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THE EXISTENCE OF EIGENVALUES EMBEDDED IN THE CONTINUOUS SPECTRU--ETC(U)
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MRC Technical Summary Report # 1688

THE EXISTENCE OF EIGENVALUES EMBEDDED IN THE CONTINUOUS SPECTRUM OF ORDINARY DIFFERENTIAL OPERATORS

M. S. P. Eastham and J. B. McLeod

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

October 1976

(Received September 8, 1976)

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ABSTRACT

In answer to two questions raised by W. N. Everitt, we show that, given $p > 1$ and any countably infinite set of points on the positive λ-axis, there is a $q(x)$ in $L^p(0,\infty)$ for which the set of points constitutes the point-continuous spectrum associated with the equation $y''(x) + \{\lambda - q(x)\} y(x) = 0 \quad (0 \leq x < \infty)$ and some homogeneous boundary condition at $x = 0$.

AMS(MOS) Subject Classification - 34B25, 47E05, 81.34

Key Words - Ordinary differential operators, eigenvalues, continuous spectrum.

Work Unit Number 1 - Applied Analysis

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THE EXISTENCE OF EIGENVALUES EMBEDDED IN THE CONTINUOUS SPECTRUM OF ORDINARY DIFFERENTIAL OPERATORS

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1. Introduction

The purpose of this paper is to answer two questions raised by W. N. Everitt concerning the spectrum σ_α associated with the differential equation

$$y''(x) + \{\lambda - q(x)\}y(x) = 0 \quad (0 \leq x < \infty) \tag{1.1}$$

and the boundary condition

$$y(0) \cos \alpha + y'(0) \sin \alpha = 0 \quad , \tag{1.2}$$

where $q(x)$ is real-valued and continuous and α is a real constant. We introduce the usual division of σ_α into the point, continuous, and point-continuous spectrum:

$$\sigma_\alpha = P\sigma_\alpha \cup C\sigma_\alpha \cup PC\sigma_\alpha$$

[1, section 43; 4], and since $PC\sigma_\alpha$ is our concern in this paper, we repeat the definition that $PC\sigma_\alpha$ consists of those eigenvalues in σ_α which are not isolated points of σ_α.

If $q \in L^p(0,\infty)$ for some $p \geq 1$, it is well-known that

$$C\sigma_\alpha \cup PC\sigma_\alpha = [0, \infty)$$

[10, p. 112, Theorem 25] and that $PC\sigma_\alpha$ is empty if $p = 1$ (see [8] for references). Specific examples have been constructed which show that

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$PC\sigma_\alpha$ is not necessarily empty when $p > 1$ and these are described later in this section. The questions raised by Everitt [8, section 1, remark 7] are whether $PC\sigma_\alpha$ can contain a countably infinite number of points and whether $PC\sigma_\alpha$ is bounded above when $p > 1$. We answer these questions in the following theorem.

THEOREM. Let $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ be any sequence of positive real numbers. Then, given $p > 1$, there exists a real number α and a real-valued continuous function $q \in L^p(0, \infty)$ such that $PC\sigma_\alpha$ for (1.1)-(1.2) is precisely $\{\lambda_n\}$ $(n = 1, 2, \ldots)$.

Here $\{\lambda_n\}$ is permitted to be either bounded or unbounded. Despite the doubts expressed in [8, section 1] about the applicability of the inverse spectral theory of Gelfand and Levitan [9], we shall nevertheless use this theory in the simplified form of Levitan and Gasymov [11] to establish our theorem. We state the results that we need from [11] in section 2 below and we prove the theorem in section 3.

The first general result on the nature of $PC\sigma_\alpha$ was obtained by Wallach [14] as follows.

Let

$$\limsup_{x \to \infty} x|q(x)| < \infty \tag{1.3}$$

and denote the value of the lim sup by k. Then $PC\sigma_\alpha$ is bounded above by k^2 for all α.

Complementing this result, there are specific examples of $q(x)$ satisfying (1.3), constructed by different methods, for which $PC\sigma_\alpha$ consists of a single point μ in $(0, k^2)$ for some α. Any $q(x)$ satisfying (1.3) is of course in $L^p(0,\infty)$ $(p > 1)$. The first examples were given by von Neumann and Wigner [13] and Wallach [14], the idea here being to construct a differential equation (1.1), with $\lambda = \mu$, by calculating $y''(x)/y(x)$ for a suitable square-integrable $y(x)$. The asymptotic theory of Atkinson [2;8] and the use of certain step-functions for $q(x)$ [6;7] provide further examples. The inverse spectral theory of [9] was first used to produce a single point μ in $PC\sigma_\alpha$ by Moses and Tuan [12] and again by Chaudhuri and Everitt [4;8]. We refer to [5] and the references therein for further related work on $PC\sigma_\alpha$.

It should be mentioned that none of the above examples produces a value of μ arbitrarily near to k^2 in any sense, the best being a value near to $4\pi^{-2}k^2$ [6]. This leads to the following open question.

(*) Let $\epsilon > 0$ be given. Is there a $q(x)$ such that

$$\limsup_{x\to\infty} x|q(x)| = k < \infty$$

and, for some α, $PC\sigma_\alpha$ contains a point μ satisfying $\mu > (1-\varepsilon)k^2$?

(The related result of Borg [3] should be noted, that if (1.3) is replaced by the condition that

$$\int_0^x |q(t)|dt \leq k \log x$$

for all sufficiently large x, then $PC\sigma_\alpha$ is bounded above by k^2, and the result is best possible. However, the proof that the result is best possible does not answer (*).)

There is a second open question which arises out of the work in this paper. It is not clear whether or not the $q(x)$ which we construct to prove our theorem satisfies (1. 3). However, as we shall point out in section 3, the estimate (3. 15) suggests that our $q(x)$ does not satisfy (1. 3). We therefore pose the following question.

(**) *Is* $PC\sigma_\alpha$ *necessarily a finite set of points when (1. 3) holds?*

Acknowledgement. The main work on this paper was done while we were at the London Mathematical Society Symposium on Partial Differential Equations at Durham in July 1976. We are grateful to Professors D. E. Edmunds and L. E. Fraenkel for the invitation to join this symposium.

2. *Inverse spectral theory*

We state here the theorem from [11] that we use.

INVERSE SPECTRAL THEOREM. *Let* $\rho(\lambda)$ *be an increasing function of* λ *in* $(-\infty,\infty)$. *Let the following conditions* (a) *and* (b) *hold.*

(a) *If* $f \in L^2(0,\infty)$ *has compact support and if*

$$\int_{-\infty}^{\infty} E^2(\lambda)\, d\rho(\lambda) = 0 \quad ,$$

where

$$E(\lambda) = \int_0^{\infty} f(x) \cos(x\sqrt{\lambda}\,)dx \quad ,$$

then $f = 0$ *p. p.*

(b) *If*

$$\sigma(\lambda) = \begin{cases} \rho(\lambda) & (\lambda < 0) , \\ \rho(\lambda) - 2\pi^{-1}\sqrt{\lambda} & (\lambda \geq 0) , \end{cases} \tag{2.1}$$

then the function

$$\int_{-\infty}^{\Lambda} \cos(x\sqrt{\lambda}\,)\, d\sigma(\lambda) = \Phi_{\Lambda}(x)$$

converges boundedly to $\Phi(x)$ in any compact x-interval as $\Lambda \to \infty$, where $\Phi(x)$ has $M + 1$ locally integrable derivatives.

Then there is a number α and a function $q(x)$ with M locally integrable derivatives such that $\rho(\lambda)$ is the spectral function of (1.1)-(1.2) with this α and $q(x)$.

The formulae in [11] from which α and $q(x)$ may be determined are as follows. Let

$$F(x,t) = \lim_{\Lambda \to \infty} \int_{-\infty}^{\Lambda} \cos(x\sqrt{\lambda}\,)\cos(t\sqrt{\lambda}\,)\, d\sigma(\lambda) \tag{2.2}$$

and let $K(x,t)$ be the unique solution of the integral equation

$$F(x,t) + K(x,t) + \int_0^x K(x,s)\,F(s,t)ds = 0 \qquad (0 \leq t \leq x) . \tag{2.3}$$

Then

$$q(x) = 2\frac{d}{dx}K(x,x) \tag{2.4}$$

and α is determined by

$$\cot\alpha = -K(0,0) .$$

The relationship between $\rho(\lambda)$ and σ_α, for the particular α and $q(x)$ given by the inverse spectral theorem, is, we recall, that σ_α consists of those points in any neighbourhood of which $\rho(\lambda)$ is not constant.

Also, $PC\sigma_\alpha$ consists of those points of discontinuity of $\rho(\lambda)$ which are not isolated points of σ_α.

In our construction, we choose $\rho(\lambda)$ to have the form

$$\rho(\lambda) = \begin{cases} 0 & (\lambda \leq 0)\ , \\ 2\pi^{-1}\sqrt{\lambda} + \sum\limits_{\lambda_n < \lambda} a_n H(\lambda - \lambda_n) & (\lambda > 0)\ , \end{cases} \tag{2.5}$$

where

$$H(\lambda) = \begin{cases} 0 & (\lambda \leq 0)\ , \\ 1 & (\lambda > 0)\ , \end{cases} \tag{2.6}$$

and the a_n are positive numbers yet to be chosen. Here the λ_n are as in the statement of our theorem in section 1, and the $PC\sigma_\alpha$ corresponding to this $\rho(\lambda)$ *is certainly precisely* $\{\lambda_n\}$. We aim to show that, given $p > 1$, the a_n can be chosen sufficiently small to ensure that $q(x)$, as given by (2.4), is $L^p(0,\infty)$. At this point we make the standing assumptions that

$$a_n < 1 \quad \text{for all } n \tag{2.7}$$

and

$$\sum_1^\infty a_n < \infty, \qquad \sum_1^\infty a_n \sqrt{\lambda_n} < \infty\ . \tag{2.8), (2.9}$$

Already (2.8) is required for (2.5) to make sense when $\{\lambda_n\}$ is unbounded.

We have to check that conditions (a) and (b) in the inverse spectral theorem are satisfied. Since $a_n > 0$ in (2.5), it is clear that in (a) we have $E(\lambda) \equiv 0$ in $(0,\infty)$, whence $f = 0$ p.p. Also, by (2.1) and (2.5), (b) is certainly satisfied with $M = 0$ since (2.9) is assumed.

3. The construction of $q(x)$

By (2.1), (2.2), (2.5) and (2.8), we have

$$F(x, t) = \sum_{n=1}^{\infty} a_n \cos(x\sqrt{\lambda_n}) \cos(t\sqrt{\lambda_n}) \quad . \tag{3.1}$$

To solve (2.3), at least formally, we try

$$K(x, t) = \sum_{n=1}^{\infty} A_n(x) \cos(t\sqrt{\lambda_n}) \quad . \tag{3.2}$$

(We note that it would also be possible to use the more explicit form

$$K(x, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} \cos(x\sqrt{\lambda_m}) \cos(t\sqrt{\lambda_n}) \quad ,$$

but there appears to be no advantage to be gained by doing this.) On substituting (3.1) and (3.2) into (2.3) and setting the coefficient of $\cos(t\sqrt{\lambda_n})$ equal to zero, we obtain

$$a_n \cos(x\sqrt{\lambda_n}) + A_n(x) + a_n \sum_{j=1}^{\infty} A_j(x)\, I_{j,n}(x) = 0 \quad ,$$

that is,

$$A_n(x)\,\{a_n^{-1} + I_{n,n}(x)\} = - \sum_{\substack{j=1,\\ j \neq n}}^{\infty} A_j(x)\, I_{j,n}(x) - \cos(x\sqrt{\lambda_n}) \qquad (n = 1, 2, \ldots) \quad , \tag{3.3}$$

where

$$I_{j,n}(x) = \int_0^x \cos(s\sqrt{\lambda_j}) \cos(s\sqrt{\lambda_n})\, ds \quad . \tag{3.4}$$

The equations (3.3) form an infinite set of equations which are to determine the $A_n(x)$.

To justify working with an infinite set of equations, we first consider the truncated set of equations obtained by setting $A_n = 0$ $(n = N+1, N+2, \dots)$ in the first N equations and treating these N equations as equations for $A_1, \dots, A_N$. This truncated set has a unique solution, since it is standard that the integral equation (2.3) has a unique solution when we take

$$F(x,t) = \sum_{n=1}^{N} a_n \cos(x\sqrt{\lambda_n}) \cos(t\sqrt{\lambda_n}) \ ,$$

and we denote this solution by $A_n^{(N)}$ $(n = 1, \dots, N)$. (The $A_n^{(N)}$ are of course functions of x.)

By (3.4), we have, for $j \neq n$,

$$\begin{aligned} |I_{j,n}(x)| &= \left|\frac{1}{2}\int_0^x [\cos\{s(\sqrt{\lambda_n}+\sqrt{\lambda_j})\} + \cos\{s(\sqrt{\lambda_n}-\sqrt{\lambda_j})\}]ds\right| \\ &\leq |\sqrt{\lambda_n} - \sqrt{\lambda_j}|^{-1} \ , \end{aligned}$$

and so

$$|I_{j,n}(x)| \leq \gamma_n (1 \leq j < \infty, \ \ j \neq n) \ , \tag{3.5}$$

where

$$\gamma_n = \sup_{j \neq n} |\sqrt{\lambda_n} - \sqrt{\lambda_j}|^{-1} . \tag{3.5A}$$

Now we can deduce from the equations for $A_n^{(N)}$ the inequalities

$$|A_n^{(N)}| \leq (a_n^{-1} + I_{n,n})^{-1} (\gamma_n \|A^{(N)}\| + 1) \ \ (n = 1, \dots, N) \ , \tag{3.6}$$

where

$$\|A^{(N)}\| = \sum_{j=1}^{N} |A_j^{(N)}| \ .$$

Summing (3.6) over N we have

$$\|A^{(N)}\| \leq \left(\sum_{n=1}^{N} a_n \gamma_n\right) \|A^{(N)}\| + \sum_{n=1}^{N} a_n \ ,$$

and if we assume, as we may, that the a_n are chosen so that

$$\sum_{n=1}^{\infty} a_n \gamma_n \leq \frac{1}{2} \ , \qquad (3.7)$$

we have

$$\|A^{(N)}\| \leq 2 \sum_{n=1}^{\infty} a_n = C \ , \qquad (3.8)$$

say, where C is finite by (2.8) and, of course, C is independent of N and x. Substituting back into (3.6), we obtain

$$|A_n^{(N)}| \leq (a_n^{-1} + I_{n,n})^{-1} (C\gamma_n + 1) \ . \qquad (3.9)$$

If we now let $N \to \infty$, for fixed n and x, the Bolzano-Weierstrass theorem assures us that, at least through some subsequence of N,

$$A_n^{(N)}(x) \to A_n(x) \ , \qquad (3.10)$$

say, and that A_n also satisfies the inequality (3.9). By the usual diagonal process, we can choose the subsequence of N so that, as $N \to \infty$, (3.10) holds for all n, and we can now show that the A_n satisfy the equations (3.3). First, the infinite series in (3.3) converges by virtue of (3.9) (with A_n in place of $A_n^{(N)}$), (3.5), (2.8) and (3.7). Thus (3.3) makes sense. Next, given $\varepsilon > 0$, we can, by (3.9), (2.8) and (3.7), find R sufficiently large that, for all $N \geq R$,

$$\sum_{j=R}^{N} |A_j^{(N)}| < \frac{1}{2}\varepsilon, \quad \sum_{j=R}^{\infty} |A_j| < \frac{1}{2}\varepsilon \ .$$

Thus, taking specifically the n-th equation in (3.3) and using (3.5), we see that the difference between the right-hand sides of the equation and its truncated form does not exceed

$$\gamma_n \sum_{j=1}^{R} |A_j^{(N)} - A_j| + \gamma_n \varepsilon < 2\gamma_n \varepsilon$$

if N is sufficiently large. Hence the n-th equation in (3.3) holds as the limit of its truncated form as $N \to \infty$. Thus we have proved that the A_n, as defined by (3.10), satisfy the equations (3.3).

With $K(x,t)$ now defined by (3.2), the convergence of the series is assumed by (3.9) (with A_n in place of $A_n^{(N)}$), as are the formal manipulations leading to (3.3). Thus $K(x,t)$ is indeed the (unique) solution of (2.3).

It remains to show that

$$\frac{d}{dx} K(x,x) \in L^p(0,\infty) \ ,$$

at least if the a_n are chosen suitably. By (3.2), we have formally

$$\frac{d}{dx} K(x,x) = \sum_{n=1}^{\infty} \{A_n'(x)\cos(x\sqrt{\lambda_n}) - A_n(x)\sqrt{\lambda_n}\sin(x\sqrt{\lambda_n})\} \ . \qquad (3.11)$$

It is therefore necessary to obtain bounds on $A_n'(x)$.

If we differentiate the truncated form of (3.3), satisfied by the $A_n^{(N)}$, we obtain easily, bearing in mind (3.4) and (3.8), that

$$|\{a_n^{-1} + I_{n,n}(x)\} \frac{d}{dx} A_n^{(N)}(x) + \sum_{\substack{j=1,\\ j\neq n}}^{N} I_{j,n}(x) \frac{d}{dx} A_j^{(N)}(x)|$$

$$\leq \sqrt{\lambda_n} + C \ . \qquad (3.12)$$

Since (2.9) is assumed, the same type of argument that was used for $A_n^{(N)}$ now establishes that, for fixed n and x, $\frac{d}{dx} A_n^{(N)}$ converges through a subsequence of N which can be taken to be the same as before. If we assume further that the a_n are such that

$$\sum_1^{\infty} a_n \lambda_n < \infty \quad \text{and} \quad \sum_1^{\infty} a_n \gamma_n \sqrt{\lambda_n} < \infty \ , \qquad (3.13)$$

then the same type of argument can also be applied to $\frac{d^2}{dx^2} A_n^{(N)}$ with a bound (which we do not need to specify) corresponding to (3.9) and which is independent of N and x. In passing, we note that, by part (b) of the inverse spectral theorem in section 2, the convergence of the first series in (3.13) implies that our $q(x)$ is absolutely continuous in $[0, \infty)$.

It can be readily verified that the bound corresponding to (3.9) which we obtain for $\frac{d}{dx} A_n^{(N)} x$ from (3.12) is

$$\left|\frac{d}{dx} A_n^{(N)}(x)\right| \leq \{a_n^{-1} + I_{n,n}(x)\}^{-1} \{(C_1 + C^2)\gamma_n + \sqrt{\lambda_n} + C\}, \qquad (3.14)$$

where $\frac{1}{2}C_1$ is the sum of the series (2.9). Thus, for any fixed n, the sequence $\{\frac{d}{dx} A_n^{(N)}(x)\}$ is, as $N \to \infty$, uniformly bounded and equicontinuous for x in $[0, \infty)$, and so the sequence (or at least a subsequence) converges uniformly as $N \to \infty$ for x in any compact interval. It follows that

$$\frac{d}{dx} A_n^{(N)}(x) \to \frac{d}{dx} A_n(x)$$

and that the estimate (3.14) holds for $A_n'(x)$.

Now that (3.11) holds, we have, by (3.9) and (3.14) (for $A_n'(x)$),

$$\left|\frac{d}{dx} K(x,x)\right| \leq C' \sum_{n=1}^{\infty} \{a_n^{-1} + I_{n,n}(x)\}^{-1} (\gamma_n + \sqrt{\lambda_n} + \gamma_n \sqrt{\lambda_n} + 1), \tag{3.15}$$

where C' is a constant. Also,

$$\begin{aligned} a_n^{-1} + I_{n,n}(x) &= a_n^{-1} + \int_0^x \cos^2(s\sqrt{\lambda_n})ds \\ &= a_n^{-1} + \frac{1}{2}x + \frac{1}{4}\lambda_n^{-1/2} \sin(2x\sqrt{\lambda_n}) \\ &\geq a_n^{-1} - \frac{1}{4}\lambda_n^{-1/2} + \frac{1}{2}x \\ &\geq \frac{1}{2}(a_n^{-1} + x), \end{aligned} \tag{3.16}$$

provided that $a_n \leq 2\sqrt{\lambda_n}$, this being a further restriction on the a_n only if $\liminf_{n\to\infty} \lambda_n = 0$. In (3.16), we use Young's inequality

$$ab \leq a^r/r + b^s/s,$$

where $a > 0$, $b > 0$, $r > 1$, $s > 1$, and $r^{-1} + s^{-1} = 1$. This gives

$$a_n^{-1} + I_{n,n}(x) \geq \frac{1}{2} r^{1/r} s^{1/s} a_n^{-1/r} x^{1/s} . \tag{3.17}$$

Then, by (2.4) and (3.15), $q(x)$ is in $L^p(0,\infty)$ if

$$\int_0^\infty x^{-p/s} \{\sum_{n=1}^\infty a_n^{1/r}(\gamma_n + \sqrt{\lambda_n} + \gamma_n \sqrt{\lambda_n} + 1)\}^p dx < \infty , \tag{3.18}$$

and this can be arranged by choosing, as we may, s so that $1 < s < p$ and $\{a_n\}$ sufficiently small that the infinite series in (3.18) converges. This completes the proof of our theorem.

In section 1 we raised the question of whether or not our $q(x)$ satisfies (1.3). That it may not do so is suggested by (3.15) and (3.16). When $x = a_N^{-1}$ for a particular N, the term with $n = N$ in the series in (3.15) is, by (3.16), roughly of the size

$$x^{-1}(\gamma_N + \sqrt{\lambda_N} + \gamma_N \sqrt{\lambda_N} + 1) \tag{3.19}$$

for large N. The factor multiplying x^{-1} in (3.19) is certainly unbounded if $\{\lambda_n\}$ is unbounded, and it is also unbounded if $\{\lambda_n\}$ is bounded because then $\{\gamma_n\}$ is unbounded by (3.5A).

On the other hand, granted the convergence of the series in (3.18), it is clear from (3.15) and (3.17) that $q(x) = O(x^{-1/s})$ as $x \to \infty$. Thus, given $c < 1$, we can arrange that $q(x) = O(x^{-c})$ as $x \to \infty$.

REFERENCES

1. N. I. Akhiezer and I. M. Glazman, Theory of linear operators in Hilbert space, 2 vols. (Ungar, New York, 1961).

2. F. V. Atkinson, The asymptotic solution of second-order differential equations, Ann. Mat. Pura Appl. (4) **37** (1954), 347-378.

3. G. Borg, On the point spectra of $y'' + (\lambda - q(x))y = 0$, American J. Math. **73** (1951), 122-126.

4. J. Chandhuri and W. N. Everitt, On the spectrum of ordinary, second-order differential operators, Proc. Roy. Soc. Edinburgh A **68** (1969), 95-119.

5. M. S. P. Eastham, On the absence of square-integrable solutions of the Sturm-Liouville equation, Proc. Dundee Conference on Ordinary and Partial Differential Equations (to appear in Lecture Notes in Mathematics, Springer, Berlin, 1976).

6. M. S. P. Eastham, ibid. II, in preparation.

7. M. S. P. Eastham and M. L. Thompson, On the limit-point, limit-circle classification of second-order ordinary differential equations, Quart. J. Math. (Oxford) (2) **24** (1973), 531-535.

8. W. N. Everitt, On the spectrum of a second order linear differential equation with a p-integrable coefficient, Applicable Analysis **2** (1972), 143-160.

9. I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spectral function, American Math. Soc. Translations **1** (1955), 253-304.

10. I. M. Glazman, Direct methods of qualitative spectral analysis of singular differential operators (I. P. S. T., Jerusalem, 1965).

11. B. M. Levitan and M. G. Gasymov, Determination of a differential equation by two of its spectra, Russian Math. Surveys **19** (2) (1964), 1-63.

12. H. E. Moses and S. F. Tuan, Potentials with zero scattering phase, Nuovo Cimento **13** (1959), 197-206.

13. J. von Neumann and E. Wigner, Über merkwürdige diskrete Eigenwerte, Physikalische Zeitschrift **30** (1929), 465-467.

14. S. Wallach, On the location of spectra of differential equations, American J. Math. **70** (1948), 833-841.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #1688	2. GOVT ACCESSION NO. (9)	3. RECIPIENT'S CATALOG NUMBER Technical
4. TITLE (and Subtitle) (6) THE EXISTENCE OF EIGENVALUES EMBEDDED IN THE CONTINUOUS SPECTRUM OF ORDINARY DIFFERENTIAL OPERATORS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) M. S. P. Eastham and J. B. McLeod		8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709	(11)	12. REPORT DATE October 1976
		13. NUMBER OF PAGES 15 (12) 18p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (14) MRC-TSR-1688		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ordinary differential operators, eigenvalues, continuous spectrum.

L superscript

GREATER THAN INFINITY LAMBDA

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In answer to two questions raised by W. N. Everitt, we show that, given $p > 1$ and any countably infinite set of points on the positive λ-axis, there is a $q(x)$ in $L^{(p)}(0, \infty)$ for which the set of points constitutes the point-continuous spectrum associated with the equation $y''(x) + (\lambda - q(x))y(x) = 0$ $(0 \leq x < \infty)$ and some homogeneous boundary condition at $x = 0$.

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