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Probability Distribution of Spectral Estimates Obtained Via Overlapped FFT Processing of Windowed Data

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NAVAL UNDERWATER SYSTEMS CENTER
Newport, Rhode Island • New London, Connecticut

PREFACE

This research was conducted under NUSC Project No. A-752-05, "Applications of Statistical Communication Theory to Acoustic Signal Processing." Principal Investigator, Dr. A. H. Nuttall (Code 313), Navy Project No. ZR000 01, Program Manager, T. A. Kleback (MAT 03521), Naval Material Command.

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The characteristic function of spectral estimates obtained via overlapped FFT processing of windowed data is presented for a random process containing a signal tone and Gaussian noise. For the special case of noise-alone, the probability distribution of the estimate is plotted and compared with an approximation utilizing only the first two moments and found to be in excellent agreement in probability over the range (.0001, .9999) for several data windows, overlaps, and time-bandwidth products. This result means that knowledge of the equivalent degrees of freedom of the spectral estimate is

next page

20. (Cont'd):

cont.

adequate for a complete probabilistic description, even when the overlap results in significant statistical dependence of the component FFT outputs.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

FFT	Fast Fourier Transform
f	Analysis frequency
$\hat{G}(f)$	Power spectral estimate
P	Number of pieces in average
$Y_p(f)$	Fourier transform of p-th weighted data segment
t	Time
$x(t)$	Available complex data process
$w(t)$	Data window
L	Length of data window
S	Shift of adjacent data windows
$s(t)$	Signal waveform
$n(t)$	Noise waveform
A	Signal tone amplitude
f_o	Signal tone frequency
θ	Signal tone phase
Y_{ps}, Y_{pn}	Signal and noise components of p-th transform
$W(f)$	Spectral window (Fourier transform of $w(t)$)
B	Bandwidth of spectral window
$C(\xi)$	Characteristic function of $\hat{G}(f)$
$\mathbf{K}$	Covariance matrix of $\{Y_{pn}\}$
λ_p	p-th eigenvalue of $\mathbf{K}$

LIST OF SYMBOLS (cont'd)

$\mathbf{m}$	Column matrix of $\left\{Y_{ps}\right\}$
$\mathbf{Q}$	Normalized modal matrix of $\mathbf{K}$
$\boldsymbol{\mu}$	Transformed means (eq. 11)
Superscript T	Transpose of matrix
$G_n(f)$	Noise spectral density at frequency f
$\phi_w(\tau)$	Autocorrelation of w(t) (eq. 14)
r_m	Normalized autocorrelation (eq. 16)
$\mathbf{R}$	Matrix of $\left\{r_m\right\}$ (eq. 15)
Var	Variance
Superscript H	Conjugate transpose of matrix
$Q_s(f)$	Output signal power of window filter centered at f
$Q_n(f)$	Output noise power of window filter centered at f
EDF	Equivalent degrees of freedom (eq. 25)
$\widehat{dB}$	Decibel equivalent of $\hat{G}(f)$ (eq. 29)
$\hat{g}$	Normalized spectral estimate (eq. 34)
$C_g(\xi)$	Characteristic function of $\hat{g}$
$\lambda_p^{(R)}$	p-th eigenvalue of $\mathbf{R}$
v	Threshold value
B_k	Coefficients (eq. 37)
t	Random variable with same mean and variance as $\hat{g}$.
$C_t(\xi)$	Characteristic function of t
b	Constant in $C_t(\xi)$

LIST OF SYMBOLS (cont'd)

K	Equivalent degrees of freedom for noise-alone
Γ	Gamma function
${}_1F_1$	Confluent hypergeometric function
BT	Product of analysis bandwidth and available record length
$G_{xy}(f)$, $\hat{G}_{xy}(f)$	Cross spectrum and its estimate
A	Amplitude of cross spectral estimate

PROBABILITY DISTRIBUTION OF SPECTRAL ESTIMATES OBTAINED VIA OVERLAPPED FFT PROCESSING OF WINDOWED DATA

INTRODUCTION

Estimation of the autospectrum of a stationary random process by means of overlapped FFT processing of windowed data (the so-called direct method) is a popular and efficient method, especially for data with pure tones present. Stable spectral estimates, as measured by the equivalent degrees of freedom of the spectral estimate, result when the product of the available record length and the desired frequency resolution (the time-bandwidth product) is large in comparison with unity. (See, for example, references 1 and 2 and the references listed therein.)

The equivalent degrees of freedom of the spectral estimate is an incomplete probabilistic descriptor, because it depends only on the mean and variance of the random variable. In this report, we address the problem of obtaining the characteristic function of the spectral estimate with overlap processing, of a signal tone present in Gaussian noise, and thence the cumulative probability distribution (perhaps by numerical means as given in references 3 and 4). For the case of signal-absent also, we will compare the exact probability distribution with an approximate distribution that uses only the first two moments of the spectral estimate, to see when the approximate distribution can be used as a valid probabilistic description. Some related work is available in reference 5 and the papers cited therein.

A discussion of the relative stability of the spectral estimates with signal tones present, and of a cross-spectral estimate, completes the presentation.

CHARACTERISTIC FUNCTION FOR SIGNAL PLUS NOISE

The method and conditions of processing are described fully in reference 1 and, for sake of brevity, will not be repeated here. The power spectral estimate at analysis frequency, f, is given by (reference 1, pp. 2-4)

$$\hat{G}(f) = \frac{1}{P} \sum_{p=1}^{P} |Y_p(f)|^2, \tag{1}$$

where P is the total number of weighted data segments. Here*

$$Y_p(f) = \int dt \, \exp(-i2\pi ft) \, x(t) \, w\left[t - \frac{1}{2} L - (p-1)S\right], \tag{2}$$

where x(t) is the available (complex) data process, w(t) is the data window of length L, and S is the amount of shift each adjacent data window undergoes. The fractional overlap is therefore 1 - S/L.

*Integrals without limits are over the range of non-zero integrand.

If we let x(t) be composed of a pure signal tone*

$$s(t) = A \exp(i2\pi f_o t + i\theta) \tag{3}$$

and zero-mean Gaussian noise n(t), (2) can be expressed as

$$Y_p = Y_{ps} + Y_{pn} , \tag{4}$$

where the variable f is suppressed for notational convenience and complex (non-random) constant

$$Y_{ps} = A\ W(f-f_o)\ \exp\left[i\theta - i2\pi(f-f_o)(\tfrac{1}{2} L + (p-1)S\right] , \tag{5}$$

where

$$W(f) \equiv \int dt\ \exp(-i2\pi ft)\ w(t). \tag{6}$$

$|W(f)|^2$ is called the spectral window (see equation (5), reference 1), and has analysis bandwidth B. Now if analysis frequency, f, is not within a bandwidth, B, of tone frequency, f_o, (5) will be zero; therefore, we limit consideration to $|f-f_o|<B$. The remaining term in (4),

$$Y_{pn} = \int dt\ \exp(-i2\pi ft)\ n(t)\ w\left[t - \tfrac{1}{2} L - (p-1)S\right] , \tag{7}$$

is complex Gaussian since n(t) is Gaussian.

Substituting (4) in (1), the spectral estimate is given by

$$\hat{G}(f) = \frac{1}{P}\sum_{p=1}^{P} |Y_{ps} + Y_{pn}|^2 , \tag{8}$$

*The generalization to several separated tones will be obvious.

where $\{Y_{ps}\}$ are complex constants and $\{Y_{pn}\}$ are complex correlated Gaussian zero-mean random variables, and the correlation is due to the overlapped processing.

In appendix A, the characteristic function of forms like (8) is evaluated; it specializes here to the form

$$C(\xi) = \prod_{p=1}^{P} \left\{ (1-i\lambda_p\xi/P)^{-1} \exp\left(\frac{i|\mu_p|^2\lambda_p\xi/P}{1-i\lambda_p\xi/P} \right) \right\}, \tag{9}$$

where $\{\lambda_p\}$ are the eigenvalues of P x P matrix

$$\mathbf{K} \equiv \left[E\left\{ Y_{pn} Y_{qn}^* \right\} \right]_1^P \quad , \tag{10}$$

and

$$\boldsymbol{\mu} = \mathbf{Q}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{m} , \tag{11}$$

where $\mathbf{Q}$ is the normalized modal matrix of $\mathbf{K}$, and

$$\mathbf{m} = \left[Y_{1s} \ldots Y_{Ps} \right]^T . \tag{12}$$

The evaluation of $\mathbf{K}$ in (10) is considered in appendix B. It is given by

$$\mathbf{K} = \left[K_{q-p} \right] = G_n(f) \phi_w(0) \mathbf{R} , \tag{13}$$

where $G_n(f)$ is the noise spectral density at analysis frequency, f;

$$\phi_w(\tau) = \int dt \; w(t) w^*(t-\tau); \tag{14}$$

and

$$\mathbf{R} = \left[r_{q-p}\right] = \begin{bmatrix} 1 & r_1 & r_2 \dots r_{P-1} \\ r_{-1} & \ddots & \\ \vdots & & \\ r_{1-P} & & 1 \end{bmatrix} , \tag{15}$$

where

$$r_m = \frac{\phi_w(mS)}{\phi_w(0)} \quad . \tag{16}$$

A Fourier transformation of (9) would yield the probability density function of the spectral estimate (8), for a tone present. This would have to be done numerically, but has not been pursued here.

MEAN AND VARIANCE FOR SIGNAL PLUS NOISE

By means of (A-16), the mean and variance* of spectral estimate, $\hat{G}(f)$, in (8) can be expressed as

$$\text{Mean}\left\{\hat{G}(f)\right\} = K_o + \frac{1}{P}\sum_{k=1}^{P} |m_k|^2 , \tag{17}$$

$$\text{Var}\left\{\hat{G}(f)\right\} = \frac{1}{P}\sum_{k=1-P}^{P-1} \left(1-\frac{|k|}{P}\right)|K_k|^2 + \frac{2}{P^2}\mathbf{m}^H\mathbf{K}\mathbf{m} , \tag{18}$$

in terms of the quantities in (12) and (13). Employing the explicit relationships in (12) and (13), there follows

$$\text{Mean}\left\{\hat{G}(f)\right\} = G_n(f)\phi_w(0) + A^2|W(f-f_o)|^2 , \tag{19}$$

*More generally, the cumulants are given by (A-7).

and

$$\operatorname{Var}\left\{\hat{G}(f)\right\} = \left[G_n(f)\phi_w(0)\right]^2 \frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)|r_k|^2$$

$$+2A^2|W(f-f_o)|^2 G_n(f)\phi_w(0)\frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)r_k \exp\left(ik2\pi(f-f_o)S\right) \quad , \tag{20}$$

where we have employed (15) and (5).

At this point, it is convenient to define the output signal power of a window filter with transfer function, W, centered at f as

$$Q_s(f) = A^2|W(f-f_o)|^2 \quad , \tag{21}$$

and the output noise power of the same filter as

$$Q_n(f) = \int d\mu\, G_n(\mu)|W(\mu-f)|^2 \cong G_n(f)\int d\mu |W(\mu-f)|^2 = G_n(f)\phi_w(0) . \tag{22}$$

Then (19) and (20) take the forms

$$\operatorname{Mean}\left\{\hat{G}(f)\right\} = Q_n(f) + Q_s(f) \tag{23}$$

and

$$\operatorname{Var}\left\{\hat{G}(f)\right\} = Q_n^2(f)\, \frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)|r_k|^2$$

$$+2Q_s(f)Q_n(f)\frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right) r_k \exp(ik2\pi(f-f_o)S) . \tag{24}$$

From (24), we see that the presence of signal ($A \neq 0$) always increases the absolute level of the variance of the spectral estimate over that for noise-alone. If the noise is absent, the variance of the estimate is zero. If the signal is absent, the equivalent degrees of freedom, defined as

$$\mathrm{EDF}_n \equiv \frac{2(\mathrm{Mean})^2}{\mathrm{Variance}} = \frac{2}{\frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)|r_k|^2}, \tag{25}$$

is identical to equation (8), reference 1, as it should be.

On the other hand, for $Q_s(f) \gg Q_n(f)$,

$$\mathrm{EDF}_s \equiv \frac{2(\mathrm{Mean})^2}{\mathrm{Variance}} \cong \frac{Q_s(f)}{Q_n(f)\,\frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)r_k\,\exp(ik2\pi(f-f_o)S)}, \tag{26}$$

When a strong signal is present, EDF_s is larger than EDF_n by approximately the ratio $\frac{1}{2}\,Q_s(f)/Q_n(f) \gg 1$, since the ratio of sums in (25) and (26) is approximately unity for $f \cong f_o$ and reasonable overlaps (see (27) below). That is, the <u>relative</u> fluctuation in the spectral estimate is reduced by the addition of signal, even though the absolute variance increases.

For Hanning weighting and 50% overlap ($S = L/2$), we find $r_o = 1$, $r_{\pm 1} = 1/6$, $r_k = 0$ for $k \geq 2$. Then the two sums in (25) and (26) take on the values

$$1 + (1-\tfrac{1}{P})\tfrac{1}{18}\ ,\ 1 + (1-\tfrac{1}{P})\tfrac{1}{3}\cos\left[2\pi(f-f_0)S\right], \tag{27}$$

respectively. The former value is slightly larger than unity, whereas the latter value varies between approximately 2/3 and 4/3, depending on the exact location of the signal tone frequency, f_o, with respect to the analysis frequency, f. For an FFT approach, at least one bin has its frequency location, f, such that $|f-f_0| \leq (2L)^{-1}$; thus, at least one frequency bin is located such that the latter value in (27) is larger than unity.

Figure 1A represents the power spectral estimate, (1), plotted on a linear scale proportional to watts. The "ribbon width" in the region of noise-alone is denoted by a. The amount of fluctuation of the estimate at f_0 is denoted by b and is larger than a. (The quantity b is observable only by rerunning the spectral estimation procedure for independent noise segments.)

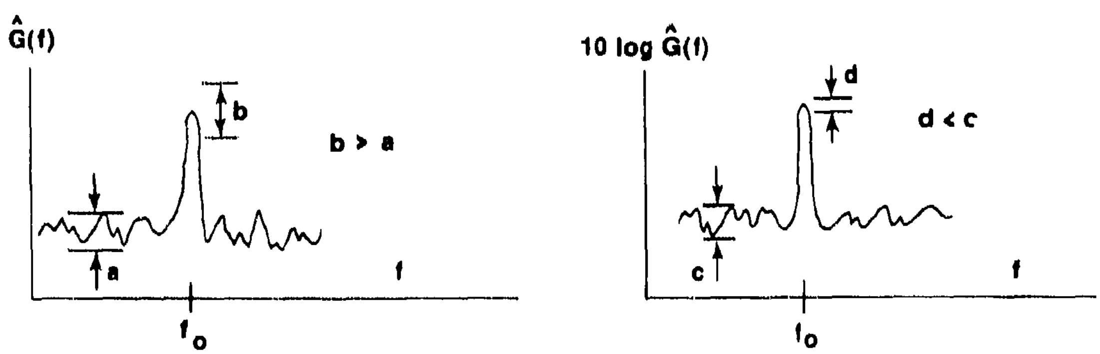


Figure 1. Spectral Estimates for Signal Plus Noise

If, instead, the power spectral estimate is plotted on a dB scale (see figure 1B), the noise-alone ribbon width, c, is larger than the fluctuation, d, of the estimate at f_0. The mathematics behind this conclusion follows. Let the spectral estimate at frequency f be expressed as

$$\hat{G}(f) = m+x, \tag{28}$$

where m is non-random and x has zero-mean and variance σ^2. Then

$$\hat{dB} \equiv 10 \log \hat{G}(f) = 10 \log m + 10 \log(1+\frac{x}{m}). \tag{29}$$

Now suppose that $\sigma/m<<1$, which could be realized by means of a large number of pieces, P, or a high signal to noise ratio; then

$$\hat{dB} \cong 10 \log m + \frac{10}{\ln 10} \frac{x}{m} . \tag{30}$$

The last term in (30) is proportional to the relative stability of the spectral estimate (28); in fact

$$\mathrm{Var}\left\{\hat{dB}\right\} \cong \left(\frac{10}{\ln 10}\right)^2 \frac{\sigma^2}{m^2} , \tag{31}$$

which can be made arbitrarily small. Thus a plot like figure 1 is easily achievable and should be anticipated for a pure tone in Gaussian noise.

PROBABILITY DISTRIBUTION FOR NOISE-ALONE

For noise-alone, the mean and variance of spectral estimate, $\hat{G}(f)$, are available from (19), (20), and (16) as

$$\mathrm{Mean}\left\{\hat{G}(f)\right\} = G_n(f)\phi_w(0),$$

$$\mathrm{Var}\left\{\hat{G}(f)\right\} = G_n^2(f) \frac{1}{P} \sum_{k=1-P}^{P-1} \left(1- \frac{|k|}{P}\right) |\phi_w(kS)|^2 , \tag{32}$$

which agree with equations (5) and (6), reference 1, respectively. More generally, the characteristic function follows from (9) as

$$C(\xi) = \left[\prod_{p=1}^{P} \left\{ 1 - i\lambda_p \xi / P \right\} \right]^{-1} \tag{33}$$

Now let us define a normalized random variable

$$\hat{g} \equiv \frac{\hat{G}(f)}{G_n(f)\,\phi_w(0)} \;; \tag{34}$$

notice that the scale factor is independent of P and the amount of overlap. Thus the mean $E\{\hat{g}\} = 1$, and the characteristic function of $\hat{g}$ is

$$C_g(\xi) = \left[\prod_{p=1}^{P} \left\{ 1 - i\lambda_p^{(R)} \xi / P \right\} \right]^{-1}, \tag{35}$$

where $\{\lambda_p^{(R)}\}$ are the eigenvalues of matrix $\mathbf{R}$ in (15). Then by a partial fraction expansion, the probability that random variable $\hat{g}$ remains below a threshold value, v, is found to be

$$\text{Prob}\ (\hat{g}<v) = 1 - \sum_{k=1}^{P} B_k \exp\left(- \frac{v}{\lambda_k^{(R)}/P} \right), \quad v>0, \tag{36}$$

where

$$B_k = \frac{\left[\lambda_k^{(R)}\right]^{P-1}}{\prod\limits_{\substack{p=1 \\ p\neq k}}^{P} \left\{ \lambda_k^{(R)} - \lambda_p^{(R)} \right\}}, \qquad 1 \le k \le P. \tag{37}$$

We have assumed all the eigenvalues of $\mathbf{R}$ to be unequal; this is the case if the overlap is greater than 0, which is the case of most practical interest. The eigenvalues are all non-negative since $\mathbf{R}$ is a non-negative definite matrix (see appendix B).

Equation (36) is an exact expression for the cumulative probability distribution in terms of the eigenvalues of matrix $\mathbf{R}$. If we consider another random variable, t, with the same mean and variance as $\hat{g}$, a candidate approximate characteristic function is (guided by form (35))

$$C_t(\xi) = (1-i\xi/b)^{-b} \ , \tag{38}$$

where, in order to maintain the same variance, we choose

$$\frac{1}{b} = \frac{1}{P^2}\sum_{p=1}^{P} \lambda_p^{(R)^2} = \frac{1}{P^2}\sum_{p,q=1}^{P} |r_{p-q}|^2 = \frac{1}{P}\sum_{k=1-P}^{P-1}\left(1-\frac{|k|}{P}\right)|r_k|^2 \ . \tag{39}$$

Equation (8), reference 1, shows that b = K/2, i.e., half of the equivalent degrees of freedom. Then the approximate probability density function is

$$p(t) = \frac{1}{\Gamma(b)} \ b^b t^{b-1} \ e^{-bt} \ , \ t>0, \tag{40}$$

and the approximate cumulative probability distribution is (equations 6.5.2 and 6.5.12, reference 6):

$$\int_0^v dt \ p(t) = \frac{1}{\Gamma(b+1)} \ (bv)^b e^{-bv} \ {}_1F_1(1;1+b;bv), \ v>0. \tag{41}$$

(A further simpler approximation, not pursued here, would be to set b_1 = integer part of b, $b_2 = b_1 + 1$, and bracket the results above by two simpler sums.)

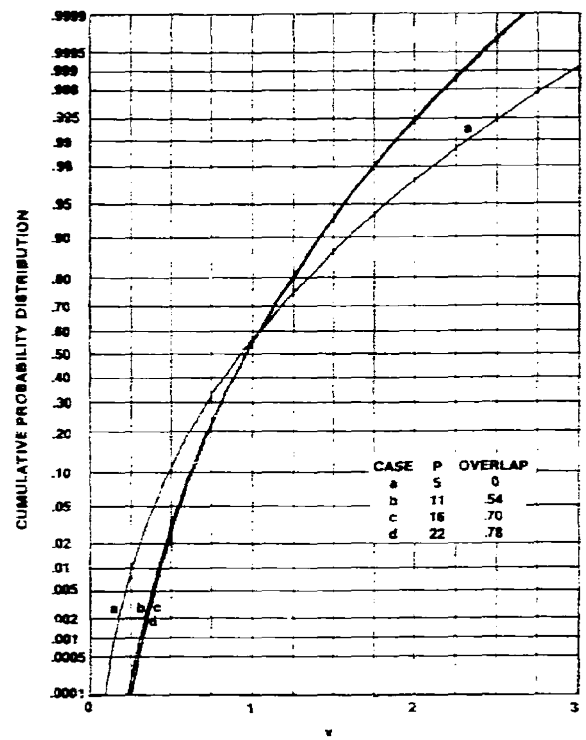
We shall now make quantitative comparisons between exact result (36) and approximation (41) which has the same mean and variance. The question is: is b in (39) and (41) a sufficient statistic to accurately quantitatively describe the exact cumulative probability distribution (36), for representative data windows, overlap, number of pieces, and time-bandwidth products, over the range of probabilities of interest to most users? If so, then attention can be confined to the equivalent degrees of freedom and its maximization alone, as was done in reference 1; this simplification would be most worthwhile and of obvious importance.

The actual numerical computation of the cumulative probability distribution Prob($\hat{g} < v$), is considered in appendix C. In figure 2, the exact cumulative probability distribution for Hanning windowing is presented for time-bandwidth product BT = 8, 16, 32, 64, where T is the available record length and B is the desired resolution bandwidth. In each plot, the overlap is varied from 0 to approximately 75%, and the distribution plotted on a normal probability ordinate covering the range (.0001, .9999). The fact that the curves are not straight lines over this range means that a Gaussian approximation to the power spectral estimate would not suffice. However, the Gaussian approximation would be a fairly good one for larger BT and P (see figure 2D, for example).

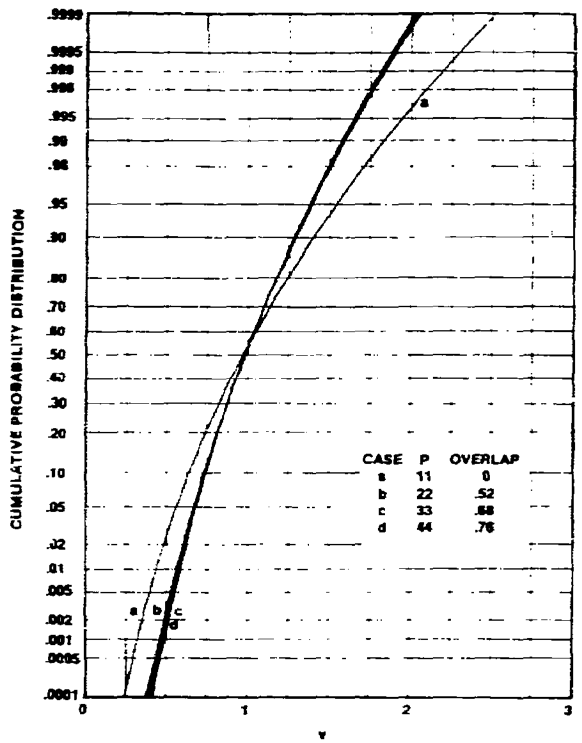
The fact that the curves in figure 2 are virtually identical for overlaps greater than 50% means that there is little point in choosing overlaps greater than this amount. This confirms the choices of overlap made in reference 1, where attention was confined to the equivalent degrees of freedom. The ideal distribution would be a vertical line at $v = 1$; the closeness of these curves to the ideal is a measure of the spread of the spectral estimate.

The corresponding results for the approximation (41) are presented in figure 3. The curves are virtually identical to those of figure 2 over the complete range of probabilities considered, for various values of BT and overlap.

For a cubic window, the exact results and the approximation are given in figures 4 and 5, respectively. The conclusions are identical to those made for the Hanning window.

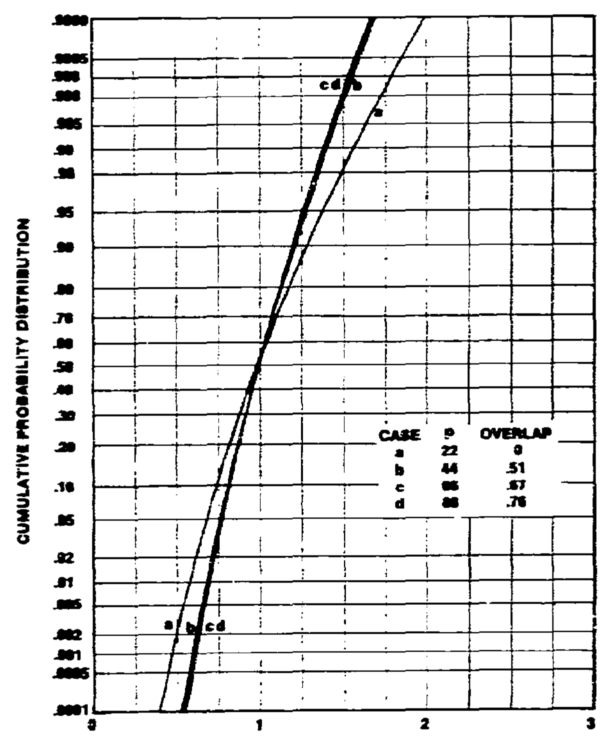


2A. BT=8

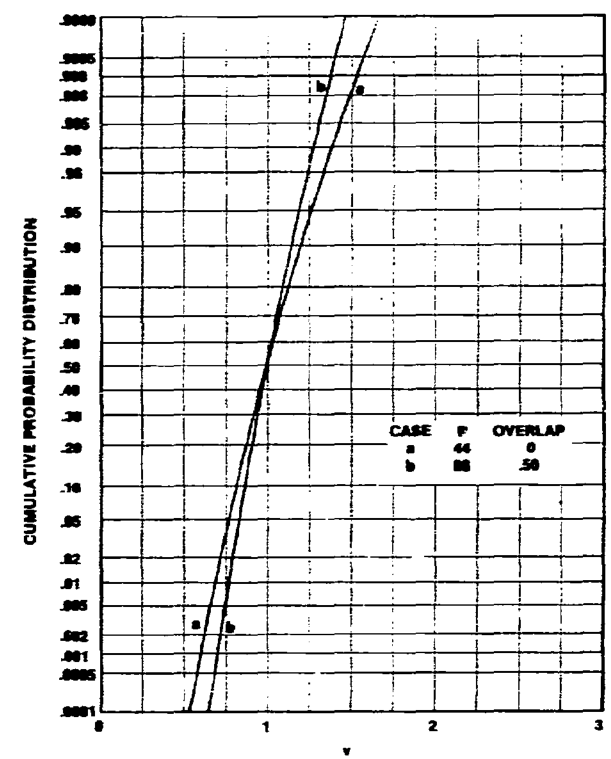


2B. BT=16

Figure 2. Exact Distribution for Hanning Window

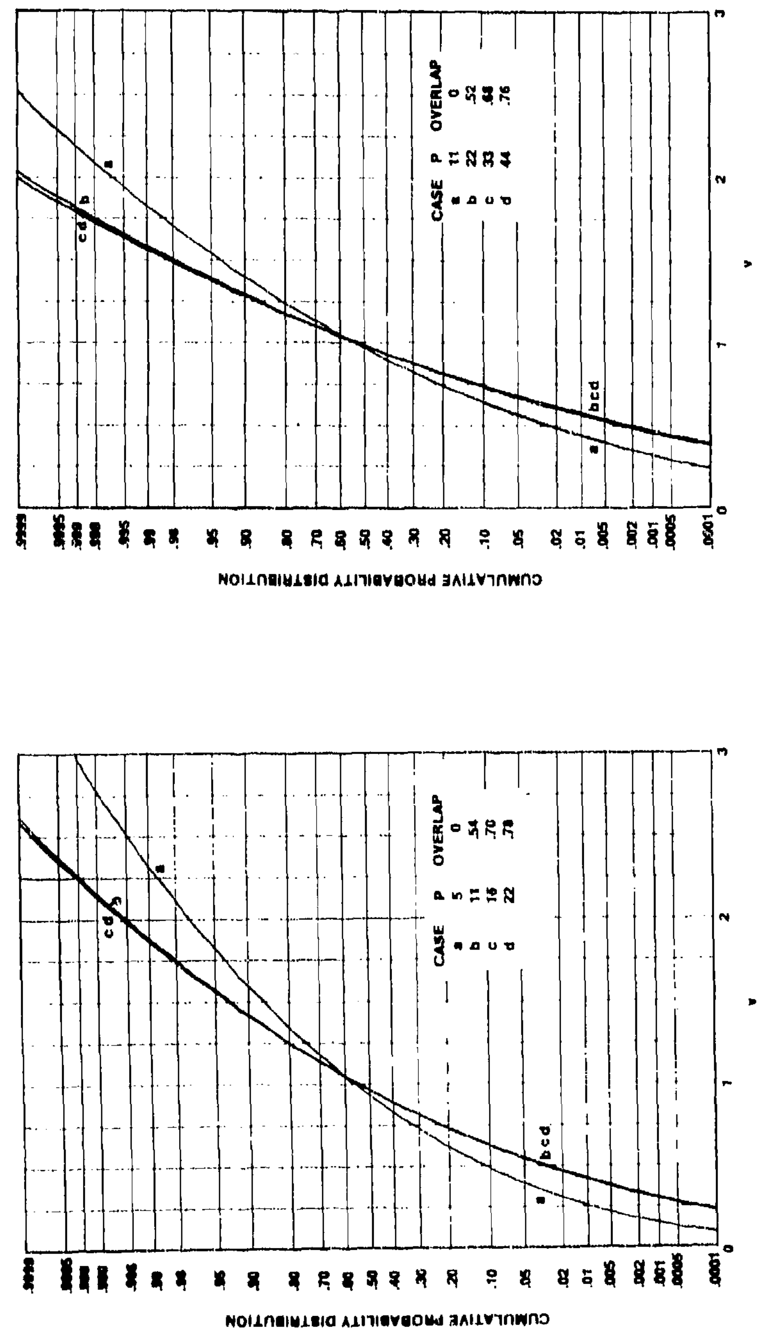


2C. BT=32



2D. BT=64

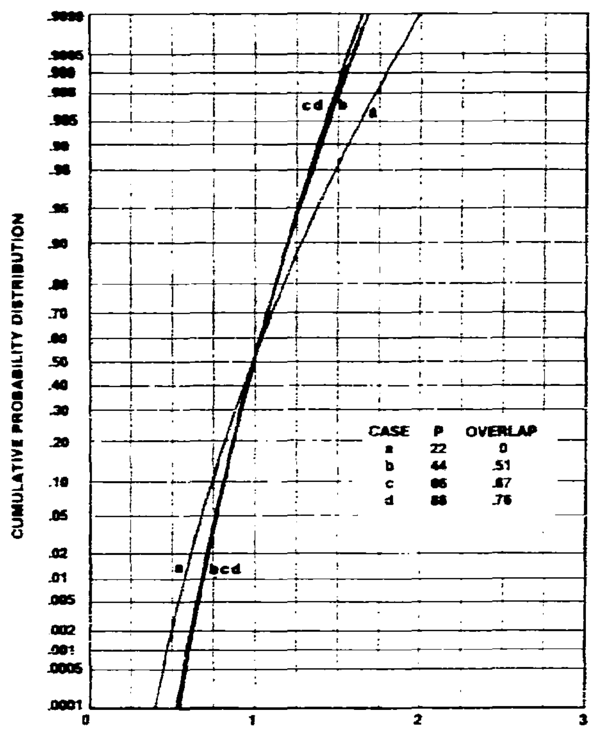
Figure 2. (Cont'd) Exact Distribution for Hanning Window



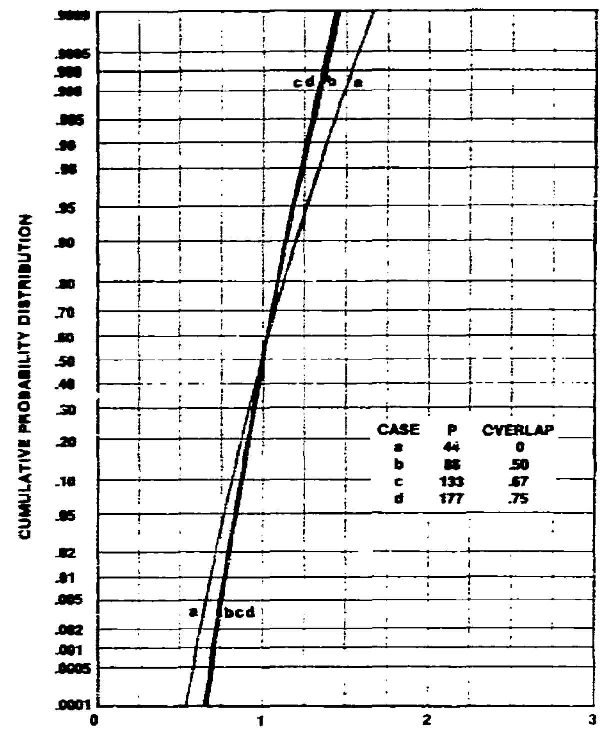
3A. BT=8

3B. BT=16

Figure 3. Approximation for Hanning Window

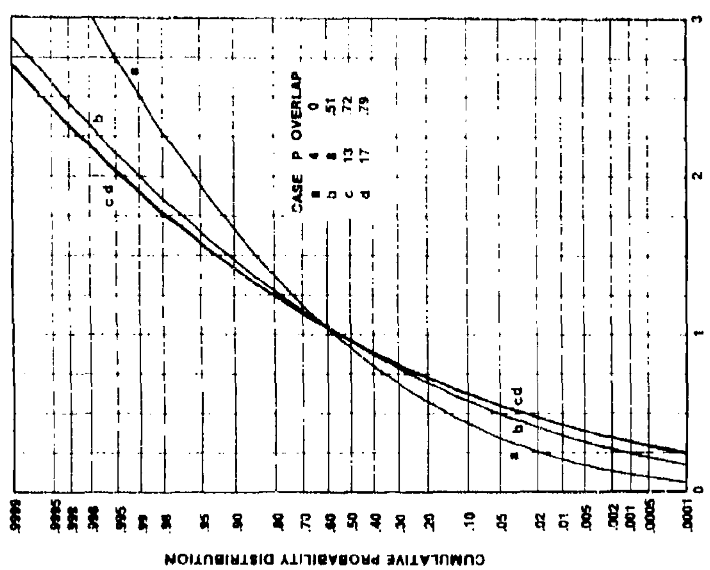


3C. BT=32

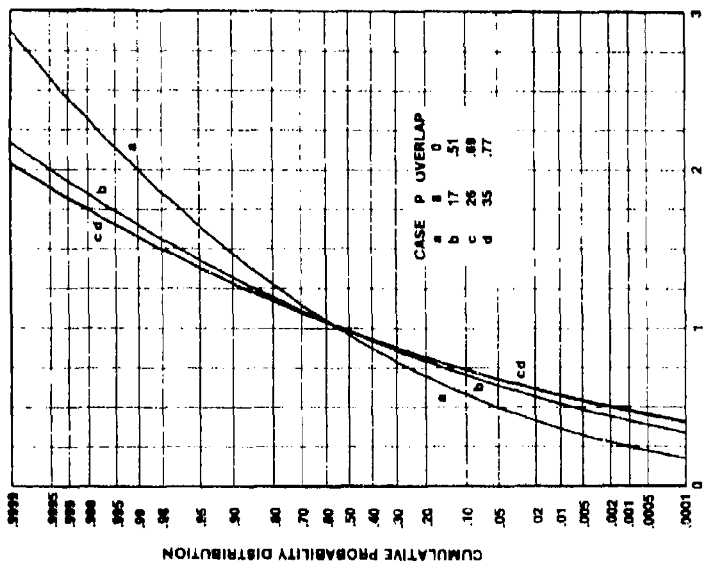


3D. BT=64

Figure 3. (Cont'd) Approximation for Hanning Window

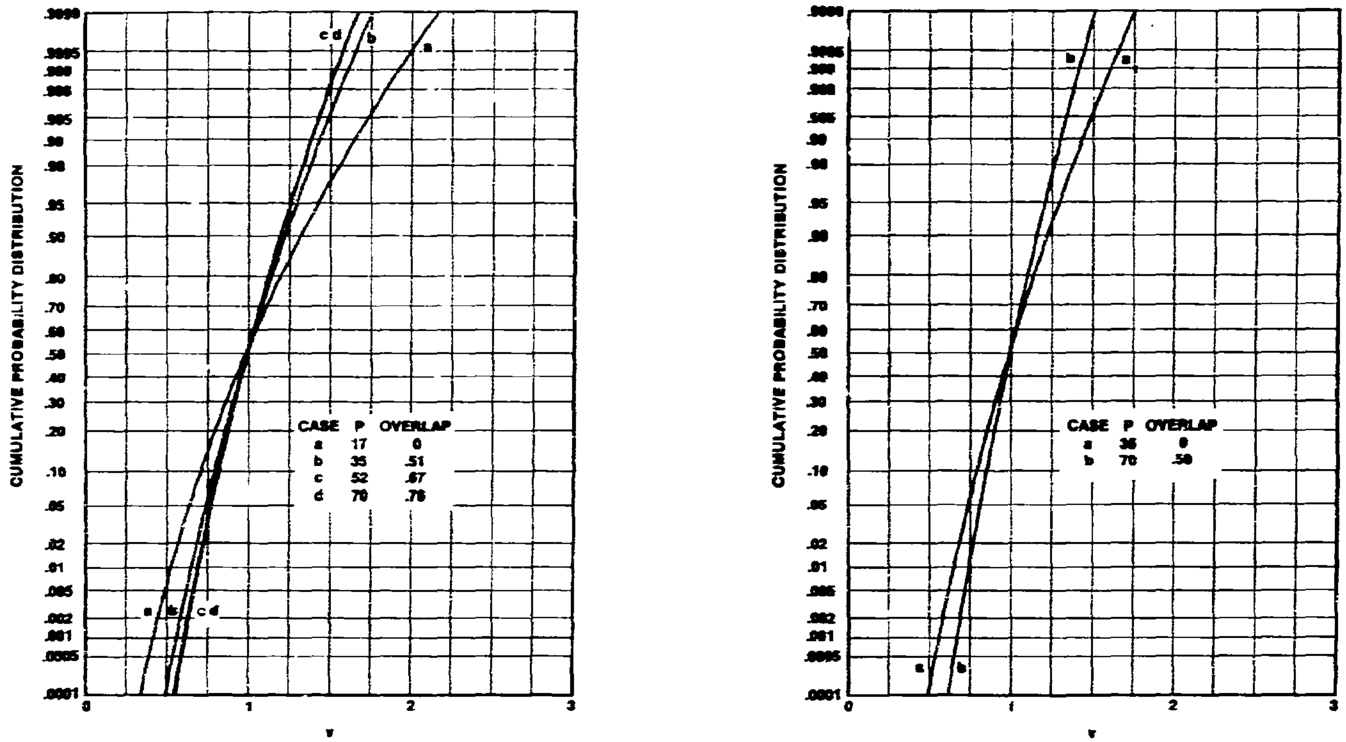


4A. BT=8



4B. BT=16

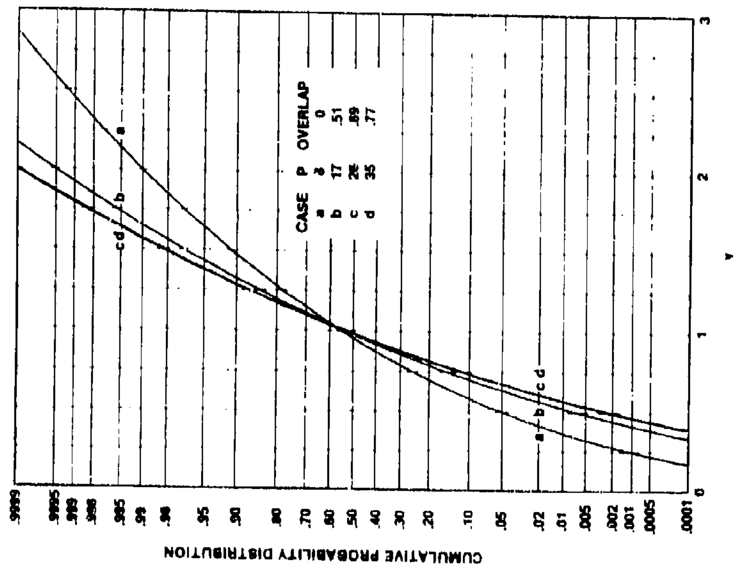
Figure 4. Exact Distribution for Cubic Window



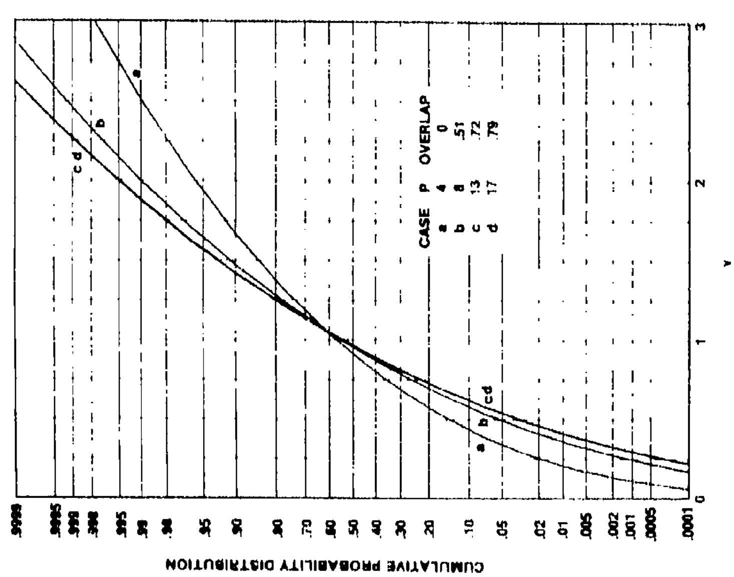
4C. BT=32

4D. BT=64

Figure 4. (Cont'd) Exact Distribution for Cubic Window

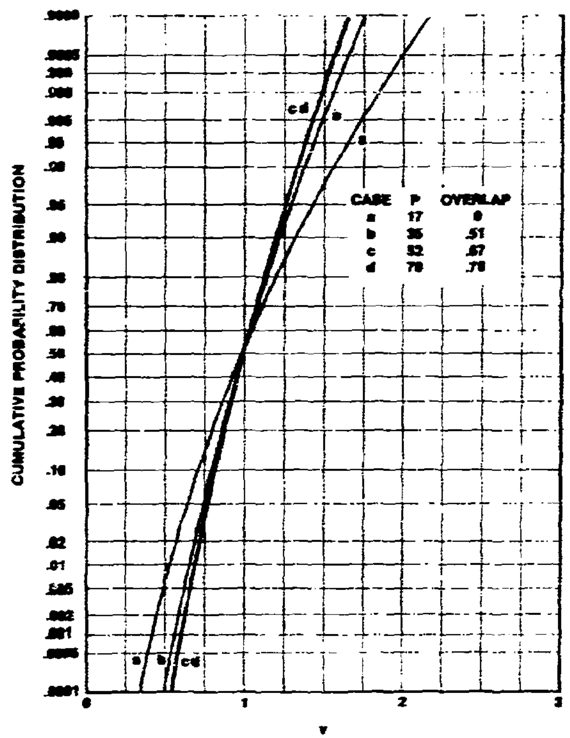


5B. BT=16

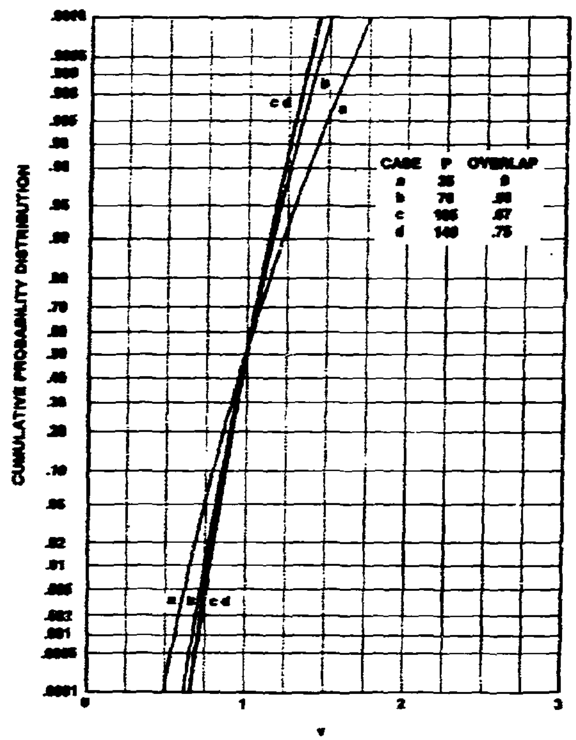


5A. BT=8

Figure 5. Approximation for Cubic Window



5C. BT=32



5D. BT=64

Figure 5. (Cont'd) Approximation for Cubic Window

FLUCTUATIONS OF CROSS SPECTRAL ESTIMATE

This topic is not directly related to the earlier material on autospectral estimation; however, it is an important observation and merits a comment. For two uncorrelated processes, x and y, the cross spectrum $G_{xy}(f) = 0$. However, the cross spectral estimate, $\hat{G}_{xy}(f)$, satisfies the equations (reference 2):

$$E\left\{\hat{G}_{xy}(f)\right\} = 0,$$

$$E\left\{\hat{G}_{xy}^{\,2}(f)\right\} = 0,$$

and

$$E\left\{\left|\hat{G}_{xy}(f)\right|^2\right\} = \frac{2}{K}\, G_{xx}(f) G_{yy}(f) \equiv 2\sigma^2, \tag{42}$$

where K is the equivalent degrees of freedom. Now, if K>>1, $\hat{G}_{xy}(f)$ is approximately complex Gaussian. Therefore, if we define the amplitude estimate

$$A \equiv \left|\hat{G}_{xy}(f)\right|, \tag{43}$$

it has probability density function

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x>0. \tag{44}$$

Then the mean of A is

$$E\left\{A\right\} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sigma = \left(\frac{\pi}{2}\, \frac{G_{xx}(f) G_{yy}(f)}{K}\right)^{\frac{1}{2}}, \tag{45}$$

which is a rather slow decay with K. Then the ratio of the mean amplitude, (45), to the square root of the product of the auto-spectra is

$$\frac{E\{A\}}{\left[G_{xx}(f)G_{yy}(f)\right]^{\frac{1}{2}}} = \left(\frac{\pi}{2K}\right)^{\frac{1}{2}} = \frac{1.253}{K^{\frac{1}{2}}} . \tag{46}$$

If, for example, K = 32, this ratio is .222 which is -6.55 dB; this is not very far down relative to unity coherence, though the two processes are uncorrelated.

Also,

$$\mathrm{Var}\ \{A\} = \left(2-\frac{\pi}{2}\right)\sigma^2 = \left(2-\frac{\pi}{2}\right)\frac{G_{xx}(f)G_{yy}(f)}{K} , \tag{47}$$

and, therefore,

$$\frac{\text{Standard deviation } \{A\}}{\text{Mean } \{A\}} = \left(\frac{4-\pi}{\pi}\right)^{\frac{1}{2}} = 0.52, \tag{48}$$

independent of K (or P). So for a zero cross-spectrum value, $A = |\hat{G}_{xy}(f)|$ will always have the same amount of <u>relative</u> variation, regardless of the number of pieces P (for large P); thus, on a dB scale, the "ribbon width" of the cross-spectral estimate is independent of P, when the two processes are uncorrelated.

DISCUSSION

An exact expression for the characteristic function of the power spectral estimate of a pure tone in Gaussian noise has been attained, and then specialized to noise-alone. In the noise-alone case, a numerical computation of the cumulative distribution function has been conducted. Comparison of the latter with an approximation utilizing only the mean and variance shows excellent agreement over a wide range of probabilities, regardless of the exact window, overlap, or the time-bandwidth product. This means that concentration on the equivalent degrees of freedom, particularly on its maximization, is sufficient for a probabilistic description of the auto-spectral estimate. Maximizing the equivalent degrees of freedom results in a narrower probability density function, as witnessed by the increased steepness of the cumulative probability distributions presented.

An entirely different method of auto- and cross-spectral estimation has been presented in references 7 and 8, and is mentioned here as a viable, attractive alternative, particularly for short data segments. Since only a few parameters are estimated, the estimates are potentially more stable, whereas the technique considered here (and in reference 1) assigns independent degrees of freedom to each and every frequency cell of interest and, therefore, requires the estimation of many more parameters.

Appendix A

DERIVATION OF CHARACTERISTIC FUNCTION

The first half of appendix C of reference 9 considers the Hermitian form

$$F = \mathbf{X}^H \mathbf{B} \mathbf{X} , \tag{A-1}$$

with mean and covariance of the complex random variable matrix $\mathbf{X}$,

$$E\{\mathbf{X}\} = \mathbf{m}, \quad \mathrm{Cov}\{\mathbf{X}\} = E\left\{(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^H\right\} = \mathbf{K} , \tag{A-2}$$

where matrix $\mathbf{X}$ is P x 1, and matrix $\mathbf{B}$ is Hermitian and P x P. Defining P x P matrix

$$\mathbf{A} = \mathbf{K}^{\frac{1}{2}} \mathbf{B} \mathbf{K}^{\frac{1}{2}} , \tag{A-3}$$

with corresponding normalized modal matrix $\mathbf{Q}$ and (diagonal) eigenvalue matrix $\boldsymbol{\lambda}$, we can express (A-1) as

$$F = \mathbf{V}^H \lambda \mathbf{V} = \sum_{k=1}^{P} \lambda_k |v_k|^2 , \tag{A-4}$$

where matrix $\mathbf{V}$ is P x 1 with mean and covariance

$$E\{\mathbf{V}\} = \mathbf{Q}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{m} \equiv \mu, \quad \mathrm{Cov}\{\mathbf{V}\} = \mathbf{I} . \tag{A-5}$$

Then a slight generalization* of the second half of appendix C of reference 9 (see also reference 10) yields the characteristic function of random variable F in (A-4) as

*We must also have $E\left\{(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^T\right\} = \mathbf{0}$, in addition to (A-2).

$$C(\xi) = \prod_{k=1}^{p} \left\{ (1-i\lambda_k\xi)^{-1} \exp\left(\frac{i\,|\mu_k|^2 \lambda_k \xi}{1-i\lambda_k\xi} \right) \right\} , \qquad \text{(A-6)}$$

where $\{\lambda_k\}$ and $\{\mu_k\}$ are the elements of matrices $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The cumulants of F follow easily from (A-6) as

$$c_n = (n-1)! \sum_{k=1}^{p} \lambda_k^n \left(1+n|\mu_k|^2 \right) . \qquad \text{(A-7)}$$

In particular, the first two cumulants are

$$\text{Mean}\,\{F\} = c_1 = \sum_{k=1}^{p} \lambda_k \,(1+|\mu_k|^2)$$

$$\text{Var}\,\{F\} = c_2 = \sum_{k=1}^{p} \lambda_k^2 \,(1+2|\mu_k|^2) . \qquad \text{(A-8)}$$

For the case of zero-mean variables, i.e., $\mathbf{m} = \mathbf{0}$, (A-5) yields $\boldsymbol{\mu} = \mathbf{0}$, and the characteristic function becomes

$$C(\xi) = \prod_{k=1}^{p} \left\{ (1-i\lambda_k\xi)^{-1} \right\} \quad \text{for zero-mean variables.} \qquad \text{(A-9)}$$

The cumulants are then

$$c_n = (n-1)! \sum_{k=1}^{p} \lambda_k^n \quad \text{for zero-mean variables.} \qquad \text{(A-10)}$$

(It is not necessary to evaluate $\mathbf{K}^{\frac{1}{2}}$ for eigenvalue purposes alone, because the eigenvalues $\{\lambda_k\}$ of matrix $\mathbf{A}$ defined in (A-3) are the same as the eigenvalues of $\mathbf{KB}$ or $\mathbf{BK}$.)

As a specific application of the general results above, we consider

$$\mathbf{B} = \mathbf{I}\ ,\ \mathbf{m} = [m_1 \dots m_P]^T,\quad \mathbf{K} = \left[K_{p-q}\right] \tag{A-11}$$

Then from (A-3), we see that $\mathbf{A} = \mathbf{K}$. In order to evaluate (A-8), we notice that

$$\sum_{k=1}^{P} \lambda_k = \sum_{p=1}^{P} A_{pp} = PK_o\ , \tag{A-12}$$

$$\sum_{k=1}^{P} \lambda_k |\mu_k|^2 = \mu^H \lambda \mu = \mathbf{m}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{Q} \lambda \mathbf{Q}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{m}$$

$$= \mathbf{m}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{A} \mathbf{K}^{-\frac{1}{2}} \mathbf{m} = \mathbf{m}^H \mathbf{m} = \sum_{k=1}^{P} |m_k|^2\ , \tag{A-13}$$

$$\sum_{k=1}^{P} \lambda_k^2 = \sum_{p,q=1}^{P} A_{pq} A_{qp} = \sum_{p,q=1}^{P} |K_{p-q}|^2 = \sum_{k=1-P}^{P-1} (P-|k|)|K_k|^2\ , \tag{A-14}$$

$$\sum_{k=1}^{P} \lambda_k^2 |\mu_k|^2 = \mu^H \lambda \lambda \mu = \mathbf{m}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{Q} \lambda \lambda \mathbf{Q}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{m}$$

$$= \mathbf{m}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{A} \mathbf{Q} \lambda \mathbf{Q}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{m} = \mathbf{m}^H \mathbf{K}^{-\frac{1}{2}} \mathbf{K}\mathbf{K}\mathbf{K}^{-\frac{1}{2}} \mathbf{m} = \mathbf{m}^H \mathbf{K} \mathbf{m}\ , \tag{A-15}$$

Then (A-8) yields

$$\text{Mean}\,\{F\} = c_1 = PK_o + \sum_{k=1}^{P} |m_k|^2\ ,$$

$$\text{Var}\,\{F\} = c_2 = \sum_{k=1-P}^{P-1} (P-|k|)|K_k|^2 + 2\,\mathbf{m}^H \mathbf{K} \mathbf{m}\ . \tag{A-16}$$

The specialization to zero-mean variables is obtained by dropping the last terms in (A-16).

Appendix B

DERIVATION OF COVARIANCE MATRIX

We are interested in deriving the two averages

$$E\left\{Y_{pn} Y_{qn}^*\right\} \quad \text{and} \quad E\left\{Y_{pn} Y_{qn}\right\} \tag{B-1}$$

because they are needed for appendix A and to see if the conditions required there are satisfied. We have, from (7),

$$E\left\{Y_{pn} Y_{qn}^*\right\} = \iint dt\ du\ \exp(-i2\pi f(t-u))\ E\left\{n(t)n^*(u)\right\}\ w\left[t-\frac{1}{2}L-(p-1)S\right] \cdot$$

$$w^*\left[u-\frac{1}{2}L-(q-1)S\right] . \tag{B-2}$$

Letting the noise correlation in (B-2) be denoted by $R_n(t-u)$, and its spectrum by G_n, (B-2) becomes

$$E\left\{Y_{pn} Y_{qn}^*\right\} = \int d\mu\ G_n(\mu) \int dt\ \exp(i2\pi(\mu-f)t)\ w\left[t-\frac{1}{2}L-(p-1)S\right] \cdot$$

$$\left\{\int du\ \exp(i2\pi(\mu-f)u)\ w\left[u-\frac{1}{2}L-(q-1)S\right]\right\}^*$$

$$= \int d\mu\ G_n(\mu)\left|W(f-\mu)\right|^2\ \exp\left[i2\pi(f-\mu)(q-p)S\right] , \tag{B-3}$$

This quantity is a function only of the difference of indices q and p.

If spectral window $|W|^2$ is narrower than the detail in noise spectrum G_n in the neighborhood of analysis frequency f, (B-3) simplifies to

$$E\left\{Y_{pn}Y_{qn}^*\right\} \cong G_n(f)\int d\mu\, |W(f-\mu)|^2 \exp\left[i2\pi(f-\mu)(q-p)S\right]$$

$$= G_n(f)\phi_w((q-p)S), \tag{B-4}$$

where

$$\phi_w(\tau) \equiv \int dt\ w(t)w^*(t-\tau) \tag{B-5}$$

is the autocorrelation function of data window w.

Now let

$$\frac{\phi_w(mS)}{\phi_w(0)} \equiv r_m . \tag{B-6}$$

Then

$$E\left\{Y_{pn}Y_{qn}^*\right\} = G_n(f)\phi_w(0)r_{q-p} , \tag{B-7}$$

and from (10),

$$\mathbf{K} = G_n(f)\phi_w(0)\,\mathbf{R} , \tag{B-8}$$

where P x P matrix

$$\mathbf{R} = \begin{bmatrix} 1 & r_1 & r_2 \cdots r_{P-1} \\ r_{-1} & \ddots & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ r_{1-P} & & 1 \end{bmatrix} \tag{B-9}$$

is Hermitian, Toeplitz, and non-negative definite.* For real weighting w, $\mathbf{R}$ is a real symmetric Toeplitz matrix. The matrix in (B-8) is the one required in (A-11) and (10).

*This property is easily proven by use of definitions (B-5) and (B-6).

The second quantity we desire is, by use of (7),

$$E\left\{Y_{pn}Y_{qn}\right\} = \iint dt\, du\, \exp(-i2\pi f(t+u))\, E\left\{n(t)n(u)\right\}\, w\left[t-\tfrac{1}{2}L-(p-1)S\right]\, .$$
$$w\left[u-\tfrac{1}{2}L-(q-1)S\right]\, . \qquad \text{(B-10)}$$

Letting the noise correlation in (B-10) be denoted by $\mathcal{R}_n(t-u)$, and its spectrum (Fourier transform) by $\mathcal{G}_n$, (B-10) becomes

$$E\left\{Y_{pn}Y_{qn}\right\} = \int d\mu\, \mathcal{G}_n(\mu) \int dt\, \exp(i2\pi(\mu-f)t)\, w\left[t-\tfrac{1}{2}L-(p-1)S\right]\, .$$
$$\int du\, \exp(-i2\pi(\mu+f)t)\, w\left[u-\tfrac{1}{2}L-(q-1)S\right]$$
$$= \int d\mu\, \mathcal{G}_n(\mu) W(f-\mu) W(f+\mu)\, \exp\left[-i2\pi f(L-2S+pS+qS)-i2\pi\mu(q-p)S\right]\, . \qquad \text{(B-11)}$$

If analysis frequency f is greater than the bandwidth B of spectral window W, then $W(f-\mu)$ and $W(f+\mu)$ do not overlap on the μ-axis, and (B-11) yields

$$E\left\{Y_{pn}Y_{qn}\right\} \cong 0 \text{ if } f>B. \qquad \text{(B-12)}$$

Thus, the property desired in appendix A (footnote to equation (A-6)) holds true if f>B.

Appendix C

NUMERICAL COMPUTATION OF CUMULATIVE PROBABILITY DISTRIBUTION

The numerical computation of the cumulative probability distribution Prob($\hat{g}<v$) is not accomplished here directly via the sum in (36). The reason is that, for large P, (36) is an alternating sum of terms of large magnitude, and accuracy is lost in the final result. Instead, the methods in references 3 and 4 are utilized on characteristic function (35): for a random variable limited to positive values, the cumulative probability distribution can be expressed as (reference 4)

$$P(v) = 1 - \frac{2}{\pi} \operatorname{Re}\left\{ \int_0^\infty d\xi \; \frac{f_i(\xi)}{\xi} \exp(-i\xi v) \right\} , \quad v>0, \qquad \text{(C-1)}$$

where $f_i(\xi)$ is the imaginary part of the characteristic function $f(\xi)$. We have $f_i(\xi)/\xi \sim E\{\hat{g}\} = 1$ as $\xi \to 0$. We approximate (C-1) according to

$$P(v) \cong 1 - \frac{2}{\pi} \operatorname{Re}\left\{ \int_0^{\xi_2} d\xi \; \frac{f_i(\xi)}{\xi} \exp(-i\xi v) \right\}, \qquad \text{(C-2)}$$

and then sample and approximate this expression according to

$$P(n\,\Delta v) \cong 1 - \frac{2}{\pi} \operatorname{Re}\left\{ \Delta\xi \sum_{k=0}^{L} w_k \frac{f_i(k\,\Delta\xi)}{k\,\Delta\xi} \exp\left[-ik\Delta\xi\, n\,\Delta v\right] \right\}, \qquad \text{(C-3)}$$

where $L\,\Delta\xi = \xi_2$, and $\left\{w_k\right\}$ are Trapezoidal weights of integration. We choose sampling increment

$$\Delta\xi = \frac{2\pi}{N\,\Delta v} , \qquad \text{(C-4)}$$

where N is chosen large enough that $f_i(\xi)/\xi$ does not change much in

$\Delta\xi$. Then

$$P(n\ \Delta v) = 1 - \frac{2}{\pi}\,\mathrm{Re}\left\{\Delta\xi \sum_{k=0}^{L} w_k \frac{f_i(k\ \Delta\xi)}{k\ \Delta\xi} \exp\left[-i2\pi kn/N\right]\right\}$$

$$= 1 - \frac{2}{\pi}\,\Delta\xi\ \mathrm{Re}\left\{\sum_{k=0}^{N-1} g_k \exp\left[-i2\pi kn/N\right]\right\} \tag{C-5}$$

where

$$g_k = \sum_{j=0}^{J} w_{k+jN} \frac{f_i((k+jN)\ \Delta\xi)}{(k+jN)\ \Delta\xi}\ ,$$

$$0 \le k \le N-1,\ \frac{L+1}{N} - 1 \le J \le \frac{L+1}{N}\quad . \tag{C-6}$$

Equation (C-5) is an N-point FFT; therefore, we choose N as a power of 2 for speed purposes.

The only remaining question is the choice of limit ξ_2 in (C-2). From (35), we know that

$$|f_i(\xi)| \le |f(\xi)| \le \frac{1}{\prod_{p=1}^{P} \max\left\{1,\ r_p\xi\right\}}, \tag{C-7}$$

where $r_p \equiv \lambda_p^{(R)}/P$. Therefore

$$|f_i(\xi)| \le \frac{1}{\prod_{p=1P}^{P} \max\left\{r_p\xi\right\}}\quad , \tag{C-8}$$

where IP can be 1 or 2 or ... or P. Therefore the error, E, in using (C-2) rather than (C-1) is bounded according to

$$E \le \frac{2}{\pi}\int_{\xi_2}^{\infty} \frac{d\xi}{\prod\limits_{p=IP}^{P}\left\{r_p\right\}} \xi^{-P+IP-2} = \frac{2}{\pi}\left[\prod_{p=IP}^{P}\left\{r_p\right\}\right]^{-1} \frac{\xi_2^{-P+IP-1}}{P+1-IP} \quad . \qquad \text{(C-9)}$$

This equation can be solved for

$$\xi_2 = P\left[\frac{2}{\pi E\prod\limits_{p=IP}^{P}\left\{\lambda_p^{(R)}\right\}(P+1-IP)}\right]^{\frac{1}{P+1-IP}} , \qquad \text{(C-10)}$$

with the guarantee that the error will be less than E if we choose ξ_2 according to (C-10). Since IP is not unique, we choose ξ_2 to be the minimum value over the range of IP=1, 2,..., P, for then the integration range in (C-2) can be kept to a minimum.

In summary, we:

specify Δv, E, P, BT

compute $\left\{\lambda_p^{(R)}\right\}$ and ξ_2

choose N = 1024 (say)

compute $\Delta\xi = \frac{2\pi}{N\ \Delta v}$

compute $L = \xi_2/\Delta\xi$

let J = (L+1)/N

compute (C-6)

compute FFT $\left\{g_k\right\}$ and printout (C-5)

choose N = 2048, go back to step 4, and observe change in printout.

A program for this procedure for the Hanning window follows. The subroutines TRIMXD and EIGVLD are presented in reference 11, and subroutines DPMCOS and DPMFFT are given in reference 12.

In order to execute the approximation (41), the line under statement number 2 is changed to CALL PROBA(BT, P, Y). This subroutine for the Cubic window is also presented below.

```
      INTEGER P
      DIMENSION X(51),Y(51),Z(200),YNORM(25)
      DATA YNORM/-3.71902,-3.29053,-3.09023,-2.87816,-2.57583,-2.32635,
     *-2.05375,-1.64485,-1.28155,-.84162,-.52440,-.25335,0.,.25335,.5244
     *,.84162,1.28155,1.64485,2.05375,2.32635,2.57583,2.87816,3.09023,
     *3.29053,3.71902/
      C=1.4405825        & HANNING
      CALL MODESG(Z,P)
      CALL SUBJEG(Z,0.,YNORM(1),3.,YNORM(25))
      CALL OBJCTG(Z,1150.,335.,2850.,2735.)
      DO 11 IBT=3,6
      BT=2.**IBT
      CALL SETSMG(Z,30,2.)
      CALL LINESG(Z,0,0.,YNORM(1))
      CALL LINESG(Z,1,0.,YNORM(25))
      CALL LINESG(Z,1,3.,YNORM(25))
      CALL LINESG(Z,1,3.,YNORM(1))
      CALL LINESG(Z,1,0.,YNORM(1))
      CALL SETSMG(Z,30,1.)
      DO 21 J=1,11
      CALL LINESG(Z,0,J*.25,YNORM(1))
21    CALL LINESG(Z,1,J*.25,YNORM(25))
      DO 22 J=2,24
      CALL LINESG(Z,0,0.,YNORM(J))
22    CALL LINESG(Z,1,3.,YNORM(J))
      CALL SETSMG(Z,30,2.)
      DO 23 I=1,51
23    X(I)=.06*(I-1)
      DO 1 IP=1,4
      P=(BT/C)*IP
      SL=(BT/C-1.)/(P-1)
      PRINT 2, BT,P,SL
2     FORMAT(////' BT ='E13.8,'      P ='I4,'      S/L ='E13.8)
      CALL PROBUP(BT,P,Y)
      PRINT 3, Y
3     FORMAT(/5E20.8)
      DO 4 I=1,51
      Q=MIN(Y(I),.999999)
      Q=MAX(Q,.000001)
4     Y(I)=TINORM(Q,$1)
      CALL LINESG(Z,51,X,Y)
1     CONTINUE
      CALL PAGEG(Z,0,1,1)
```

```
11    CONTINUE
      CALL EXITG(2)
      END
      SUBROUTINE PROBDP(BT,P,ANS)
      PARAMETER M=100  @ MAXIMUM NUMBER OF PIECES
      PARAMETER N=2048,N41=N/4+1
      DOUBLE PRECISION R(M,M),D(M),B(M),E(M),W(M),F(M),GR(N),GI(N),CO(N4
     $1),C,ERROR,DELV,PI,SL,TPE,XI2,PR,AT,DELXI,S,U,WFIDXI
      INTEGER P,P1
      DIMENSION ANS(1)
      C=1.44058258D0  @ HANNING
      IF(P.LE.M) GO TO 1
      J=M
      PRINT 2, P,J
2     FORMAT(/' P =',I4,' IS GREATER THAN M =',I3/)
      DO 3 J=1,51
3     ANS(J)=-1.
      RETURN
1     ERROR=1.D-12
      DELV=.06D0
      PI=3.14159265358979324D0
      P1=P-1
      SL=(BT/C-1.D0)/P1
      DO 4 K=0,P1
4     D(K+1)=U(K*SL)
      DO 5 J=1,P
      DO 5 K=1,P
      L=ABS(J-K)+1
5     R(J,K)=D(L)
      CALL TRIMXD(P,M,R,D,B)
      CALL EIGVLD(P,E,D,B,W,F)
      TPE=2.D0/(PI*ERROR)
      XI2=1.D100
      PR=0.D0
      DO 6 J=P,1,-1
      PR=PR+LOG(E(J))
      AT=1.D0/(P-J+1.D0)
      S=P*EXP(AT*(LOG(TPE*AT)-PR))
6     XI2=MIN(XI2,S)
      NF=N/2
7     DELXI=2.D0*PI/(NF*DELV)
      S=XI2/DELXI
      JC=(S+1.D0)/NF
      N1=NF-1
      DO 8 K=0,N1
      S=0.D0
      DO 9 J=0,JC
9     S=S+WFIDXI((K+J*NF)*DELXI)
      GR(K+1)=S
8     GI(K+1)=0.D0
      CALL DPMCOS(CO,NF)
      J=1.4427*LOG(NF)+.5
      CALL DPMFFT(GR,GI,CO,J,-1)
      S=2.D0*DELXI/PI
      DO 10 J=1,[illegible]
```

```
10    ANS(J)=1.D0-S*GR(J)
      IF(NF.EQ.N) RETURN
      DO 11 J=1,46,5
11    PRINT 12, ANS(J),ANS(J+1),ANS(J+2),ANS(J+3),ANS(J+4)
      PRINT 12, ANS(51)
12    FORMAT(/5E20.8)
      NF=N
      GO TO 7
      FUNCTION U(T)  @ HANNING
      DOUBLE PRECISION T,SI
      IF(T.GE.1.D0) GO TO 1
      SI=2.D0*PI*T
      U=2.D0/3.D0*(1.D0-T)*(1.D0+.5D0*COS(SI))+(.5D0/PI)*SIN(SI)
      RETURN
1     U=0.D0
      RETURN
      FUNCTION WFIDXI(X)
      DOUBLE PRECISION X,XTOP,AL,BE,BI,TEMP,SQ
      IF(X.GT.0.D0) GO TO 1
      XTOP=1.D100
      WFIDXI=.5D0
      RETURN
1     IF(X.GT.XTOP) GO TO 3
      AL=1.D0
      BE=-E(P)*X/P
      DO 2 JI=1,P1
      BI=E(JI)*X/P
      TEMP=AL+BE*BI
      BE=BE-AL*BI
2     AL=TEMP
      SQ=AL*AL+BE*BE
      IF(SQ*(X*ERROR)**2.GT.10.D0) XTOP=MIN(XTOP,X)
      WFIDXI=-BE/(SQ*X)
      RETURN
3     WFIDXI=0.D0
      RETURN
      END
      SUBROUTINE PROBA(BT,P,ANS)
      DOUBLE PRECISION GD,D,BV,F11L
      INTEGER P,P1
      DIMENSION ANS(1)
      C=1.82009566   @ CUBIC
      P1=P-1
      SL=(BT/C-1.)/P1
      B=1.
      DO 1 K=1,P1
1     B=B+2.*(1.-FLOAT(K)/P)*U(K*SL)**2
      CAPK=2.*P/B
      PRINT 101, CAPK
101   FORMAT(/' CAPK IS 'E15.8)
      B=P/B
      IB=B
      FB=B-IB
      CALL GAMMA(1.+FB,G,$2,$2)
      GD=LOG(DBLE(G))
      DO 5 K=1,IB
      D=FB+K
```

```
5     GD=GD+LOG(D)
      DO 3 K=1,51
      V=.06*(K-1)
      IF(V.GT.0.) GO TO 6
      ANS(K)=0.
      GO TO 3
6     BV=B*V
      ANS(K)=EXP(B*LOG(BV)-BV+F11L(DBLE(B+1.),BV)-GD)
3     CONTINUE
      RETURN
2     PRINT 4, B
4     FORMAT(/' PROBLEM AT B  = 'E15.8)
      RETURN
      FUNCTION F11L(A,XD)
      DOUBLE PRECISION SD,TD,AD,XD,A
      SD=1.D0
      TD=1.D0
      AD=A-1.D0
      DO 1 K=1,1000
      TD=TD*XD/(AD+K)
      SD=SD+TD
1     IF(ABS(TD).LE.1.D-8*ABS(SD)) GO TO 2
      PRINT 3,
3     FORMAT(/' 1000 TERMS'/)
2     F11L=LOG(SD)
      RETURN
      FUNCTION U(T)  @ CUBIC
      IF(T.GE.1.) GO TO 1
      U=1024./151.*(1.-T)**7
      IF(T.GE.0.75) RETURN
      U=U-8192./151.*(.75-T)**7
      IF(T.GE.0.5) RETURN
      U=U+28672./151.*(.5-T)**7
      IF(T.GE.0.25) RETURN
      U=U-57344./151.*(.25-T)**7
      RETURN
1     U=0.
      RETURN
      END
```

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