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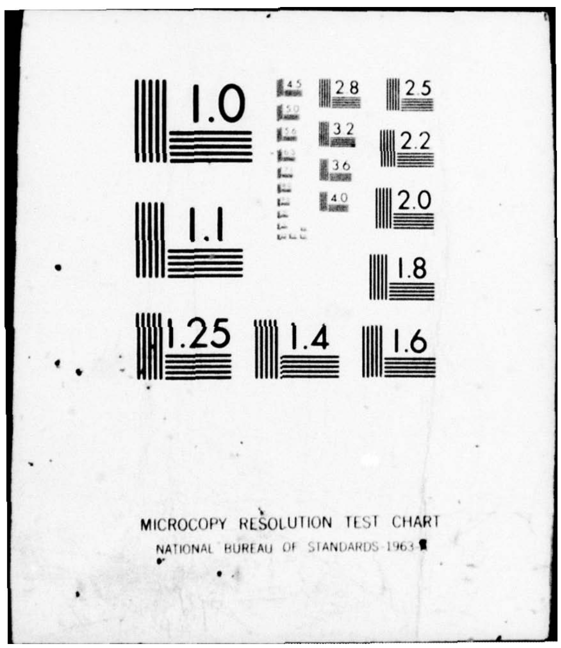
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PROJECT ON EFFICIENCY OF DECISION MAKING

IN

ECONOMIC SYSTEMS

MANIFOLDS OF PREFERENCES AND EQUILIBRIA

by

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Preface

This work is concerned with the study of local and global properties of spaces of preferences and their applications to general equilibrium, utility and demand analysis. Differential topology and global analysis tools are used to study the mathematical aspects of these problems, adding further results and techniques to an approach introduced in mathematical economics by Gerard Debreu in 1970.

Spaces of smooth, not necessarily convex or increasing, preferences are shown here to be representable as differentiable Hilbert manifolds. These structures on spaces of preferences are then used to extend major results on the regularity of equilibria, based on Sard's theorem and Abraham-Thom transversality theory, to economies where agents are described by their preferences and endowments. Certain topological properties of these manifolds of preferences are studied. Applications are also given to the study of the utilities and the demands of the agents in relation to the underlying preferences. The results point to extensions and indicate new branches of research of both economic and mathematical interest; these are written as suggestions or conjectures in the text. Further possible applications of the results to the study of regularity properties of economic aggregates are also indicated.

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Chapter 1

INTRODUCTION

a. Motivation of the Results and Discussion of Applications

Preferences are primitive concepts of the theory of economic behavior. An economic agent is usually represented by a preference relation and certain initial data, such as a vector of endowments of commodities, wealth, production possibilities sets, etc. Thus, spaces of economic agents are usually represented as products of spaces of preferences and subsets of Euclidean spaces. As the theory of economies with many agents grew, more and more structure was required on the spaces of economic agents in order to formalize and study certain concepts of the theory. For instance, in order to investigate continuity of the equilibria and the core with respect to the agent's characteristics, a topological (metric) structure was introduced on spaces of preferences in Kannai [24], Debreu [10] and Hildenbrand [21]. For the study of certain economic aggregates, such as the aggregate or mean demand, spaces of preferences became metric measurable spaces [10], [21] and an economy was represented by a measure which described a distribution of its agent's characteristics.

More recently, further investigation of certain global properties of equilibria, such as finiteness continuity and stability, has developed by the use of tools of differential topology, starting from the leading article by Debreu [12]. In Debreu's theorem an economy is represented by the C^1 demand functions of the agents, satisfying a boundary condition; the main result is that for almost all initial allocation of commodities of the n agents, there are only a finite number of equilibria which depend continuously on the allocations. Such an economy is also called regular. In Smale [38], the results of Debreu are extended: it is proven that for almost all initial allocations and utility functions of the agents, the "extended" equilibria (which include and may not coincide in general with the classical equilibria) are locally unique and stable.

The results of [12] and [38] rely on differential topology techniques; in [12] basically the theorem of Sard, and in [38] Abraham-Thom's transversality theorem and other infinite dimensional differential topology techniques. In order to be able to use these types of techniques one needs differentiable structures on the spaces one works on. Smale's results extended those of Debreu, describing or parametrizing an economy by the utilities of the agents as well as by their initial endowments of commodities as in [12]. Utility functions are elements of linear function spaces, which have enough structure to work on, in particular, to apply transversality theory.

However, utilities are considered unsatisfactory as primitive concepts [13], and the results of [38] cannot be translated from spaces of utilities to spaces of preferences unless more structure for spaces of preferences is given. This is also discussed in [16]. Furthermore, as has been known in economic theory for a long time, the whole analysis of equilibria and demand behavior ultimately rests on the indifference surfaces of the preferences which are the level surfaces of the utility functions. As pointed out, for instance, by Smale [38] the utility functions are mostly used as a convenient description of these indifference surfaces. So it seems also methodologically more adequate to work on spaces of preferences directly. Since Abraham-Thom's transversality theorem is available on infinite dimensional manifolds which admit some C^k representation into a function space [1], if such a representation is obtained for spaces of preferences, one can take advantage of these techniques and extend the results of [12] and [38] for economies where the agents are identified by smooth (not necessarily convex) preferences and initial endowments. In this work, a manifold structure for spaces of preferences is constructed. We show that under certain conditions spaces of preferences can be represented by certain spaces of foliations of the commodity space; the leaves of the foliation are the indifference surfaces of the preferences. We give here a Hilbert manifold structure to these spaces of foliations, with the topology induced by Sobolev's norms. Sobolev's inequality theorem, which relates the Sobolev's norms to the C^k norms, is then used to provide the C^k representation referred to above, and then we obtain the desired extension of the results of regularity of equilibria of [12] and [38] for economies where agents are identified by preferences and initial endowments.

We shall now briefly discuss possible further applications of the manifold structures of preferences constructed here, to problems of regularity of economic aggregates. Differential topology tools have also proven useful for the study of regularity properties of economic aggregates. An economy is in this context usually represented by a measure on the space of economic agents indicating a distribution of the agent's characteristics [21]. (Such as preferences, endowments, etc.). For the study of aggregate properties of the economy it helps to have an adequate structure on the space of agents in order to be able to use well-studied measure theories. For instance, in [36], the admissible spaces of preferences are restricted to have finite dimensional manifold structures, and under these conditions, differential topology techniques and Lebesgue-like measures on spaces of preferences are used to obtain results of continuity of the mean or aggregate demand. Lebesgue-like measures, which can be defined on locally Euclidean spaces such as finite dimensional manifolds, provide "suitable diffused" distributions on spaces of economic agents; in general metric spaces there exist no counterpart for these measures. As pointed out by Debreu [13] a natural next step would be to endow spaces of preferences with an algebraic structure. Such an (local or global) algebraic structure is obtained when the spaces of preferences are given manifold structures. As discussed in [13] these structures should be of help in the study of specific classes of measures on spaces of agents and of certain desirable properties of economic aggregation obtained by the use of such measures as, for example, the smoothing effect of the aggregation of individual demands with "suitably diffused" measures. The smoothness of the excess demand function of an economy is an important property. Since the equilibria are the zeroes of the excess demand function, its smoothness allows to use differential topology tools for the study of properties of the set of equilibria of the economy. For a summary of the existing literature on economies with smooth excess demands, see Debreu [14].

Another desirable property of economic aggregates that could be studied if one has sufficient structure on the spaces of economic agents, or preferences, is the uniqueness of equilibria, that can be obtained by aggregation of "suitable concentrated" measures. This latter point refers to the following intuitive idea: in a many agent economy if all agents have identical endowments and identical preferences, under certain conditions, the economy will exhibit a unique equilibrium [3]. But such an economy can be thought of as one described by a measure on the spaces of endowments and preferences which is completely concentrated on (or supported by) one point. (We shall denote this measure μ_0.) If there is to be any continuity of the properties of the equilibria with respect to the characteristics of the economy (the economy in these cases is usually described by a measure on the spaces of preferences and of endowments), then if an economy (a measure) is "sufficiently close" to the economy represented by μ_0, one can expect that the property of uniqueness of the equilibria will be preserved. For the study of these continuity properties of the manifold of equilibria as depending on the parameter or measure that represents the economy, it is, of course, useful to have as much structure on the underlying space as possible, in order to use limit or convergence theorems, etc. If the economies, for instance, are described by measures which are absolutely continuous with respect to a certain basic measure (say a Lebesgue measure if the underlying space has a locally Euclidean structure), and thus can be represented by certain classes of functions, one can, for instance, actually try to further study the "bifurcation" values (of these functions) that represent the economies at which the corresponding set of equilibria changes from being a one point set to a many point set, with corresponding implications for the stability of the economy.

To the above motivations for the construction of differentiable manifold structures for spaces of preferences we can also add another provided by recent results on majority voting or aggregation rules for which it is required to have an underlying algebraic structure of the spaces of preferences similar to the one we construct here, see Jean-Michel Grandmont [19].

It is of interest that one can actually endow spaces of preferences with the differentiable manifold structures needed by the recent advances of economic theory. However, as we shall see in what follows, these manifolds are by nature infinite dimensional. This restricts the validity of results such as those of [40] which are obtained assuming finite dimensional differentiable manifold structures on the admissible spaces of preferences. In general, this infinite dimensionality precludes the use of finite dimensional differential topology techniques on spaces of preferences, but may open up a whole new range of applications of infinite dimensional differential topology and global analysis tools which have been developed in recent years.

From the point of view of measure theory, Lebesgue-like measures would not anymore be available for these spaces of preferences: infinite dimensional Banach spaces do not admit non-trivial translation and rotation invariant σ-additive measures [37] (even bounded balls in infinite dimensional normed space do not admit such rotation and translation invariant probability measures). However, a theory of probability measures on infinite dimensional spaces is available. In the case of Hilbert spaces, there exists certain well studied standard classes of rotational invariant measures, which can be described by limits of sequences of their restrictions to finite dimensional spaces, since Hilbert spaces have countable base. In particular, there exist Gaussian-type measures called Wiener measures, commonly used in mathematical physics, whose restrictions to finite dimensional subspaces are Gaussian distributions, and such that any rotational invariant measure can be represented as an average of such measures on pre-Hilbert spaces [37]. These measures have many advantages; perhaps the most noticeable one in our context is that their restrictions to finite dimensional subspace are absolutely continuous with respect to the Lebesgue measure and have well known properties. Thus the techniques which require measures continuous with respect to the Lebesgue measure for the study of aggregation problems described above can be used on each finite dimensional subspace of the space of preferences. Since the whole space of preferences is given here a Hilbert manifold structure, and since Hilbert spaces have the particular property of having countable base and can be described as a limit of an increasing sequence of subspaces (this is not true of most Banach spaces), one can expect that many aggregation results proved in finite dimensional subspaces will go through in the limit of a sequence of subspaces, i.e., will carry through to the whole space. Another advantage of these measures is that, since they are rotationally invariant, aggregation or voting rules for the types of societies with the distribution of preferences they represent, would under certain conditions be consistent with majority rules [19].

With such measure theoretical applications in mind, we study Hilbert manifold structures on spaces of preferences, induced by Sobolev's norms [39]. In order to justify the introduction of Sobolev's norms in mathematical economics, we make use of the Sobolev's inequalities theorem [39] to show the relationship between these norms and the C^k-sup norms and the Whitney topology on spaces of C^k (k-times continuously differentiable) functions which have been used in the mathematical economic literature. By Sobolev's theorem one can also show conditions under which Sobolev's spaces of preferences, while being Hilbert spaces, and thus selfdual, are made up of preferences representable by continuous or C^k functions, thus combining two characteristics which are very useful for economic theory.

Further applications of the techniques and results to properties of the utilities and the demands as depending on the underlying preferences are given in Chapter 3 and are described in the following summary.

b. Summary of the Results

In Chapter 2, Part a, we introduce an approach to the study of smooth preferences. We show conditions under which smooth preferences (not necessarily convex or increasing) can be represented as subspaces of a space of retractions from the commodity space to a submanifold, and we show conditions for this space to be not empty. We then endow the space of preferences with a Sobolev norm, and show that it is a Hilbert manifold. We discuss other norms and their relations. In Part b, we show that the subspace of the above space of preferences which give codimension-one foliations of the commodity space is a submanifold. Subspaces of convex and increasing smooth preferences are also shown to be submanifolds. We then indicate how to extend these results to a much larger space of preferences, represented by all codimension one foliations of the commodity space given as retractions into some (not necessarily the same) smooth one dimensional submanifold of the commodity space. The above is restricted to compact commodity spaces.

In Chapter 3, Part b, we give sufficient conditions for a manifold of preferences constructed in Chapter 2 to be a contractible space. We show that the results of Chapter 2 can be extended to produce manifold structures for spaces of many-agents preferences, which are described each by a retraction onto a submanifold of the commodity space, more general and of higher dimension than those of Chapter 2. These preferences also define foliations of the commodity space, of higher codimension.

We study properties of the indifference surfaces of the utilities as related to the critical points of the vector field normal to the foliation determined by the underlying preference at the boundary of the commodity space. We study genericity of the set of preferences (not necessarily convex) that yield C^1 demand functions.

In Chapter 4 we extend the results on genericity of regular economies of Debreu and Smale to economies with agents represented by their preferences and endowments, and discuss extensions to a non-compact commodity space, the positive quadrant in R^n. Existence of equilibria is briefly discussed. Necessary conditions for the local uniqueness and continuity with respect to the parameters of the equilibria are also given.

Chapter 2

MANIFOLDS OF PREFERENCES

a. Introduction

We shall first discuss the geometrical motivation underlying the construction of manifold structures for spaces of preferences. An initial segment of this construction links with a method which has been used in mathematical economics for representing one convex monotone preference by one utility indicator studied rigorously first by Hermann Wold [45], but introduced probably long before that. We now describe our construction briefly by means of an example. Let f be a preference representable by a monotone concave utility function g defined on the unit cube. The preference is completely described by the "indifference surfaces", the level surfaces of the utility g (in the sense that two utilities with the same indifference surfaces describe the same preference, and two utilities with different indifference surfaces will induce a different preference). By monotonicity one can completely describe the set of all indifference surfaces of g as the set of the inverse images (under g) of the (real) values of g on points of the diagonal D of the cube. If one now maps each point in the cube into that point of the diagonal D which has the same utility value, one can completely describe the preference as a map from the unit cube onto D, which is the identity when restricted to D. With this motivation in mind, one can show that such preferences as described above correspond to certain types of retractions from the cube onto D, i.e., maps from the cube onto D which restricted to D are the identity. Thus the space of such preferences can be thought of as a space of retractions from the cube onto the diagonal. Without the convexity and the monotonicity constraint, but under other, much weaker, regularity conditions on the utilities representing the preferences, one can show that certain spaces of preferences can be represented as certain subspaces of spaces of retraction from the cube to a more general type of submanifold in the cube. One then studies these subspaces of retractions and certain distinguished subspaces representing convex and monotone preferences and also preferences that define codimension-one foliations of the cube (the leaves of the foliations being the indifference surfaces of the preferences), and shows conditions under which they are themselves representable by manifolds. We also indicate a method to give a manifold structure to more general types of spaces of preferences: those given by retractions of the commodity space onto a submanifold that may be different for each preference. This is the general plan of this chapter. We now describe the procedure in more detail for very special cases, to motivate the general approach.

b. An Approach to the Study of Smooth Preferences

Let I^n denote the n-dimensional unit cube in R^n, and $\leq$ the standard vector order of R^n. In its most general form a preference f on I^n is defined to be a subset of $I^n \times I^n$; $(a,b) \in f$ will also be written $a \leq_f b$. If $(a,b) \in f$ and $(b,a) \in f$, we shall say that a is equivalent to b, and denote it $a \approx_f b$. f is monotone if $a \leq b$ in the vector order of R^n implies $(a,b) \in f$. f is called convex when, for all $a \in I^n$, the set $\{b \in I^n$ with $(a,b) \in f\}$ is convex.

For heuristic reasons, we shall first motivate the construction by discussing convex and monotone preferences. Then convexity and monotonicity assumptions are subsequently shown to be inessential to the construction and dropped, and we study more general cases.

By the results of [11], if for all $a \in I^n$, the sets $\{b \in I^n$ with $b \leq_f a\}$, and $\{b \in I^n$ with $a \leq_f b\}$ are closed (in I^n), then there exists a continuous real valued order preserving function $g: I^n \to R$, i.e., $g(b) \geq g(a)$ when $(a, b) \in f$ and $g(a) = g(b)$ when $a \approx_f b$. Such a function g is said to represent f and is also called from here on a utility function representing the preference f. If f is convex, g will be quasi-concave, but, as is well known [32], there are certain convex preferences not representable by concave functions. If f is monotone, g is increasing in the order $\leq$ of I^n. Then one can visualize I^n as the union of the convex sets

$$\bigcup_{a \in I^n} \overline{f_a}$$

where $\overline{f_a}$ denotes the set

$$\{b \in I^n : g(b) \geq g(a)\} ,$$

(see Figure 1). Let D denote the diagonal in I^n, i.e., $D = \{a \in I^n : a_1 = a_2 = \cdots = a_n\}$. I^n can also be visualized as the union over a in D of the sets $I_a = \{b \in I^n : b \approx_f a\}$, where a is in D, i.e.,

$$I^n = \bigcup_{a \in D} \{b \in I^n : b \approx_f a\} = \bigcup_{a \in D} I_a .$$

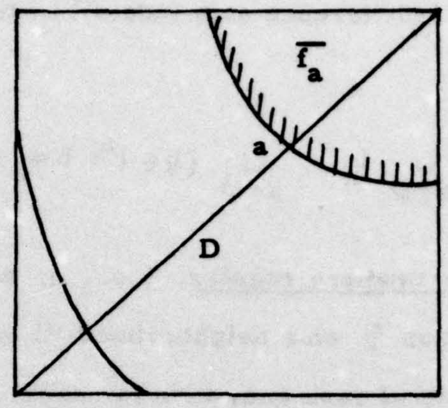

Figure 1

In a sense, the set of all "indifference sets" I_a for a in D completely describe the preference f; if two such families of indifference sets are not identical, they shall describe two different preferences. Thus, one can intuitively think of the space of convex monotone preferences as being represented or described by the space of such families of indifference sets.

We shall now look at differentiable preferences which are not necessarily convex, but still keep the monotonicity assumption. This monotonicity assumption will subsequently be dropped also. A preference f shall be said to be of class C^k, or k-times continuously differentiable ($k \geq 2$), if there exists a real valued C^k utility function g defined on a neighborhood of I^n which represents f --this definition corresponds to Definition (iii) of Debreu [13] when, for all $x \in I^n$, $Dg(x) > 0$, in which case f is said to be a strictly monotone increasing preference.

As in the case of a convex preference, one can express I^n as a union of a family of indifference sets indexed by the elements of the diagonal D, i.e.,

(1) $$I^n = \bigcup_{a \in D} I_a = \bigcup_{a \in D} \{b \in I^n : b \approx_f a\} .$$

If g is also <u>everywhere regular</u>, i.e., g admits an extension to a real valued function $\tilde{g}$ on a neighborhood U of I^n such that for all $a \in I^n$, $Dg(x)$ is of rank one, then for each $r \in R$, $\tilde{g}^{-1}(r)$ is an $n-1$ dimensional C^k submanifold of R^n, or else $g^{-1}(r) = \phi$. (1) can also be stated as

(2) $$I^n = \bigcup_{r \in R} \{g^{-1}(r)\} .$$

Since g is increasing, $g(1,\ldots,1) \geq g(x)$ for all x in I^n. Without loss of generality, one can assume that g is positive valued, and that $g(0,\ldots,0) = 0$. Then, for any a in I^n, there exists a point b in D with $g(a) = g(b)$, by continuity of g and connectedness of D. Therefore (2) can be rewritten as

(3) $$I^n = \bigcup_{a \in D} \{g^{-1}(g(a))\} ,$$

or, equivalently,

(4) . $$I^n = \bigcup_{a \in D} \{x : g(x) = g(a)\} .$$

Furthermore, for all r in R, the set $g^{-1}(r)$ intersects D at most once, since g is increasing, at a point a in D with $g(a) = r$. If g is everywhere regular, the manifold $\tilde{g}^{-1}(r)$ is transversal to the manifold D in the neighborhood U of I^n, since g is increasing. (See the definition in the appendix.) Thus, in general, by (4) the "utility function" g which represents the preference f can be then identified with a map φ_g from I^n onto D, defined by

(5) $$\varphi_g(a) = b \quad ,$$

where $b \in D$ and $g(a) = g(b)$.

Since g is increasing, φ_g is well defined, and, by the definition of φ_g, $\varphi_g(a) = a$ for all a in D. Note that for each C^k preference f there exist infinitely many utilities representing f, actually an infinite dimensional family within the linear space of real valued C^k utility functions on I^n, denoted $C^k(I^n, R^1)$.[1] The space of preferences can be considered as a quotient space of $C^k(I^n, R^1)$ with the following equivalence relation: if g_1 and g_2 are in $C^k(I^n, R^1)$, g_1 is equivalent to g_2 if and only if for each a in D, the sets

$$\{b \in I^n: g_1(b) = g_1(a)\}$$

and

$$\{b \in I^n: g_2(b) = g_2(a)\}$$

are equal, i.e., if g_1 and g_2 induce the same preference of I^n. Ideally one would like to obtain a linear structure on the quotient space $C^k(I^n, R^1)/\sim$ induced by or derived from the linear structure of $C^k(I^n, R^1)$. However, the equivalence relation defined above does not have nice

[1] $C^k(I^n, R^1)$ is the space of all maps from I^n onto R^1 which admit a C^k extension to some neighborhood of I^n.

properties, in particular it is not preserved under the linear structure of $C^k(I^n, R^1)$ and hence the equivalence relation $\sim$ does not induce, even locally, a linear structure on the space of preferences or quotient space $C^k(I^n, R^1)/\sim$. Therefore one has to take a different line of approach.

In the following we shall give an intuitive description of a special case of the construction of manifolds of preferences and a description of its structure for the special case of strictly increasing preferences.

As we saw above, if g_1 and g_2 are two functions in $C^k(I^n, R^1)$ which represent the same preference f (or, in other words, are in the same equivalence class as described above), then the corresponding maps φ_{g_1} and φ_{g_2} defined in (5) are the same. One can thus *define a one-to-one map* ϕ *on the space of strictly* monotone increasing C^k preferences, with values on the space of increasing functions from I^n onto D, denoted $F(I^n, D)$, given by

$$\phi(f) = \varphi_g \quad , \tag{6}$$

where g represents f. Intuitively, the idea is that in the case of an increasing preference f there exists a one dimensional submanifold, the diagonal D of I^n, such that "D represents f in terms of utility", i.e., for every a in I^n, there exists a point b in D with $a \sim_f b$ and if g is a function representing f, the level surfaces of g intersect D only once (and transversally) as we saw above. Once the value of the function g on D is known, and the indifference surfaces of g are known, the function g is completely determined. However, any other increasing function on D which has the same indifference surfaces will yield the same preference. So what actually counts in the representation of f is the indifference surfaces of g, or, in other words, since g is increasing, the sets $\{g^{-1}(g(a))\}$, for points a in D. With the aid of these sets, the function φ_g is constructed as in (5) above, and for each preference f, if g represents f as above, φ_g is a uniquely given element within a space of maps $F(I^n, D)$. Thus the space of preferences can be mapped injectively (<u>not</u> bijectively) into $F(I^n, D)$ by ϕ. By a suitable transformation, the interior of D can be mapped diffeomorphically into the real line R. Since the space of maps $F(I^n, R^1)$ has a linear structure induced by the addition in R^1, this structure can induce a linear structure on $F(I^n, D)$. This structure, in turn, could be thought of as inducing a linear structure on the space of increasing preferences: the sum of two preferences could then be given, for instance, by that induced by ϕ and the sum in $F(I^n, R)$:

$$
\begin{aligned}
f_1 + f_2 &= \phi(f_1) + \phi(f_2) \\
&= \varphi_{g_1} + \varphi_{g_2} \quad ,
\end{aligned}
$$

where g_i represents f_i, $i = 1, 2$. However, there is a problem for this procedure. Not any function on $F(I^n, D)$ describes what we understand by a preference. We have to restrict ourselves, as we discussed above, to a subset of maps in $F(I^n, D)$ that, restricted to D, are the identity on D. The usual addition in $F(I^n, D)$ will not make this subset into a linear subspace of $F(I^n, D)$, since if two maps f_1 and f_2 in $F(I^n, D)$ satisfy $f_1|_D = f_2|_D =$ identity on D, in general, $f_1 + f_2$ as defined above, fails to satisfy $f_1 + f_2|_D =$ identity on D. Hence, a further argument is needed if one desires to consider the space of monotone preferences as a linear subspace of $F(I^n, D)$. This is discussed in Part c of this section. As for the topological structure on the space of increasing preferences, one can similarly consider one induced by the injection map ϕ, from a suitable norm on $F(I^n, R^1)$, chosen from some of the norms which are available in linear function spaces. Even if a linear, or locally linear, structure is given to the space of preferences, one still has the problem that the topology induced on the space of preferences by that on $F(I^n, R^1)$ may fail to make the subspace of preferences complete. Linear subspaces of complete infinite dimensional spaces may fail to be closed. Hence, to the study of the algebraic properties of the space one has to add that of a natural and adequate topology with the desired properties. This is done in Part c of this section. As is frequently the case with infinite dimensional spaces, the adequate choice of topologies is a delicate point here. In addition to the mathematical adequacy criteria, one should also add economic considerations. It is desirable that the space of preferences has a norm and is complete, so that one starts by trying with norms that yield Banach space structures. But the map ϕ between the space of preferences and the space of utilities imposes a restriction from the economic viewpoint: topologies on spaces of utility functions have already been introduced in the literature, and they were chosen so as to conform to economic intuition and also to provide the right mathematical framework for the study of certain economic properties, for instance, in general equilibrium theory [38]. So one should look for norms on spaces of preferences which, through ϕ, relate adequately to the norms already in use in the literature for spaces of utilities. Furthermore, if, as discussed in the Introduction, one would like to have inner products also, at least locally, on spaces of preferences, i.e., to give the spaces of preferences a Hilbert manifold structure, then a closer look at the procedure is required to justify the introduction of such further structure.

As the above indicates, the existence of a diffeomorphism from $\mathring{D}$ into R is important for the construction of the topological and the linear structure of preferences on I^n. As we see in what follows, under certain conditions, when the preferences considered are more general preferences of type C^k, no longer assumed to be increasing, D can be replaced by a more general type of one dimensional submanifold I of I^n, $\mathring{I}$ diffeomorphic to a segment, and a similar construction can be obtained. Furthermore, we shall indicate how a still more general space of preferences can be given a manifold structure, one in which each agent might have a different submanifold playing the role of D.

With all the above considerations in mind, we now proceed to the technical aspects and proofs. For definitions and other complementary material, see also the appendix.

c. Construction of Hilbert Manifold Structures for Spaces of Preferences

In order to simplify matters in the following we shall assume that the commodity space S is a ball $B^n \subset R^n$, with two "antipodal" points distinguished in its boundary, which shall be assumed to be the points $(0, \ldots, 0)$ and $(1, \ldots, 1)$, denoted 0 and 1, respectively. This is done for technical reasons: B^n has the structure of a C^∞ manifold with boundary, while I^n does not. The results can then be translated to any subset of R^n homeomorphic to B^n, and whose interior is C^∞ diffeomorphic to the interior of B^n --such as, for instance, $S = I^n$. We now need some definitions. For complementary mathematical definitions, see also the appendix.

If N is an n-dimensional smooth compact manifold, the Sobolev norm $\|\cdot\|_s$ is defined on the space of maps $C^\infty(N, R^m)$ by:

$$\|f\|_s^2 = \int_N \sum_{0 \le |k| \le s} |D^k f(x)|^2 \, dx$$

$H^s(N, R^m)$ is defined as the completion of $C^\infty(N, R^m)$ under the $\|\cdot\|_s$ norm. These H^s spaces are Hilbert spaces with the inner product defined by

$$(f, g)_s = \int_N \sum_{0 \le |k| \le s} D^k f \cdot D^k g \, dx \quad ,$$

(where $D^k f$ denotes the k-th derivative of the map f, a linear map from R^n to R^m) and are called Sobolev spaces.

The C_k norm on the space $C^k(N, R^m)$ of k-times continuously differentiable functions is defined by

$$\|f\|_k = \sup_{x \in N} \|f(x), Df(x), \ldots, D^k f(x)\|$$

$C^k(N, R^m)$ with the $\|\cdot\|_k$ norm is a Banach space. The relationship between the H^s spaces and the $C^k(N, R^m)$ spaces is given by Sobolev's theorem. This theorem is important to us because it will be used to relate the Sobolev norms used here to others used in the mathematical economic literature such as the C^k norms and the Whitney topology, and we state it here:

SOBOLEV THEOREM (for a proof, see [39]). Let $s > n/2 + k$. Then $H^s(N, R^m) \subset C^k(N, R^m)$ and the inclusion is a continuous and compact map where $C^k(N, R^m)$ has the C^k norm.

We shall start by working with a class $\mathscr{P}$ of preferences defined on S satisfying certain conditions: $f \in \mathscr{P}$ if and only if

(C1) these exists a C^k $(k \geq 2)$ real valued function g defined on a neighborhood of S representing f.

(C2) there exists a compact connected strictly $\leq$ ordered C^∞ one dimensional neat submanifold[2] I of S, with $\partial I = \{0 = (0, \ldots, 0) \cup 1 = (1, \ldots, 1)\}$ and such that for all f in $\mathscr{P}$, if g represents f, $g(1) \geq g(x) \geq g(0)$ for all x in S, and g is strictly increasing along I.

Note that the preferences in $\mathscr{P}$ are not necessarily increasing or convex.

The fact that I is strictly $\leq$ ordered will be used in the following construction. However, this condition can be weakened to assume that I is contractible; this will be proven later on.

We now show that for any preference f in $\mathscr{P}$, I represents S in terms of utility, much the same way that the diagonal D represents the increasing preferences in terms of utility:

LEMMA 1. If f is a preference in $\mathscr{P}$, and g represents f, then for any x in S, there exists a unique point b in I with $g(x) = g(b)$.

---

[2] I is said to be strictly $\leq$ ordered as a subset of R^n when, if x and y are in I, $x \neq y \Rightarrow x < y$ or $y < x$. A submanifold N of a manifold M is called neat if and only if its boundary ∂N is the intersection of the boundary of M with N, $\partial N = N \cap \partial N$, and N is also transversal to M at ∂N. (See the appendix for further definitions.)

<u>Proof.</u> This is immediate from the assumptions: for any x in S, $g(1) \geq g(x) \geq g(0)$ by (C2), and thus, by connectedness of I, there exists a $b \in I$ with $g(x) = g(b)$. Since g is increasing along I by (C2), b is also unique.

Because of Lemma 1, the description of Part b which was given for increasing preferences only applies now to all preferences in $\mathscr{P}$: any preference f in $\mathscr{P}$ can be thought of as a retraction from S onto I. Thus, $\mathscr{P}$ can be thought of as a subset of a space $\mathscr{F}$ of maps from S onto I. We shall choose as $\mathscr{F}$ the Sobolev space $H^s(S,I)$, ($s > n/2 + k$, and $k \geq 2$) defined above. Given this choice of space, if the preferences in $\mathscr{P}$ are assumed to be H^s (by Sobolev theorem $C^k \subset H^s$, since $s > n/2 + k$) then $\mathscr{P} \subset H^s(S,I)$, and this inclusion in injective. We shall use this inclusion to induce a representation of $\mathscr{P}$ as a differentiable manifold. Let $h: S \to I$ be in $H^s(S,I)$ such that h restricted to I is the identity map on I, i.e., $h|_I = id_I$. Then h defines a preference f on S as follows: for any $a \in S$, let the set of points b in S "preferred to a" according to f, i.e., $\{b: (a,b) \in f\}$ be given by

$$\{b \in S: h(b) \geq h(a) \text{ on } I\}$$

and let

$$\{b \in S: h(b) \leq h(a) \text{ on } I\}$$

be the set of points in S to which a is preferred according to f, i.e., $\{b \in S: (b,a) \in f\}$. The above can be formalized by the following exact sequence

$$(7) \qquad 0 \to \mathscr{P} \overset{\phi}{\hookrightarrow} H^s(S,I) \overset{R}{\to} H^s(I,I) \to 0$$

where R is the restriction map defined by $R(h) = h|_I$, $\phi: \mathscr{P} \to H^s(S,I)$ is defined by $\phi(f) = \varphi_g \in H^s(S,I)$ defined as in (5) and (6) above: $g \in H^s(S,R^1)$ represents f, and $\varphi_g(a) = b \in I$, for $a \in \{g^{-1}(g(b))\}$.

Since (7) is exact,

$$\mathscr{P} \approx \phi(\mathscr{P}) \approx R^{-1}(id_I) .$$

So, provided the structure of the spaces $H^s(S,I)$ and $H^s(I,I)$ is appropriate for the use of infinite dimensional implicit function theorems [27], one can then study the structure of the space $\mathscr{P}$ as the inverse image under R of an element, $R^{-1}(id_I)$. If R is sufficiently smooth, and (id_I) is a regular value of R, then $\mathscr{P}$ can be shown to be a manifold by the implicit function theorem. This procedure is intrinsic, i.e., it does not depend on the choice of the diffeomorphism between I and R^1. This is the procedure we shall follow.

In view of the above, we shall take as model space for the space $\mathscr{P}$ the space of H^s $(s > n/2 + k,\ \ k \geq 2)$ retractions from S onto I: an H^s retraction is an H^s map from S to I which, restricted to I, is the identity on I. For the technical reasons discussed above, in order to obtain a Hilbert manifold structure on the space $\mathscr{P}$, we shall consider Sobolev norms on the spaces of functions we work on. One can alternatively work with C^k topologies on these function spaces. In that case, one obtains Banach manifold structures for the spaces of preferences. This is discussed in Part d.

We now briefly discuss the structures of the spaces $H^s(S,I)$ and $H^s(I,I)$. Manifold structures for spaces of C^∞ and C^k maps between manifolds have been introduced and studied in the mathematical literature by Eells in 1958 and Smale and Abraham in 1961; the H^s case was studied by Elliason in 1967 and by Palais in 1968. Marsden[28], [29] and Palais [34] are good references. In the appendix a description is given of manifold structures of spaces of maps between manifolds. In particular, it can be seen that if M and N are smooth (C^∞) compact manifolds with boundary, and N a submanifold of M and $\partial N \subset \partial M$, then the spaces $H^s(M,N)$ are Hilbert manifolds with the Sobolev norms defined analogously as above, but in local charts. Sobolev theorem also applies: If M is n-dimensional, the space $H^s(M,N)$ is contained in $C^k(M,N)$ when $s > n/2 + k$. So in what follows we assume $s > n/2 + k$, where n is the dimension of the commodity space S to insure enough differentiability of the preferences we work with; since $s > n/2 + k$ (and $k \geq 2$) the preferences will be of class C^2.

First we show that the restriction map $R: H^s(S,I) \to H^s(I,I)$ defined by $R(f) = f|_I$ is a C^∞ map. Then we study $R^{-1}(id_I)$ where id_I denotes the identity map from I to itself. Following the previous discussion, the subspace of $R^{-1}(id_I)$ is identified with $\mathscr{P}$. We then prove that id_I is a regular value of R, and thus $\mathscr{P}$ is a submanifold of $H^s(S,I)$ which inherits the (local) Hilbert space structure of $H^s(S,I)$.

The next results are stated for general C^∞ manifolds with boundary M and N, N a submanifold of M. A special case is $M = S$ and $N = I$. This is done for technical reasons that are discussed at the end of this section and in Chapter 3, where more general spaces of preferences are studied. For complementary definitions, see the appendix.

LEMMA 2. <u>Let</u> M <u>and</u> N C^∞ <u>manifolds with boundary</u>, N <u>a neat submanifold of</u> M, $\partial N \subset \partial M$. <u>Then the restriction map</u> $R: H^s(M,N) \to H^s(N,N)$ <u>defined by</u> $R(f) = f|_N$ <u>is a</u> C^∞ <u>map</u>.

<u>Proof</u>. By the definition of $H^s(M,N)$ and the assumptions about $\mathscr{P}$, without loss of generality we can assume that M and N are manifolds without boundary. Recall that a map $\varphi: H^s(M,N) \to H^s(N,N)$ is C^k (for all $k \geq 0$) at $f \in H^s(M,N)$ if there exists a chart (U, ϕ_1) at f and a chart (V, ϕ_2) at $R(f)$ such that $R(U) \subset V$ and

$$\overline{R} \equiv \phi_2 \circ \varphi \circ \phi_1^{-1}: \phi_1(U) \to \phi_2(V)$$

is C^k. Here $\phi_1(U)$ and $\phi_2(V)$ are open subsets of $T^s(M,N)$, the model space for $T_f(H^s(M,N))$, the tangent space of $H^s(M,N)$ at f. By definition (see appendix), if $v = \phi_1(f)$ and $h \in T^s(M,N)$, then the value at the vector h of the linear map representing the derivative of the map R at the point f, denoted $DR(f;h)$, if it exists, must satisfy

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \|\overline{R}(v+h) - \overline{R}(v) - DR(f;h)\| = 0 \quad .$$

For any h in $T^s(M,N)$ define $\widehat{DR}(f;h)$ as the restriction of h to N, denoted $\overline{h}$. Since $h \in T^s(M,N)$, by definition h makes the following diagram commutative:

$$\begin{tikzcd} & TN \arrow[d, "\Pi_N"] \\ M \arrow[ur, "h"] \arrow[r] & N \end{tikzcd}$$

(where Π_N is the natural projection of the tangent bundle of N, TN onto N), i.e., $h(m) \in T_{f(m)}N$ for all $m \in M$. So, let $\widehat{DR}(f;h) = \overline{h}$. Then, since the diagram

$$\begin{tikzcd} & TN \arrow[d, "\Pi_N"] \\ N \arrow[ur, "\overline{h}"] \arrow[r, "f"'] & N \end{tikzcd}$$

commutes; $\bar{h}$ belongs to $T^s(N,N)$, and it satisfies $\bar{R}(v+h) - \bar{R}(v) - \widetilde{DR}(f;h) = 0$, by linearity of $\bar{R}$ on $T^s(M,N)$, and thus $\widetilde{DR}(f;h)$ is the derivative of R, i.e., $\widetilde{DR}(f;h) = DR(f;h)$.

Also, since $DR(f;h)$ is independent of f, DR is continuous, so that R is C^1. The same proof shows R is C^k, for all $k \geq 0$.

PROPOSITION 1. *Under the assumption of Lemma* 2, *the identity map* $id_N : N \to N$ *is a regular value of the restriction map* $R: H^s(M,N) \to H^s(N,N)$.

Proof. Let $f \in H^s(M,N)$ such that $f|_N = id_N$ (if there exists no retraction from M to N, id_N is a regular value by definition). Note that if v is a tangent vector at f, i.e., $v \in T_f(H^s(M,N))$ then, since by definition the diagram

$$\begin{tikzcd} & TN \arrow[d, "\Pi_N"] \\ M \arrow[ur, "v"] \arrow[r, "f"'] & N \end{tikzcd}$$

commutes (i.e., $v(m) \in T_{f(m)}N$), v is actually given by a family of vector fields on N, indexed by the set $f^{-1}(n)$ for each $n \in N$. We shall check that for any s in $T_{id_N}(H^s(N,N))$ there is a v in $T_f(H^s(M,N))$ with $f|_N = id_N$ and $DR(f;v) = s$. If $s \in T_{id_N}(H^s(N,N))$, then s is actually a vector field on N, since by definition, s "covers" the identity, i.e., $s(n) \in T_n(N)$. In the lemma above we saw that $DR(f;v) = v/N$. Let v be the element of $T_f(H^s(M,N))$ defined by $v(m) = s(f(m))$. Then $v \in T_f(H^s(M,N))$, since it is given by the (right) composition of two H^s maps, and $DR(f;v) = s$, which completes the proof.

THEOREM 1. Under the assumptions of Lemma 2, the space of H^s retractions of M onto N form an H^s Hilbert manifold.

Proof. This follows from Lemma 2 and Proposition 1 above, the implicit function theorem for Hilbert spaces [27], and the fact that $H^s(M,N)$ and $H^s(N,N)$ are Hilbert manifolds (see the appendix).

Remark. The space of H^s retractions from M to N is called a Hilbert manifold because it has locally a Hilbert space structure inherited from the (local) Hilbert space structure of $H^s(M,N)$. However, this local Hilbert space structure depends on the choice of charts, i.e., the inner product and the norms may vary with the choice of charts, they are not canonical. Such spaces are also sometimes called Hilbertable spaces, i.e., spaces on which some complete inner product exists.

We next study sufficient conditions on N so that there exists a retraction from M onto N.

LEMMA 3. Under the assumptions of Lemma 2, if N is contractible, then there exists a continuous retraction from M to N.

Proof. Because by assumption N is a neat submanifold of M, it has a tubular neighborhood (see [22]). This implies that the pair (M,N) has the "homotopy extension property", which allows standard homotopy theory to be used [41]. It follows from contractibility of N that every continuous map $u: N \to N$ extends to a continuous map from M to N (see, for example, [41], page 56, Exercise 6). Taking $u = id_N$ gives the retraction.

<u>Remarks</u>

1. Using arguments of partitions of unity, as in Lemma 4 below, one can then prove the existence of a C^∞ extension.

2. A much simpler proof of Lemma 3 can be given for the special case $M = S$ and $N = I$ by using Tietze's extension theorem to prove the existence of a continuous extension j for the map $id: N \to N$, $j: M \to N$. Lemma 3, however, can be used for constructions of spaces of preferences on more general manifolds M, where the preferences are represented in terms of utility by more general submanifolds than I.

We now extend Lemma 3 to prove the existence of a C^∞ retraction. This will imply existence of a C^k retraction, i.e., non emptiness of the space of preferences $\mathscr{P}$.

LEMMA 4. <u>Let</u> M <u>and</u> N <u>be as in Lemma</u> 3. <u>Then there exists a</u> C^∞ <u>retraction</u> $f_1: M \to N$.

Proof. Let $f: M \to N$ be the continuous retraction whose existence is proven in Lemma 3. Let W be a tubular neighborhood of N. Let $p: W \to N$ be the C^∞ retraction of W onto N which exists by [22]. Let g be a C^∞ approximation of $f, g: M \to N$, g sufficiently close to f so that the segment in R^n given by $[p(x), g(x)]$ is contained in W. Take λ to be a C^∞ function from M into $[0, 1]$ such that

$$\lambda = 0 \quad \text{on } N$$

$$\lambda = 1 \quad \text{outside } W$$

Define

$$h(x): M \to W$$

by

$$h(x) = \begin{cases} g(x) & \text{if } x \in M - W \\ \lambda(x)g(x) + (1 - \lambda(x))\,p(x) & \text{if } x \in W \end{cases}$$

when $x \in N$, $\lambda = 0 \Rightarrow h(x) = p(x) = x$. Since p is C^∞ and λ is C^∞, so is h. So the map $f_1 = p \circ h: V \to N$ is a C^∞ retraction from M to N.

From Lemmas 3 and 4 it follows, in particular

CORollary 1. <u>The space of preferences</u> $\mathcal{P}$ <u>is non-empty</u>.

Proof. This follows immediately from Lemma 4, since by (C2) I is strictly $\leq$ ordered, which implies that I is contractible by the classification theorem for one dimensional manifolds [30], since it cannot have any loops.

In view of Lemma 4, we can now weaken assumption (C2) as follows:

(C2') There exists a compact connected contractible one dimensional neat submanifold I of S, with $\partial I = \{0, 1\}$, and such that for all f in $\mathscr{P}$ if g represents f, $g(1) \geq g(x) \geq g(0)$, for all x in S, and g is strictly increasing along I.

COROLLARY 2. Under condition (C2'), $\mathscr{P} \neq \phi$.

We can thus assume that the definition of $\mathscr{P}$ is given by (C1) and (C2').

Note that the space of preferences $\mathscr{P}$ can still be defined as the retractions from S onto I even if I is not strictly $\leq$ ordered in R^n; one considers a C^∞ diffeomorphism between $\mathring{I}$ and (0,1) which induces a complete order structure on I and repeats the above construction to show that each element in $\mathscr{P}$ is a preference on S. In view of Sobolev's theorem and Theorem 1 we can now prove:

THEOREM 2. The space of preferences $\mathscr{P}$ has a Hilbert manifold structure.

Proof. Note that by Corollary 2 $\mathscr{P}$ is not empty. The set of maps in $H^s(S,I)$ which are increasing along I form an open set. This follows from the following facts. The inclusion map

$i: H^s(S, I) \to C^k(S, I)$ is continuous when $s > n/2 + k$ by Sobolev's theorem. Since $k \geq 2$, the set U of maps in $C^k(S, I)$ which are increasing along I (i.e., $U = \{g \in C^k(S, I): \psi \circ g(x) > 0$ where ψ is the C^∞ diffeomorphism of R and $\mathring{I}\}$ form an open set in $C^k(S, I)$ by compactness of S and I. Hence, by continuity of i, $i^{-1}(U)$ is open in $H^s(S, I)$. Hence, in view of Theorem 1, since $\mathcal{P} = R^{-1}(id_I) \cap i^{-1}(U)$, $\mathcal{P}$ is a manifold. This completes the proof.

d. Manifolds of Foliations and other Submanifolds of Preferences

In Part c we proved that $\mathcal{P}$ is a Hilbertable manifold. We shall now study the structure of certain subspaces of preferences of $\mathcal{P}$ that satisfy further specifications and other larger spaces of preferences $\mathcal{Q}$ which include $\mathcal{P}$. One may want to consider, for instance, as in Antonelli [2] and Debreu [13], an alternative definition of smooth preferences as foliations of the commodity space S, i.e., smoth regular locally integrable normalized vector fields on S. A normalized (unit length) C^k $(k \geq 2)$ vector field v on S assigns to each commodity-vector a vector which indicates the preferred direction of the agent, in a C^k manner. If such a vector field v, or preference, can be locally described as the gradient of a C^k utility function g, then it is called integrable. If g is regular, i.e., $Dg(x) \neq 0$ for all x in S (or equivalently, the vector field v has no singular points, i.e., it is a regular vector field), then the indifference surfaces of g are also called "leaves" of the foliation and each describes a C^k submanifold of the commodity space of codimension 1. In particular, there are no "thick" indifference surfaces for the preference f (for instance, indifference surfaces of positive measure in the usual Lebesgue measure of S). We shall first study a subspace of $\mathscr{P}$ made up of preferences that satisfy this second definition.

Definition. Let $\mathscr{P}_0$ be the subspace of preferences f in $\mathscr{P}$ such that the normal unit vector field to the indifference surfaces of f describe a foliation of S. Since for all f in $\mathscr{P}$, f is representable by a C^k utility function, $g: S \to R$, and since by assumption (C2) (or (C2')) g is increasing along I, then g is transversal to I, $g \pitchfork I$, $\mathscr{P}_0$ coincides with the subspace of f in $\mathscr{P}$ representable by C^k functions $g: S \to R$ with $Dg(x) \neq 0$ for all x in S, i.e., $\mathscr{P}_0$ is made up of the preferences in $\mathscr{P}$ which are representable by everywhere regular utilities. Hence, for any $r \in R$, $g^{-1}(r)$ is either an empty, or an n-1 dimensional C^k submanifold of S, and, in particular, since $k \geq 2$, $g^{-1}(r)$ has Lebesgue measure zero in S, i.e., there are no "thick indifferences" in f. Since, for any $d \in I$, $f^{-1}(d) = g^{-1}(r)$, for some r in R, and g is increasing along I, it follows that the n-1 dimensional submanifold $f^{-1}(d)$, for all d in I, is transversal to I. This transversality of the surfaces $f^{-1}(r)$ and of I eliminates cases such as that of Figure 2.

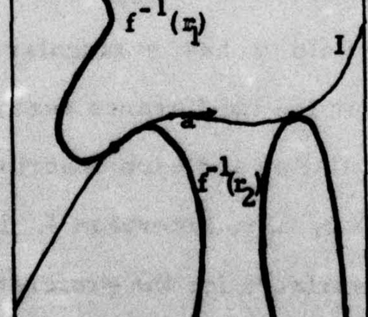

Figure 2

We now prove:

THEOREM 3. <u>The space $\mathscr{P}_0$ of preferences in $\mathscr{P}$ which define codimension one foliations of the commodity space</u> S <u>is an open submanifold of $\mathscr{P}$, and, in particular, $\mathscr{P}_0$ has a Hilbert manifold structure</u>.

<u>Proof</u>. In view of the above discussion, $\mathscr{P}_0 = \{f \in \mathscr{P}$ such that if g represents f, $Dg(x) \neq 0$ for all x in $S\}$. By Sobolev's theorem, $\mathscr{P} \subset C^2(S, I)$, since $s > n/2 + k$, and $k \geq 2$. Since S is compact, the set

$$V = \{g \in C^2(S, I): Dg(x) \neq 0 \text{ for all } x \text{ in } S\}$$

is open in $C^2(S, I)$. By Sobolev's inequality, the inclusion map

$$i: H^s(S, I) \to C^2(S, I)$$

is continuous. Hence, $i^{-1}(V)$ is open in $H^s(S, I)$, and thus $\mathscr{P}_0 = i^{-1}(V) \cap \mathscr{P}$ is a submanifold of $\mathscr{P}$. This completes the proof.

We now study other important subspaces of $\mathscr{P}$. Let $\mathscr{P}_1$ be be the space of strictly monotone increasing preferences in $\mathscr{P}$, and let $\mathscr{P}_2$ be the set of strictly convex preferences in $\mathscr{P}_0$.

THEOREM 4. <u>The spaces $\mathscr{P}_1$ and $\mathscr{P}_2$ of strictly increasing and of strictly convex preferences in $\mathscr{P}$ respectively, and their intersection $\mathscr{P}_3 = \mathscr{P}_1 \cap \mathscr{P}_2$ are open submanifolds of $\mathscr{P}$. In particular, they have Hilbert manifold structures</u>.

<u>Proof</u>. Note that both $\mathscr{P}_1$ and $\mathscr{P}_2$ are contained in $\mathscr{P}_0$. Hence, by Theorem 3 it suffices to show that they are open subsets of $\mathscr{P}_0$. Note that $\mathscr{P}_1$ and $\mathscr{P}_2$ are open subsets of $\mathscr{P}_0$ with the C^2 norm on $C^2(S, I)$, which $\mathscr{P}_0$ inherits from the inclusion

$$\mathscr{P}_0 \subset \mathscr{P} \subset H^s(S, I) \subset C^2(S, I)$$

given by Sobolev's inequalities. By compactness of S, $\mathscr{P}_1$ and $\mathscr{P}_2$ are both open sets with the C^2 topology (see, for instance [38]). Hence, by the same arguments of Theorems 2 and 3, $\mathscr{P}_1$ and $\mathscr{P}_2$ are submanifolds of $\mathscr{P}_0$. This completes the proof.

We should note that all that was said about the structures of $\mathscr{P}$, $\mathscr{P}_0$, $\mathscr{P}_1$ and $\mathscr{P}_2$ can be rephrased for C^k norms as follows:

COROLLARY 3. <u>If, in the above, the spaces</u> $H^s(S, I)$ <u>and</u> $H^s(I, I)$ <u>are replaced by</u> $C^k(S, I)$ <u>and</u> $C^k(I, I)$, $k \geq 2$, <u>then all the results of Theorems</u> 1, 2, 3, <u>and</u> 4 <u>carry through and yield corresponding Banach manifold structures of type</u> C^{k-1} <u>for the spaces of</u> C^k <u>preferences defined as follows</u>:

(1) $\tilde{\mathscr{P}} = \{f$ satisfying (C1) and (C2'), $f \in C^k(S, I)$, $f \in R^{-1}(id_I)\}$

(2) $\tilde{\mathscr{P}}_0 = \{f \in \tilde{\mathscr{P}}$ and f defines a codimension one foliation of $S\}$

(3) $\tilde{\mathscr{P}}_1 = \{f \in \tilde{\mathscr{P}}_0$ and f is strictly increasing$\}$

(4) $\tilde{\mathscr{P}}_2 = \{f \in \tilde{\mathscr{P}}_0$ and f is strictly convex$\}$

(5) $\tilde{\mathscr{P}}_3 = \tilde{\mathscr{P}}_1 \cap \tilde{\mathscr{P}}_2$.

e. <u>More General Spaces of Foliations of the Commodity Space</u>

We now indicate how to study a much larger space of preferences than the space $\mathscr{P}_0$ studied above. Detailed proofs of the results described here will be given elsewhere, since otherwise we would make this work too long. However, the ideas are simple and worth pointing out here.

By definition, $\mathscr{P}_0$ is given by the codimension one foliations on the commodity space S that are retractions onto some fixed submanifold I. In order to indicate the dependence of $\mathscr{P}_0$ on I we shall denote it here $\mathscr{P}_0(I)$. We have shown that conditions (C1) and (C2) (or (C2')) together with an appropriate choice of topologies, made $\mathscr{P}_0(I)$ into an H^s Hilbert manifold. We shall now indicate how to extend the results to a space of smooth preferences $\mathscr{Q}_0$, the space of all C^∞ codimension one foliations of the commodity space S which are given by retractions into <u>some</u> C^∞ one dimensional neat contractible submanifold of S, this submanifold being allowed to vary from preference to preference. Formally, $f \in \mathscr{Q}_0$ if and only if

(C3) f is representable by a C^∞ utility $g: S \to R$ defined on a neighborhood of S,

and

(C4) There exists a compact connected contractible C^∞ neat one dimensional submanifold I of S, which, in general, depends on f, with $\partial I = \{p\} \cup \{q\} \subset \partial S$, and such that if g represents f, $g(q) \geq g(x) \geq g(p)$ for all x in S and g is increasing along I.

The manifolds I satisfying (C4) are also called admissible. The space $\mathscr{Q}_0$ can be visualized as the union of the spaces $\mathscr{P}_0(I)$, i.e.,

$$\mathscr{Q}_0 = \bigcup_I \mathscr{P}_0(I)$$

where I is any admissible submanifold.

We have now restricted ourselves to C^∞ preferences, or preferences representable by C^∞ utilities for the following reason: In the proof of the manifold structure for the space $\mathscr{Q}_0$ (which will be modelled on spaces of C^∞ maps and hence it will be a Fréchet space rather than a Banach or Hilbert space) one uses the fact that each $\mathscr{P}_0(I)$ is a C^∞ manifold, and then attempts to paste the $\mathscr{P}_0(I)$'s together to get the C^∞ structure for the union $\cup_I \mathscr{P}_0(I) = \mathscr{Q}_0$. It is in this "pasting" that C^∞ of the functions is required. For, if f can be represented as a retraction from S onto I_1 and also a retraction from S onto I_2 (i.e., f is in the intersection of $\mathscr{P}_0(I_1)$ and $\mathscr{P}_0(I_2)$), then the natural change of coordinates from $\mathscr{P}_0(I_1)$ to $\mathscr{P}_0(I_2)$ will be induced by composing f in the left, with a diffeomorphism ϕ between I_1 and I_2, induced by the leaves or level surfaces of f. See Figure 3 below.

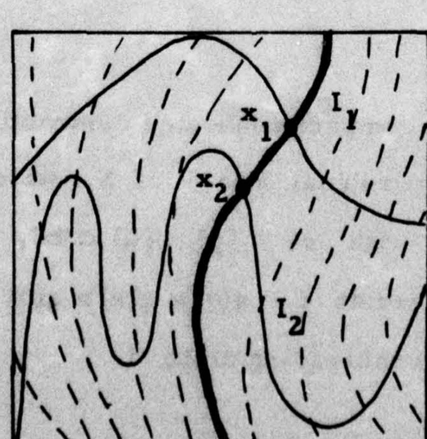

Figure 3.

The dotted lines in Figure 3 represent the leaves of f. The heavy line is a leaf that retracts onto x_1 in I_1, and onto x_2 in I_2 according to f. So if ϕ is the map that sends x_1 into x_2, ϕ determines a C^∞ diffeomorphism between I_1 and I_2 if f is C^∞, and is a C^k map if f is C^k.

Since, for $f \in \mathscr{P}_0(I_1) \cap \mathscr{P}_0(I_2)$, the change of coordinates from $\mathscr{P}_0(I_1)$ to $\mathscr{P}_0(I_2)$ is actually $f \to \phi \circ f$ (it is easy to check that indeed, $\phi \circ f$ is a retraction from S onto I_2) and since ϕ is C^{k-1} if f is a C^k preference, so will be $\phi \circ f$. However, the map of function spaces $f \mapsto \phi \circ f$ (composition on the left) will, in general, <u>not</u> be even C^1, even if ϕ is C^k. For a discussion see [28] or [29]. (If f is in C^k and ϕ is in C^{k+s}, then $f \to \phi \circ f$ is a C^s map of function spaces.) However, if f and ϕ are C^∞, then the left composition map of function spaces $f \to \phi \circ f$ is C^∞. This is a main reason for using a C^∞ (Fréchet) topology in this larger space of foliations $\mathscr{D}_0$. The procedure described above of changes of coordinates between $\mathscr{P}_0(I_1)$ and $\mathscr{P}_0(I_2)$ is actually the way of inducing the manifold structure for $\mathscr{D}_0$. There is, however, a technical detail here that cannot be overlooked. In the proof of the result that $\mathscr{P}_0(I)$ was a Hilbert manifold we used the isomorphism

$$\mathscr{P}_0(I) \approx R^{-1}(id_I)$$

and the implicit function theorem for Hilbert spaces. If $\mathscr{P}_0(I)$ is modelled by C^∞ retractions, the implicit theorem is no longer valid, since spaces of C^∞ maps are Frechet spaces (not Banach spaces) and for these spaces the theorem is known not to be true in general.

However, another method than the implicit function theorem is available in this case to prove that $\mathscr{P}_0(I)$ is a C^∞ Frechet manifold. Basically, all admissible I's are diffeomorphic to the line R^1, say, through diffeomorphisms $\phi_I: I \to R^1$. So one can study the space $\mathscr{P}_0(I)$ through the diffeomorphism ϕ_I as a subset of the space of C^∞ maps from S into R^1. This $C^\infty(S, R^1)$ is a linear space, with addition induced by addition in R^1. The question is whether the $\mathscr{P}_0$, which is the set of retractions from S onto I can be represented as a C^∞ manifold. The answer is affirmative, but it requires a proof with a method less appealing than the one we gave here by the use of the implicit function theorem, which is not available in Fréchet spaces such as spaces of C^∞ maps.

Remarks.

1. Note that all the preferences in $\mathscr{P}_0$ and in $\mathscr{Q}_0$ are globally integrable, i.e., representable by globally defined utilities. If v is a locally integrable vector field which defines a monotone preference on S, then by [13] and [43], it is actually globally integrable as well; this monotonicity assumption is sufficient but clearly not necessary for the equivalence between local and global integrability. M. Hirsch and the author have found a necessary and sufficient condition for the equivalence of local and global integrability of codimension one foliations on S that will appear elsewhere. In any case, there is a difficult and important open question here: If $\mathscr{Q}$ is a (nonempty) space of C^k codimension one foliations of S, under what conditions will $\mathscr{Q}$ be a manifold?

A subspace $\mathscr{P}_0$ of $\mathscr{Q}$ made up of globally integrable ones is proven here to be a manifold--how much can one improve this result? And, further, what can be said of the space $\mathscr{Q}_k$ of all codimension k foliations of a C^∞ manifold M with boundary if $\mathscr{Q}_k \neq \phi$?

2. An extension of the results of this section to unbounded commodity spaces seems difficult to obtain for the following reasons: topologies such as those given by the Sobolev norm and the C^k norm are not defined on unbounded regions unless one uses a weight factor or finite measure as in [5]. This complicates the geometry of the spaces at infinity. If one attempts to use Whitney topologies [38] which are adequate for spaces of maps defined on unbounded regions, one loses a crucial tool; the implicit function theorem is not valid any longer since spaces of C^k maps with the Whitney topology are not Banach spaces. However, the fact that on each compact set of the type of the set S the results carry may be all that is needed in many cases. The positive orthant R^n_+ of R^n used frequently as a commodity space can be represented as a countable union of an increasing family of sets of the type of S, and as we see in Chapter 4, this allows many results to go through.

Chapter 3

PREFERENCES, UTILITIES AND DEMANDS

a. Introduction

In this chapter we study properties of manifolds of preferences and also properties of utilities and demands of the agents in relation to the underlying preferences. In Part b we study a topological property of the manifold of preferences $\mathcal{P}$ (represented as a space of H^s retractions from the commodity space into a submanifold) defined in Chapter 2. We show that under the conditions of Chapter 2, $\mathcal{P}$ is a contractible space, and we indicate possible extensions of this result. We also discuss more general spaces of preferences, which we call many agent preferences. In Part c we study properties of the utilities of the agents derived from properties of the underlying vector fields defined by preferences in the boundary of the commodity spaces. In Part d we also show that a generic set of preferences in $\mathcal{P}$ yields demand functions which are locally well defined and of class C^1 on prices and incomes. Also, for a generic class of preferences in $\mathcal{P}$, the demand vectors depend in a C^1 manner on the underlying preferences, on an open dense set of price and income vectors. These results are local, and they are proven without convexity or monotonicity assumptions on the preferences; for the special cases of $\mathcal{P}_3$ (convex and monotone preferences) analogous global results are proved.

b. Properties of Manifolds of Preferences

We shall now study a little of the topology of $\mathcal{P}$ and of other related manifolds of preferences. Assume that M and N satisfy the conditions of Lemma 2 of Chapter 2. Recall that R is the restriction map, $R: H^s(M,N) \to H^s(N,N)$ given by $R(f) = f/N$. Then one has the following

PROPOSITION 2. *The space of H^s retractions from M onto N, $R^{-1}(id_N)$ is a contractible space.*

Proof. First assume that N is convex. Fix a retraction $f: M \to N$. For any retraction $g: M \to N$ define a homotopy

$$g_t(x) = (1-t)\,g(x) + tf(x) \;, \qquad 0 \leq t \leq 1 \;.$$

Define the map $\chi: R^{-1}(id_N) \times I \to R^{-1}(id_N)$ by $\chi(g,t) = g_t$. Then $\chi(g,0) = g$, $\chi(g,1) = f$. Note that g_t is in $H^s(M,N)$ and is a retraction, i.e., $g_t(x) = x$, for each t. Hence $R^{-1}(id_N)$ is contractible.

If η is a C^∞ diffeomorphism $\eta: N \to N^1$, η induces a C^∞ diffeomorphism (since $s > n/2 + k$) between the space of H^s retractions from M to N (see [29], Lemma 2.2.1), and the space of H^s retraction from M to N^1 and thus the above extends to any N^1 which is diffeomorphic to a convex N. This completes the proof.

From Proposition 2 we obtain:

THEOREM 5. *The space $\mathcal{P}$ of preferences is contractible.*

Proof. This follows directly from Theorem 2 in Chapter 2 and Proposition 2 above applied to $M = S$ and $N = I$.

Remark. Note that the proof of Proposition 2 is based on the fact that N is contractible itself. A natural mathematical question is how the topological structure of the space of retractions from M onto N is related to the topological structure of N for more general manifolds N. Since M is retractible onto N, the topology of M is also dependent on that of N.

As discussed in Chapter 2, Part a, $\mathscr{P}$ can be visualized as a subset of maps in $F(S, R^1)$, by mapping a preference f into $\phi\ f: S \to R^1$, where ϕ is a diffeomorphism between I and R^1. However, under the usual addition in F, induced by the addition of the values of the map in R^1 (i.e., $\phi \circ f + \phi \circ g(x) = \phi \circ f(x) + \phi \circ g(x)$) $\mathscr{P}$ is not a linear subset. But, for any $0 \leq \lambda \leq 1$, the addition of two elements f and g in $\mathscr{P}$ given by the convex combination with factor λ and $1 - \lambda$ in F (i.e., $\phi \circ f + \phi \circ g(x) = \lambda(\phi \circ f(x)) + (1-\lambda)(\phi \circ g(x))$ is an element of $\mathscr{P}$, since $\lambda(\phi\ f(x)) + (1-\lambda)(\phi\ g(x)) = x$ if x is in I. Therefore, we have

COROLLARY 4. The space $\mathscr{P}$ of preferences can be identified, for each C^∞ diffeomorphism $\phi: I \to R^1$ with a convex subset of the space of $C^k(S, R^1)$.

Proof. It follows from the above observations and from the fact that the left composition of an H^s map with a C^∞ map is C^k (since $s > \frac{n}{2} + k$).

The results of Theorem 5 may have interesting economic applications as well. For instance, in dynamic models where the time dependent or choice variables are preferences (such as optimal advertisement models, or, more in general, political-economic models where peoples' preferences can be influenced or made to vary) the property that $\mathscr{P}$ be convex or contractible and the fact that the inclusion $H^s \subset C^k$ is compact if $s > \frac{n}{2} + k$ may be useful to prove existence of equilibria by fixed point arguments. In this light, it may be of interest to consider the following extension of the definition of $\mathscr{P}_0$ to spaces of many agent preferences $\mathscr{P}^j$.

Consider a space of families of j C^k unit vector fields $v = (v_1, \ldots, v_j)$ defined on the commodity space $S \subset R^n$, with $j < n$. For each point x in S, the i-th vector field indicates a preferred direction of the i-th member of the economy, $i = 1, \ldots, j$. Assume that $\mathscr{P}^j$ can be imbedded in the space of retractions $R^{-1}(id_N)$ where $R: H^s(M,N) \to H^s(N,N)$ is the restriction map, and $\mathscr{P}^j \neq \phi$. This can be deduced, for example, from analogous conditions to (C1) and (C2) of Chapter 2.

<u>Definition</u>. Let $\mathscr{P}^j$ be the family of such codimension $n-j$ foliations on S with the H^s topology inherited from the inclusion $\mathscr{P}_j \subset H^s(M,N)$. These $\mathscr{P}_j$ define a family of preferences of j agents on S and are thus called a space of many (j) agent preferences.

From Theorems 4 and 5 above, one has:

THEOREM 6. <u>The space of many agent preferences</u> $\mathscr{P}_j$ <u>with the</u> H^s <u>structure inherited from</u> $H^s(S, I)$ <u>is a contractible Hilbert manifold</u>.

<u>Proof</u>. This is proved in the same way as Theorems 4 and 5.

c. <u>Properties of Utilities</u>

Next we discuss the structure of the hypersurface (or <u>indifference surfaces</u>) of the utilities representing preferences in $\mathscr{P}_0$. It is of interest to study when each indifference surface is connected, i.e., when the agent can move along the indifference surface continuously from one bundle to any other bundle which is equally preferred. Related questions were studied, for instance, in [25]. As the example in Figure 4 shows, in general a retraction does not have this property.

The retraction from the square to I is indicated by the arrows on the level surfaces. The dotted paths indicate the value of the retraction for the level surfaces on the bottom right hand side of the square.

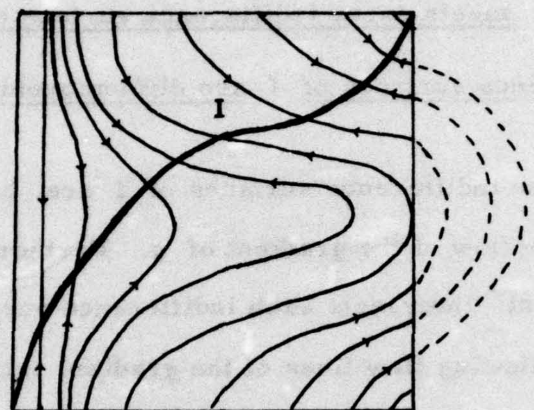

Figure 4

We shall next study sufficient conditions for the indifference surfaces to be connected and contractible, which are subsequently weakened in Corollary 4.

The condition (2) of Proposition 3 below can be described by: Starting at the zero indifference level and following any path in the direction of maximal increases of utility at each point, the agent with preference f can reach all utilities levels. Recall that $\mathscr{P}_0$ is the subspace of preferences in $\mathscr{P}$ which are codimension-one foliations of the commodity space.

PROPOSITION 3. <u>Let</u> f <u>be a preference in the space</u> $\mathscr{P}_0$. <u>Let</u> $g: S \to R$ <u>represent</u> f <u>and assume</u>:

(1) <u>there exists a neighborhood</u> U <u>of</u> $(0, \ldots, 0)$ <u>such that</u> $U \cap \partial S$ <u>is the indifference class of</u> $(0, \ldots, 0)$ <u>and</u>

(2) each flow line of the gradient of g starting at a point in $U \cap \partial S$ meets every indifference surface of f.

Then all indifference surfaces of f are diffeomorphic to $U \cap \partial S$.

Proof. The indifference surfaces of f are, by definition, transversal to the flow of the gradient of g. Furthermore, each flow line of the gradient[3] intersects each indifference surface in at most one point. By following flow lines of the gradient vector field then, one can induce a diffeomorphism from each indifference surface to $U \cap \partial S$. This is proved in Proposition 2.2 of V-2-6 of [22]. Thus all indifference surfaces are C^1 diffeomorphic to $U \cap \partial S$. This completes the proof.

COROLLARY 5. If conditions (1) and (2) of Proposition 3 are satisfied, and at least one indifference surface is (connected and) contractible, all indifference surfaces are (connected and) contractible.

Proof. It follows Proposition 3, since all indifference surfaces are diffeomorphic.

Since the preferences in $\mathscr{P}_0$ are given by regular retractions, we shall next study what properties of f as a retraction from S onto I are sufficient to obtain the same result. Let $Df(x)$ denote the derivative of the retraction map $f: S \to I$ so that if $x \in \partial S$, $Df|_{\partial S}(x): T_x \partial S \to T_{f(x)} I$.

[3] For a definition of a flow line, see [22] (V-2-3).

PROPOSITION 4. <u>Let</u> $f \in \mathscr{P}_0$, <u>and let</u> $g: S \to R$ <u>represent</u> f. <u>Assume that there exist only two critical points of the projection on</u> ∂S <u>of the vector field given by the gradient of</u> g, <u>say</u> $(0) = (0, \ldots, 0)$ <u>and</u> $(1) = (1, \ldots, 1)$ <u>and they are not degenerate</u>. <u>Then all indifference surfaces of</u> f (<u>except</u> (0) <u>and</u> (1)) <u>are connected and contractible</u>. <u>Actually, they are all</u> C^1 <u>diffeomorphic to</u> n-1 <u>dimensional discs</u>.

<u>Proof</u>. Recall that for f in $\mathscr{P}_0$, any g representing f has rank one everywhere, in a neighborhood of S. Choose one such g. We now define an auxiliary C^k vector field: let w be a vector field defined on a neighborhood N_0 of ∂S such that

(1) on ∂S, w is equal to the projection of the gradient of g.

(2) The inner product $\langle w(x), \text{grad } g(x)\rangle \geq 0$

For example, let w(x) be the orthogonal projection of grad g(x) on the sphere of radius $|x|$ concentric with S, when S is the usual metric sphere. Take λ to be a C^∞ real valued function on a neighborhood of S, $0 \leq \lambda \leq 1$, and $\lambda = 0$ exactly on ∂S. Define v_λ a vector field on a neighborhood of S, by

$$v_\lambda = (1-\lambda)w + \lambda(\text{grad } g)$$

For all λ, v_λ is C^k and v_λ is transversal to the level surfaces (except (0) and (1)) of g, for

$$\langle v_\lambda(p), \text{grad } g(p)\rangle = (1-\lambda(p))\langle w(p), \text{grad } g(p)\rangle + \lambda(p)|\text{grad } g(p)|^2$$

Each summand is ≥ 0, and $\lambda(p)|\text{grad } g(p)|^2 = 0$ only if $\lambda(p) = 0$; but then, v_λ is grad g(p) if $p \neq (0), (1)$ (w(p) and grad g(p) make an acute angle).

Figure 5

Flow lines of v_λ

Note also that v_λ has only two singularities for all λ (at (0) and (1)) by construction.

There exist coordinates near (0) such that $g(x^1, \ldots, x^n) = x^n$. In these coordinates, near (0), ∂S is the graph of a function $h: R^{n-1} \to R^1$, that is ∂S is defined by the relation

$$x^n = h(x^1, \ldots, x^{n-1}) \quad ,$$

and S is defined by $x^n \geq h(x^1, \ldots, x^{n-1})$.

The nondegeneracy assumption on the gradient of $g|_S$ implies that h is a Morse function [31]. By Morse's lemma [31] there are new coordinates in $\mathbb{R}^{n-1}$, say $(n^1, \ldots, n^{n-1})$ so that

$$h(u^1, \ldots, u^{n-1}) = (u^1)^2 + \cdots + (u^{n-1})^2 \quad .$$

Therefore, the indifference surfaces near (0) are n-1 discs. The other indifference surfaces (except (0) and (1)) are all diffeomorphic to these by following gradient lines of v_λ (similar to the proof of Proposition 3). This completes the proof.

Remark 1. We conjecture that in two commodities if there exist exactly n points $x_1, \ldots, x_n$ in ∂S such that the gradient of g does not have rank one, then each indifference surface of the preference f in $\mathscr{P}_0$ represented by g has at most n-1 connected components.

2. More generally, for n-dimensional commodities spaces S, there is a connection between the topological type of the indifference surfaces of the preference f in $\mathscr{P}_0$ and the critical points of grad g in the boundary, which can be studied using Morse theory.

We now discuss further relations between the properties of utilities and the underlying preferences. In the next result we give a sufficient condition for an everywhere regular C^k utility function u on S to induce a preference f on S which is actually representable by an element of a space $\mathscr{P}_0$, for some I, i.e., a retraction from S onto some submanifold I of S, I contractible and neat. We then use this result to show an extension of Proposition 4: we give necessary and sufficient conditions for a utility function to induce a preference whose indifference surfaces are all connected and contractible.

Let f be a preference representable by a C^k regular utility function u on S. Then there exist two points x and y in the boundary of S, ∂S, with the gradient of u orthogonal to ∂S at x and y, f increasing towards the interior of S at x, and decreasing towards the interior of S at y. This is immediate; since S is compact, u assumes a maximum x and a minimum y in S. Since u is everywhere regular, the gradient of u in $\mathring{S}$ cannot vanish, and thus both x and y lie in the boundary of S. Therefore, at both x and y the gradient of v projected on the tangent space of ∂S must be zero. If the preference f is given by a retraction of S onto a submanifold I, the points x and y correspond to the points where I intersects ∂S. If f is increasing, I is the diagonal, and $x = (0, \ldots, 0)$, $y = (1, \ldots, 1)$. Conversely, under certain conditions, if f is represented by a regular C^k utility $u: S \to R^+$, then f can actually be representable by a retraction from S *onto* some one dimensional (neat) submanifold I of S.

PROPOSITION 5. Under the conditions of Proposition 4, f is representable by a C^k retraction u from S onto a one dimensional C^k contractible neat submanifold of S denoted I.

Proof. We shall construct the C^k manifold I using flow lines of vector fields given by the preference. Ideally, one would want to take as I a flow line of the gradient of a C^k function g representing $f \in \mathcal{P}_0$. Such a function has both its maximum and its minimum at (0) and (1), since by the assumptions of $\mathcal{P}_0$, g is everywhere regular. However, it is not necessarily true that a flow line of grad g (denoted I^1) should begin at (0) and end at (1). (See Figure 5 below.) Instead, we modify the vector field grad g as in the proof of Proposition 4, to obtain a vector field v_λ. This vector field v_λ has a flow line starting at (0) and ending at (1); v_λ is transversal to all level surfaces of g other than (0) and (1) and it intersects each exactly once. Hence f can be given as a retraction u from S onto a submanifold I, namely a flow line of v_λ for some $0 \le \lambda \le 1$. That u is a C^k retraction follows from the implicit function theorem.

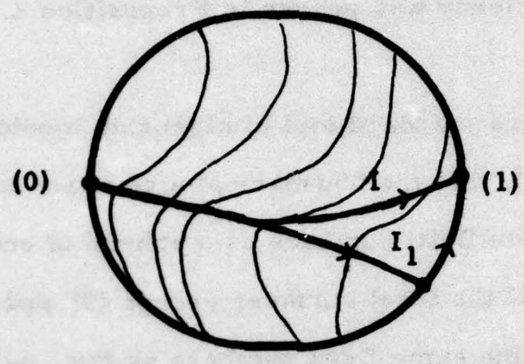

Figure 6

Definition. f is called a Morse preference if $f \in \mathscr{P}_0$ and f is represented by a utility function $g: S \to R$ such that $g|_{\partial S}$ is a Morse function (i.e., g restricted to ∂S is everywhere regular and the zeros of grad $(g|_{\partial S})$ are nondegenerate [31]). It is important to remark that if f is a Morse preference, then every C^2 utility function representing f is also a Morse function. This is because locally, the nondegeneracy of the critical points of the gradient of the utility function on ∂S is determined by the underlying foliation (i.e., the preference relation).

THEOREM 7. Let f be a Morse preference in $\mathscr{P}_0$. Then a necessary and sufficient condition for f to be representable by a C^k retraction from the commodity space S to a one dimensional neat contractible C^k submanifold I of S, with all indifference surfaces connected and contractible, is that the gradient of the utility u representing f has only two critical points (denoted (0) and (1)) when projected to the tangent space of the boundary of S.

<u>Proof</u>. Sufficiency was proven in Proposition 4. We now show necessity.

It follows from a standard tool of algebraic topology, the Poincaré duality theorem [41], that the boundary of a compact contractible manifold has the same Betti numbers as a sphere of one lower dimension. Therefore, all the level surfaces except (0) and (1) on the boundary ∂S have the same Betti numbers as the $n-2$ sphere. It follows from Morse theory [31] that there cannot be any other critical points except (0) and (1) (the minimum and maximum of u) since otherwise the Betti numbers of the level surfaces would change as a critical level is passed which contradicts the hypothesis.

d. Properties of Demands

In this section we study certain properties of the demands in relation to the underlying preferences. For references, see, for instance [3]. Recall that if $f \in \mathcal{P}$ is represented by a concave increasing utility $g: S \to R$, then for each pair (p, y), p a price vector and y income, the value of the demand function at (p, y), $d(p, y)$ is that vector h in S which satisfies

$$(1) \qquad g(h) = \max_{x \in B(p,y)} g(x)$$

where the budget set $B(p,y)$ of S is defined by

$$B(p,y) = \{x \in S : x \cdot p \leq y\} \quad .$$

If g is not concave or increasing, $d(p,y)$ is in general not well defined as a function.

The next result proves that under certain conditions, for a generic set of preferences, the demand function $d(p,y)$ is locally well defined and of class C^1 on prices and income.

Let $\bar{\mathscr{P}}$ be a subset of $\mathscr{P}_2$ (the increasing preferences in $\mathscr{P}_0$) which is bounded in the H^{s+1} norm. Let $p \in P \subset R^{n+}$, $y \in Y \subset R^+$, P and Y compact sets representing the space of prices and of incomes, respectively. Assume that the interior of P is C^∞ diffeomorphic to an open ball in R^n, and the interior of Y to an open interval in R. For related results in the context of utilities rather than preferences, see, for instance [6].

THEOREM 8. <u>For an open and dense set of preferences in</u> $\bar{\mathscr{P}}$, <u>the interior solutions of problem (1) above define locally unique</u> C^1 <u>demand functions</u> $d_f(p,y)$. <u>on a subset of</u> $P \times Y$ <u>which contains an open and dense set</u>.

Proof. For f in $\bar{\mathscr{F}}$ let $g = \phi \circ f \in C^k(S, R^1)$ represent f, where ϕ is a C^∞ diffeomorphism between $\mathring{I}$ and R. Define the map

$$\psi: \bar{\mathscr{F}} \times P \times Y \to C^{k-1}(S \times R^1, R^n \times Y)$$

by

$$\psi(f, p, y)(x, \lambda) = \left(\frac{\partial}{\partial x}(\phi \circ f) + \lambda p,\ p \cdot x - y\right)$$

where $\lambda \in R^1$. Note that, for each p, y in $P \times Y$, $\psi(\cdot, p, y)$ is continuous as a function on $\bar{\mathscr{F}}$ since the map

$$\partial: C^k(S, R^1) \to C^{k-1}(S, R^1)$$

defined by

$$g \to \frac{\partial}{\partial x} g$$

is continuous in the respective C^k and C^{k-1} topologies, and the inclusion map

$$\bar{\mathscr{F}} \subset H^s(S, I) \subset C^k(S, R^1)$$

is continuous by Sobolev's theorem. Thus ψ itself is a continuous map. Consider now the restriction of ψ on $S \times B_0$, where B_0 is a compact interval in R^1 which contains the λ's in the kernel of $\psi(f, p, y)(x, \cdot)$ for x in S. Such B_0 exists by the results of [35], page 30: for all x in S and for all (f, p, y) in $\bar{\mathscr{F}} \times P \times Y$ the respective λ's in the kernel of $\psi(f, p, y)(x, \cdot)$ are contained in such a compact set. For simplicity, denote $\psi(f, p, y)|_{S \times B_0}$ by $\psi(f, p, y)$.

Let $B_1 = S \times B_0$. Thus

$$\psi(f,p,y) \in C^{k-1}(B_1, R^n \times Y) \quad .$$

Let θ be the set of maps ξ in $C^{k-1}(B_1, R^n \times Y)$ such that ξ has zero as a regular value (denoted $\xi \pitchfork 0$). Since B_1 is compact by openness of the transversality property on compact sets [1], θ is an open set.

Consider now the restriction of the C^{k-1} norm on the subset I of $C^{k-1}(B_1, R^n \times Y)$, where I is the image of $\bar{\mathcal{F}} \times P \times Y$ under ψ. Let $\bar{\theta} = \theta \cap I$, and give I the relative topology. Let $\bar{\psi}$ be defined as equal to ψ on the domain of ψ, but having I as its range. Then $\bar{\theta} = \theta \cap I$ is open in the relative topology of I, and by continuity of $\bar{\psi}$, $\bar{\psi}^{-1}(\bar{\theta})$ is also open in $\bar{\mathcal{F}} \times P \times Y$. Note that $\bar{\psi}^{-1}(\theta)$ is contained in the set of elements in $\bar{\mathcal{F}} \times P \times Y$ such that the corresponding interior optimal solutions of (P) define locally a unique C^1 demand function, by the implicit function theorem (since $\frac{\partial}{\partial x, \lambda} \psi(f,p,y)$ is regular at the kernel of $\psi(f,p,y)$ if and only if it is invertible). Hence, for an open set of elements in $\bar{\mathcal{F}}$, and an open set of price and income pairs in $P \times Y$ the interior solutions of (1) define locally unique C^1 demand functions. By Sard's theorem (see [1]), since $k \geq 2$, the set of regular values of $\bar{\psi}(f,p,y)$ is dense in $R^n \times Y$. Then, for any $\epsilon > 0$, let $(q,k) \in R^n \times Y$ be a regular value of the map $\bar{\psi}(f,p,y)$, with $\|q,k\| < \epsilon$. Define $\bar{\psi}^\epsilon$ by

$$\bar{\psi}^\epsilon(f,p,y) = \psi(f,p,y) - (q,k) \quad .$$

Note that $\bar{\psi}^\epsilon(f,p,y) \pitchfork 0$ if and only if $(q,k) \in R^n \times Y$ is a regular value of $\bar{\psi}(f,p,y)$. If $\phi \cdot \bar{f} = \phi \cdot f - q \cdot x$, and $\bar{y} = y - k$, then

$$\overline{\psi}^{\epsilon}(f,p,y) \;=\; \overline{\psi}(\overline{f},\, p, \overline{y}) \quad .$$

Since S is compact and ϕ is a C^{∞} diffeomorphism, $\overline{f}$ can be taken to be arbitrarily close to f in the C^k norm by choosing ϵ small enough. Since the inclusion $\overline{\mathcal{P}} \subset H^s(S,I) \subset C^k(S,I)$ is continuous, $\overline{f}$ can be taken to be arbitrarily close to f also in the H^s norm of $\overline{\mathcal{P}}$. Similarly, $\overline{y}$ can be chosen arbitrarily close to y. Hence, since 0 is a regular value of $\overline{\psi}^{\epsilon}(f,p,y)$, then $(\overline{f},p,\overline{y}) \in \overline{\psi}^{-1}(\overline{\theta})$ and thus $\overline{\psi}^{-1}(\overline{\theta})$ is also dense in $\overline{\mathcal{P}} \times P \times Y$. This completes the proof.

<u>Remarks</u>. 1) The results of Theorem 8 are valid in the C^k norm of spaces of preferences (as defined in Chapter 2) as well.

2) Note that there might be elements f in $\overline{\mathcal{P}} \times P \times Y$ such that the corresponding $d_f(p,y)$ define a C^1 function, and are not contained in $\overline{\psi}^{-1}(\theta)$, since $\frac{\partial}{\partial x, \lambda}\psi(f,p,y)$ may be singular. Also, the boundary solutions to (P) may not be contained in $\overline{\psi}^{-1}(\overline{\theta})$.

3) If f is a convex preference, Theorem 8 yields global instead of local results. In these case, the demand is a globally defined generically C^1 function.

4) Sard's theorem actually can be used to prove that the open and dense set in $P \times Y$ where, for an open dense set of preferences f in $\overline{\mathcal{P}}$ the corresponding demand is locally a well defined C^1 function, has actually measure one (see [1]).

5) A natural question that remains to be answered is the nature of the map that assigns to a convex preference f the corresponding demand function d(p,y). For instance, in [10] this map is shown to

be upper hemi-continuous, with the Hausdorff metric as the topology given to the space of preferences (closed graph relations). The H^s norm is finer than the Hausdorff metric on the space $\mathscr{P}$ since it is finer than the C^k norm if $s > n/2 + k$, and hence these results of Debreu will also hold in this context. This remains to be formalized. However, $\mathscr{P}$ has a much richer structure with the H^s norm than with the Hausdorff metric, with the H^s norm differentiability can be defined. Of functions defined on preferences, an open question is whether the demand function $d_f(p, y)$ depends in a differentiable manner on the underlying preference f in $\mathscr{P}$.

Chapter 4

PREFERENCES AND EQUILIBRIA

a. Introduction

Our aim in this chapter is to show an application of the results of Chapter 2 to the theory of general equilibrium. We shall extend results on finiteness and stability of equilibria of Debreu [12] and Smale [38] to economies where the agents are represented by their preferences and endowments.

In Debreu's theorem [12] an economy is represented by the C^1 demand functions of the agents, satisfying a boundary condition; for *almost* all initial allocation of commodities to the n agents there are only a finite number of equilibria, which depend continuously on the allocations. Such an economy is also called regular.

In Smale [38] the results of Debreu are extended to economies where the C^1 demand functions of the agents may not be well defined, working with C^2 (not necessarily convex) utility functions of the agents instead; it is proven that for almost all initial allocations and utility functions of the agents, the "extended" equilibria (which do not coincide but, in general, contain the classical equilibria) are locally unique and stable. Under certain boundary conditions and convexity assumptions existence of equilibria is also proven [38].

The results of [12] and [38] rely on differential topology techniques; in [12] basically the theorem of Sard, and in [38] Abraham-Thom's transversality theorem and other infinite dimensional differential topology techniques.

In order to be able to use these types of techniques, one needs *some differentiable structures on the spaces of parameters* one works on. Smale's results extended those of Debreu, describing an economy by the utilities as well as by the initial endowments of commodities of the agents--utility functions are elements of linear function spaces, which have enough structure to work on, in particular, to apply transversality theory. However, utilities are considered unsatisfactory as primitive concepts [13], and the results of [38] cannot be translated to spaced of preferences unless more structure for spaces of preferences is given. Furthermore, as has been known in economic theory for a long time, the whole analysis of equilibria and demand behavior ultimately rests on the indifference surfaces; as pointed out by Smale [38], the utility functions are mostly used as a convenient description of the indifference surfaces. So it would seem also methodologically more adequate to work on spaces of preferences directly.

Since Abraham-Thom's transversality theorem is available on some infinite dimensional manifolds, and we showed in Chapter 2 that spaces of preferences can be given such structures, one can now take advantage of the techniques and the results of Debreu [12] and Smale [38] for economies where the agents are identified by *smooth* (not necessarily convex) preferences and initial endowments.[3] This is what we intend to do in this section.

[3] An extension only valid for spaces of convex smooth preferences of the results of [12] is obtained in H. Dierker [16] with different techniques. Dierker states in [16] that in her work the results of [38] cannot be extended in all generality *for the lack of enough structure on* the spaces of preferences. Here we show how the structure given in Chapter 2 can be used to overcome this, and also to yield generic results about regularity *of* equilibria without such convexity assumptions.

We shall now give a brief discussion of the problem. Each C^k preference (with the definition of Chapter 2) is, in general, represented by an infinite class of utilities. And it is the "indifference surfaces" of the preferences (or the level surfaces of the utilities), rather than the utility function as a whole, which determines the agent's demand behavior.[4] So if an agent is identified with a preference, rather than a utility function, a natural question is whether the results of [38] carry over. In a sense, an extension of the results of [12] and [38] to preferences is intuitively clear: one would expect that the open density of the class of utilities that yield regular equilibria on an open dense set of initial allocations would not be all "used up" in a set of utilities which represent very few preferences.[5] That this is the case for the representation of spaces of preferences given in Chapter 2, as proved below, seems to give further support to the intuitive naturality of this representation.

This section is organized as follows: In (b) we prove that the results of Chapter 2 can be applied to obtain a C^k representation of spaces of C^k preferences defined on the positive cone of R^ℓ, $R^{\ell +}$, which is basically an inclusion map. The spaces of preferences are given here two alternative topologies: the Whitney topology and a Sobolev norm. Since with both these topologies the inclusion of the spaces of preferences into utility function spaces is not an open map, one cannot "pull back" the results of density of regular economies of [38] on utility function spaces to spaces of preferences--further reasonings are needed. In

[4] That is, two utilities which represent the same preference exhibit the same demand behavior.

[5] That is, that the preferences induced by these open dense sets of utilities are also an open dense set in the space of preferences, with the right topologies.

Theorem 9 we show that the results of [38] can be extended in this framework, in view of the representation of the spaces of preferences given here. Finally, in Part c we discuss very briefly the question of existence of equilibria in this context and necessary conditions for the local uniqueness of equilibria, which are a partial converse to the sufficient conditions given in Part b.

b. Genericity of Regular Economies

In this section we shall consider as a space of commodities the positive cone of R^ℓ,

$$\overline{P} = \{x = (x^1, \ldots, x^\ell) \in R^\ell \text{ with } x^i \geq 0 \text{ for each } i\} \quad .$$

As a model for the space of preferences we shall take a space of C^k, $k > 2$ retractions from $\overline{P}$ to a strictly $\leq$ ordered one dimensional C^∞ submanifold I of $\overline{P}$ with $\partial I = \{0\}$ and I C^∞ diffeomorphic to $[0, \infty)$.

We shall restrict ourselves to the case of everywhere regular retractions, the analog of $\mathscr{P}_0$ in Chapter 2. We next discuss the topologies on these spaces. The spaces of preferences are endowed with two different topologies: a Whitney topology and an H^s topology. The Whitney topology on the space of C^k retractions from $\overline{P}$ to I is given for any C^∞ diffeomorphism $\varphi: I \to [0, \infty)$ by an inclusion map ϕ of $C^k(\overline{P}, I)$ into $C^k(\overline{P}, [0, \infty))$ induced by the composition with the map φ, i.e.,

$$\phi: C^k(\overline{P}, I) \to C^k(\overline{P}, [0, \infty))$$

is defined by

$$\phi(f) = \varphi \circ f \quad ;$$

since φ is C^∞, $\varphi \circ f \in C^k(\overline{P}, [0,\infty))$. ϕ is an injective map and the image of $C^k(\overline{P}, I)$ in $C^k(\overline{P}, [0,\infty))$ inherits the Whitney topology of $C^k(\overline{P}, [0,\infty))$. To define the Whitney topology on $C^k(\overline{P}, [0,\infty))$ one gives a family of neighborhoods of zero: a neighborhood N_h is defined for each strictly positive continuous function $h: \overline{P} \to R$ as follows: $f \in N_h$ if

$$\sup\left(|f(x)|, \|Df(x)\|, \ldots, \|D^k f(x)\|\right) < h(x) \quad ,$$

for all x in $\overline{P}$ (see also [38]). For a given C^∞ diffeomorphism $\varphi: I \to [0,\infty)$ the space of everywhere regular C^k retractions from $\overline{P}$ to I is an open subset of the space of all C^k retractions from $\overline{P}$ to I, with the Whitney topology induced by ϕ described above. We denote this open set $\mathscr{P}_W$; it will be one model space for the space of preferences on $\overline{P}$.

Next we study a Sobolev norm on a space of C^k retractions. We consider a finite measure on $\overline{P}$, with a C^∞ density function μ such as $\mu(x) = e^{-\lambda\|x\|}$, $0 < \lambda < 1$.[6] The space of H^s functions from $\overline{P}$ (with the measure given by μ) to R is defined, as in Chapter 2, as the completion under the $\|\cdot\|_s$ norm of $C^\infty(P, [0,\infty))$. Here the $\|\cdot\|_s$ norm is defined with respect to the measure induced by μ. The space of H^s retractions from $\overline{P}$ to I is similarly given a metric derived from the $\|\cdot\|_s$ norm, and it becomes a Hilbert (or Hilbertable) manifold (by using the implicit function theorem, see Chapter 2, Theorem 1). By Sobolev's theorem, $H^s(\overline{P}, I)$ is included in $C^k(\overline{P}, I)$, if $s > \ell/2+k$ (i.e., the functions in $H^s(\overline{P}, I)$ are k-times continuously differentiable).

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[6] These type of measures are given naturally in certain infinite horizon models such as those of optimal growth, see Chichilnisky [5], where λ represents a "discount factor".

However, the inclusion of H^s in C^k is not continuous with the Whitney topology on $C^k(\overline{P}, I)$. For analogous reasons, the subspace $\mathscr{P}_H$ of everywhere regular retractions in $H^s(\overline{P}, I)$ will not be open, and will thus nor form a Hilbert manifold itself. However, when the preferences are restricted to a compact region Ω of $\overline{P}$, the space of preferences with the above structure, denoted $\mathscr{P}_{H\Omega}$, is a Hilbertable manifold since it coincides with the manifold $\mathscr{P}_0$ studied in Chapter 2. That is all that is needed in order to be able to use Abraham-Thom's transversality theory to extend the results of [38] to these spaces of preferences, in order to obtain results on genericity of regular economies.

We need a few technical results. See the Appendix for the definition of $H^s(\Omega, \mathbb{R}^m)$.

LEMMA 5 (Calderon's Extension Theorem). <u>Let</u> $\Omega \subset R^\ell$ <u>be an open bounded set with</u> C^∞ <u>boundary</u>. <u>If</u> $f \in H^s(\Omega, R^m)$, <u>then</u> f <u>has an extension</u> $\tilde{f} \in H^s(R^\ell, R^m)$.

LEMMA 6. <u>For any open bounded set</u> Ω <u>with</u> C^∞ <u>boundary in</u> $\dot{P}$ <u>the restriction map</u> $R: H^s(\overline{P}, R) \to H^s(\Omega, R)$ <u>defined by</u>

$$R(f) = f|_\Omega$$

<u>is a continuous and open map</u>.

<u>Proof</u>. Note that $H^s(\Omega, R)$ with the measure μ on Ω coincides with $H^s(\Omega, R)$ with the Lebesgue measure on Ω, since $\overline{\Omega}$ is compact. Next, note that if $f \in H^s(\overline{P}, R)$, $R(f) \in H^s(\Omega, R)$, and that by definition of the topology on the space $\mathscr{P}_H$, R is continuous. From Calderon's

extension theorem (see [26], Theorem 1.1.1), we know that if $f \in H^s(\Omega, R)$, then there exists an extension of f to a map $\tilde{f} \in H^s(\overline{P}, R)$. Given any g in a neighborhood of f in $H^s(\Omega, R)$ with the $\|\cdot\|_s$ norm, one can find an extension of g to a map $\tilde{g}$ in $H^s(\overline{P}, R)$ such that $\|\tilde{f} - \tilde{g}\|_s$ is as small as wanted in $H^s(\overline{P}, R)$, by using arguments of C^∞ partitions of unity and Calderon extension theorem.

Therefore, the image under the restriction map of an open neighborhood of an element h in $H^s(\overline{P}, R)$, $h|_\Omega$ will be an open neighborhood of $h|_\Omega$ in $H^s(\Omega, R)$ with the induced $\|\cdot\|_s$ norm. Thus the map R is open, which completes the proof.

Let $C^k(\overline{P}, R)$ be endowed with the C^k Whitney topology, and let φ be the C^∞ diffeomorphism between I and $[0, \infty)$ defined above. Then φ induces, as seen above, an inclusion denoted ϕ of the space of preferences $\mathscr{P}_W$ into the space $C^k(\overline{P}, [0, \infty))$. The next lemma gives a property of this inclusion.

LEMMA 7 (Left Composition of Maps). <u>Let</u> U <u>be a bounded open set in</u> R^ℓ, <u>and</u> $h: R^n \to R^m$ <u>be</u> C^∞. <u>Then</u>

$$w_h: C^k(U, R^n) \to C^k(U, R^m)$$

<u>defined by</u>

$$w_h(f) = h \circ f$$

<u>is a</u> C^∞ <u>map. Further,</u>

$$\overline{w}_h: H^s(U, R^n) \to H^s(U, R^m)$$

In the following, we assume that in the definition of $\mathscr{P}_H$, $s > \ell/2 + k$ and $k \geq 2$. Let Ω be a bounded region with a smooth boundary and assume that $\partial(I \cap \Omega) \subset \partial\Omega$. Let $I \cap \Omega$ be denoted I_Ω, and let $\mathscr{P}_{H\Omega}$ denote the space of everywhere regular retractions from Ω onto I_Ω with the H^s norm. (This space coincides with the space of preferences $\mathscr{P}_0$ discussed in Chapter 2 when $\Omega = S$). Define maps

$$\Phi_\Omega : \mathscr{P}_{H\Omega} \to C^k(\Omega, R)$$

by

$$\Phi_\Omega(f) = \varphi \circ f \quad .$$

__Definition.__ A map $\rho: A \to C^r(X, Y)$ from a manifold A to the space of C^r maps from a C^∞ manifold X to a C^∞ manifold Y is called a C^r __representation__ if the evaluation map $ev_\rho : A \times X \to Y$ given by $ev_\rho(a, x) = \rho(a)(x)$ for $a \in A$ and $x \in X$, is a C^r map from $A^\rho \times X$ to Y (see [1]).

LEMMA 8. $\phi_\Omega : \mathscr{P}_{H\Omega} \to C^k(\Omega, R)$ __is a__ C^∞ __map__. __In particular__, ϕ_Ω __defines a__ C^k __representation of__ $\mathscr{P}_{H\Omega}$.

__Proof.__ First note that if $f \in H^s(\Omega, I_\Omega)$, then f is a C^k map by Sobolev's inequality, since s is assumed to be strictly larger than $\ell/2 + k$, so that $\phi_\Omega(f) = \psi \circ f$ is indeed in $C^k(\overline{\Omega}, R)$. By Sobolev's theorem the inclusion j

$$j: H^s(\Omega, I) \to C^k(\Omega, I)$$

is continuous. Since φ is C^∞, the inclusion i

$$i: H^s(\Omega, I) \to C^k(\Omega, R) \quad ,$$

given by $i(f) = \psi \; j(f)$, is also continuous--since it is (locally) linear, i is also C^∞. Finally, note that by Lemma 7 on left composition of maps, since the map $\bar{\Phi}_\Omega$

$$H^s(\Omega, I_\Omega) \xrightarrow{\bar{\Phi}_\Omega} H^s(\Omega, R)$$

defined by $f \to \varphi \circ f$, is given by the composition with the C^∞ map φ, then it is also a C^∞ map from $H^s(\Omega, I_\Omega)$ into $H^s(\Omega, R)$. Since $\Phi_\Omega = i \circ \bar{\Phi}_\Omega$, then ϕ_Ω is C^∞. This completes the proof.

Using these lemmas we shall now show an extension of the results of [12] and [38] on regular economies. We need some definitions. An economy is identified by a vector of initial allocations for each of the m agents, $r = (r_1, \ldots, r_m) \in (\bar{P})^m$, and a vector of the preferences of each agent $f = (f_1, \ldots, f_m) \in (\mathscr{P})^m$ where $\mathscr{P}$ denotes either $\mathscr{P}_W$ or $\mathscr{P}_H$. So an economy is a pair

$$(r, f) \in (\bar{P})^m \times (\mathscr{P})^m \quad .$$

Now let $S_+ = \{p \in \bar{P}: \|p\| = 1\}$ where $\|p\|^2 = \Sigma_{j=1}^{\ell} (p^j)^2$. As in [34], the space of states $\mathscr{S}$ is the product space $\mathscr{S} = (\bar{P})^m \times S_+$, where m is the number of agents in the economy. $(\bar{P})^m$ is the space of initial resources and S_+ is the space of prices. From now on, the letter p is used to indicate prices.

An <u>attainable state</u> of the economy (r, f) is a vector $(x, p) \in \mathscr{S}$ satisfying

$$\sum x_i = \sum r_i \quad .$$

An attainable state is an <u>extended price equilibrium</u> if, for each i, x_i is in the set $B_{i,p} = \{y \in \overline{P}: p(y) = p(r_i)\}$ and x_i is a critical point of the restriction

$$\varphi \circ f_i \big| B_{i,p} \quad ,$$

i.e.,

$$D\varphi \circ f_i \big| B_{i,p} (x_i) = 0 \quad .$$

For each $f = (f_1, \ldots, f_m) \in (\mathscr{S})^m$, let Γ be a map from $(\mathscr{S})^m$ into $C^{k-1}(\mathscr{S}, S^{m+1})$ defined by

$$\Gamma(f_i)(x, p) = \left(\frac{Du_1(x_1)}{\|Du_1(x_1)\|} \quad , \ldots , \quad \frac{Du_m(x_m)}{\|Du_m(x_m)\|} \quad , \ p \right) \quad ,$$

where $u = (u_1, \ldots, u_m) = (\varphi \circ f_1, \ldots, \varphi \circ f_m)$. Let Δ be the diagonal in $(S)^{m+1}$, i.e.,

$$\Delta = \{(y_1, \ldots, y_{m+1}) \in (S)^{m+1} \mid y_1 = y_2 = \cdots = y_{m+1}\} \quad .$$

PROPOSITION 6. Let Z be the subset of m-tuples of preferences f in $(\mathscr{P}_W)^m$ such that

$$\Gamma(f): \mathscr{P} \to (S)^{m+1}$$

is transversal to Δ. Then Z is an open and dense set which contains the m-tuples of preferences which define C^{k-1} demand functions.

Proof. Let $C_0^k(\overline{P}, R)$ be the set

$$\{u \in C^k(\overline{P}, R): Du(x) \neq 0 \text{ for all } x \text{ in } \overline{P}\}$$

By Lemma 8 above, the map

$$\phi: \mathscr{P}_W \to C^k(\overline{P}, R)$$

defined as before by $\phi(f) = \varphi \circ f$, is C^∞. Let $\psi: (C^k(\overline{P}, R))^m$ into $C^{k-1}(\mathscr{P}, (S)^{m+1})$ be defined as in [38], (Section 2) by

$$\psi(u)(x, p) = \left(\frac{Du_1(x_1)}{|Du_1(x_1)|}, \ldots, \frac{Du_m(x_m)}{|Du_m(x_m)|}, p \right)$$

thus $\Gamma = \psi \circ \phi$, so that

$$ev_\Gamma = ev_{\psi \circ \phi} = ev_\psi \circ (\phi^m \times id)$$

where

$$\phi^m \times id: (\mathscr{P}_W)^m \times \mathscr{P} \to (C^k(\overline{P}, R))^m \times \mathscr{P}$$

is defined by

$$\phi^m \times \mathrm{id}((f_1, \ldots, f_m), x) = (\varphi \circ f_1, \ldots, \varphi \circ f_m, x)$$

Thus, since by [38], Proposition 1, $\overline{\psi}$ is a C^1 representation, ev_Γ is a C^1 representation also, by Lemma 7. Since $\mathscr{S}$ is compact, by Theorem 18.2 of [1] (openness of transversal intersection) Z is an open set.

We now check that $ev_\Gamma \pitchfork \Delta$. Note that for any f in $(\mathscr{P}_W)^m$, the map

$$(D\phi): T_f(\mathscr{P}_W) \to T_{\phi(f)}(C^k(\overline{P}, R))$$

is onto, since φ is a diffeomorphism, thus, since by [38] (Proposition 1), the evaluation map ev_ψ is transversal to Δ, and since $ev_\Gamma = ev_\psi \circ (\phi^m \times \mathrm{id})$, ev_Γ is also transversal to Δ. Also, the dimension of $\mathscr{S}$ is $m \cdot \ell + \ell - 1$, the codimension of Δ in $(S)^{m+1}$ is $(m+1)(\ell-1) - 1 = m \cdot \ell - m + \ell - 2$. Thus

$$\max (0, \dim (\mathscr{S}) - \mathrm{codim}\, (S)^{m+1}) = 0 \quad .$$

Hence, since $k \geq 2$, the conditions of Abraham and Thom's transversality theorem ((19.1 of [1]) are satisfied and thus Z is also dense in $(\mathscr{P}_W)^m$. The remarks after Proposition 1 of [38] apply and thus Z contains the set of m-tuples of preferences which define C^{k-1} demand functions.

Let Ω be as in Lemma 3 above. Let $\mathscr{S}_\Omega$ be the space of states such that the commodity coordinates are in Ω, i.e., $\mathscr{S}_\Omega = (\Omega)^m \times S_+$. By similar arguments to those used in Proposition 1 and by applying Lemmas 8 and 9, we obtain:

PROPOSITION 7.

(a) <u>Let</u> $\overline{Z}_\Omega$ <u>be the subset of m-tuples of preferences</u> f <u>defined on</u> Ω <u>in the space</u> $(\mathscr{P}_{H\Omega})^m$, <u>such that</u> $\Gamma(f)$: $\mathscr{P}_\Omega \to (S)^{m+1}$ <u>is transversal to</u> Δ. <u>Then</u> $\overline{Z}_\Omega$ <u>is an open and dense set which contains the m-tuples of preferences which define</u> C^1 <u>demand functions</u>.

(b) <u>Let</u> $\overline{Z}$ <u>be the subset of m-tuples of preferences in</u> $(\mathscr{P}_H)^m$ <u>such that</u> $\Gamma_f : \mathscr{P} \to (S)^{m+1}$ <u>is transversal to</u> Δ. <u>Then</u> $\overline{Z}$ <u>is a residual</u> (<u>and hence dense</u>) <u>set in</u> $(\mathscr{P}_H)^m$.

We now need further definitions. We follow the notation of [38]:

For an initial allocation r, let the set of attainable states relative to r with a budget condition be defined by:

$$\Sigma_r = \{(x,p) \in \mathscr{P} \mid \Sigma x_i = \Sigma r_i, \; p(x_i) = p(r_i), \; i = 1,\ldots,m-1\}$$

Σ_r contains the classical and extended price equilibria.

Let K be the subset of $(\overline{P})^m \times \mathscr{P}$:

$$K = \{(r,s) \in (\overline{P})^m \times \mathscr{P} \mid s \in \Sigma_r\}$$

for f in $(\mathscr{P})^m$, define the set:

$$G = \{(r,x,b) \in K : \Gamma_f(x,b) \in \Delta\}$$

where Γ is defined above.

(r,f) is called a <u>regular economy</u> if $f \in Z$ and Π restricted to G has r as a regular value, where $\Pi: (\overline{P})^m \times \mathscr{P} \to (\overline{P})^m$ is the projection. For a regular economy the equilibria are finite and stable in a global sense [38]. The proof of [38] can now be applied, and one obtains the following result:

THEOREM 9. <u>For an open and dense set</u> Z <u>of m-tuples of preferences in</u> $(\mathscr{P}_W)^m$, <u>if</u> $f = (f_1, \ldots, f_m) \in Z$, (r, f) <u>is a regular economy for almost all initial allocations</u> r. <u>For a residual set</u> $\bar{Z}$ <u>of m-tuples of preferences in</u> $(\mathscr{P}_H)^m$ <u>if</u> $f = (f_1, \ldots, f_m) \in \bar{Z}$, (r, f) <u>is a regular economy for almost all initial allocations.</u>

<u>Proof</u>. It follows from Propositions 6 and 7 above, and Proposition 4 and Theorem 2 of [38]. (Note that the results of Proposition 4 of [38] are immediately translated into this context as referring to the m-tuples of utilities

$$u = (u_1, \ldots, u_m) = (\varphi \circ f_1, \ldots, \varphi \circ f_m) \quad .)$$

c. __Existence of Equilibria and Necessary Conditions for Local Uniqueness and Stability__

We now give a very brief discussion on the problem of existence of equilibria based on the results of the Appendix of [34]. Recall that in [38] if $u \in U$, the set of everywhere regular utilities in $C^2(\overline{P}, R)$ with the boundary condition (BC) of the Appendix of [38] and the Monotonicity Hypothesis (M_H) (there is an open halfspace H in R^{ℓ} and $Du(x) \in H$ for all x in $\overline{P}$) are satisfied, then for the set $\mathscr{F}$ of all economies (r, u) in $(\overline{P})^m \times U^m$ such that for each agent j, u_j satisfies M_H and (r_j, u_j) satisfies (BC), there exists an extended price equilibrium. (See Appendix [38].) Furthermore, if $\mathscr{F}_D$ is the subset of $\mathscr{F}$ where for each $x \in \overline{P}$ the restriction of the second derivative $D^2 u_j(x)|_{\ker Du_j(x)}$ is negative definite, then for each economy in $\mathscr{F}_D$ there exists a classical price equilibrium.

These results extend in this context as follows:

Let $\mathscr{H}$ be the set of economies (r, f) with $\phi(f) \in \mathscr{F}$, and $\mathscr{H}_D$ be the set of economies (r, f) with $\phi(f) \in \mathscr{F}_D$. Then it follows from the above theorem of [38] that __for an economy__ (r, f) __in__ $\mathscr{H}$ __there exists an extended price equilibrium__, __and for any economy__ (r, f) __in__ $\mathscr{H}_D$ __there exists a classical price equilibrium__.

We end this section with a brief discussion of necessary conditions for the local uniqueness of equilibria, motivated by discussions in [3].

Let $f: U \to R^{\ell}$ be a C^2 function representing an aggregate demand of an economy, restricted to an open set U of the price space. The zeros of f represent the equilibria. Let $x \in f^{-1}(0)$. As is well known, if $Df(x)$ is not singular, then x is a locally unique equilibrium. Also, as discussed in [12], [38] and in Part b above, if the economy is regular in the sense discussed in Part b, the set of equilibria is a discrete set, which moves continuously with the parameters (r, u) which define the economy r representing the initial resources, and u the utility functions which determine the aggregate demand f.

These results follow basically from the implicit function theorem. However, it is known that the condition that $Df(x)$ be nonsingular is sufficient, but not necessary, for local uniqueness and local stability of the equilibrium x. Here we discuss simple necessary conditions for the equilibria to be locally unique, and related sufficient conditions for the (nonlocally unique) equilibria to determine a submanifold of some codimension which moves continuously with the parameters. These conditions were studied following a remark given by Arrow and Hahn in [3] to the effect that such necessary conditions seem hard to obtain.

PROPOSITION 8. Let $f: U \to V$ be C^2, U and V open subsets of R^N, $0 \in V$. Then if $x \in f^{-1}(0)$ is a locally unique zero of f, then either $Df(x)$ is nonsingular or else $Df(x)$ changes rank at every neighborhood of x, i.e., if rank $Df(x) = k$, $k < n$, then not all $k + 1$ minors of Df can vanish identically in a neighborhood of f.

Example. We first illustrate the conditions by means of an example. Consider $f: R^3 \to R^3$. Assume $x \in f^{-1}(0)$ and that $Df(x)$ is not invertible. Then either rank $Df = 3$ at some point in every neighborhood of x, or else determinant of $Df \equiv 0$ in some neighborhood of x.[7]

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[7] Note that if Df has rank k at x, then Df must have rank at least k in some neighborhood of x.

In the first case x is a locally unique zero by the inverse function theorem. In the latter case, for the rank of Df not to be constantly equal to two, the nine two by two minors of Df must have a common zero in every neighborhood of x. Then either Df has rank 2 at some point in every neighborhood of x, in which case x is not an isolated zero of f, or else all two by two minors vanish identically in some neighborhood U of x. In the latter case, for x to be an isolated zero of f the entries of Df must have a common zero in every neighborhood of x, because Df is <u>not</u> identically zero. Note that these conditions are necessary but not sufficient for the local uniqueness of a zero. A simple example is given by $f: R^2 \to R$, $f(x,y) = (x^2 - xy^2, x^2 - xy^2)$. $(0,0)$ is a zero of f. Since the rank of Df at $(0,0)$ is zero, and rank of Df at $(x,y) \neq (0,0)$ is not zero, Df changes rank at every neighborhood of $(0,0)$, but $(0,0)$ is not an isolated zero of f.

<u>Proof</u>. If the rank of Df is constantly equal to k $(k \leq N)$ at a neighborhood U of x, then $f^{-1}(0) \cap U$ is an $N-k$ dimensional submanifold of U, by a version of the implicit function theorem called also the constant rank theorem (see [36]).

COROLLARY 6. <u>Let</u> x <u>be an equilibrium of the</u> C^2 <u>excess demand function restricted to an open set</u> U <u>of the price space</u> $f: U \to R^n$. <u>Then if</u> x <u>is a locally unique equilibrium</u> $(x \in f^{-1}(0))$, <u>either</u> Df <u>is nonsingular at</u> x <u>or else</u> Df <u>changes rank in every neighborhood of</u> x. <u>If</u> Df <u>has constant rank</u> $k \leq n$ <u>in some neighborhood</u> U <u>of</u> x, <u>then the set of equilibria in</u> U <u>determines an</u> $n-k$ <u>dimensional submanifold</u>

<u>of</u> U <u>which moves continuously with the parameters</u> (r,u), <u>the initial resources and utility functions of the economy</u>.

Remarks. 1. The inverse function theorem admits a global version also called the Monodromy type theorem (see [33]). This would allow an extension of the results of Corollary 2 above to global properties of equilibria. Other results on global uniqueness of equilibria are discussed in [3].

2. The above discussion describes the structure of the set of equilibria in some cases: where the derivative (or Jacobian) of the excess demand function is invertible and when it has a constant rank smaller than the dimension of the commodity space. These results can be studied either locally or globally. The natural next step is to study what happens to the cases in between these two and to try to give sufficient conditions that may describe further the set of equilibria in these other cases. These types of questions are probably best studied by the use of tools of bifurcation theory.

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APPENDIX

Definitions and Tools from Global Analysis and Differential Topology

Definitions (See for instance [28], [34] and [39]).

Let $\Omega \subset R^n$ be an open bounded set with C^∞ boundary. Let $\overline{\Omega}$ be the closure of Ω. Define $C^\infty(\Omega, R^m)$ to be the set of functions from Ω to R^m that can be extended to a C^∞ function on some open set in R^n containing $\overline{\Omega}$.

If k is a vector of n nonnegative numbers, $k = (k_1, \ldots, k_n)$ let $|k| = k_1 + k_2 + \cdots + k_n$. If $u \in C^\infty(\Omega, R^m)$ define $D^k(u)$ by the formula

$$D^k(u) = (\partial^{|k|} u / \partial x_1^{k_1} \ldots \partial x_n^{k_n})$$

and $D^0 u = u$.

Let $u \in C^k(\Omega, R^m)$, the space of all k-times continuously differentiable maps defined on a neighborhood of $\overline{\Omega}$ with values on R^m. The C^k norm $\|\cdot\|_k$ is defined by

$$\|u\|_k = \sup_{x \in \Omega} (|u(x)|, \ldots, |D^j(x)|), \; 0 \le |j| \le k \; .$$

For $u \in C^\infty(\Omega, R^m)$ define

$$\|u\|_s^2 = \int_\Omega \sum_{0 \le |k| \le s} |D^k u(x)|^2 \; dx$$

Now let $H^s(\Omega, R^m)$ be defined as the completion of $C^\infty(\Omega, R^m)$ under the $\|\cdot\|_s$ norm. These H^s spaces are called Sobolev spaces. Note that $H^0(\Omega, R^m) = L^2(\Omega, R^m)$.

Sobolev Theorem

(a) Let $s > n/2 + k$. Then $H^s(\Omega, R^m) \subset C^k(\Omega, R)$ and the inclusion is a continuous and compact map where C^k has the C^k topology.

(b) If $s > 1/2$, and $f \in H^s(\Omega, R^m)$ then $f/\partial\Omega \in H^{s-1/2}$, where $\partial\Omega$ denotes the boundary of Ω.

Manifolds and Tangent Bundles

For the following definitions see for instance [22], [41], [42] and [43].

Let M be a Hausdorff topological space such that each point in M has a neighborhood homeomorphic to an open subset of Euclidean space R^d. If φ is a homeomorphism[1] of a connected open set $U \subset M$ onto an open subset of R^d, the pair (U,φ) is called a coordinate system, and φ a coordinate map. A differentiable structure $\mathscr{F}$ of class C^k $(1 \leq k \leq \infty)$ on M is a collection of coordinate systems $\{U_\alpha, \varphi_\alpha, \alpha \in A\}$ which satisfy:

(a) $\bigcup_\alpha U_\alpha = M$

(b) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^k for all α, β in A.

A d-dimensional differentiable manifold of class C^k is a pair $(M,\mathscr{F})$, where M is second countable, and $\mathscr{F}$ is a C^k differentiable structure with coordinate systems which map open subsets of M onto open subsets of R^d.

A tangent vector v at a point m in M is a mapping that assigns to each function f defined and differentiable of class C^k on a neighborhood of m, a real number v(f) such that if (U,φ) is a coordinate system on a neighborhood of m, there exists a list of real numbers $(a,\ldots,a_d)$ (depending on φ) such that

$$v(f) = \sum_{i=1}^{d} a_i \left. \frac{\partial(f\circ\varphi^{-1})}{\partial r_i} \right|_{\varphi(m)}$$

where $r_1 \ldots r_d$ is the canonical coordinate system on R^d. The space of all tangent vectors at $m \in M$ is denoted TM_m and called the tangent space of M at m. TM_m turns out to be d-dimensional, with a basis $\{\partial/\partial x_i|_m\}$. Let M be a C^∞ manifold with differentiable structure $\mathscr{F}$. Let

$$T(M) = \bigcup_{m\in M} TM_m$$

[1] A continuous one to one onto map with a continuous inverse.

There is a natural projection

$$\Pi : T(M) \to M, \quad \Pi(v) = m \quad \text{if} \quad v \in TM_m .$$

Let $(U,\varphi) \in \mathscr{F}$, with coordinate functions $x_i = r_i \circ \varphi$, $i = 1,\ldots,d$. Define $\tilde{\varphi} : \Pi^{-1}(U) \to R^{2d}$ by

$$\tilde{\varphi}(v) = (x_1(\Pi(v),\ldots,x_d(\Pi(v)), dx_1(v),\ldots dx_d(v))$$

for all $v \in \Pi^{-1}(U)$. $\tilde{\varphi}$ is one to one onto an open subset of R^{2d}. Then it can be checked that

(1) if (U,φ) and $(V,\psi) \in \mathscr{F}$ then $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is C^∞.

(2) the collection $\{\tilde{\varphi}^{-1}(W);\ W \text{ open in } R^{2d},\ (U,\varphi) \in \mathscr{F}\}$ forms a basis for a topology on $T(M)$ which makes $T(M)$ into a 2d-dimensional C^∞ manifold. $T(M)$ with this differentiable structure is called the <u>tangent bundle</u> of M. Any point of $T(M)$ can be written as a pair (m, v), were $m \in M$ and $v \in TM_m$.

A C^1 map $f : M \to N$ is an <u>embedding</u> if for all $x \in M$ $T_{xf} : T_xM \to T_{f(x)}N$ is one-one, and f maps M homeomorphically onto its image. Let N be a C^∞ submanifold of M. A <u>tubular neighborhood</u> of N in M is a tuple (ξ, W, f) where ξ is a C^∞ vector bundle, with projection p, fiber E and base space N, $W \subset E$ is an open neighborhood of the zero section $Z(N)$ of this bundle, and $f : W \to M$ is a C^∞ embedding onto an open neighborhood of N such that $f \circ Z = id_N$. A tubular neighborhood of N can be described by saying that there is a neighborhood of N in which N "looks like the zero section of a vector bundle." (See for instance [22]).

A map $f : X \to Y$ is <u>homotopic</u> to a map $g : X \to Y$ if there exists a continuous map

$$\phi : X \times I \to Y$$

such that

$$\phi(x,0) = f(x) \quad \text{and} \quad \phi(x,1) = g(x)$$

for all $x \in Y$.

A topological space X is contractible when the identity map $id_X : X \to X$ is homotopic to any constant map $C_x : X \to x$, $x \in X$.

Manifolds with boundary (see [22]).

A half space of R^n is a subset of the form

$$H = \{x \in R^n \mid \lambda(x) \geq 0\}$$

where $\lambda : R^n \to R$ is a linear map. If $\lambda \equiv 0$ then $H = R^n$, otherwise H is called a proper half space. If H is proper the boundary of H is the set $\partial H = \text{kernel}\ \lambda$, which is a linear subspace of dimension $n-1$. If $H = R^n$, $\partial H = \phi$. We now extend the definition of chart on a space M to mean a map $\psi : U \to R^n$ which maps the open set $U \subset M$ homeomorphically onto an open subset of a half space in R^n, this includes all charts defined before, since R^n is itself a halfspace. Using this definition of chart one extends the meaning of a C^r differentiable structure for M to manifolds with boundary. Similarly one defines a submanifold of a manifold with boundary (see [22]).

∂M (the boundary of M) is defined as follows $x \in \partial M$ if $x \in \psi^{-1}(\partial H)$, for some chart ψ, where H is a proper halfspace. This condition is independent of the chart (see [22]).

A is a neat submanifold of M if it is a submanifold and $\partial A = A \cap \partial M$, and $A \pitchfork M$ at ∂A.

If $M \subset R^m$ and $N \subset R^n$ are manifolds with boundary, a map $f : M \to N$ is in $H^s(M, N)$ when there exists an H^s map

$$g : U \to V$$

where $U \subset R^m$ is an open set containing M, V is an open manifold containing N, and $g/M = f$.

<u>Bundles</u> (see for instance [42] and [44]).

A <u>bundle</u> consists of a <u>bundle space</u> B, a <u>base space</u> X, a continuous onto map $p : B \to X$ called a <u>fibration</u> and a space Y called the <u>fiber</u>. In addition there is a group G of homeomorphism of Y and for each $x \in X$ a family of homeomorphisms G_x of Y into Y_x such that

(i) $\xi, \xi' \in G_x \Rightarrow \xi^{-1} \xi' \in G$ and

(ii) $\xi \in G_x$, $g \in G \Rightarrow \xi g \in G_x$.

(iii) A family $\{V_j\}$ (called <u>coordinate neighborhoods</u>) of open sets covering X such that for each j there is a homeomorphism $\phi_j : V_j \times Y \to p^{-1}(Y)$ (called a <u>coordinate function</u>) and

$$p \cdot \phi_j(x, y) = x \quad \text{for } x \in V_j, \quad y \in Y$$

(iv) if the map $\phi_{j,x} : Y \to p^{-1}(x)$ is defined by $\phi_j, x(y) = \phi_j(x, y)$ then for each pair i, j in J and each $x \in V_i \cap V_j$ the homeomorphism $\phi_{j,x}^{-1} \phi_{i,x} : Y \to Y$ coincides with the operation of an element of G and for each i, j the map $g_{ji} : V_i \cap V_j \to G$ defined by $g_{ji}(x) = \phi_{j,x}^{-1} \phi_{i,x}$ is continuous.

Let M be an n-dimensional manifold of class r. A <u>foliated structure</u> or <u>foliation</u> $\mathscr{F}$ <u>of class</u> r and of codimension p is defined by giving a system of charts h_i which are homeomorphisms of class r of open sets U_i of R^n over open sets of the manifold V satisfying the following properties:

(1) $U_i\, h_i(U_i)$ cover M.

(2) The changes of coordinates or charts $h_j^{-1}\, h_i$ are local homeomorphisms of R^n of class r which are locally of the form

$$h(x, y) = (x^1, y^1)$$

$$\begin{cases} x^1 = h_1(x) \\ y^1 = h_2(x, y) \end{cases}$$

for all z in M, where

$$z = (x, y) = (x_1, \ldots, x_p, y_1, \ldots, y_{n-p})$$

A <u>vector bundle</u> is a bundle in which the fiber is a real vector space and G is a group of linear transformations.

A cross-section of a bundle is a continuous map $f : X \to B$ such that $pf(x) = x$ for each $x \in X$.

Let $\eta : X \to X^1$ be a continuous map, and let (B^1, X^1, Y, p) be a bundle with group G. The <u>induced</u> or <u>pull back</u> bundle can be defined having base space X, fiber Y, group G and bundle space $\eta^{-1}B^1$.

The coordinate neighborhoods are the inverse image of those of B^1: $V_j = \eta^{-1} V_j^1$ the coordinate transformations are given by

$$g_{ji}(x) = g_{ji}^1(\eta(x)), \quad x \in V_i \cap V_j .$$

It can be seen that this construction defines a unique bundle.

Let M be an m-manifold, $m \in M$. Let $Y \cong R^m$, and let L be the group of linear transformations of R^m. Let $T(M) = \bigcup_{m \in M} T_m(M)$. Then there is a vector bundle $(T(M), M, R^n, p)$ with group structure L where $p : T(M) \to M$ assigns to each tangent vector its origin, called the tangent bundle. The system of coordinates constructed for this bundle is made up of maps $\psi_j^1 : U_j \times R^n \to T(M)$ defined by $\psi^1(u, y) = \phi_j(\psi_j(u), y)$, where $\{U_j, \psi_j\}$ are a system of coordinates charts for the manifold M, and where ϕ_j is the coordinate function (see [42], pp. 15).

Let M be a compact manifold of dimension n. Let E be a vector bundle of M and let $\Pi_M : E \to M$ denote the canonical projection. Then for each $m \in M$, $\Pi_M^{-1}(m) \cong R^m$ for some m and there exists a finite open cover $\{U_i\}$ of M such that each U_i is a chart of M and $\Pi_M^{-1}(U_i) \cong U_i \times R^m$ for each i. Such a cover is called a trivialization (see [42]). A section of a vector bundle E of M is a map $h : M \to E$ such that $\Pi_M \circ h = id_M$.

In view of the above, a section of E can locally be thought of as a map from $R^n \to R^m$, where n is the dimension of M. One can thus put a Hilbert structure locally on the space of sections of E whose derivatives up to order s are in L_2 (denoted $H^s(E)$) as defined above. In view of the compactness of M one can check that the definition of $H^s(E)$ is independent of the trivialization for $s > n/2$, see [29]. However, the Hilbert space structure depends on the choice of charts, and although the space $H^s(E)$ is well defined, the norms may vary with the choice of charts, (i. e., they are not canonical) so that $H^s(E)$ is called a <u>Hilbert manifold</u> or Hilbertible space, i. e., a space on which some complete inner product exists.

The Sobolev theorem has an analog for $H^s(E)$. Let M and N be compact manifolds. Let n be the dimension of M, and ℓ the dimension of N. We say that $f \in H^s(M, N)$ if for any $m \in M$ and any chart (U, φ) containing m and any chart (V, ψ) of f(m) in N, the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to R^\ell$ is in $H^s(\varphi(U), R^\ell)$. If $s > n/2$ this can be shown to be a well defined notion independent of the choice of charts (see for instance [28], [29]).

<u>Manifold of maps.</u> Let M and N be compact Riemannian manifolds, i. e., manifolds with a compatible metric, defined on the tangent spaces [44].

We now sketch the definition of a <u>manifold structure for $H^s(M, N)$</u> (see for instance [28], [29] and [34]).

This is done by first finding charts for $H^s(M, N)$ and then showing that the changes of charts are well defined (i. e., map into the right spaces).

Let $f \in H^s(M, N)$. We first define $T_f H^s(M, N)$, the <u>tangent space at f of $H^s(M, N)$</u> to be the linear space of sections X from M to TN (TN can be identified with the vector bundle $f^*(TN)$, the "pull back" of TN by $f : M \to N$, which is a bundle over M, as defined above) which satisfy $\Pi_N \circ X = f$, i. e., X is a section that maps $m \in M$ into a vector v which is in the tangent space of $n = f(m)$.

So formally

$$T_f H^s(M, N) = \{X \in H^s(M, TN) : \Pi_N \circ X = f\}$$

there is a map denoted $\overline{\exp_f}$ that maps the linear space $T_f H^s(M, N)$ onto a neighborhood of f in $H^s(M, N)$, taking 0 to f, and thus can define a chart in $H^s(M, N)$. We sketch the definition of this map.

If $v_p \in T_pN$, there is a unique geodesic σ_{v_p} through p whose tangent vector at p is v_p. Then $\exp_p(v_p) = \sigma_{v_p}(1)$. In general, $\exp_p$ is a diffeomorphism from some neighborhood of 0 in T_pN onto a neighborhood of p in N. If N is compact, $\exp_p$ is defined over all of T_pN. This map can be extended to a map $\exp : TN \to N$ such that if $v_g \in TN$ then $\exp(v_g) = \exp_p(v_g)$. We now define $\overline{\exp_f}: T_f H^s(M, N) \to H^s(M, N)$ by $X \to \exp \circ X$. It can be seen that $\overline{\exp_f}$ maps the linear space $T_f H^s(M, N)$ onto a neighborhood of f in $H^s(M, N)$ taking 0 to f and hence it gives a chart for $H^s(M, N)$ at f.

It can be seen that inspite of the use of the geodesics σ_p for the definition of $\overline{\exp}$, the structure is independent of the metric, and also that the changes of charts are well defined and smooth (see [28] and [29]).

Transversality (See, for instance [1])

Let M_1 and M_2 be C^1 submanifolds of a C^1 manifold X, and $x \in X$ a point. We say M_1 and M_2 are transversal at x, in symbols, $M_1 \pitchfork M_2$ at x, if and only if either $x \notin M_1 \cap M_2$ or $x \in M_1 \cap M_2$ and $T_xX = T_xM_1 + T_xM_2$. We say M_1 is tranversal to M_2, $M_1 \pitchfork M_2$ if and only if $M_1 \pitchfork M_2$ at all x in X. Let X and Y be C^1 manifolds and $f: X \to Y$ a C^1 map, and $M \subset Y$ a submanifold, X and Y finite dimensional. We say that f is transversal to M at a point x in X, in symbols, $f \pitchfork M$ at x if and only if where $y = f(x)$, either $y \notin M$ or $y \in M$ and the inverse image of $(T_xf)(T_xX)$ contains a complement to T_yM in T_yY. We say f is transversal to M, in symbols $f \pitchfork M$ if and only if $f \pitchfork M$ at x, for all $x \in X$.

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13. ABSTRACT

Local and global properties of spaces of preferences are studied, with applications to general equilibrium, utility and demand analysis. Spaces of smooth, not necessarily convex or increasing preferences are proven to be representable as differentiable Hilbert manifolds. These structures of spaces of preferences are then used to extend results on the regularity of equilibria to economies where the agents are described by their preferences and endowments. Subspaces of preferences which give foliations of the commodity space and also subspaces of convex and increasing smooth preferences are shown to be submanifolds. Topological properties of these manifolds, and local and global properties of the demands and the utilities of the agents in relation to the underlying preferences are also studied.