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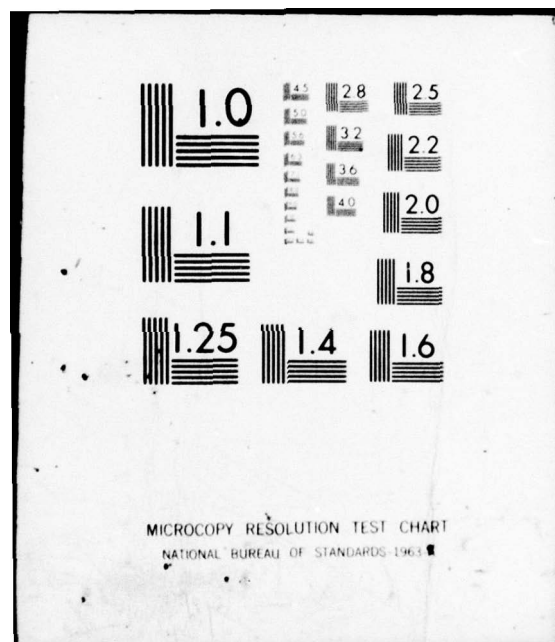
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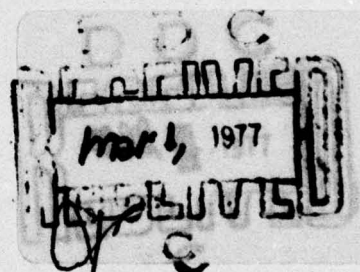
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PROBLEM OF CALCULATING GROUND FREEZING

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A.I. Pekhovich



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Calculating ground freezing is of considerable theoretical and practical interest; thus, knowledge of the depth of freezing is necessary in solving numerous problems in construction and agriculture, for instance: determining the depth at which water-supply and sewage pipes are laid, determining depths for laying the foundations of buildings, the initial depths for digging drains when carrying out reclamation projects, etc. Therefore, during the last 20 years a number of studies have appeared devoted to this problem.

This article gives an analysis of some of the existing methods of calculation; in this process we have selected only those which at the present time should be recommended for engineering and whose area of application of each method is indicated. In addition, new computation functions have been derived which should be used in a number of cases associated with a considerable thermal influx from thawed ground.

1. Some Observations on Existing Computation Methods

We will examine theoretical solutions to the problem of ground freezing as given in the works of G. Gerber, N. N. Petrunichev and G. S. Shadrin, I. A. Charnyy and B. V. Proskuryakov.

Greber [1] formulates the problem which was previously solved Stefan, as follows:¹ "at the initial moment in time all points of a humid soil have temperature T . Beginning at a certain moment in time, on the soil surface constant temperature θ is always maintained. Freezing of the soil takes place at temperature t_0 . In order to freeze one volumetric unit of soil, a quantity of heat equal to w is required. We will find an equation for the temperature field and the speed with which the freezing boundary moves" (see Figure 1).

The solution has the following form:

$$x = q\sqrt{\tau}, \quad (1)$$

where x is the depth of freezing,

τ is time,

q is a coefficient.

In order to find the coefficient q which determines the freezing speed, it is necessary to solve the following equation:

¹The designations are shown in accordance with those adopted in this article.

$$q = k_1 \frac{e^{-\frac{q^2}{4a_1}}}{G\left(\frac{q}{\sqrt{4a_1}}\right)} - k_2 \frac{e^{-\frac{q^2}{4a_2}}}{1 - G\left(\frac{q}{\sqrt{4a_2}}\right)},^2 \quad (2)$$

where

$$k_1 = \frac{2\sqrt{\lambda_1 c_1 \gamma_1}}{w\sqrt{\pi}} (t_0 - \theta), \quad (3)$$

$$k_2 = \frac{2\sqrt{\lambda_2 c_2 \gamma_2}}{w\sqrt{\pi}} (T - t_0). \quad (4)$$

In equations (1), (2), (3) and (4) the letters λ , c , γ and a designate the thermal conductivity, heat capacity, specific weight and temperature conductivity of the ground; index "1" refers to frozen ground, and index "2" refers to thawed ground.

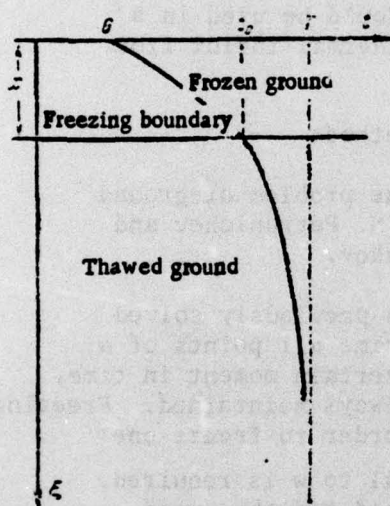


Figure 1.

The following should be noted about the above-cited solution to the problem of ground freezing:

1. The solution is given for conditions which considerably restrict its practical application; in particular, the solution is not applicable in the presence of one of the following conditions:

- a) in the presence of a snow covering,
- b) if during the freezing process the temperature of the soil surface changes,
- c) in the presence of a filtration flow.

2. For practical purposes it is necessary to introduce the assumption that the temperature of the soil surface is equal to the air temperature.

3. The performance of the calculations is associated with cumbersome mathematical computations which are required to determine coefficient q .

4. Within the boundaries of the stated conditions the problem solution is analytical. Therefore this method should be used wherever there is no snow covering or filtration flow and where the change in air temperature can be averaged in time and the initial temperature of the ground can be averaged in depth, and these factors can be assumed to be constant in the calculation.

²In his derivation Greber made an error which leads to a plus sign in front of the last term in the equation. We draw the reader's attention to this fact especially because this error has been consistently repeated up to the present time by other authors.

In order to overcome the difficulty associated with determining coefficient q , we propose to make use of nomograms which we have devised (see Figures 2, 3 and 4).

The unknown proportionality coefficient q is presented in the form of two terms:

$$q = q_1 - q_2, \quad (5)$$

From equation (2) it follows that:

$$q_1 = k_1 \frac{e^{-M_1^2}}{G(M_1)}; \quad (6)$$

$$q_2 = k_2 \frac{e^{-M_2^2}}{1 - G(M_2)}, \quad (7)$$

where

$$M_1 = \frac{q}{\sqrt{4a_1}}; \quad (8)$$

$$M_2 = \frac{q}{\sqrt{4a_2}}. \quad (9)$$

From these two equations, we find

$$M_2 = \sqrt{\frac{a_1}{a_2}} \cdot M_1; \quad (10)$$

$$q = \sqrt{4a_1} \cdot M_1. \quad (11)$$

The nomogram in Figure 2 relates to the case of ground freezing in the absence of heat flow from the thawed ground, i.e., when $T = t_0$ and consequently $q_2 = 0$. The nomogram in Figure 3 relates to the case where the ground thaws under the action of the heat influx from the thawed ground in the absence of a thermal flow in the frozen ground, i.e., when $\theta = t_0$ and $q_1 = 0$. The nomogram in Figure 4 relates to the general case, i.e., when there are thermal flows in both the frozen and thawed ground, when $\theta \neq t_0 \neq T$. The method of utilizing the nomograms in Figures 2 and 3 and the principles of their construction do not require explanation. They should be used when q_1 is considerably greater (or less) than q_2 . With regard to the nomogram in Figure 4, some remarks are in order.

Figure 4-a (the attachment) shows two families of curves: one curve is plotted according to equation (6), and the second is plotted according to equation (7). In Figure 4-b (attachment at the end of the book) there are also two families of curves: one is plotted according to equation (10), and the second according to equation (11). The value of the unknown coefficient q is determined in the following fashion.

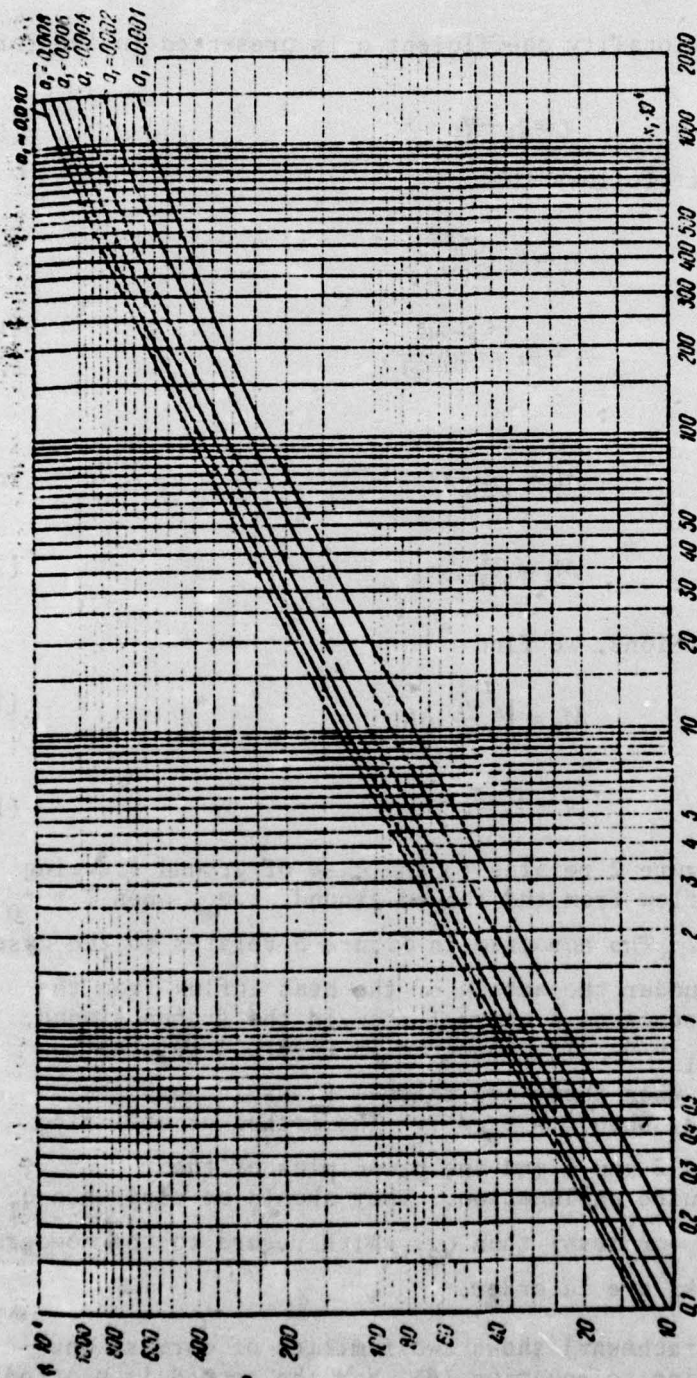


Figure 2. Freezing-Thermal Influx from Frozen Ground Is Equal to Zero:

$$q_1 = k, \frac{q_1}{x} = \frac{q_1}{x} \cdot \frac{1}{1 + \frac{q_1}{kx}}$$

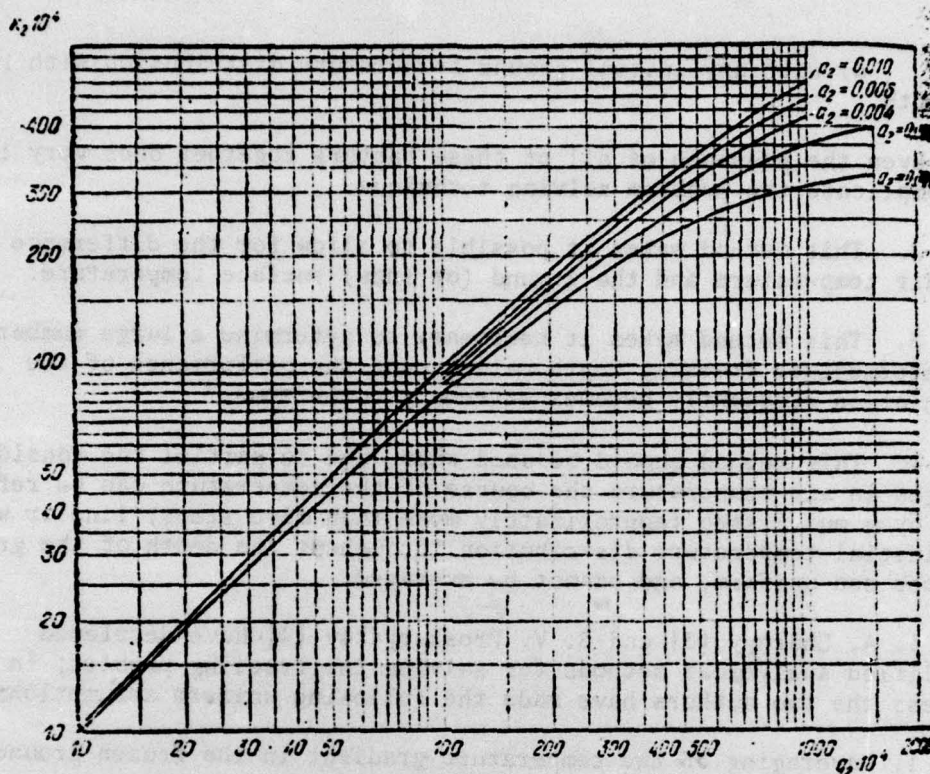


Figure 3. Thawing -- Thermal Influx Only From Thawed Ground:

$$q_2 = k_2 \frac{e^{-\frac{q_2}{k_2 a_2}}}{1 - G\left(\frac{q_2}{\sqrt{4} a_2}\right)}; \quad \text{if } k_2 < 0.001, \text{ then } q_2 = k_2.$$

We use the initial data to find the values k_1 , k_2 , a_1 and a_1/a_2 . We will set an arbitrary value of M_1 , and then in Figure 4-b we will determine the values q and M_2 ; then, in Figure 4-a we find the values: q_1 , corresponding to M_1 and q_2 corresponding to M_2 . [Translator's Note: one word obscured] difference should be equal to q ; when these values do not coincide, it is necessary to select another value for M_1 .

N. N. Petrunichev and G. S. Shadrin solved the freezing problem by utilizing the method of finite differences [2]. Let us note the following about this solution:

1. this method can be applied not only for the conditions of the problem formulated by Greber, but also

- a) in the presence of a snow-covering,
- b) if during freezing the air temperature changes,
- c) in the presence of a filtration flow,

d) with any initial ground temperature distribution with regard to depth.

Even the presence of all of these factors together does very little to complicate the problem solving technique.

2. This method makes it possible to allow for the difference between the air temperature and the ground (or snow) surface temperature.

3. This method makes it necessary to determine a large number of intermediate freezing depth values, and the performance of the calculation frequently becomes extremely cumbersome.³

4. This method should be used when, due to partial and considerable changes in air temperature the course of the temperature can be reflected only by a multi-step (approximately more than five steps) line or when the initial temperature distribution throughout the depth of the ground becomes non-uniform, and cannot be averaged.

I. A. Charnyy [3] and B. V. Proskuryakov [4] have developed simplified analytical methods for solving the freezing problem; in this process the two authors have made the following uniform assumptions:

1. averaging of the temperature gradient in the frozen ground;

2. when calculating the advance of the freezing boundary the depth of freezing x_1 is determined which occurs in the absence of a thermal influx from the thawed ground; then layer x_2 , which thaws under the action of the thermal influx from the thawed ground, is subtracted, i.e.,

$$x = x_1 - x_2. \quad (12)$$

The basic difference in performing the calculation according to the method of I. A. Charnyy and D. V. Proskuryakov arises in connection with determining x_1 . The heat balance equation which is related to a vertical section of ground with a cross-section area equal to unity, according to Charnyy has the following form:

$$Q_f = Q_0 - Q_g - Q_h, \quad (13)$$

and according to Proskuryakov it has the form

$$Q_f = Q_0 - Q_h, \quad (14)$$

³This drawback is alleviated by utilizing Petrunichev's proposed method of performing tabular integration of the Fourier equation instead of the graphic method which is used; this makes the performance of the computation somewhat faster and more convenient.

where Q_f is the quantity of heat liberated when the ground within the limit of the section freezes,

Q_o is the quantity of heat which escapes to the atmosphere,

Q_g is the change in the heat content of the frozen ground,

Q_h is the thermal influx from the thawed ground.

In solving the equation, Charnyy obtained:

$$x_1 = \sqrt{\frac{-\lambda_1(\theta - t_a)}{w - \gamma_1 c_1 \frac{(\theta - t_a)}{2}} (\tau - \tau_0) + x_0^2}. \quad (15)$$

Proskuryakov found:

$$x_1 = \sqrt{\frac{-2\lambda_1(\theta - t_a)(\tau - \tau_0)}{w} + (\beta + x_0)^2} - \beta, \quad (16)$$

where

$$\beta = h \frac{\lambda_1}{\lambda_s} - \frac{\lambda_1}{a}, \quad (17)$$

where h is the thickness of the snow covering,

λ_s is the thermal conductivity of the snow,

a is the coefficient of snow-air thermal emission,

x_0 is the initial depth of freezing.

It must be emphasized that the expressions for Q_0 are not identical in equations (13) and (14). This is due to the fact that according to Charnyy and Greber, θ is the temperature of the ground surface⁴; according to Proskuryakov θ is the temperature of the air and the thermal resistance of the snow and the inequality between the temperature of the snow surface and that of the air are taken into account.

In the absence of a filtration flow Charnyy and Proskuryakov determined [Translator's Note: word obscured] by the same means, using the solution from the theory of heat transmission to the problem of a loss of heat by a rod which is limited on one side and which at the initial moment of time has a uniform temperature at all its points and when there is no heat emission from its lateral surface; on this assumption the authors obtain:

"In his work Charnyy made no special examination of ground freezing, but rather "the advance of a flat division boundary between two phases"; therefore it is natural that the problem of what influence the snow covering had was not stated there.

$$x_2 = k_2(\sqrt{\tau} - \sqrt{\tau_0}). \quad (18)$$

For the case of a filtration flow Proskuryakov found:

$$x_2 = \frac{1}{2} k_2 \sqrt{\frac{v}{y}} (\tau - \tau_0), \quad (19)$$

where y is the length of the filtration flow path under the frozen ground to the point where the depth of freezing is determined;

v is the speed of the filtration flow's motion.

With regard to these two assumptions, we should point out:

1. First assumption, averaging of the temperature gradient in the frozen ground, causes an increased temperature gradient at the freezing boundary and a reduction of this gradient at the ground surface; the value Q_f which is calculated according to heat balance equation (13) is reduced, but according to equation (14) this value is increased. As a result computation formula (15) gives understated values of x , and formula (16) gives exaggerated results; it is known only that the true values are found within these limits which also limit the possible error. As calculations show, this error does not exceed 30% for practical purposes. Taking into account the usual imprecision of initial computation data, we must acknowledge that the introduction of this assumption is completely justified.

2. With regard to the second assumption which is expressed by equation (12), it is obvious that it causes some increase in the influence of the thermal flow from the thawed ground; a qualitative evaluation of this factor is difficult, and therefore we will return to this problem later after we obtain new computation equations which are derived without the aid of this assumption.

With regard to determining the freezing of ground by means of simplified analytical methods, the following should be said:

1. The methods can be applied if the initial ground temperature distribution can be averaged in depth and if the course of the air temperature is replaced by a staggered line within the limits of three-five steps.

2. Formulas (16) and (18) make it possible to perform calculations in the presence of a snow layer and with allowance for the presence of a difference between the temperature of the ground surface and the temperature of the air. In case there is a filtration flow, formula (18) should be replaced by formula (19).

3. In case there is no snow covering and the temperature of the soil surface is equal to the air temperature, it is possible to replace formula (16) with formula (15). As has already been indicated, computation according to formula (16) gives somewhat exaggerated results, and computation according to formula (15) provides correspondingly reduced values of the freezing depth.

In solving the problem of freezing in the presence of a snow covering and with allowance for a difference between the air temperature and the temperature on the ground surface, as is done by Proskuryakov, but in contrast to him, by utilizing equation (13) we obtain:

$$\tau - \tau_0 = A(x_1^2 - x_0^2) + N(x_1 - x_0) + R \ln \frac{x_0 + \beta}{x_1 + \beta}, \quad (20)$$

where

$$A = \frac{w}{2\lambda_1(\theta - t_0)}; \quad (21)$$

$$N = \frac{\beta}{2a_1} + 2A\beta; \quad (22)$$

$$R = \frac{c_s \gamma_s h \left(\beta + \frac{\lambda_1}{a} \right)}{2\lambda_1}. \quad (23)$$

We neglected to derive formula (20) since in principle it is similar to the course of the derivations carried out by Proskuryakov and Charnyy when they obtain formulas (15) and (16); let us simply point out that for us

$$Q_g = - \frac{c_1 \gamma_1 (\theta - t_0)}{2} \left[\frac{\beta}{(x + \beta)^2} x + \frac{x}{x + \beta} \right] dx - \frac{c_s \gamma_s h (\theta - t_0) \left(\beta + \frac{\lambda_1}{a} \right)}{2(x + \beta)^2} dx. \quad (24)$$

Like equation (15) equation (20) provides smaller values of x_1 , but in contrast to equation (15) the problem is solved with allowance for the difference between the air temperature and the temperature on the ground surface, as well as in the presence of a snow covering.

11. Deriving Refined Computation Formulas

The simplified analytical method cannot be used in some cases since sometimes it is not adequately precise. If we proceed on the basis of heat balance equation (14) and do not introduce assumption (12), then the formulas obtained should precisely allow for the action of the heat flow from the thawed ground. Otherwise all of the advantages and disadvantages of the above-described simplified analytical method will also be noted here.

Let us examine two possible cases. First case. The problem can be formulated as follows.

At the initial moment of time the moist ground has temperature T at all points. The surface of the ground is covered with a layer of snow of thickness h . Beginning at a certain moment in time, the air temperature is kept equal to θ . The ground freezes at temperature t_0 .

A quantity of heat equal to w is required to freeze a volumetric unit of ground. The physical constants of the ground and the snow and the snow-air thermal emission factor are given. It is necessary to determine the speed with which the freezing boundary moves. Under these conditions

the intensity of the thermal influx from the thawed ground q_T is inversely proportional to time.

The heat influx term from the frozen ground will be:

$$Q_f = q_f d\tau, \quad (25)$$

where

$$q_f = \sqrt{\frac{\lambda_2 c_2 \rho_2}{\pi}} \cdot \frac{(T - t_0)}{\sqrt{\tau}}. \quad (26)$$

The other terms of heat balance have the following form:

$$Q_f = w dx, \quad (27)$$

$$Q_0 = -\lambda_1 \frac{\theta - t_0}{x + \beta} d\tau. \quad (28)$$

The latter equation allows only for convection heat exchange on the snow surface; however, it is not particularly difficult also to make allowance for the terms of solar radiation R and the radiation of the snow surface S . For this purpose it is necessary in all subsequent computations to replace the true temperature of the air θ by a fictitious one which is calculated according to the equation:

$$\theta_f = \theta + \frac{R - S}{a}.$$

By substituting expressions (25), (26), (27) and (28) into equation (14), after transformation we obtain:

$$dx + \frac{k_2}{2} \cdot \frac{d\tau}{\sqrt{\tau}} + \frac{k_2}{2} \cdot \frac{d\tau}{(x + \beta)} = 0, \quad (29)$$

for k_2 , see expression (4).

$$k_2 = \frac{2\lambda_1(\theta - t_0)}{w}. \quad (30)$$

We will introduce a new variable

$$r = \frac{\tau^{1/2}}{x + \beta}; \quad (31)$$

then

$$\tau = r^2(x + \beta)^2, \quad (32)$$

$$d\tau = 2r(x + \beta)^2 dr + 2r^2(x + \beta) dx. \quad (33)$$

By substituting (32) and (33) into (29), after simple transformations we will have

$$\frac{dx}{x+\beta} = -\frac{k_2+k_3r}{1+k_2r+k_3r^2} dr.$$

By integrating this equation with allowance for the fact that $4k_3 - k_2^2 < 0$:

$$\ln(x+\beta) = \ln C \frac{\left(\frac{2k_2r+k_2-\sqrt{k_2^2-4k_3}}{2k_2r+k_2+\sqrt{k_2^2-4k_3}} \right)^{-\frac{k_3}{k_2^2-4k_3}}}{\sqrt{1+k_2r+k_3r^2}};$$

from this we find:

$$x = C \frac{1}{\sqrt{(r+p)(r+n)}} \left(\frac{r+p}{r+n} \right)^m - \beta, \quad (34)$$

where

$$n = \frac{k_2}{2k_3} \left(1 - \sqrt{1 - \frac{4k_3}{k_2^2}} \right); \quad (35)$$

$$p = \frac{k_2}{2k_3} \left(1 + \sqrt{1 - \frac{4k_3}{k_2^2}} \right); \quad (36)$$

$$m = \frac{1}{2\sqrt{1 - \frac{4k_3}{k_2^2}}}. \quad (37)$$

The integration constant is determined from the following initial condition; when $\tau = \tau_0$, we have

$$x = x_0, \\ r_0 = \frac{x_0^{3/2}}{x_0 + \beta};$$

from this we find

$$C = (x_0 + \beta) \sqrt{(r_0 + p)(r_0 + n)} \cdot \left(\frac{r_0 + n}{r_0 + p} \right)^m. \quad (38)$$

By substituting the expression for the integration constant into equation ([Translator's Note: number obscured]) we finally find:

$$x = (x_0 + \beta) \frac{\sqrt{(r_0 + p)(r_0 + n)}}{\sqrt{(r + p)(r + n)}} \left(\frac{r_0 + n}{r_0 + p} \right)^m \left(\frac{r + p}{r + n} \right)^m - \beta. \quad (39)$$

Second case. The problem conditions are the same as in the first case, but the intensity of the heat influx from the thawed ground has a constant value like, for instance [word obscured] by means of the filtration flow from the lower surface of the frozen ground, as

$$q_t = \frac{\sqrt{\lambda_2 c_2 \gamma_2} \sqrt{\frac{\theta}{y}}}{\sqrt{\pi}} (T - t_2). \quad (40)$$

Then heat balance equation (14) will have the form

$$\lambda_1 \frac{\theta - t_0}{x + \beta} d\tau + w dx + q_t d\tau = 0,$$

or

$$d\tau = -w \frac{x + \beta}{A + q_t x} dx, \quad (41)$$

where

$$A = \lambda_1(\theta - t_0) + q_t \beta. \quad (42)$$

We integrate equation (41):

$$\tau = -\frac{w}{q_t} x - w \frac{\beta q_t - A}{q_t^2} \ln(A + q_t x) + C. \quad (43)$$

We determine the integration constant from the initial condition: when $\tau = \tau_0$, we have $x = x_0$; consequently

$$C = \tau_0 + \frac{w}{q_t} x_0 + w \frac{\beta q_t - A}{q_t^2} \ln(A + q_t x_0).$$

Finally we find:

$$\tau = w \frac{(\beta q_t - A)}{q_t^2} \ln \frac{A + q_t x_0}{A + q_t x} - \frac{w}{q_t} (x - x_0) + \tau_0. \quad (44)$$

By utilizing equations (39) and (44) and equations (16), (18) and (19) of the simplified analytical method, we perform numerous calculations which showed that the introduction of computations carried out according to equations (39) and (44) refined the solution by no more than

$$\eta = \frac{x_2}{x_1 - x_2} 100\%. \quad (45)$$

In actuality the refinement is less significant, but the latter can only be shown by pairing the results of calculations carried out according to both methods. Consequently, if for example $\eta > 20\%$, then the calculations have to be carried out according to equations (39) or (44), although the refinement may also prove to be small.

III. Areas of Application for the Computation Methods

The selection of the computation method which should be used depends on the specific given conditions under which freezing takes place and on the required precision of the problem solution.

We can recommend:

a) the analytic method -- formulas (1) and (2) (for simplified boundary conditions): in case there is no snow covering, there is no

filtration flow, when the air temperature can be averaged in time and assumed to be constant in the calculation and when the initial ground temperature can also be averaged in depth.

b) Analytical method -- formulas (39) and (44) (for refined boundary conditions): in case the initial ground temperature can be averaged in depth but the temperature current of the air is replaced by a line which has no more than 3-5 plateaus, and if the value of factor η calculated according to equation (45) exceeds 20%.

c) Simplify analytical method -- formulas (16), (18), (19) and (20) (for refined boundary conditions): if $\eta \leq 20\%$. The rest is the same as in item b.

d. Method of finite differences: in cases of special and significant changes in air temperature, where the temperature curve can be replaced only by a multi-stage curve (more than five plateaus) or when the initial ground temperature cannot be averaged in depth.

IV. Example of Calculating Ground Freezing

The following are given:

$$\lambda_1 = \lambda_2 = 1 \frac{\text{kcal}}{\text{m} \cdot \text{hr} \cdot ^\circ\text{C}}; \quad c_2 = 1 \frac{\text{kcal}}{^\circ\text{C} \cdot \text{kg}}; \quad \gamma_2 = 1000 \frac{\text{kg}}{\text{m}^3}; \quad \beta = 0.54 \text{ m};$$

$$w = 24000 \frac{\text{kcal}}{\text{m}^3}; \quad \theta = -22^\circ\text{C}; \quad T = 3^\circ\text{C}; \quad t_0 = 0^\circ\text{C}; \quad y = 100 \text{ m};$$

$$v = 0.85 \frac{\text{m}}{\text{hr}}; \quad \tau = 210 \text{ hrs}; \quad \tau_0 = 0; \quad x_0 = 0.$$

According to equation (4) we find [the thermal capacity and specific weight of water should be substituted into expressions (4) and (40)]:

$$k_2 = \frac{2\sqrt{1 \cdot 1 \cdot 1000}}{24000 \cdot 1.73} \cdot 3 = 0.0046 \frac{\text{m}}{\text{hr}^{1/2}}.$$

[Translator's Note: due to the fact that the equation numbers are obscured in the foreign text on this page, they will not be given].

If we carry out calculation according to the simplified analytical method, i.e., according to formulas (12), (16), and (19), we obtain

$$x_1 = \sqrt{0.54^2 + \frac{2 \cdot 1 \cdot 22 \cdot 210}{24000}} - 0.54 = 0.28 \text{ m};$$

$$x_2 = \frac{0.0046}{2} \sqrt{\frac{0.85}{100}} \cdot 210 = 0.04 \text{ m};$$

$$x = 0.28 - 0.04 = 0.24 \text{ m}.$$

By utilizing expression (45), we find

$$\eta = \frac{0.04}{0.28 - 0.04} 100 = 16.7\%.$$

Consequently, there is no need to refine the result obtained (we simply point out that the refinement would prove to be equal to 13%).

If, however, we adopt in the same example: $v = 1.7$ m/hr and $y = 20$ m, we obtain

$$x_1 = 0.28 \text{ m}; \quad x_2 = \frac{0.0046}{2} \sqrt{\frac{1.7}{20}} 210 = 0.14 \text{ m};$$

$$x = 0.28 - 0.14 = 0.14 \text{ m};$$

then:

$$\eta = \frac{0.14}{0.28 - 0.14} \cdot 100 = 100\%.$$

In this case it is necessary to utilize a more precise solution.

First, according to equations (40) and (42) we determine:

$$q_t = \frac{\sqrt{1 \cdot 1 \cdot 1000}}{1.73} \sqrt{\frac{1.7}{20}} 3 = 15.9 \text{ kcal/m}^2 \cdot \text{hr};$$

$$A = -1.22 + 15.9 \cdot 0.54 = -13.4 \text{ kcal/m} \cdot \text{hr}.$$

According to computation equation (44) we find:

$$\tau = 24000 \frac{0.54 \cdot 15.9 + 13.4}{15.9^2} \ln \frac{-13.4}{-13.4 - 15.9 \cdot 0.14} - \frac{24000}{15.9} 0.14 = 170 \text{ hrs.}$$

The refinement proved to be equal to:

$$\frac{210 - 170}{170} \cdot 100 = 24\%.$$

In conclusion it seems appropriate to us to note that in his work the author was provided by many valuable comments by B. V. Proskuryakov and N. N. Petrunichev.

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