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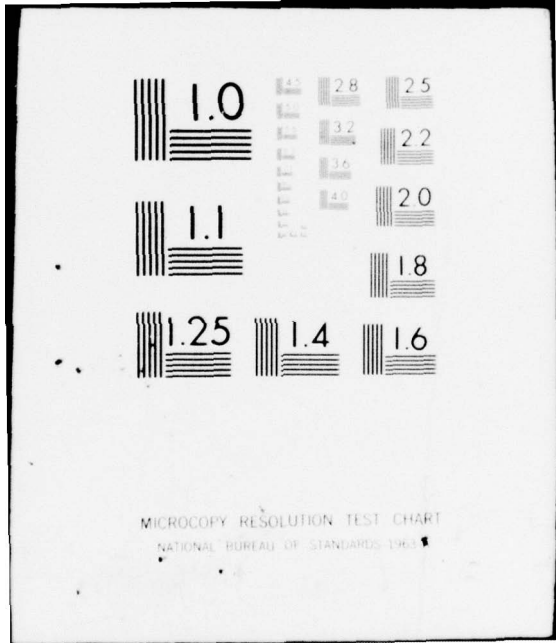
SINGULAR SINGULAR-PERTURBATION PROBLEMS

by

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Introduction

Let us consider the nonlinear system

(1) $$\epsilon \dot{z} = f(z, t, \epsilon)$$

of M nonlinear equations on some finite subinterval of $t \geq 0$ subject to the initial condition

(2) $$z(0) = z^0(\epsilon)$$

in the limit as the small positive parameter $\epsilon \to 0$. Any acquaintance with singular perturbations would lead one to expect that the unique solution of this initial value problem might converge as $\epsilon \to 0$ to a solution Z_0 of the limiting equation

(3) $$f(Z_0, t, 0) = 0$$

away from any "boundary layer" regions of nonuniform convergence. In the singular situation that $f_z(Z_0, t, 0)$ is a singular matrix, this limiting equation has an infinite number of solutions, so the limiting equation (3) is not adequate for the purpose of determining the reduced problem satisfied by the limiting solution (assuming that such a limit exists). Traditional methods apply whenever f_z is everywhere stable. The solution to

(1)-(2) then converges to the unique solution of (3) away from the initial point $t = 0$ (cf., e.g., O'Malley (1974)). Indeed, the solution can then be uniformly asymptotically represented as the sum of a function of t and a decaying function of the stretched variable $\tau = t/\epsilon$.

Many singular perturbation problems can be written in the form (1)-(2). Indeed, partial differential equations are often disguised as such ordinary differential equations in appropriate Banach spaces (cf. Friedman (1969)) and boundary value problems for ordinary differential equations are often conveniently studied via shooting techniques (i.e., by using solutions of auxiliary initial value problems) (cf. Bernfeld and Lakshmikantham (1974)). Singular problems (i.e., those where f_z is singular) occur throughout the literature, though they do not seem to have been analyzed in a systematic fashion. We note that Fife (1972) studied problems "whose degenerate forms have many solutions". His work was motivated by the need to analyze the dispersion of flow in small tubes (cf. Fife and Nicholes (1975)). Likewise, Gordon (1975) considered the quasilinear problem (1) in a Banach space where the matrix $f_z(z,t,0) = f_z(0,t,0)$ had a one-dimensional null space. His applications include a nonlinear stability analysis of the convection state in a Benard problem (cf., Gordon and Hoppensteadt (19)). Our interest was generated by studying singular arc problems in optimal control (cf. O'Malley (1976)). The solution of these singular singular-perturbation problems involves projections and a resulting reduction to certain algebraic systems and lower dimensional systems of differential equations, both for the outer solution (asymptotically valid away from regions of nonuniform convergence) and for the boundary layer corrections (which allow nonuniform convergence). Such considerations are basic to our general plan of attack and to the related approach of Butuzov and Vasil'eva (1970) and Vasil'eva (1975).

We shall consider the initial value problem (1)-(2) under the primary assumptions that the matrix

$$(4) \qquad\qquad f_z(z,t,0) \equiv f_{0z}(z,t)$$

has constant rank k, $0 < k < M$, that its null space is spanned by $M - k$ linearly independent eigenvectors, and that f_z has k stable eigenvalues (counting multiplicities) for all z and t. Assuming sufficient differentiability of f and asymptotic expansions for both f and z^0, we shall attempt to construct an asymptotic solution of the form

$$(5) \qquad\qquad z(t,\epsilon) = Z(t,\epsilon) + \Pi(\tau,\epsilon)$$

where Z and Π have asymptotic power series expansions in ϵ and the terms of Π all tend to zero as the stretched variable

$$(6) \qquad\qquad \tau = t/\epsilon$$

tends to infinity. Since $Z(t,\epsilon)$ provides the asymptotic solution for $t > 0$, it is called the outer solution, while $\Pi(\tau,\epsilon)$ is called the boundary layer correction since it is asymptotically negligible away from the initial region of nonuniform convergence. Simple examples show that the form of solution (5) would differ if there were a deficiency of eigenvectors in the null space of f_{0z} (cf. Butuzov and Vasil'eva (1970)).

To proceed, it is helpful to realize that the matrix $f_{0z}(z,t)$ can be put into its row-reduced echelon form by an orthogonal matrix $E(z,t)$. Thus, we have

$$(7) \qquad\qquad E f_{0z} = \begin{bmatrix} U \\ 0 \end{bmatrix}$$

row-reduced with U being a $k \times M$ matrix of rank k. (Ever since Levin (1957), it has been common in the singular perturbations literature to use block diagonalization in such situations. It seems more convenient, however, to instead reduce to triangular form. For purposes of efficient computation, we note that it is important to use iterative procedures to compute E (cf. Golub (1965)). Defining

$$(8) \qquad\qquad A = (0 \ I_{M-k})E \quad \text{and} \quad B = (I_k \ 0)E,$$

the matrix $A(z,t)$ annihilates f_{0z} and $B(z,t)$ is its orthogonal complement, i.e.,

$$(9) \qquad\qquad A f_{0z} = 0, \quad AB' = 0, \quad AA' = I_{M-k},$$

$$BB' = I_k \quad \text{and} \quad A'A + B'B = I_M.$$

Such matrices are convenient since they allow us to represent any M vector w in the form $w = A'u + B'v$ and separately examine $u = Aw$ and $v = Bw$. (We note that Clasen et al. (1976) used such matrices to develop numerical schemes for solving stiff differential equations.) It is important to

realize that our assumptions on f_{0z} imply that the matrix

$$(10) \qquad\qquad S = Bf_{0z}B'$$

is stable, hence nonsingular, because

$$Ef_{0z}E' = \begin{bmatrix} Bf_{0z}B' & Bf_{0z}A' \\ 0 & 0 \end{bmatrix}$$

has k eigenvalues with negative real parts. The stability condition (10) will be especially critical in obtaining the boundary layer behavior needed at $t = 0$.

We will completely solve several large classes of problems, but the original problem must be studied further. We shall show how the outer solution and the boundary layer correction can be generally obtained, but our matching procedure for determining $Z(0,\epsilon)$ is not yet satisfactory.

The Outer Solution

Since the boundary layer correction Π is asymptotically negligible for $t > 0$, the representation (5) implies that the outer solution

$$(11) \qquad\qquad Z(t,\epsilon) \sim \sum_{j=0}^{\infty} Z_j(t)\epsilon^j$$

must asymptotically satisfy the system

$$(12) \qquad\qquad \epsilon\dot{Z} = f(Z,t,\epsilon) \sim \sum_{j=0}^{\infty} f_j(Z,t)\epsilon^j$$

for $t > 0$. When $\epsilon = 0$, then, we have the reduced system

$$(13) \qquad\qquad f_0(Z_0,t) = 0.$$

This fails to determine Z_0 because f_{0z} is singular. However, (9) and (13) imply that

$$(14) \qquad B(Z_0,t)f_0(A'(Z_0,t)A(Z_0,t)Z_0 + B'(Z_0,t)B(Z_0,t)Z_0,t) = 0,$$

and since the Jacobian S with respect to BZ_0 is nonsingular, we can determine $B(Z_0,t)Z_0$ in terms of the many remaining variables. We shall assume that the system (13) is consistent. (An example where the reduced

system is inconsistent is provided by $\epsilon \dot{z}_1 = -z_1 + b$, $\epsilon \dot{z}_2 = b_2$, $b_2 \neq 0$. Then, the solution is $O(\frac{1}{\epsilon})$.)

We shall obtain differential equations for Z_0 and later terms Z_j in a termwise fashion. Rewriting (12), we have

(15)
$$\epsilon \dot{Z} = B'(Z_0,t)B(Z_0,t)f_{0z}(Z_0,t)(Z - Z_0)$$

$$+ (f(Z,t,\epsilon) - f_{0z}(Z_0,t)(Z - Z_0)),$$

since $f_{0z} = B'Bf_{0z}$ by (9). Differentiating (12) with respect to t, however, provides the Lagrangian derivative

$$f_z(Z,t,\epsilon)\dot{Z} + f_t(Z,t,\epsilon) = \epsilon \ddot{Z}.$$

Thus, multiplication by B implies that

(16)
$$B(Z_0,t)f_{0z}(Z_0,t)\dot{Z} = -B(Z_0,t)f_t(Z,t,\epsilon) + B(Z_0,t)(f_{0z}(Z_0,t)$$

$$- f_z(Z,t,\epsilon))\dot{Z} + \epsilon B(Z_0,t)\ddot{Z}.$$

Using (15), (16), and the invertibility of S, then, implies that

(17)
$$\frac{1}{\epsilon}B(Z_0,t)f_{0z}(Z_0,t)(Z - Z_0)$$

$$= S^{-1}(Z_0,t)B(Z_0,t)\{-f_t(Z,t,\epsilon) + \epsilon \ddot{Z} + (f_{0z}(Z_0,t) - f_z(Z,t,\epsilon))\dot{Z}$$

$$- \frac{1}{\epsilon}f_{0z}(Z_0,t)(f(Z,t,\epsilon) - f_{0z}(Z_0,t)(Z - Z_0))\}.$$

Finally substituting into (15), we have

(18)
$$\dot{Z} = A(Z_0,t)[-f_t(Z,t,\epsilon) + \epsilon \ddot{Z} + (f_{0z}(Z_0,t) - f_z(Z,t,\epsilon))\dot{Z}]$$

$$+ \frac{1}{\epsilon}B(Z_0,t)[f(Z,t,\epsilon) - f_{0z}(Z_0,t)(Z - Z_0)]$$

where

$$A(Z_0,t) = B'(Z_0,t)S^{-1}(Z_0,t)B(Z_0,t)$$

(19)   and

$$B(Z_0,t) = I - A(Z_0,t)f_{0z}(Z_0,t).$$

We note that the projection $B = B^2$ satisfies $BB' = 0$ and $Bf_{0z}B = 0$.

When $\epsilon = 0$, then, we find that Z_0 must satisfy the nonlinear equation

$$(20) \qquad \dot{Z}_0(t) = - A(Z_0,t)f_{0t}(Z_0,t) + B(Z_0,t)f_1(Z_0,t).$$

Likewise, the coefficient of ϵ in (18) implies that Z_1 must satisfy the nonlinear equation

$$(21) \qquad \dot{Z}_1(t) = A(Z_0,t)[-f_{0tz}(Z_0,t)Z_1 - f_{1t}(Z_0,t) + \ddot{Z}_0$$

$$- (f_{0zz}(Z_0,t)Z_1 + f_{1z}(Z_0,t))\dot{Z}_0]$$

$$+ B(Z_0,t)[\tfrac{1}{2}f_{0zz}(Z_0,t)Z_1^2 + f_{1z}(Z_0,t)Z_1 + f_2(Z_0,t)]$$

where the quadratic term must be interpreted appropriately. Succeeding terms Z_j will satisfy linear equations obtained as the coefficient of ϵ^j in (18). Using the resulting differential equations (20), (21), etc., it follows that Z_0, Z_1, ... will be uniquely obtained on some subinterval of $t \geq 0$ once their initial values are specified. These initial values by (12), must necessarily satisfy

$$(22) \qquad \epsilon \dot{Z}(0,\epsilon) = f(Z(0,\epsilon),0,\epsilon).$$

Since the prescribed initial value $z^0(0)$ will generally not satisfy (22) at $\epsilon = 0$, the need for an initial boundary layer is obvious.

Expanding (22) and introducing $I_M = A'A + B'B$, we have

$$(23) \qquad f_0(Z_0(0),0) = 0$$

and

$$(24) \qquad B_0(Z(0,\epsilon) - Z_0(0)) = - S_0^{-1}B_0[f_{0z}^0 A_0 A_0'(Z(0,\epsilon) - Z_0(0))$$

$$+ (f(Z(0,\epsilon),0,\epsilon) - f_{0z}^0(Z(0,\epsilon) - Z_0(0)) - \epsilon \dot{Z}(0,\epsilon)]$$

where

$$(25) \qquad (B_0, S_0, f_{0z}^0, A_0) = (B, S, f_{0z}A)\Big|_{(Z_0(0), 0)}.$$

Equating coefficients of ϵ^j, $j > 0$, in (24) implies that

$$B_0 Z_j(0) = -S_0^{-1} B_0 f_{0z}^0 A_0 A_0' Z_j(0) + \alpha_{j-1}$$

where α_{j-1} is known in terms of the initial values $Z_\ell(0)$ of earlier terms. Thus, (9) and (19) imply that

$$(26) \qquad Z_j(0) = B_0 A_0' A_0 Z_j(0) + B_0' \alpha_{j-1}, \qquad j > 0,$$

where $B_0 = B(Z_0(0), 0)$, so only the initial values

$$(27) \qquad B_0 A_0' A_0 Z_j(0)$$

need be specified for each $j > 0$. The equation (23) for $Z_0(0)$ also remains. (Equation (14) implies that $B_0 Z_0(0)$ can be uniquely obtained in terms of S_0, B_0, and $A_0' A_0 Z_0(0)$, but the relationship will generally be much more complicated than the linear relationship (26).)

The Boundary Layer Correction and Matching

Since the outer solution Z satisfies (12), (5) implies that the boundary layer correction $\Pi(\tau, \epsilon)$ must be a decaying solution of the boundary layer equation

$$(28) \qquad \frac{d\Pi}{d\tau} = f(Z + \Pi, \epsilon\tau, \epsilon) - f(Z, \epsilon\tau, \epsilon).$$

Further, since $\Pi \to 0$ as $\tau \to \infty$, we must have

$$(29) \qquad \Pi(\tau, \epsilon) = -\int_\tau^\infty [f(Z(\epsilon s, \epsilon) + \Pi(s, \epsilon), \epsilon s, \epsilon) - f(Z(\epsilon s, \epsilon), \epsilon s, \epsilon)]ds.$$

Before reconsidering the general problem, we consider some special cases.

a. The quasilinear problem

We'll consider the quasilinear problem with

$$(30) \qquad f_0(z, t) = F_0(t)z + G_0(t)$$

(cf., also, Butuzov and Vasil'eva (1970)). Here, f_{0z}, E, A, B, S,

A, and B are all independent of z, so

(31) $$Z(t,\epsilon) = A'(t)A(t)Z(t,\epsilon) + B'(t)B(t)Z(t,\epsilon)$$

(cf. (9), noting that A and B are now completely specified). Further, $f_0(Z_0,t) = 0$ implies that

(32) $$B(t)Z_0(t) = -S^{-1}(t)[B(t)F_0(t)A'(t)A(t)Z_0(t) + G_0(t)]$$

and there remains a nonlinear initial value problem for $A(t)Z_0(t)$ (cf. (31) and (20)). For higher order terms Z_j, we'll now obtain BZ_j as a linear function of AZ_j and a linear differential equation for AZ_j will remain with nonhomogeneous terms being functions of the preceding Z_ℓ's.

Here, the boundary layer correction Π will satisfy

(33) $$\frac{d\Pi}{d\tau} = F_0(\epsilon\tau)\Pi + \{f(Z + \Pi,\epsilon\tau,\epsilon) - f(Z,\epsilon\tau,\epsilon) - f_0(Z + \Pi,\epsilon\tau)$$

$$+ f_0(Z,\epsilon\tau)\}.$$

Thus, when $\epsilon = 0$, (9) implies that

(34) $$\frac{d\Pi_0}{d\tau} = F_0(0)(A'(0)A(0) + B'(0)B(0))\Pi_0$$

and $A(0)F_0(0) = A(0)f_{0z}(0,0) = 0$ implies that

(35) $$A(0)\Pi_0(\tau) = 0$$

since $\Pi_0 \to 0$ as $\tau \to \infty$. From (34), it follows that

$$\frac{d}{d\tau}(B(0)\Pi_0) = S(0)B(0)\Pi_0,$$

hence

$$B(0)\Pi_0(\tau) = e^{S(0)\tau}B(0)\Pi_0(0)$$

and, by (5), (9), and (35),

(36) $$\Pi_0(\tau) = B'(0)e^{S(0)\tau}B(0)(z^0(0) - Z_0(0)).$$

We note that (35) implies that

$$(37) \qquad A(0)Z_0(0) = A(0)z^0(0),$$

so we can now integrate the initial value problem for AZ_0. Then, (32) will provide BZ_0 which by (31) and (36) imply the leading terms Z_0 and Π_0 of both the outer solution and the boundary layer correction. We note, in particular, that our formulas imply that

$$(38) \qquad Z_0(0) = B(0)A'(0)A(0)z^0(0) - B'(0)S^{-1}(0)G_0(0).$$

Higher order terms follow analogously. Thus, we have

$$\frac{d\Pi_1}{d\tau} = F_0(0)\Pi_1 + \gamma_1(\tau)$$

where $\gamma_1(\tau)$ is known and exponentially decaying. Clearly, $A(0)F_0(0) = 0$ implies that

$$(39) \qquad A(0)\Pi_1(\tau) = A(0)\int_\tau^\infty \gamma_1(s)ds$$

and this provides the initial value $A(0)Z_1(0) = A(0)(z_{01} - \Pi_1(0))$ needed to completely determine $A(t)Z_1(t)$ and $Z_1(t)$. Continuing, we have

$$(40) \qquad B(0)\Pi_1(\tau) = e^{S(0)\tau}B(0)(z_{01} - Z_1(0))$$

$$+ \int_0^\tau e^{S(0)(\tau-s)}B(0)[F_0(0)A'(0)A(0)\int_s^\infty \gamma_1(r)dr + \gamma_1(s)]ds$$

and

$$(41) \qquad \Pi_1(\tau) = A'(0)A(0)\Pi_1(\tau) + B'(0)B(0)\Pi_1(\tau)$$

is completely specified. In this way, all terms of the asymptotic solution to the quasilinear problem (30) are completely and uniquely determined.

b. The classic problem

A much-studied nonlinear initial value problem

$$(42) \quad \begin{cases} \epsilon\dfrac{dy}{dt} = v(x,y,t,\epsilon), \quad y(0,\epsilon) = y^0(\epsilon) \\[2mm] \dfrac{dx}{dt} = u(x,y,t,\epsilon), \quad x(0,\epsilon) = x^0(\epsilon) \end{cases}$$

can be analyzed under the principal assumption that the reduced problem

$$(43) \quad \begin{cases} 0 = v_0(x,y,t) \\[2mm] \dfrac{dx}{dt} = u_0(x,y,t), \quad x(0) = x^0(0) \end{cases}$$

has a unique solution $(X_0(t), Y_0(t))$ along which $v_{0y}(X_0(t), Y_0(t),t)$ < 0 for $t \geq 0$ (cf., e.g., O'Malley (1974)). For simplicity, we'll assume that x and y are scalars and that

$$(44) \quad v_{0y}(x,y,t) < 0$$

everywhere. The problem (42) can be put into the form (1)-(2) by introducing $z = \begin{pmatrix} y \\ x \end{pmatrix}$, so that we have

$$(45) \quad \epsilon\dot{z} = f(z,t,\epsilon) \equiv \begin{pmatrix} v(x,y,t,\epsilon) \\ \epsilon u(x,y,t,\epsilon) \end{pmatrix}, \quad z(0,\epsilon) = z^0(\epsilon).$$

The limiting system $f(Z_0,t,0) = 0$ reduces to

$$(46) \quad v_0(X_0,Y_0,t) = 0$$

and the Jacobian matrix

$$(47) \quad f_{0z} = \begin{pmatrix} v_{0y} & v_{0x} \\ 0 & 0 \end{pmatrix}$$

has rank one under hypothesis (44). (Here, there cannot be a deficiency of eigenvectors, since the null space of f_{0z} and its transpose are both one-dimensional.)

Because f_{0z} is already row-reduced, we can take $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = (0 \ 1)$, and $B = (1 \ 0)$, so that $y = Bz$ and $x = Az$. The reduced problem (46) can be solved for $Y_0 = \phi(X_0,t)$ since v_{0y} is nonsingular,

but (46) fails to determine X_0. The critical assumption (44) corresponds to the earlier hypothesis (10), since $S = v_{0y} < 0$.

For this problem, the leading term Z_0 of the outer solution satisfies the equations

(48)
$$\dot{Y}_0 = -v_{0y}^{-1}(v_{0t} + v_{0x}u_0), \quad \dot{X}_0 = u_0$$

(a specialization of (20)). Indeed, we can rewrite this system as $\frac{d}{dt}v_0(X_0,Y_0,t) = 0$ with $\dot{X}_0 = u_0$, so the initial condition $v_0(X_0(0), Y_0(0),0) = 0$ reduces the system to the familiar reduced problem (43) provided $X_0(0) = x^0(0)$.

If we further set the boundary layer correction $\Pi = \begin{Bmatrix} \eta \\ \xi \end{Bmatrix}$, the limiting form of the boundary layer system (28) reduces to the pair of equations

(49)
$$\frac{d\eta_0}{d\tau} = v_0(X_0(0) + \xi_0, Y_0(0) + \eta_0, 0), \quad \frac{d\xi_0}{d\tau} = 0.$$

Moreover, the decay required as $\tau \to \infty$ implies that $\xi_0(\tau) \equiv 0$, so we must have $X_0(0) = x^0(0)$ and the limiting outer solution $(X_0(t), Y_0(t))$ is uniquely specified. Thus, there remains the problem

(50)
$$\frac{d\eta_0}{d\tau} = v_0(x^0(0), \phi(x^0(0),0) + \eta_0, 0), \quad \eta_0(0) = y^0(0) - \phi(x^0(0),0).$$

The Dini-Hukahara theorem (cf. Hille (1969)) and the negativity of v_{0y}, imply the existence of a unique decaying solution $\eta_0(\tau)$. Thus, the leading boundary layer correction term Π_0 is completely determined. Succeeding terms follow in a straightforward manner.

We note that a considerably more complicated vector problem would result if v_{0y} had zero eigenvalues.

c. The original problem

For the problem (1)-(2), the initial value $B_0 Z_0(0)$ of the B_0 portion of the leading term of the outer solution is restricted by (14). It restricts the leading boundary layer term Π_0 since it must satisfy $B_0\Pi_0(0) = B_0(z^0(0) - Z_0(0))$ (cf. (5)). Alternatively the initial value $A_0 Z_0(0)$ is free as far as the outer solution is concerned, though it would be determined through $A_0\Pi_0(0) = A_0(z^0(0) - Z_0(0))$.

Multiplying (28) by B_0 and integrating from 0 to τ implies that

(51)
$$B_0'B_0\Pi(\tau,\epsilon) = B_0'e^{S_0\tau}(z^0(\epsilon) - Z(0,\epsilon)) + B_0'\int_0^\tau e^{S_0(\tau-s)}B_0$$

$$[f(Z(\epsilon s,\epsilon) + A_0'A_0\Pi(s,\epsilon) + B_0'B_0\Pi(s,\epsilon),\epsilon s,\epsilon)$$

$$- f(Z(\epsilon s,\epsilon),\epsilon s,\epsilon) - f_{0z}^0 B_0'B_0\Pi(s,\epsilon)]ds$$

while (29) implies that

$$(52) \qquad A_0'A_0\Pi(\tau,\epsilon) = -A_0'A_0\int_\tau^\infty [f(Z(\epsilon r,\epsilon) + A_0'A_0\Pi(r,\epsilon) + B_0'B_0\Pi(r,\epsilon),$$

$$\epsilon r,\epsilon) - f(Z(\epsilon r,\epsilon),\epsilon r,\epsilon)]dr.$$

Putting these together, we get the new integral equation

$$(53) \qquad \Pi(\tau,\epsilon) = \Pi^0(\tau) + F(\Pi)$$

where

$$\Pi^0(\tau) = B_0' e^{S_0\tau} B_0(z^0(\epsilon) - Z(0,\epsilon))$$

and

$$F(\Pi) = B_0'\int_0^\tau e^{S_0(\tau-s)} B_0[f(Z(\epsilon s,\epsilon) + \Pi(s,\epsilon),\epsilon s,\epsilon) - f(Z(\epsilon s,\epsilon),\epsilon s,\epsilon)$$

$$- f_{0z}^0 B_0'B_0\Pi(s,\epsilon)]ds - A_0'A_0\int_\tau^\infty [f(Z(\epsilon r,\epsilon) + \Pi(r,\epsilon),\epsilon r,\epsilon)$$

$$- f(Z(\epsilon r,\epsilon),\epsilon r,\epsilon)]dr.$$

We recall that for any initial value $Z(0,\epsilon)$, consistent with (12), $Z(t,\epsilon)$ is completely determined. Likewise, the exponential decay of $e^{S_0\tau}$ implies that for any arbitrary initial vector $B_0\Pi(0,\epsilon)$ a k-dimensional manifold of exponentially decaying solutions of (53) is obtainable by successive approximations. We simply define

$$(54) \qquad \begin{cases} \Pi^{(j+1)}(\tau,\epsilon) = \Pi^0(\tau) + F(\Pi^{(j)}), \quad j \geq 0 \\ \text{and} \\ \Pi(\tau,\epsilon) = \lim_{j\to\infty} \Pi^{(j)}(\tau,\epsilon) \end{cases}$$

(cf. Erdélyi (1964)). As for the quasilinear and classic problems, we are prescribing the vector $B_0'B_0\Pi(0,\epsilon)$ (of rank $\leq k$) and letting the

vector $A_0'A_0\Pi(0,\epsilon)$ (of rank $\leq M - k$) be determined by the exponentially decaying solution constructed (cf. (51) and (52)). Vasil'eva (1975) expresses this relationship by writing an expression like $A_0'A_0\Pi(0,\epsilon)$ $= \Phi(B_0'B_0\Pi(0,\epsilon),Z(0,\epsilon))$. This actually relates initial values of the outer solution since it implies that

$$(55) \qquad A_0'A_0(z^0(\epsilon) - Z(0,\epsilon)) = \Phi(B_0'B_0(z^0(\epsilon) - Z(0,\epsilon)), Z(0,\epsilon)).$$

It is instructive to further consider the asymptotic expansion

$$(56) \qquad \Pi(\tau,\epsilon) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\epsilon^j$$

for the boundary layer correction. Clearly, Π_0 must satisfy the nonlinear system

$$(57) \qquad \frac{d\Pi_0}{d\tau} = f_0(Z_0(0) + \Pi_0,0)$$

and we have

$$(58) \qquad \Pi_0(\tau) = \Pi^0(\tau) + \int_0^{\tau} B_0'e^{S_0(\tau-s)} B_0[f_0(Z_0(0) + \Pi_0(s),0)$$

$$- f_{0z}^0 B_0'B_0\Pi_0(s)]ds$$

$$- A_0'A_0\int_{\tau}^{\infty} f_0(Z_0(0) + \Pi_0(r),0)dr.$$

We note, in particular, that we'll generally have $A_0\Pi_0(\tau) \neq 0$, unlike for the quasilinear or classic problems where $A_0 f_{0z} = 0$. Moreover, Π_1 will satisfy a linear system

$$(59) \qquad \frac{d\Pi_1}{d\tau} = f_{0z}(Z_0(0) + \Pi_0,0)\Pi_1 + \alpha_0(\tau)\Pi_0$$

where

$$\alpha_0(\tau)\Pi_0 = \lim_{\epsilon \to 0} \{-\frac{1}{\epsilon}\{f(Z + \Pi_0,\epsilon\tau,\epsilon) - f(Z_0(0) + \Pi_0,0,0) - f(Z,\epsilon\tau,\epsilon)\}.$$

Since the coefficient matrix f_{0z} has k stable eigenvalues, we can anticipate a k-dimensional manifold of initial values $\Pi_1(0)$ corresponding to decaying solutions as $\tau \to \infty$. This allows us to prescribe $B_0'B_0\Pi_1(0)$ while the remaining components $A_0'A_0\Pi_1(0)$ of the initial

vector become specified. (We recall that Vasil'eva (1975) seems to relate these initial values in terms of the $M - k$ eigenvectors spanning the null space of f'_{0z}.) Analogous considerations apply for higher order terms.

A further problem remains; namely, the initial value $Z(0,\epsilon)$ of the outer solution is still imbedded in the nonlinear equations for the outer solution and for the boundary layer correction (cf. (12) and (14) with $t = 0$ and (55)). We shall, for the time being, simply assume that the resulting nonlinear equations have a unique solution $Z(0,\epsilon)$ as they do in the special cases already considered (cf. also Vasil'eva (1975)). More critically, it is important to develop numerical methods for stiff systems based on this analysis. Such work is in progress.

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epsilon approaches 0 $t >$ or $= 0$ $\tau_{au} = t/epsilon$

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