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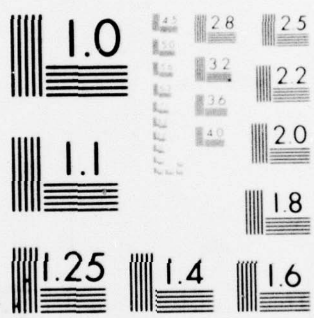

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by
ROLF FÄRE
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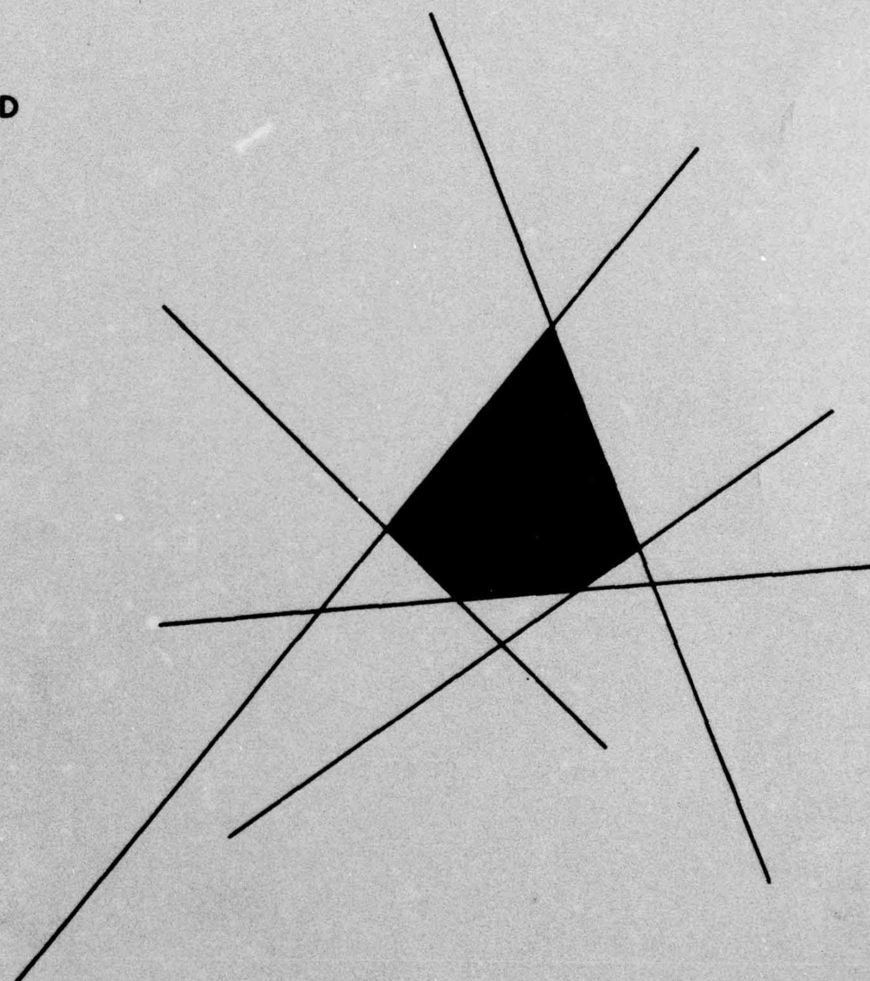

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NOTE ON PRODUCTION CORRESPONDENCES WITH RAY-HOMOTHETIC INPUT AND OUTPUT STRUCTURE

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ABSTRACT

It is shown that, if both input and output correspondence are ray-homothetic, they are semi-homogeneous.

NOTE ON PRODUCTION CORRESPONDENCES WITH RAY-HOMOTHETIC INPUT AND OUTPUT STRUCTURE

by

Rolf Färe and Ronald W. Shephard

An output correspondence $x \to P(x)$, respectively input correspondence $u \to L(u)$, that is mappings $P : x \in R^n_+ \to P(x) \in 2^{R^m_+}$, respectively $L : u \in R^m_+ \to L(u) \in 2^{R^n_+}$, under weak axioms (see [4]) were defined in [3] to be Ray-Homothetic if

$$P(\lambda x) = \frac{F(\lambda x)}{F(x)} \cdot P(x) \ , \ \lambda \in (0,+\infty) \ , \ P(x) \neq \{0\}$$

respectively

$$L(\theta u) = \frac{G(\theta u)}{G(u)} \cdot L(u) \ , \ \theta \in (0,+\infty) \ , \ L(u) \neq \emptyset$$

hold. These relations are equivalent to

$$P(\lambda x) = \Delta(\lambda,x) \cdot P(x) \ , \ \lambda \in (0,+\infty) \ , \ P(x) \neq \{0\}$$

$$L(\theta u) = \delta(\theta,u) \cdot L(u) \ , \ \theta \in (0,+\infty) \ , \ L(u) \neq \emptyset$$

with

$$\Delta : R_{++} \times R^n_+ \to R_{++} \ , \ \Delta(1,x) = \Delta(\lambda,0) = 1$$

$$\delta : R_{++} \times R^m_+ \to R_{++} \ , \ \delta(1,u) = \delta(\theta,0) = 1 \ .$$

If both output and input correspondence for the same production structure are ray-homothetic, and $\theta \to \delta(\theta,u)$ and $\lambda \to \delta(\lambda,x)$ are strictly increasing, it is implied that the production structure has both semi-homogeneous input and output structure (see [4]).

This result was not shown in [3] and is proven here in this note.

Let x and u be a feasible pair of vectors, i.e. $x \in L(u)$. By the weak axiom L.4 of the correspondences $x \to P(x)$, $u \to L(u)$, it follows that for all $\theta \in (0,+\infty)$ there exists a positive scalar λ_θ such that $(\lambda_\theta \cdot x) \in L(\theta u)$. Using the ray-homotheticity of $u \to L(u)$ and $x \to P(x)$:

$$(\lambda_\theta x) \in \delta(\theta,u) \cdot L(u) \Leftrightarrow \frac{\lambda_\theta x}{\delta(\theta,u)} \in L(u) \Leftrightarrow$$

$$u \in P\left(\frac{\lambda_\theta x}{\delta(\theta,u)}\right) = \Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right) \cdot P(\lambda_\theta x) \Leftrightarrow$$

$$\frac{u}{\Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right)} \in P(\lambda_\theta x) \Leftrightarrow$$

$$\lambda_\theta x \in L\left(\frac{u}{\Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right)}\right) = \delta\left(\frac{1}{\Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right)},u\right) \cdot L(u) .$$

Thus,

$$\delta\left(\frac{1}{\Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right)},u\right) = \delta(\theta,u)$$

$$\frac{1}{\theta} = \Delta\left(\frac{1}{\delta(\theta,u)},\lambda_\theta x\right) , \theta \in (0,+\infty)$$

and

$$(1) \qquad \Delta^{-1}\left(\frac{1}{\theta}, \lambda_\theta x\right) \cdot \delta(\theta, u) = 1 \ , \ \theta \in (0, +\infty) \ .$$

By repeating the same argument starting with $u \in P(x)$, noting that for all $\theta \in (0,+\infty)$ there exists a positive scalar σ_θ such that $(\sigma_\theta u) \in P(\theta x)$, one obtains

$$(2) \qquad \delta^{-1}\left(\frac{1}{\theta}, \sigma_\theta u\right) \cdot \Delta(\theta, x) = 1 \ , \ \theta \in (0, +\infty) \ .$$

Equations (1) and (2) can be written

$$(1)' \qquad \delta\left(\theta \ \frac{1}{\theta}, u\right) = \delta(\theta, u) \cdot \Delta^{-1}\left(\frac{1}{\theta}, \lambda_\theta x\right) \ , \ \theta \in (0, +\infty)$$

$$(2)' \qquad \delta\left(\theta \ \frac{1}{\theta}, u\right) = \Delta(\theta, x) \cdot \delta^{-1}\left(\frac{1}{\theta}, \sigma_\theta u\right) \ , \ \theta \in (0, +\infty)$$

to observe that they are functional equations of the form $f(w \cdot z) = f(w) \cdot g(z)$, the general solutions of which are: (see [1])

$$\delta(\theta, u) = \theta^{\alpha(u)} \ , \ \alpha(u) > 0 \ , \ L(u) \neq \emptyset \ , \ \theta \in (0, +\infty)$$

$$\Delta(\theta, x) = \theta^{\beta(x)} \ , \ \beta(x) > 0 \ , \ P(x) \neq \{0\} \ , \ \theta \in (0, +\infty) \ .$$

But, since

$$\begin{aligned} L(\theta\sigma u) &= (\theta\sigma)^{\alpha(u)} \cdot L(u) \\ &= \theta^{\alpha(\sigma u)} \cdot L(\sigma u) = \theta^{\alpha(\sigma u)} \cdot \sigma^{\alpha(u)} \cdot L(u) \ , \end{aligned}$$

it follows that

$$\theta^{\alpha(\sigma u)} = \theta^{\alpha(u)}$$

for all $\sigma \in (0,+\infty)$, implying

$$(3) \qquad \delta(\theta,u) = \theta^{\alpha\left(\frac{u}{|u|}\right)} , \ \alpha\left(\frac{u}{|u|}\right) > 0 \ , \ L(u) \neq \emptyset \ , \ \theta \in (0,+\infty) \ .$$

Similarly

$$(4) \qquad \Delta(\lambda,x) = \lambda^{\beta\left(\frac{x}{|x|}\right)} , \ \beta\left(\frac{x}{|x|}\right) > 0 \ , \ P(x) \neq \{0\} \ , \ \lambda \in (0,+\infty) \ .$$

Consequently the input and output correspondences of the given production structure are semi-homogeneous (see [4]). By substituting (3) and (4) into (1) and (2) respectively, one observes that

$$\alpha\left(\frac{u}{|u|}\right) = \frac{1}{\beta\left(\frac{x}{|x|}\right)}$$

for every feasible pair $u \in P(x) \Leftrightarrow x \in L(u)$. Along a ray segment $\{\lambda x \mid \lambda \geqq 0\}$, $\beta\left(\frac{x}{|x|}\right)$ is constant for all $x \in L(u)$. Thus for connected input sets $L(u) \cap L(v) \neq \emptyset$, both $\alpha\left(\frac{u}{|u|}\right)$ and $\beta\left(\frac{x}{|x|}\right)$ are reciprocal constants. However, under the weak axioms for the correspondence $u \in R^m_+ \rightarrow L(u) \in 2^{R^n_+}$, not all input sets need be connected.

It is of interest to consider a second proof of the problem, utilizing the functional equation

$$f\left(\alpha \cdot \beta, \frac{z}{|z|}\right) = f\left(\alpha, \frac{z}{|z|}\right) \cdot f\left(\beta, \frac{z}{|z|}\right)$$

where $\alpha, \beta \in (0,+\infty)$ and $z \in R^r_+$. The solution of this equation is shown by Eichhorn [2] to be

$$f\left(\alpha, \frac{z}{|z|}\right) = \alpha^{h\left(\frac{z}{|z|}\right)}$$

where $h(\cdot)$ is positive finite and scalar valued. To pursue the issue note that from the assumption of weak disposability i.e., $L(\mu \cdot u) \subset L(u)$ for $\mu \in (1,+\infty)$ or equivalently $L(u) \subset L(\theta \cdot u)$ for $\theta \in (0,1)$, it follows that there exists a scalar λ_θ such that

$$\lambda_\theta \cdot x \in L(\theta \cdot u) \subset L(\mu \cdot \theta \cdot u) \ , \ \mu \in (0,1] \ .$$

As above one obtains

$$\text{(5)} \qquad \Delta^{-1}\left(\frac{1}{\theta}, \lambda_\theta \cdot x\right) \cdot \delta(\theta, \mu \cdot u) = 1 \ , \ \theta \in (0,+\infty) \ , \ \mu \in (0,1]$$

Thus by (1) and (5),

$$\text{(6)} \qquad \delta(\theta,u) = \delta(\theta,\mu \cdot u) \ , \ \theta \in (0,+\infty) \ , \ \mu \in (0,1] \ .$$

If $|u| \geqq 1$ take $\mu = \frac{1}{|u|}$ in (6) thus

$$\text{(7)} \qquad \delta(\theta,u) = \delta\left(\theta, \frac{u}{|u|}\right), \ \theta \in (0,+\infty) \ , \ |u| \geqq 1 \ .$$

Now if $|u| \varepsilon (0,1]$, take $\lambda \geqq 1$ such that $|\lambda \cdot u| \geqq 1$, and it follows from (6) that

$$(8) \qquad \delta(\theta,\lambda \cdot u) = \delta(\theta,\lambda \cdot \mu \cdot u) \ , \ \mu \text{ and } |u| \ \varepsilon \ (0,1] \ , \ \lambda \geqq 1 \ .$$

Now take $\mu = 1/|\lambda \cdot u|$ in (8) where $|u| \varepsilon (0,1]$, and

$$(9) \qquad \delta(\theta,\lambda \cdot u) = \delta\left(\theta,\frac{u}{|u|}\right) \ , \ \theta \ \varepsilon \ (0,+\infty) \ , \ |u| \ \varepsilon \ (0,1] \ , \ \lambda \geqq 1 \ .$$

Thus by (6), (7) and (9),

$$(10) \qquad \delta(\theta,\mu \cdot u) = \delta(\theta,u) = \delta\left(\theta,\frac{u}{|u|}\right) \quad \textit{for} \ \ \theta \text{ and } \mu \ \varepsilon \ (0,+\infty) \ .$$

Moreover, consider $L(\mu \cdot \theta \cdot u)$, μ and $\theta \varepsilon (0,+\infty)$, then by ray-homotheticity of the input correspondence it follows that the scaling function $\delta(\theta,u)$ obeys the functional equation

$$(11) \qquad \delta(\theta \cdot \mu,u) = \delta(\theta,\mu \cdot u) \cdot \delta(\mu,u) \ .$$

Now it is clear from expressions (10) and (11) that the scaling function $\delta(\theta,u)$ obeys the functional equation

$$(12) \qquad \delta\left(\theta \cdot \mu,\frac{u}{|u|}\right) = \delta\left(\theta,\frac{u}{|u|}\right) \cdot \delta\left(\mu,\frac{u}{|u|}\right)$$

with the solution

$$\delta\left(\theta,\frac{u}{|u|}\right) = \delta^{\alpha\left(\frac{u}{|u|}\right)}$$

i.e., the input structure is semi-homogeneous.

Similar arguments apply to show that the output correspondence $x \to P(x)$ is also semi-homogeneous i.e.,

$$P(\lambda \cdot x) = \lambda^{\beta\left(\frac{x}{|x|}\right)} \cdot P(x)$$

and as pointed out above, $\alpha\left(\frac{u}{|u|}\right) \cdot \beta\left(\frac{x}{|x|}\right) = 1$.

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