad (32)$$

We estimate $\overline{\theta u_3}$ by $\mu \ell (\partial \overline{T}/\partial x_3)$ and where μ and ℓ are representative velocity and length,

$$\mu \sim \sqrt{k_2\, \phi(k_2)} \qquad (33)$$

$$\ell \sim k_2^{-1} \quad . \qquad (34)$$

The miniaturized $(\partial \overline{T}/\partial x_3)$, that is, the gradient of the "larger eddy" acting in the "smaller eddy" is

$$\text{(gradient of temp)} = \sqrt{k_1^3\, \phi_{TT}(k)} \qquad (35)$$

$k_2 > k_1$. Let Q_θ be the flux past k in the temperature cascade. We than have

$$Q_\theta \approx k_2^{-1} \sqrt{k_2\, \phi(k_2)}\, k_1^3\, \phi_{TT}(k) \quad . \qquad (36)$$

Assuming $k_1 \approx k_2$

$$Q_\theta = c\, k^{5/2}\, \phi_{TT}(k) \sqrt{\phi(k)} \quad . \qquad (37)$$

This derivation shows the important point that the scaler temperature cascade is due directly to the action of the velocity field on the temperature field. This important point will be used below.

5. K-H TURBULENCE AND APPROXIMATE CANCELLATION OF KE PRODUCTION TERMS

The only way to explain an anomalous -5/3 spectrum is to assume that one has a situation which is equivalent to the conservative steady cascade. In the present case we argue that in K-H turbulence, terms II and V in Eq. (8) essentially cancel each other over all wavelengths from microscale to outer scale. This can be true only if the two terms are approximately equal and opposite and in addition have the same dependence on k. To prove the latter, it is necessary to show that the temperature field is determined by the velocity field. This, however, has already been done by Lumley[21] and the necessary assumptions are: (1) both Reynolds number and Peclet number must be high enough that molecular effects can be ignored, (2) $(\partial \overline{T}/\partial t) = 0$; and (3) $\{\sqrt{2}\, \partial^2 \overline{T}/\partial x_i^2\, L_E\}/(\partial \overline{T}/\partial x_i) \ll 1$ where L_E is the Eulerian space integral scale, $L_E \simeq u' T_L$ (Corrsin[36]) where u' is the r.m.s. velocity and T_L the Lagrangian time scale. Strictly speaking, the mean temperature is changed in time in K-H turbulence. However, we make the approximation, for an interval of interest, that we have stationarity.

Briefly, Lumley's proof begins with

$$\frac{\partial \theta}{\partial t} + \tilde{u}_i \left(\frac{\partial \theta}{\partial x_i} + \frac{\partial \overline{T}}{\partial x_i} \right) = 0 \tag{38}$$

which is Eq. (5) with $\mu_T = 0$, $T = \overline{T} + \theta, \tilde{u}_i \equiv \overline{U_i} + u_i$ and $\partial \overline{T}/\partial t = 0$. Equation (38) is in Eulerian form. Lumley solved (38) under his above stated assumptions to arrive at

$$\theta(x, t) = - \left(\frac{\partial \overline{T}}{\partial x_i} \right) \left(x_i - a_i(\vec{x}, t) \right) \tag{39}$$

where $\vec{a}(\vec{x}, t)$ is the position at $t = 0$ of that fluid particle which will arrive at $\vec{x}$ at time t. In our problem, where we assume a vertical temperature gradient constant in space and time, we have $((\partial \overline{T}/\partial x_3) \equiv \overline{T}')$

$$\theta(\vec{x}, t) = \overline{T}'(z - z_o) \quad . \tag{40}$$

Physically, Eq. (40) is easy to understand. The temperature fluctuation is all due to the vertical displacement of parcels from their initial position to new positions where, by virtue of the gradient, they are at temperatures different from the mean (compare Section 4.2 above).

The conclusion of Lumley's proof is that, from Eq. (39) we can infer

$$\begin{aligned} \overline{\theta(x, t)\, u_j(x, t)} &= \left(\frac{-\partial \overline{T}}{\partial x_i} \right) \overline{(x_i - a_i)\, u_j(\overline{x}', t)} \\ &\equiv (-\partial \overline{T}/\partial x_i)(A_{ij}) \quad . \end{aligned} \tag{41}$$

Since A_{ij} depends only on the velocity field, the proof is completed.

Using the fact that the velocity field controls the temperature field Lumley[21] went on to show, by means of a similitude argument that

$$\phi_{wT} = -C_1 \frac{\partial \overline{T}}{\partial x_3} \varepsilon^{1/3} k^{-7/3} \tag{42}$$

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36. Corrsin, S. (1963) Estimates of the relations between Eulerian and Lagrangian scales in large Reynolds number turbulence, J. Atm. Sci. 20:115-119.

where C_1 is a constant of order one. This result is an application of the Kovasney[37] formalism which says the spectral forms are determined by the local spectral flux.

It should be mentioned that Lumley's proof of Eq. (42) rested on one more assumption, namely "local inertiality" (cf. Lumley[38]):

$$(\varepsilon/k)/(\partial\varepsilon/\partial k) \gg 1 \quad . \tag{43}$$

Physically, this assumption means that the transfer of energy in or out of the cascade is but a small fraction of the throughput. This assumption is consistent with the conclusion which follows from it below.

In a similar way, one can show

$$\phi_{uw} = -C_2 \left(\frac{\partial \overline{U}_1}{\partial x_3}\right) \varepsilon^{1/3} k^{-7/3} \tag{44}$$

where, hopefully, $C_1 \approx C_2$.

In Eq. (8) let us assume that at the value of k of interest, we can ignore molecular dissipation; that except for the effects of production terms, the cascade is steady and conservative; and that $\phi \cong 0$. In this case, using $F \equiv (-\partial\varepsilon/\partial k)$,

$$\frac{\partial\varepsilon}{\partial k} = -\phi_{uw} \frac{d\overline{U}_1}{dx_3} + \frac{g}{\overline{T}} \phi_{wT} \cong \varepsilon^{1/3} k^{-7/3} C(1 - Ri) \left(\frac{d\overline{U}_1}{dx_3}\right)^2 \tag{45}$$

where $\partial\varepsilon/\partial k$ is the change of throughput with respect to k, and Ri* is the gradient Richardson number. Recalling $Ri \equiv [(g/\overline{T}) \cdot d\overline{T}/dx_3]/(d\overline{U}_1/dx_3)^2$ we see

$$0.25 \le Ri \le 1 \tag{46}$$

for K-H turbulence in view of the previous discussion. This implies that $(\partial\varepsilon/\partial k)$ will remain small except for large wavelengths. In order that $(\partial\varepsilon/\partial k) \ll \varepsilon/k$, we have as a lower limit on k, k_c,

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*It is assumed that Ri = Rf.

37. Kovasznay, L. S. (1948) Spectrum of locally isotropic turbulence, J. Aeron. Sci. 15:741-753.

38. Lumley, J. L. (1967a) Theoretical aspects of research on turbulence in stratified flows, in Atmospheric Turbulence and Radio Wave Propagation, edited by A. M. Yaglom, and V. I. Tatarsky, Publishing House "Nauka," Moscow.

$$k_c^{-7/3}(\overline{U}_1')(1 - Ri)\,\varepsilon^{1/3} \approx \frac{\varepsilon}{k_c} \tag{47}$$

where

$$(\overline{U}_1') \equiv d\overline{U}_1/dx_3 \quad . \tag{48}$$

The following estimates will be employed:

$$(\overline{U}_1') \sim \overline{U}/L \tag{49}$$

where L is the outer length. Also

$$\varepsilon \sim \frac{u^3}{L} \cdot C_1 \tag{50}$$

where u is the rms value of the velocity fluctuations and C_1 is a constant which takes the buoyancy into account.

According to Woods and Wiley,[12] the rate at which turbulence loses energy to viscous dissipation is much greater (up to a factor of 10) than the rate at which the turbulence does work against gravity by changing the density (temperature) profile. This implies C_1 is near unity.

Defining C_2 as

$$u = C_2 |U_1| \tag{51}$$

and Eqs. (49), (50) and (51) in (47), we obtain

$$(k_c L)^{4/3} = \frac{(1 - Ri)}{C_1^{2/3} C_2^2} \quad . \tag{52}$$

Letting λ_c be the maximum wavelength for inertiality, that is, $k_c \equiv 2\pi/\lambda_c$, we have

$$\lambda_c = (2\pi L C_1^{1/2} C_2^{3/2})/(1 - Ri)^{3/4} \quad . \tag{53}$$

From this relation it follows that, as long as C_1 and C_2 are not zero (that is, assuming there is turbulence and thus $u \neq 0$) there always exists a value of Ri less than unity such that $\lambda_c = L$. Thus, the -5/3 range can extend to the outer length for Ri sufficiently close to unity. The effect described by Gifford[23] will make the -5/3 range extent to about 5 λ_c in the one-dimensional spectrum.

In addition to all the above mentioned explicit assumptions, it was also implicitly assumed that the dynamic effects of the inertial cascade are not small in their effects in comparison to mean shear effects, in other words we assume that the mean strain rate is not large compared to the eddy strain rates. In our judgment, all the above assumptions are appropriate for the approximate description of K-H turbulence. This weak interaction assumption is justified on the basis that, unlike boundary layer or wall turbulence, the mean structure is too weak to permanently "feed" the turbulence directly. In actual fact, the layer is fed, to a large degree, at the turbulent-laminar interface, by layer spreading (c.f. Woods[8, 7]).

6. THE DUAL CASCADE MODEL AND THE ANOMALOUS, K-H "-5/3" SPECTRUM

6.1 Cascade Model

We now put the spectral equations into a form more amenable to their resolution and visualization. First we integrate Eqs. (8) and (9) from 0 to k and then integrate them from 0 to ∞ and subtract the former from the latter. Using

$$\varepsilon = \int_0^\infty 2\nu k^2 \phi \, dk = \text{(KE dissipation rate)} \tag{54}$$

and

$$N = \int_0^\infty 2\nu_T k^2 \phi_{TT} \, dk = \text{(temperature fluctuation (PE) dissipation rate)} \tag{55}$$

(Hinze,[26] p. 179), we obtain

$$\overset{\text{I}}{\int_k^\infty \dot{\phi}\, dk} = \overset{\text{II}}{-\varepsilon} + \overset{\text{III}}{2\nu \int_0^k k^2 \phi \, dk} - \overset{\text{IV}}{\int_0^k F \, dk} + \overset{\text{V}}{\frac{g}{\overline{T}} \int_k^\infty \phi_{wT} \, dk} - \overset{\text{VI}}{\overline{U}_1' \int_k^\infty \phi_{wu} \, dk}$$

$$- \overset{\text{VII}}{\int_0^k \left(k_1 \frac{\partial E_{i,i}}{\partial k_3} \right)_{SP.AV.} dk \, \overline{U}_1'} \tag{56}$$

for the KE spectrum, and

$$\overset{\text{I}}{\int_k^\infty \dot{\phi}_{TT}\, dk} = \overset{\text{II}}{(-N)} + \overset{\text{III}}{2\nu_T \int_o^k k^2 \phi_{TT}\, dk} - \overset{\text{IV}}{\int_o^k F_{TT}\, dk} - \overset{\text{V}}{\overline{T}' \int_k^\infty \phi_{wT}\, dk}$$

$$\overset{\text{VI}}{- \overline{U}'_1 \int_o^k \left(k_1 \frac{\partial E_{ii}}{\partial k_3}\right)_{SP.AV.} dk} \qquad (57)$$

for the PE spectral equation, where we have used

$$\int_o^k F\, dk = -\int_k^\infty F\, dk \quad , \qquad (58)$$

$$\int_o^k \left(k_1 \frac{\partial E_{ii}}{\partial k_3}\right)_{SP.AV.} dk = -\int_k^\infty \left(k_1 \frac{\partial E_{ii}}{\partial k_3}\right)_{SP.AV.} dk \quad . \qquad (59)$$

Referring back to the identification of terms for Eqs. (8) and (9), the physical meaning of most of the above is straightforward. As for the "k-transfer" terms IV and VII in Eq. (56) and IV and VI in Eq. (57), they refer to the energy (PE or KE) transferred out from the range 0 to k into the range k and higher.

In the following, we shall consider Eq. (57) as having been multiplied by $(g/\overline{T})(\overline{T}')^{-1}$ so that we can regard it literally as the PE spectrum. Equations (56) and (57) thus describe two energy cascades that are coupled through the (except for sign) common term V. Figure 3 gives a schematic diagram of the dual cascade represented by these equations. We imagine the cascades as being analogous to pipes, large at the large scale (or small k) end, (symbolizing the large amount of energy stored there), and small at the high k end. For the most part, the "narrowing of the pipes" is due to the acceleration of the cascade discussed previously. But, at the high k end we imagine the pipes as becoming increasingly porous to symbolize the dissipation from the molecular effects, N and ε.* The process of "going down the pipes" is the cascade.

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*Woods and Wiley[12] point out the ε is significantly larger than N. This is clear from the fact that Ri is presumably below 0.25 at the start of the turbulence.

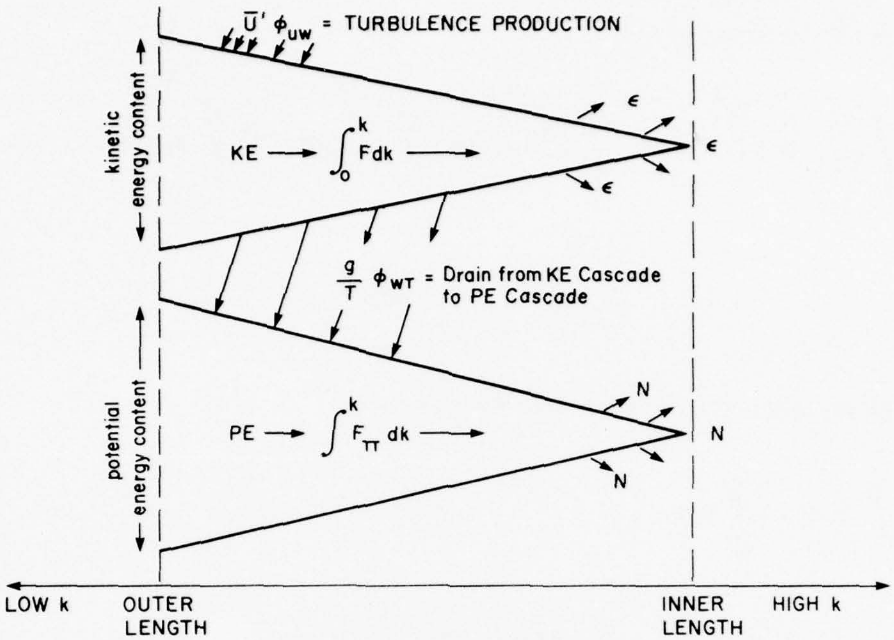

Figure 3. Dual Cascade Model

The source, if we ignore the $\dot{\phi}$ term, is the turbulent production at the large end of the KE cascade. We imagine the energy as entering through the wall of the pipe. The PE cascade is fed by draining from the KE cascade through the "porous pipe walls." This can be considered as a generalization based upon Lumley's[21] "porous pipe" concept. Note that between the "fed" scale and dissipation scale there is a conservative region for both pipes, and this of course represents the inertial range.

The most important point to note about the above picture is that it suggests the physical explanation of the anomalous -5/3 range. If we imagine that, in the KE cascade, the input due to production and output due to "buoyancy drain" are approximately equal to each other for all values of k where they are important, then their net effects will approximately compensate or cancel, and the "shape of the KE pipe," that is, the slope of the turbulence spectrum, must approximate the -5/3 shape. This is because the "flow" would then only depend on the cascade term, F, which accelerates, as Onsager demonstrated, in just such a manner as to give -5/3 as slope.

The above explanation leaves out a number of things. In the first place, the direct spectral transfer due to mean shear has been omitted. More importantly, if production and buoyancy terms exactly cancelled, the KE spectrum would have no source at all. In K-H turbulence, however, if we assume Ri = 0.25 for instability to occur, then we are guaranteed that during the initial phase of the turbulence, production will exceed "drainage" so that there will be a net feeding of the "KE pipe," and relation (43) assures us that "local inertiality" will exist and hence to a good approximation the -5/3 shape will prevail. As for mean shear effects on the cascade, we have already assumed that the interaction between mean shear and turbulence vorticity is "weak" in K-H turbulence, and, even including" shear transfer" effects, it will be shown below that we would still expect -5/3 for the slope in the "anisotropic" region.

Once Ri = 1 and the decay phase of the turbulence begins, the "source" of the turbulence will be the slowly decaying large eddies ($\dot{\phi}$) and, since both shear and temperature gradient within the layer will have been destroyed by the mixing process, we would expect a -5/3 slope on the basis that local inertiality would hold even more accurately than during the initial phase. In this second phase, both KE and PE cascade should have a -5/3 slope which is in contrast to the initial phase where the PE slope, in the anisotropic region, would be significantly flattened (if not in fact reversed in sign) due to the effects of "production" without a compensating drain.* On the other hand, at the very start of the turbulence regime, we might expect a k^{-1} dependence in the KE spectrum due to "strong interaction" (cf. Gisina[39]).

6.2 The Anomalous -5/3 Range

We now proceed in the spirit of Section 5 to derive the KE spectrum in the manner used by Lumley[21] in his derivation of the buoyancy range equations. We start with Eq. (45) and integrate to obtain

$$\varepsilon^{2/3} = \varepsilon_o^{2/3}\left(1 - \frac{C}{2}(\overline{U_1'})^2(1 - \mathrm{Ri})\,\varepsilon_o^{-2/3}\,k^{-4/3}\right) \tag{60}$$

where the constant of integration has been chosen to make $\varepsilon \to \varepsilon_o$ when $k \to \infty$ (ε_o is the viscous dissipation). Now the condition given by Eq. (43) assures us that the spectrum will be determined locally by the spectral flux and that the

*A log-log plot of the temperature spectrum derived from HICAT showed (Vinnichenko and Dutton[40]) a -5/3 power law on the average. The implication of the above reasoning is that most of the data were taken in the decay phase.

39. Gisina, F.A. (1966) The effect of mean velocity and thermal gradients on the spectral characteristics of turbulence, Izvest. Atm. and Oceanic Phys. 2:804-813.

40. Vinnichenoko, N.K., and Dutton, J.A. (1969) Empirical studies of atmospheric structure and spectra in the free atmosphere, Radio Sci. 4:1115-1126.

spectrum will remain approximately isotropic since anisotropic inputs or outputs are a small fraction of the "throughput" ε. In other words, we have the condition where the cascade is "locally inertial in wavenumber space" (a phase attributed to Corrsin by Lumley). This allows the use of all the customary forms familiar in the inertial subrange, but with "variable ε" as Lumley[21] has put it.

Thus we find that ,

$$\phi(k) = \alpha k^{-5/3} \varepsilon_o^{2/3} \left[1 - \frac{C}{2} (\bar{U}')^2 (1 - Ri) \varepsilon_o^{-2/3} k^{-4/3}\right] \quad . \tag{61}$$

From this we immediately arrive at the conclusion that: (1) for high enough k we have the ordinary inertial range relationships, and (2) under the restriction of Eqs. (43), the -5/3 relation holds for all scales up to the outer length which is also indicated by Eq. (47) whenever we are concerned with K-H turbulence at a stage where Ri is sufficiently close to unity. It should be mentioned in passing that Lumley[41] also explained an anomalous -5/3 spectrum, in a different context, by means of this constant flux mechanism.

Another approach to the anomolous -5/3 slope is the following. This time we use Eq. (56) to derive the -5/3 slope of the KE spectrum of K-H turbulence under the assumptions of weak interaction between turbulence and mean profile, and stationarity (of both mean and turbulent quantities). We shall make use of the Heisenberg concept mentioned previously. In addition, we shall make use of Tchen's idea of using the Boussinesq's concept to represent terms VI and VII together as a "dissipative effect" (see Tchen[42] and Hinze[26]). We shall also treat term V in a similar way (cf Gisina[39]).

Thus, for terms VI and VII we have:

$$\bar{U}' \int_k^\infty \left\{ \phi_{wu} - \left(k_1 \frac{\partial E_{ii}}{\partial k_3} \right) \right\} dk = -\nu_m(k) (\bar{U}')^2 \tag{62}$$

where

$$\nu_m(k) = \alpha'\alpha \int_k^\infty \sqrt{\frac{\phi(k)}{k^3}}\, dk = \alpha'\alpha\, K(k) \tag{63}$$

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41. Lumley, J. L. (1967b) The inertial subrange in nonequilibrium turbulence, in Atmospheric Turbulence and Radio Wave Propagation, edited by A. M. Yaglom and V. I. Tatarsky, Nauka, Moscow.

42. Tchen, C. M. (1953) On the spectrum of energy in turbulent shear flow, J. or Res. of NBS 50:51-62.

and for term V

$$\frac{g}{\overline{T}} \int_k^{\infty} \phi_{wT}\, dk = \frac{g}{\overline{T}} \left(-\nu_H(k)\right)\overline{T}' \tag{64}$$

where

$$\nu_H(k) = \alpha''\alpha \int_k^{\infty} \sqrt{\frac{\phi(k)}{k^3}}\, dk = \alpha''\alpha K(k) \; . \tag{65}$$

We can thus write Eq. (56) as

$$\varepsilon = 2\left(\nu + \alpha K(k)\right) \int_o^k k^2 \phi\, dk + aK(k) \tag{66}$$

where

$$a \equiv \alpha'\alpha\, (\overline{U}')^2 - \frac{g}{\overline{T}} \alpha''\alpha \overline{T}' \quad . \tag{67}$$

To solve (66), one uses the technique of Agostini and Bass[43] and Chandresekhar[44] namely, that of defining

$$\Gamma(k) \equiv \int_o^k k^2\, \phi(k)\, dk \; , \tag{68}$$

or

$$\phi = \frac{1}{k^2} \frac{d\Gamma}{dk} \quad . \tag{69}$$

In the resulting differential equation, which can be solved by direct quadrature, the boundary condition is $\Gamma = \varepsilon/2\nu$ when $k \to \infty$, giving for constant of integration

$$\mathrm{CONST} = \frac{1}{6}\left(\frac{\nu}{\varepsilon + a\nu}\right)^3 \quad . \tag{70}$$

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43. Agostini, L., and Bass, J. (1950) Publ. Sci. et Tech. Ministere Air 237.
44. Chandrasekhar, S. (1949) On Heisenberg's elementary theory of turbulence, Proc. Roy. Soc. London, 200A:20.

Thus, as Gisina[39] has shown in a different context,

$$\phi(k) = \left(\frac{8}{9\alpha}\right)^{2/3} (\varepsilon + a\nu)^{2/3} \left[1 + \frac{8\nu^3 k^4}{(3\alpha^2)(\varepsilon + a\nu)}\right]^{-4/3} k^{-5/3} \quad . \tag{71}$$

For values of k of interest, this gives a $k^{-5/3}$ dependence. The most important part of this result to notice, other than the -5/3 slope, is the fact that the "constant" no longer simply depends on ε, as it would in a classical inertial range. Rather, it also contains $(a\nu)$ which involves the production and bouyant dissipation terms. As mentioned in the introduction, this could lead to an erroneous determination of ε if the investigator makes the mistake of perceiving an anomolous -5/3 range as an inertial range.

It should not be overlooked that Eqs. (61) and (71) are not consistent. Only the -5/3 exponent is consistent. This lack of consistency is excusable, perhaps, on the grounds that the Heisenberg approach usually does give physically incorrect results outside of the wavenumber range of primary interest (cf Hinze's[26] discussion of the Heisenberg result for the dissipation range).* In view of the necessity for an inertial range between the anisotropic production range and the dissipation range, we should choose the relation (61) as being more correct at large λ since it includes the inertial range as a high wavenumber limit in contrast to (71) which does not. From (61), the remark above about determination of ε still holds.

7. SPECTRAL NONLINEARITIES

In both HICAT data (Crooks et al[45]) and in the tropospheric data of Reiter and Burns[46] it has been noted that there are "humps" and "dips", sometimes called "gaps" in the KE spectra. Various attempts have been made to explain these (Cf. Reiter and Burns[46]). One such explanation, Pao[47], was based on a numerical simulation in which it was found that, with high degrees

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*The explanation for the somewhat incorrect nature of the Heisenberg formalism is that it reflects distant wavenumber interactions, whereas close wavenumber interactions are in reality much more important.

45. Crooks, W.M., Hoblit, F.M., and Prophet, D.T., et al (1967) Project HICAT: An Investigation of High Altitude Clear Air Turbulence, Tech. Rept. AFFDL-TR-67-123. AD824 865, AD824 904, AD825 369 (1967). See also AD846 086, AD847 497 (1968).

46. Reiter, E.R. and Burns, A. (1966) The structure of clear-air turbulence derived from "TOPCAT" aircraft measurements, J. Atm. Sci. 23:206-212.

47. Pao, Y.H. (1968) Turbulence Velocity and Scalar Spectra in Stably Stratified Fluids, Boeing Scientific Research Laboratories, D 1-82-0680, AD677-585.

of stability, the nonlinearities would occur. Unfortunately in these simulations the production term due to shear was omitted.

Another possible explanation is the following. In Eq. (56), let us ignore terms I, III, and VII, and in Eq. (57) omit I, III, and VI. We then have

$$\overset{\text{II}}{\varepsilon} = - \overset{\text{IV}}{\int_0^k F\,dk} + \overset{\text{V}}{\frac{g}{\overline{T}} \int_k^\infty \phi_{wT}\,dk} - \overset{\text{VI}}{\overline{U}' \int_k^\infty \phi_{wu}\,dk} \qquad (72)$$

$$\overset{\text{II}}{N} = - \overset{\text{IV}}{\int_0^k F_{TT}\,dk} - \overset{\text{V}}{\overline{T}' \int_k^\infty \phi_{wT}\,dk} \ . \qquad (73)$$

These equations have been treated by Lin et al[25, 48] and by taking the limiting form of the generalized eddy-viscosity approximation due to Panchev[35] they arrived at the following nondimensionalized representation

$$\overset{\text{II}}{1} = \overset{\text{IV}}{\left\{ X^{5/2}\, \Phi(X)^{3/2} \right\}} - \overset{\text{V}}{\left\{ \Gamma_1\, X^{\left(-\frac{1}{2}\right) + \left(\frac{3C_4}{2}\right)}\, \Phi(X)^{1/2}\, \Phi_{TT}(X)^{C_4/2} \right\}} \qquad (74)$$

$$+ \overset{\text{VI}}{\left\{ |\Gamma|\, X^{\left(-\frac{1}{2}\right) + \left(\frac{3C_2}{2}\right)}\, \Phi(X)^{(1/2) + (C_2/2)} \right\}}$$

$$\overset{\text{II}}{1} = \overset{\text{IV}}{\left\{ X^{5/2}\, \Phi^{1/2}\, \Phi_{TT}(X) \right\}} + \overset{\text{V}}{\left\{ |\Gamma_T|\, X^{-\frac{1}{2} + \frac{3C_4}{2}}\, \Phi(X)^{1/2}\, \Phi_{TT}(X)^{C_4/2} \right\}} \qquad (75)$$

where Eq. (74) corresponds to (72), and (75)* corresponds to (73). The numbering of terms is retained. Also, Φ and Φ_{TT} are the nondimensionalized spectra, and X is the nondimensionalized wavenumber. Γ = a parameter characterizing turbulence production due to mean shear, Γ_1 = a parameter characterizing the bouyancy dissipation which depends on mean temperature gradient, and Γ_T = a parameter characterizing the production of temperature fluctuations which depends temperature gradient. Parameters C_2 and C_4, both of which can range from 0 to 1, represent a generalization of Tchen's concept of strong and weak interaction between turbulence and mean shear and temperature gradient respectively. "0" means weak interaction and "1" means strong interaction in the sense that the vorticity

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*Compare term IV of Eq. (75) with Eq. (37).

48. Lin, J.T., Panchev, S., and Cermak, J.E. (1969b) A modified hypothesis on turbulence spectra in the bouyancy subrange of stably stratified shear flow, Radio Sci. 4:1333-1337.

squared term (in the Heisenberg expression for "dissipation") is the square mean gradient or shear in the "weak" case, and the square vorticity* or equivalent form for temperature fluctuations in the range 0 to k in the "strong" case (cf Hinze[26] pp. 264, 265). Finally, note that in Eqs. (74) and (75), the spectral transfer terms are those of Onsager and Kovasnay.

Lin et al[25] solved these equations numerically for a number of cases. One particular case is of special interest here, and it is given in Fig. 4 which is copied from their Fig. 6. It shows both the KE and temperatures spectra. This example has the following values for the parameters: $C_4 = 0.3$, $C_2 = 1$, $\Gamma_1 = 0.01$, $\Gamma = 0.001$, and $\Gamma_T = 0.001$. From these parameter values, it can be seen that shear effects in this example are less important than bouyancy effects. The figure displays the -5/3 slope in the KE spectrum at both high and low ends, but a -2.75 in the midrange where bouyancy dissipation effects become dominant. This brings out two important points. First, it clearly illustrates the anomalous -5/3 range effect. In that range, the spectral transfer term, $X^{-5/3}\Phi^{3/2}$ is essentially constant in the numerical simulation, and this is in accordance with our previous discussion. Second, this figure may provide an explanation for some of the spectral nonlinearities seen in the data. No doubt, by adjusting the parameters so that production is relatively dominant over bouyancy, one could cause the intermediate range slope to be flatter (rather than steeper) than -5/3. Note that "jogs" also appear in some HICAT data.

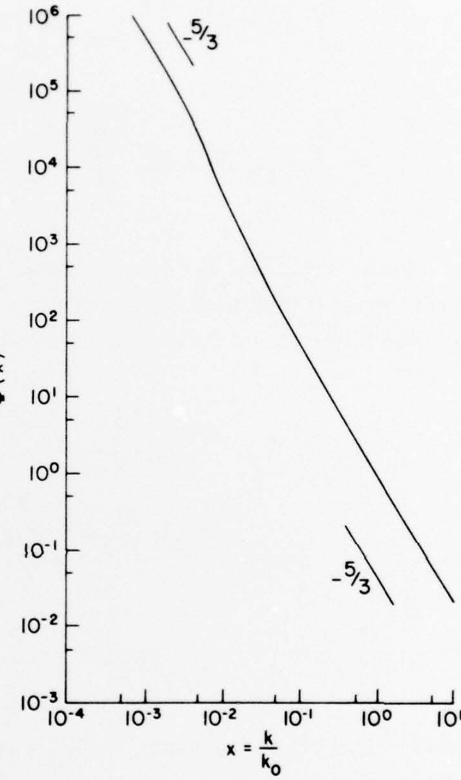

Figure 4. Minus 5/3 Spectrum With "Jog" Due to Turbulence Production (after Lin et al, 1969a)

It should be emphasized that the idealized treatment that we have given for the spectra of K-H turbulence would predict a -5/3 slope for all k between the inner and outer length (assuming $a = 0$, that is, a perfect balance between produc-

*This refers now to velocity fluctuations.

tion and bouyancy). * However, in reality, neither "a" will be zero nor will strengths or production and bouyant dissipation interactions (between fluctuations and mean quantities) match perfectly. Hence, spectral nonlinearities should hardly come as a surprise. Perhaps the more surprising aspect of the data is the extent to which it is linear in the majority of cases published. In any case, it is exactly such nonlinearities which could lead the unwary to an erroneous value for ε.

8. TURBULENT DECAY AND GENERATION OF LARGE EDDIES

8.1 Introduction

Section 8 examines two further aspects of stratospheric turbulence in relation to the anomalous -5/3 spectrum. The first concerns the possibility that turbulence persists for a relatively long time after the energy input has stopped, that is, after Ri = 1. It can then be argued that in situ measurements of stratospheric turbulence are much more likely to be of decaying turbulence rather than turbulence with energy input. The second aspect to be discussed concerns the generation of large horizontal eddies by the mean shear during the "energy production" phase of the Kelvin-Helmholtz event. The possibility will be discussed that such large eddy structures could play the role of an outer length during the decay period. This would provide an additional rationale for the presence of -5/3 slopes at considerably large wavelengths.

If in fact the -5/3 spectra are only observed well beyond the time when there is energy input, then the above theory (based upon compensation between production and bouyant dissipation) would be replaced by a theory based on the idea that production and bouyancy in the layer are negligible.

In the discussions below, we shall assume that radiation effects can be neglected. This is reasonable since, according to McClatchey et al,[49] there would be no more than 1°K change per day from radiation effects in the stratosphere. A second assumption is that atmospheric billow events result in fairly complete mixing within the layer so that during decay, mean potential temperature gradient and mean shear are negligible (Turner[50]). This assumption is important, not only for present purposes, but also for the purpose of explaining vertical transport in the stratosphere by turbulence (see Rosenberg and Dewan[1]). It would, therefore, be very useful to validate this assumption by in situ measurements.

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*Note that there is a k^{-7} dependence at large values of k beyond the inner scale.

49. McClatchey, R., Kuhn, W.R., Bailey, P., and Ellington, R. (1975) The Natural Stratosphere of 1974 CIAP Monograph 1, Final Report, DOT, CIAP, DOT-TS7-75-51, p. 4-77.

50. Turner, J.S. (1973) Bouyancy Effects in Fluids, Cambridge University Press.

However, one can interpret the radar study of Browning and Watkins,[51] which is mentioned below, as lending some support to our assumption.

8.2 Decay

From the above we see that an important question is "How long does stratospheric turbulence persist?" In the initial stages, the turbulence consists of a breaking K-H wave which takes less than 5 min to turn over (5 min is about the period for a bouyancy oscillation cf Woods and Wiley[12]). In the ocean, where this process occurs at low Reynolds number, one would estimate the turbulence duration as being on the order of the turnover time (Woods[7]). Such is not the case in the atmosphere due to high Reynolds number there. We would therefore be interested to know how long the turbulence persists beyond the time it is fed by the mean shear through layer spreading etc. (the latter being, presumably on the order of 5 min).

First, let us consider experimental observations. Apparently, the only available direct measurements of turbulent persistence in the stratosphere are those of the HICAT program. The longest observation to be found there was 14 min and 14 sec. In contrast to the stratosphere, there are many observations of CAT in the troposphere which are of interest provided we remember that the turnover time there is about double that of the stratosphere. These involve both radar and radiosonde measurements. Boucher[52] has published a radar observation of tropospheric CAT which occupies a 500 m thick layer and which persisted for 1 hr. Perhaps the most useful radar study in this connection is that of Browning and Watkins[51] who, as mentioned above, described a typical K-H billow sequence as consisting of 20 min of visible billow structure followed by a double layer of turbulence persisting for 2 hr. This observation is consistent with the idea that beyond the initial 20 min the turbulence continues for about 2 hr without bouyancy damping (mean gradients inside the layer having been destroyed by mixing). At the top and bottom edges of the layer, enough entrainment continues to occur with sufficient intensity as to cause (for 2 hr) an observable reflection due to the inhomogeneities of index refraction thus generated. If this interpretation is correct, one can expect that stratospheric turbulence occurs for hours rather than minutes.

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51. Browning, K.A., and Watkins, C.D. (1970) Observations of clear air turbulence by high power radar, Nature 227:260-263.

52. Boucher, R.J. (1973) Mesoscale history of a small patch of clear air turbulence, J. of Appl. Meteorol. 12:814-821.

53. Vinnichenko, N.K., Pinus, N.Z., Shmeter, S.M., and Shur, G.N. (1973) Turbulence in the Free Atmosphere, Consultants Bureau, New York.

The radiosonde observations reported in Vinnichenko et al[53] are consistent with the idea that the decay of atmospheric K-H turbulence takes hours. They say that the probability that the turbulent situation remains unchanged at 12 to 20 km altitude is less than 50 percent over the period 1.5 to 6.0 hr.

In the troposphere, persistence can be due to factors other than decay, for example, continually increased mean shears due to the deformation effects of moving fronts. In the stratosphere one can imagine that mountain waves could prolong turbulence. The above must be viewed in this context.

A crude theoretical estimate of decay time can be obtained as follows. (Compare problem 1.2 of Tennekes and Lumley[20]). Let us imagine a cubical box of volume L^3 containing atmosphere in turbulent motion. Assuming no energy source, the turbulent decays, but, since L is fixed, we can assume the length scale is constant and equal to L. We derive an expression for the decay of kinetic energy, $\frac{3(u)^2}{2}$, as follows:

$$\frac{d\left(\frac{3(u)^2}{2}\right)}{dt} = -\varepsilon = \frac{-u^3}{L} \quad (76)$$

Integrating, we find the decay time is

$$\Delta t = 3L\left(\frac{1}{u_f} - \frac{1}{u_i}\right) \quad (77)$$

where u_i is the initial turbulent velocity and u_f is the final velocity.

In this crude model let us set $L = 1000$ m, the observed HICAT turbulence layer thickness mentioned earlier. For u_i and u_f, let us use the value for "strong" turbulence and $\frac{1}{2}$ the value for "weak" turbulence, respectively, as found in Vinneshenko et al[53], p. 190 namely, $u_i = 2.5$ m/s and $u_f = 0.25$ m/s. In this case, $\Delta t = 3$ hr* which is consistent with the above considerations. Obviously, Δt depends greatly on what we choose for u_f and the above choice is perhaps conservative.

*This value for duration disagrees with the results of Badgley[54] who calculated decay time on the basis of constant ε. In general, however, ε decreases with time and this prolongs the duration of decay. On the other hand, the value calculated for duration depends very much upon the final turbulence velocity; thus, a question arises concerning what final velocity should be considered reasonable. Zimmerman and Loving[22] calculated several values of ε from the HICAT spectra which ranged from 262 to 24 cm^2/sec^3. Converting these to initial and final velocities and again assuming $L = 1000$m, we find $\Delta t = 20$ min, and this is four times longer than the turn over time. Thus, even by this conservative estimate, it is reasonable to suppose that most of the HICAT turbulence data is in the decay phase.

54. Badgley, F.I. (1969) Large scale processes contributing energy to clear air turbulence, in Clear Air Turbulence and Its Detection, edited by Y.H. Pao and A. Goldburg, Plenum Press, New York.

8.3 Generation of Large Horizontal Eddies by Shear, and Cascade

There is a possibility that, in the anisotropic shear turbulence which we are here considering, the turbulence cascade starts at a scale much larger than the "outer scale" of turbulent layer thickness. If such were the case, it would not be possible to treat this situation theoretically with the help of present theories of turbulence. On the other hand, if such an effect were actually present, it would further explain an anomolous -5/3 range to unexpectedly large scales. In the following, we shall state the case for this possibility. We shall assume that the effect would be seen only during the decaying stage of the turbulence. But, as the above discussion indicates, we would expect that this stage is significantly more probable than the initial "fed" stage, in any in situ observation. This implies that the anomalous -5/3 slope could be more often due to the absence of both production and bouyant dissipation terms rather than their mutual cancellation. In such cases, the measured value of ε would become reliable even in the anomalous range.

Terms VII in Eq. (56) and VI in Eq. (57) describe the rotation and deformation effects of mean shear on turbulence. The latter consist of compression at 135^{o} and stretching at 45^{o} where 0^{o} is taken as downstream and 90^{o} is vertical, and a positive value for shear is assumed. This effect of shear is duscussed in a number of places, eq Phillips[55], Townsend[56], Lumley[57], Moffatt[58], and Lumley and Panofsky[27]. In the above discussion, we have referred to it as a transfer of energy in k- space due to mean shear. At 135^{o}, the transfer is from lower to higher wavenumbers and at 45^{o} it is from higher to lower wavenumbers. Lumley and Panofsky[27] showed that the spherically averaged effect is a transfer to higher wavenumbers. However, the geometry of the situation clearly indicates that, for horizontal eddies, the transfer is to larger scale.

As discussed in Phillips[55] the generation of large and elongated eddies in the horizontal direction is not only theoretically expected but has been observed experimentally in wall turbulence. Panofsky and Deland[59] have studied such

55. Phillips, O. M. (1969) Shear-flow Turbulence, Annual Rev. of Fluid Dynamics Vol. I, edited by W. R. Sears, and M. Van Dyke, Annual Reviews, Inc., Palo Alto, California.

56. Townsend, A. A. (1956) The Structure of Turbulent Shear Flow, Cambridge University Press.

57. Lumley, J. L. (1964b) Spectral energy budget in wall turbulence, Phys. Fluids 7:190-196.

58. Moffatt, H. K. (1967) The interaction of turbulence with strong wind shear, in Atmospheric Turbulence and Radio Wave Propagation, edited by A. M. Yaglom, and V. I. Tatarsky, Nauka, Moscow.

59. Panofsky, H. A., and Deland, R. J. (1957) Structure of turbulence of O'Neill, Nebraska and its relation to the structure at Brookhaven, National Laboratory, Upton, Long Island, AFCRC-TR-58-201, AD-146 843, J. Geophys. Res. 64:2226-2229.

elongated eddies in the boundary layer with co-spectral techniques. It is perhaps fair to say that large persistent horizontal eddies should be found in the turbulent shear layers of the stratosphere (having been generated during the strong interaction, early time of the turbulence) and that for this reason, significant amounts of energy may be found at scales exceeding layer thickness. During the decay phase, shear and temperature gradient will be absent, and term I of Eqs. (56) and (57) would be the only source of energy. The unanswered question is, can some anisotropic analogue of "F" and "F_{TT}" describe the cascade processes in such a way that an approximate -5/3 slope of the one-dimensional horizontal spectra occur out to scales exceeding layer thickness during the decay? In other words, could the large eddies generated by the shear at the start of the turbulence act later, during decay, as the prime energy source of the turbulence, and could an anisotropic cascade result which approximates the inertial cascade? Unfortunately, these questions must remain unanswered at present. If the answers were affirmative, however, it would probably also help one to explain why the slope of the vertical velocity spectra are flatter at large scale than the horizontal velocity spectra in the HICAT data.

9. CONCLUSIONS

The HICAT data contain spectra which have an approximately -5/3 slope out to wavelengths orders of magnitude larger than expected for an inertial range. In order to explain this we have employed treatments based on those of Lumley[21], Gisina[39], and Lin[25], all of which lead to a -5/3 slope for K-H turbulence the outer length when Ri is close to unity. A key assumption in some of these treatments was "local inertiality." Since current theory relies upon spherical averages, it cannot explain the large wavelengths for -5/3 that are observed. To bridge the gap, we have invoked the concept of Gifford which accounts for such effects in the one-dimensional spectra on the basis of spatial aliasing.

In addition, some remarks were made regarding the generation, by mean shears, of large elongated horizontal eddies which might, during the decay phase, act as the energy source for the turbulence. The speculation was advanced that an anisotropic cascade effect might help to explain extremely long wavelength -5/3 regions.

The practical impact of these considerations to the determination of ε from measured spectra was also noted. In addition, it can be concluded that one need not resort to wave theory to explain -5/3 slopes at large wavelengths.

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Notes Added in Proof

1 - The Kelvin Helmholtz type of turbulence discussed in this paper is intrinsically a non-stationary, developing flow with weak interaction between turbulence and the mean shear (as has already been pointed out in the text). For this reason it is, strictly speaking, not correct to omit the contribution to the kinetic energy cascade made by the decay of the larger eddies. As mentioned in the text, when R_f approaches the value 1, there would be virtually no net input of energy to the cascade if large eddy decay were ignored, and, Townsend has shown that [J. Fluid mech, 3, 361-372(1958)] R_f could not reach the value of unity but would be constrained by the relation

$$(1-R_f) = \epsilon/(\overline{uw}\ U').$$

The energy balance with decay can be described as follows. Let D be defined as the rate of kinetic energy input to the inertial cascade by the decay of the largest eddies and let P be turbulent production through mean shear, B the bouyant dissipation, $F \equiv \frac{g}{\overline{T}\ \overline{T'}}$, i.e. the conversion factor from temperature fluctuation dissipation to potential energy dissipation, ϵ_θ the temperature fluctuation dissipation rate, and ϵ the kinetic energy dissipation rate. We then have

$$B = F\epsilon_\theta \quad \text{and}$$

$$D + P = B + \epsilon.$$

The second of these equations can be derived from Eq. 6 by retaining term number I. It should also be mentioned that keeping this term would modify our interpretation of Fig. 3, p. 30 in the sense that the large eddies would be considered to be slowly emptying reservoirs of energy which then cascades down the scale, i.e. *down the more narrow parts of the pipes.*

As R_f approaches unity, P approaches $-F\varepsilon_\theta$, hence ε approaches D. We therefore can have quasi-inertiality at large wavelengths when D is significant. Thus, when P and B are significant in size, and when R_f is near unity, then the condition for a -5/3 sloped spectrum is $\partial\varepsilon/\partial k << D/k$. Eq. 61 for the spectrum should be viewed in this light.

Note also that we have assumed that $R_i \approx R_f$. Here again D would have to be significant in value, since otherwise, as Townsend (op. cit.) has shown

$$\frac{\varepsilon}{\varepsilon_\theta} R_i = (1 - R_f)(R_f)\left(\frac{\overline{uw}}{\overline{\theta w}}\right)^2$$

From all these considerations we now see that the assumption of weak shear-interaction has a double significance: (a) the decay rate, D, must be assumed significantly large in order to allow quasi-inertiality, and (b) a strong interaction would give a k^{-1} spectrum rather than a $k^{-5/3}$ spectrum as was shown by Gisina, op. cit.

In Eq. 71, when a = 0 we would have $\varepsilon = 0$ if D were equal to zero. This again emphasizes the need for the condition that $D \neq 0$ if the -5/3 formalism is to be consistent. However, we have not taken into formal consideration the input of energy due to layer spreading. This could be done superficially by incorporating that input in the term D above. On the other hand there exists no theory at present that gives the spectral distribution of this input due to layer spreading.

2 - It should be emphasized that the more sophisticated model of Lin, which was described in the text, makes possible a number of nonlinear spectral shapes which would make a measurement of ε, by means of the spectrum, incorrect. On the other hand, when both the turbulent production and bouyancy dissipation terms are very small, then such measurements of ε should be correspondingly accurate. As seen from the arguments given, such a state of affairs may well be the more probable one for stratospheric measurements. Nevertheless the most feasible way to check this is by measurements of the spectrum at sufficiently high k such that the wavelengths are well within the true inertial range.

3 - The text mentions the need to find experimental evidence for adiabatic lapse rates in the stratosphere following turbulent breakdown. Fig. 6.33 on P. 6-76 from Loving (1975), [The Natural Stratosphere of 1974, CIAP Monograph I, Final Report, DOT, CIAP., DOT-TST-75-51] seems to provide a published example of this.