AD-A034 467 WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 5/9
SELF SIMILAR SOLUTION OF PLASMA EQUATIONS.(U)
NOV 76 K E LONNGREN DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1698 NL

1 OF 1
ADA034467

END
DATE
FILMED
2-77

ADA034467

MRC Technical Summary Report #1698

SELF SIMILAR SOLUTION OF PLASMA EQUATIONS

K. E. Lonngren

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

November 1976

(Received November 1, 1976)

DDC
RECEIVED
JAN 18 1977
A

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

SELF SIMILAR SOLUTION OF PLASMA EQUATIONS[‡]

K. E. Lonngren[†]

Mathematics Research Center
University of Wisconsin
Madison, Wisconsin 53706

Technical Summary Report #1698
November 1976

ABSTRACT

This paper describes the method of self similar solution of partial differential equations and reviews its application to several problems found in plasma physics.

AMS(MOS) Subject Classification - 35K05, 35K55

Key Words - Heat Equation, Nonlinear Equations Systems

Work Unit Number 3 - Applications of Mathematics

ACCESSION for
NTIS White Section
DDC Buff Section
UNANNOUNCED
JUSTIFICATION
BY
DISTRIBUTION/AVAILABILITY CODES
Dist. AVAIL. and/or SPECIAL
A

† On leave from the University of Iowa.

‡ Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation Grant No. ENG 76-15645.

Presented at the NSF sponsored work shop on Plasma Physics, Ahmedabad, India, November 29 – December 11, 1976.

SELF SIMILAR SOLUTION OF PLASMA EQUATIONS[‡]

K. E. Lonngren[†]

Mathematics Research Center
University of Wisconsin
Madison, Wisconsin 53706

Near the end of the nineteenth century, Boltzmann (1894) noted in his study of the linear diffusion equation, that the two independent variables space x and time t could be combined into a new independent variable ξ where $\xi = \xi(x, t)$. With this relation, the diffusion equation which is a partial differential equation (PDE) could be transformed into an ordinary differential equation (ODE). Boltzmann had the "Ansatz" that the new variable should be $\xi = x/\sqrt{t}$.

At approximately the same time, Sophus Lie (1881) attempted to construct a general integration theory for differential equations utilizing the theory of algebra. Using the idea of continuous groups of transformations, he was able to reduce the order of an ODE and in some cases obtain a solution. Although not developing a general integration theory, he also examined some first and second order PDE.

In 1952, Morgan (1952) and Michal (1952) presented an interesting simplification of Lie's work to obtain a complete picture of the group structure of partial differential equations. Using their ideas, as

---

† On leave from the University of Iowa.

‡ Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation Grant No. ENG 76-15645.

Presented at the NSF sponsored work shop on Plasma Physics, Ahmedabad, India, November 29 – December 11, 1976.

summarized in the books by Ovsjannikov (1962), Ames (1965, 1972) and Bluman and Cole (1974), it is now possible to remove the mystique surrounding Boltzmann's "Ansatz" and, in fact, have a methodical procedure to construct self similar variables $\xi = \xi(x, t)$ and to examine the self-similar behavior of partial differential equations.

An alternative approach to examine the self similar behavior of PDE is to use the ideas of dimensional analysis to construct the self similar variables. This approach, as outlined in Sedov (1959), does not, however, yield the extensive opportunities that the group theory approach affords and will not be discussed further.

To date, the major application of self similar analysis has been restricted to the areas of fluid mechanics and heat transfer where extensive application to boundary layer phenomena has been made. Application to physiological and electrical circuit problems has also recently been discussed (Shen (1976), Lonngren et al. (1975)). Plasma physics is rich in phenomena which fall into the self-similar category and should receive equivalent attention. To date, it has not. It is to this end of stimulating our colleagues that this review is directed.

In Section II, the detailed procedure to obtain the self-similar variables and the resulting ODE will be presented. We shall focus on the linear diffusion equation as being the vehicle to lead to an understanding of the procedure. Certain constraints imposed by either boundary or initial conditions or by conservation laws will be discussed.

In Section III, this procedure shall be applied to several problems which are found in plasma physics and that we have treated, mainly with colleagues at Iowa. They include:

(1) One-dimensional diffusion where the diffusion coefficient is nonlinear or inhomogeneous as might be found in an electron cloud expansion, multipole experiments, or in studying the evolution of the distribution function of particles in plasmas supporting various aperiodic instabilities;

(2) Diffusion in more than one dimension with the inclusion of a preferred direction of drift;

(3) The Korteweg-de Vries equation which can be used to model ion acoustic waves;

(4) Sets of ion acoustic wave equations;

(5) The Vlasov model of a plasma.

In each of the problems that is discussed, references to other theoretical and experimental work will be made.

Self-similarity is not the panacea to solve all problems. Some difficulties that we've encountered are that the ODE with transformed boundary conditions may not be amenable to solution, neither analytical nor numerical. Second, if solvable mathematically, the solution may not describe a physically interesting phenomena. Third, the technique is limited to problems where no scale length nor time scale such as fixed boundaries exist in the problem.

II Self-Similar Procedure

In order to find the similarity variables, we make use of a theory growing from the Lie theory of groups where it has been shown that the similarity variables are identical to the invariants of a particular one (or more) parameter group of transformations. We shall briefly outline the procedure, details and references can be found in the texts by Ames (1965, 1972). We shall examine the one-dimensional linear diffusion equation:

$$\rho_{xx} - \rho_t = 0 \tag{1}$$

where the subscripts denote differentiation with respect to x and t.

We shall define a one parameter ("a", a is positive and real) group G as:

$$G = \begin{cases} \rho = a^{\alpha} \bar{\rho} \\ x = a^{\beta} \bar{x} \\ t = a^{\gamma} \bar{t} \; . \end{cases} \tag{2}$$

This is called the "linear" group. Other groups exist and there is a "most general" group called the "infinitesimal group". This latter group yields all possible similarity variables but one is soon lost in a sea of algebraic formulas that it is difficult to discern the procedure in the reams of scratch paper. One can refer to Ames (1965, 1972), Bluman and Cole (1974) or Shen (1976) for examples. It is not, however, entirely hopeless, as computer programs can now be written to handle large arrays of simultaneous symbolic algebraic equations.

In (2), α, β, and γ are constants which are determined such that (1) is "(absolutely) constant conformally invariant" under the group G (Ames 1965, 1972). A function F(y) is said to be "constant conformally invariant" (CCI) under G if $F(y) = f(a)\, F(\bar{y})$ where f(a) is some function of the parameter a . If $f(a) \equiv 1$, the constant conformal invariance is called "absolute". (ACCI)

Substituting (2) in (1), we write

$$a^{\alpha-2\beta}\, \bar{\rho}_{\bar{x}\bar{x}} - a^{\alpha-\gamma}\, \bar{\rho}_{\bar{t}} = 0 \; . \tag{3}$$

For (3) to be ACCI under the transformation group G one requires

$$\alpha - 2\beta = \alpha - \gamma \tag{4}$$

or $\gamma = 2\beta$. We shall defer until later the further specification of these constants.

Instead, we now seek to determine the "invariants" of the transformation group G . This is achieved by employing a theorem from group theory (Ames (1965), (1972)). The invariants are obtained from $QI \equiv 0$ where I is an invariant and Q is the operator

$$Q \equiv \left.\frac{\partial \bar{\rho}}{\partial a}\right|_{a=1} \frac{\partial}{\partial \rho} + \left.\frac{\partial \bar{x}}{\partial a}\right|_{a=1} \frac{\partial}{\partial x} + \left.\frac{\partial \bar{t}}{\partial a}\right|_{a=1} \frac{\partial}{\partial t}$$

$$= -\alpha \rho \frac{\partial}{\partial \rho} - \beta x \frac{\partial}{\partial x} - \gamma t \frac{\partial}{\partial t} \; . \tag{5}$$

The solutions of $QI \equiv 0$ are obtained by solving the Lagrange subsidiary equations

$$\frac{d\rho}{-\alpha\rho} = \frac{dx}{-\beta x} = \frac{dt}{-\gamma t} . \tag{6}$$

According to the theorem developed by Morgan (1952), these "invariants" are the self similar variables. Solutions of (6) are

$$\phi(\xi) = \frac{\rho(x,t)}{t^{\alpha/\gamma}} \quad \text{and} \quad \xi = \frac{x}{t^{\beta/\gamma}} . \tag{7}$$

From (4), we found that $\beta/\gamma = 1/2$. Therefore the Boltzmann transformation is recovered. Note that we could have combined (6) in a different order and obtained

$$\phi'(\xi') = \frac{\rho(x,t)}{x^{\alpha/\beta}} \quad \text{and} \quad \xi' = \frac{t}{x^{\gamma/\beta}} \tag{8}$$

Having found the self-similar variables, let us transform the PDE (1) using the self similar variables (7) into an ODE. The result is

$$\phi_{\xi\xi} + \frac{\xi}{2}\phi_{\xi} - \frac{\alpha}{\gamma}\phi = 0 \tag{9}$$

or

$$\phi_{\frac{\xi}{2}\frac{\xi}{2}} + 2(\xi/2)\,\phi_{\xi/2} - 4\frac{\alpha}{\gamma}\phi = 0 . \tag{10}$$

Writing the ODE in the form (10) allows us to recognize that its solution can be written in terms of complementary error functions (Gautschi (1964))

$$\phi = A i^{2\frac{\alpha}{\gamma}} \operatorname{erfc}(\tfrac{\xi}{2}) + B i^{2\frac{\alpha}{\gamma}} \operatorname{erfc}(-\xi/2) \tag{11}$$

where $i^{2\frac{\alpha}{\gamma}}$ is an ordering parameter and

$$
\begin{aligned}
i^{-1} \operatorname{erfc} \xi/2 &= \frac{2}{\sqrt{\pi}} \, \epsilon^{-\xi^2/4} \\
i^{0} \operatorname{erfc} \xi/2 &= \operatorname{erfc} \frac{\xi}{2} \\
i^{n} \operatorname{erfc} \xi/2 &= \int_{\xi/2}^{\infty} i^{n-1} \operatorname{erfc} t\,dt \qquad n = 0, 1, 2, \ldots\ .
\end{aligned}
\tag{12}
$$

At this stage, the parameter α/γ is still arbitrary. We shall specify it to satisfy boundary conditions or a conservation law. We note that two boundary conditions on $\rho(x, t)$ have necessarily "consolidated" into one for ϕ, namely

$$
\left.
\begin{aligned}
\rho\,(x = \infty, t) &= 0 \\
\rho(x, t = 0) &= 0
\end{aligned}
\right\}. \Longrightarrow \phi(\xi = \infty) = 0
\tag{13}
$$

The third boundary condition could have one of two forms which would yield self similar solutions. They are

$$\rho(x = 0, t) = \text{constant} \tag{14}$$

$$\int_0^\infty \rho dx = \text{constant}\ . \tag{15}$$

Since $\rho(x = 0, t)$ transforms via (7) to $\phi(\xi = 0)$, we note that (14) requires that $\alpha/\gamma = 0$. The self similar solution (11) for this boundary condition is

$$\rho = \phi = A\,\mathrm{erfc}(\xi/2) = A\,\mathrm{erfc}\left(\frac{x}{2\sqrt{t}}\right) \tag{16}$$

where the constant $B = 0$ in order to satisfy (13).

Although not proven to our knowledge but generally accepted as a "Similarity Postulate" (Moran and Gaggioli (1969)), the conservation law given in (15) should be invariant under the group transformation in order to have similarity solutions. Applying (2) to (15), we write

$$\int_0^\infty \rho\, dx = a^{\alpha+\beta} \int_0^\infty \bar{\rho}\, d\bar{x} . \tag{17}$$

For this to be ACCI, we have

$$\frac{\alpha}{\gamma} = -\frac{\beta}{\gamma} = -\frac{1}{2} .$$

The self similar solution (11) that satisfies this conservation law and (13) is

$$\rho(x,t) = \frac{\phi}{\sqrt{t}} = \frac{2A'}{\sqrt{\pi}\,\sqrt{t}}\,\epsilon^{-x^2/4t} \tag{18}$$

where the constant A' can be determined by evaluating (15) using (18).

The procedure, as described in this section will be applied to several examples found in plasma physics in the next section.

III Applications to Plasma Physics

Plasmas are a particularly interesting and fruitful area in which to apply the self similar technique. This is particularly true when it can be argued that there is no scale length that is important. However, there usually is a relevant scale length such as the Debye length in most plasma problems, at least in some initial stage. The self similar analysis may, in those cases, yield valuable information for some asymptotic state (Barrenblatt and Zel'dovich (1972)).

In the following, we shall select several examples from plasmas which fall into the self-similar class. To analyze each problem, we follow the procedure given in Section II and only describe the physical phenomena, list the PDE, the self similar variables, the ODE, and if possible, the solution to the ODE without repeating the procedure for each problem. References for each problem are also presented.

A. Linear 1-D Diffusion Equation, Inhomogeneous Diffusion Coefficient

In studies of the evolution of the distribution function of particles in plasmas supporting various aperiodic instabilities or in calculations involving the forward scattering of photons by plasmons, it has been found that the problem could be modeled with a diffusion equation (Peyraud and Coste (1976)).

$$\rho_t = [x^m \rho_x]_x \quad . \tag{19}$$

Using the linear group $G(2)$, we find the self similar variables to be

$$\xi = \frac{x}{\frac{1}{t^{2-m}}} \quad \text{and} \quad \phi = \frac{\rho}{t^{\alpha/\gamma}} \ . \tag{20}$$

Equation (19) transforms to

$$\frac{\alpha}{\gamma} \phi + \frac{\xi}{m-2} \phi_{\xi} = (\xi^{m} \phi_{\xi})_{\xi} \ . \tag{21}$$

The requirement of "consolidation" (13) specifies that $\frac{\beta}{\gamma} = \frac{1}{2-m} > 0$ or $m < 2$.

The constant α/γ can be specified to satisfy two boundary conditions, namely (14) and (15). The solutions for the two cases $\rho(0,t) =$ constant and $\int_0^{\infty} \rho \, dx =$ constant are (Lonngren (1976)):

$$\rho(x,t) = K \left\{ \int \frac{1}{\xi^{m}} \exp\left[-\left(\frac{\xi^{2-m}}{(2-m)^{2}} \right) \right] d\xi - 1 \right\} \tag{22}$$

and

$$\rho(x,t) = \frac{K}{t^{1/(2-m)}} \exp\left[- \frac{x^{2-m}}{t(2-m)^{2}} \right] \tag{23}$$

respectively and where K is a constant.

As the diffusion equation can be also used to model a distributed RC transmission line (eg. Lonngren, et al. (1975)), there can be a practical application of this result. In particular, let us assume that all the capacitors have the same value and the resistors are distributed inhomogeneously such that $R \sim R_0 \, x^{-m}$. If such a line were constructed and it were sufficiently short such that exponential term in (23) remained approximately constant, then the voltage response of the line to an impulse source would be proportional to $1/t^{(1/2-m)}$. Such a line has been used in biomedical applications for the value $m = 1$ (Gagné and Poussart (1976)).

B. *Nonlinear 1 - D Diffusion Equation*

Several phenomena in plasma physics can be modeled with the nonlinear diffusion equation.

$$[\rho^n \rho_x]_x - \rho_t = 0 . \tag{24}$$

For example: (1) Recent experiments in multipoles have confirmed that in certain regions, the diffusion coefficient for particles across the magnetic field depends on (the density of the particles)$^{-1/2}$. (Drake (1973), Berryman (1976)); (2) In studying the skin current penetration into turbulent plasmas where the conductivity depends on (the local electric field)^n (Hirose et al. (1970)), it has been shown that the governing equation is of the form of (24) (Hirose and Alexeff (1973)). An examination of this equation allowed us to comment on penetration times under various stages of ion acoustic turbulence (Lonngren, et al. (1974)). (3) As the conductivity of the turbulent plasma saturated with increasing field (Hirose et al. (1970)), we could model the conductivity as $\sigma \sim \sigma_0 \exp(-E)$ and have been able to obtain (24) with $n = -1$. (Ahmadi, et al. (1976)). A similar equation describes the expansion of a Maxwellianized electron cloud into a vacuum (Lonngren and Hirose (1976)).

For these problems, the self similar variables are of the form

$$\xi = \frac{x}{t^{\beta/\gamma}} \quad \text{and} \quad \phi = \frac{\rho}{t^{\frac{1}{n}(2\frac{\beta}{\gamma} - 1)}} \tag{25}$$

where the parameter β/γ is chosen to satisfy the boundary conditions or conservation laws and ϕ satisfies

$$(\phi^n \phi_\xi)_\xi + \frac{\beta}{\gamma} \xi \phi_\xi - \frac{1}{n}(2\frac{\beta}{\gamma} - 1)\phi = 0 \ . \tag{26}$$

Again the boundary conditions (14) and (15) can be applied. Using (15), we find that $\beta/\gamma = -\frac{1}{n}(2\,\beta/\gamma - 1)$ and (26) can be directly integrated (Ames (1965), Gilding and Peletier (1976)) to yield

$$\phi(\xi) = \begin{cases} = \left\{1 - \left[\frac{n}{n+2}\right]\frac{\xi^2}{2}\right\}^{\frac{1}{n}} & \xi < \xi_0 \\ = 0 & \xi > \xi_0 \end{cases} \tag{27}$$

where $\xi_0 = \left[\frac{2(n+2)}{n}\right]^{1/2}$, $n > 0$.

This could be considered a "sharpfront" solution in that

$$\phi^{n+1}(\xi = \xi_0) = 0$$

$$\left.\frac{d\phi^{n+1}}{d\xi}\right|_{\xi = \xi_0} = 0 \ . \tag{28}$$

For $-2 < n < 0$, say $n = -1$; the solution is

$$\phi = \frac{1}{K + \xi^2/2} \quad \text{or} \quad \rho = \frac{1}{t\left[\frac{x^2}{2t^2} + K\right]} \tag{29}$$

which is valid for $\frac{1}{\rho}\frac{\partial \rho}{\partial x} \to 0$ as $x \to \infty$ (Lonngren and Hirose (1976)).

To apply the second boundary condition (14), it is convenient to let $\phi = \epsilon^{-\psi}$ and note that the constant $\beta/\gamma = 1/2$. Equation (26) becomes

$$(\epsilon^{-n\psi}\epsilon^{-\psi}\psi_\xi)_\xi + \frac{1}{2}\epsilon^{-\psi}\psi_\xi = 0 . \tag{30}$$

The solution for $n = -1$ with the boundary conditions $\psi(\xi=0) = 1$ and $\psi(\xi=\infty) = 0$ has been treated by Fujita and is described in the classic book by Crank (1964). Approximate solutions were recently discussed by Ahmadi, et al. (1976).

Solutions of other one-dimensional diffusion equations have been given by e.g. Phillip (1960), Singh (1967), Swan (1976) and Tuck (1976) and should be referred to.

C. Diffusion In More Than One Dimension

In several cases in plasma physics, it is important to examine diffusion of particles in more than one dimension and in cases where the parallel and perpendicular diffusion coefficients are not equal, parallel could be interpreted as being along an external magnetic field. In such cases, the linear diffusion equation becomes

$$\rho_t + w\rho_x = D_{11}\rho_{xx} + D_\perp \tilde{\nabla}_\perp^2 \rho \tag{31}$$

where w is a drift velocity. As D_{11} and $D_{\perp}$ are assumed at this stage to be linear, we can renormalize the transverse coordinates such that

$$D_{\perp} \tilde{\nabla}_{\perp}^{2} \rho = D_{11} \nabla_{\perp}^{2} \rho \ . \tag{32}$$

In addition, letting

$$\begin{aligned} \rho &= N \exp\left[\frac{w}{2D_{11}} x - \frac{w^2}{4D_{11}} t\right] \\ \tau &= D_{11} t \end{aligned} \tag{33}$$

(31) becomes

$$N_{\tau} = N_{xx} + \nabla_{\perp}^{2} N \ . \tag{34}$$

Using the procedure outlined in Section II with the addition of two terms to the group G (2), it follows that the group invariants are

$$\xi = \frac{x}{\sqrt{\tau}} \ , \quad \zeta = \frac{y}{\sqrt{\tau}} \ , \quad \mu = \frac{z}{\sqrt{\tau}}, \quad \phi = \frac{N}{\tau^{\alpha/\gamma}} \tag{35}$$

where y and z are in the transverse direction. Substituting (35) in (34), we obtain

$$\frac{\alpha}{\gamma} \phi - \frac{1}{2}[\xi\phi_{\xi} + \zeta\phi_{\zeta} + \mu\phi_{\mu}] = \phi_{\xi\xi} + \hat{\nabla}_{\perp}^{2} \phi \tag{36}$$

where $\hat{\nabla}_{\perp}^{2}$ indicates $\nabla_{\perp}^{2}$ in the transformed transverse variables ζ and μ.

We shall examine the conservation law only (15). For this problem, it generalizes to

$$\int N \, dx\,dy\,dz = \text{constant} \quad (37)$$

which implies that $\alpha/\gamma = -3/2$. Using this in (36) suggests that we write

$$-\frac{1}{2}[(\xi\phi)_\xi + (\zeta\phi)_\zeta + (\mu\phi)_\mu] = \phi_{\xi\xi} + \hat{\nabla}_\perp^2\phi \quad (38)$$

which is amenable to a treatment using separation of variables and integration. Note that if the problem were posed in cylindrical coordinates, the proper conservation law would be

$$\int N \, dr\,dx = \text{constant} \quad (39)$$

as there is a fixed scale length of 2π in the third coordinate. This implies that $\alpha/\gamma = -1$.

The procedure that follows is straight forward and will not be reproduced here. In cylindrical coordinates, we find

$$\rho = \frac{\text{const}}{(D_{11}t)} I_0\left(\frac{r^2}{8D_\perp t}\right) \exp\left[-\frac{r^2}{8D_\perp t} - \frac{x^2}{4D_{11}t} + \frac{w}{2D_{11}}\left(x - \frac{wt}{2}\right)\right]. \quad (40)$$

This agrees with a result of Eastlund (1966), who examined diffusion in a Q-machine.

This diffusion equation (31) has found importance in several non-plasma areas also. We cite as examples: Dispersion in uniform porous media flow of ground water (Shen (1976)) and in the spread of cancer in the uterus (Swan (1975)). Extensions to nonlinear and inhomogeneous problems would be worthwhile as we've observed in the one dimensional case.

D. Korteweg de Vries Equation

Considerable attention has been given to understanding the Korteweg-de Vries equation.

$$\rho_t + \rho\rho_x + \delta^2 \rho_{xxx} = 0 \ . \tag{41}$$

For example, Washimi and Taniuti (1966) showed that the low frequency ion acoustic wave could be modeled with this equation as a first approximation for including nonlinear effects.

Using the linear group G, given in (2), one computes the self similar variables to be

$$\xi = \frac{x}{t^{1/3}} \quad \text{and} \quad \phi = \frac{\rho}{t^{-2/3}} \tag{42}$$

where ϕ satisfies

$$-\frac{2}{3}\phi - \frac{1}{3}\xi\,\phi_\xi + \phi\phi_\xi + \delta^2 \phi_{\xi\xi\xi} = 0 \ . \tag{43}$$

A discussion of the solution of (43) is given in Berezin and Karpman (1964).

Two extensions to this solution have been given. For cases where the nonlinear term $\rho\rho_x$ in (41) can be neglected, the self similar solutions for (41) can be written in terms of Airy functions or integrals of Airy functions which have the argument of the similarity variable (42). These two solutions satisfy the conservation law (15) and the boundary condition (14) respectively (Ikezi, et al. (1974)).

The second extension by Shen and Ames (1974) uses the infinitesimal group rather than the linear group that we've used and has ascertained all possible similarity variables for the KdV equation. They find

$$\phi = (at + \delta)^{2/3} (\alpha - \frac{2}{3} a\rho)\frac{2}{3} a^{5/3}$$

$$\xi = \frac{1}{2} a^{-5/3}[2a^2x - 3\alpha(at+\delta) + b(\alpha\beta-\alpha\delta)]/(at+\delta)^{1/3} \tag{44}$$

where a, α, β, and δ are constants and ϕ satisfies (43)

E. Ion Acoustic Wave Equations, Fluid Model

In a series of two papers (Hsuan, et al. (1974) and Shen and Lonngren (1976a)), the similarity properties of four sets of fluid equations that have been used to describe the propagation of ion acoustic waves in a plasma are discussed. The sets of equations are: (a) multiple species fluid equations truncated at the third moment plus Poisson's equation; (b) massless isothermal linear electron fluid and cold nonlinear ion fluid plus Poisson's equation; (c) massless isothermal nonlinear electron fluid and cold nonlinear ion fluid plus Poisson's equation; and (d) massless isothermal electron fluid and cold nonlinear ion fluid with a quasineutrality assumption.

Set (c) is the most general and the infinitesimal group was applied in addition to the linear group G (2). (Shen and Lonngren (1976a) Shen (1976)). As a result several possible similarity variables were found which included those quoted by Zhmudsky (1975) and the traveling wave variables. Numerical difficulties prevented a complete integration of the resulting of ODE (Shen (1976)). Also a scale length (the Debye length) existed in the problem.

Set (d) has been frequently studied in plasma problems (eg. Alexeff et al. (1971), Allen and Andrews (1970)). The equations are identical to those in ordinary fluids.

Finally, set (c) was examined in a nonneutral plasma approximation where the effect of the electrons was neglected (Shen and Lonngren (1976b)). Such a model extends a paper by Gintsburg (1974) and is germane to relativistic electron beam devices.

Below, we tabulate the sets of PDE, the similarity variables and the resulting ODE. The results for sets (a) and (d) were obtained using the "linear" group, (b) was obtained using the "spiral" group and (c) was obtained using the "infinitesimal" group. Details appear in (Hsuan et al. (1974), Shen (1976) and Shen and Lonngren (1976a)).

a) The PDE are:

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x}(n_1 \nu_1) = 0 \ ,$$

$$\frac{\partial \nu_1}{\partial t} + \frac{\partial \nu_1}{\partial x} + \frac{1}{m_1}\frac{1}{n_1}\frac{\partial P_1}{\partial x} = \frac{q_1}{m_1} E \ ,$$

$$\frac{\partial P_1}{\partial t} + \gamma_1 P_1 \frac{\partial \nu_1}{\partial x} + \nu_1 \frac{\partial P_1}{\partial x} = 0 \ , \tag{45}$$

$$\frac{\partial E}{\partial x} = \sum_1 \frac{q_1}{\epsilon_0} n_1 \ .$$

The similarity variables are:

$$\xi = \frac{x}{t^A}\ , \quad N_l = n_l t^2, \quad U_l = \nu_l t^{1-A}\ ,$$
$$P_l = P_l t^{4-2A}, \quad \epsilon = E t^{2-A}\ , \tag{46}$$

The ODE are:

$$\frac{d}{d\xi}(N_l U_l) - A\xi \frac{d}{d\xi} N_l - 2N_l = 0\ ,$$
$$-A\xi \frac{d}{d\xi} U_l + (A-1) U_l + U_l \frac{d}{d\xi} U_l + \frac{1}{m_l N_l} \frac{d}{d\xi} P_l + \frac{e_l}{m_l} \epsilon = 0\ ,$$
$$(2A-4)P_l - A\xi \frac{d}{d\xi} P_l + \gamma_l P_l \frac{d}{d\xi} U_l + U_l \frac{d}{d\xi} P_l = 0\ , \tag{47}$$
$$\frac{d}{d\xi}\epsilon = \sum_l \frac{e_l}{\epsilon_0} N_l\ .$$

b) The PDE are:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i \nu_i) = 0\ ,$$
$$\frac{\partial \nu_i}{\partial t} + \nu_i \frac{\partial \nu_i}{\partial x} + \frac{V_a^2}{n_0} \frac{\partial n_e}{\partial x} = 0\ ,$$
$$\lambda_D^2 \frac{\partial^2 n_e}{\partial x^2} = n_e - n_i\ , \tag{48}$$
$$E = -\frac{KT_e}{n_0 e} \frac{\partial}{\partial x} n_e\ ,$$

The similarity variables are:

$$\xi = t \exp(-\alpha x), \quad N_i = n_i \exp(2\alpha x)\ ,$$
$$N_e = n_e \exp(2\alpha x), \quad U = \nu_i \exp(\alpha x)\ , \tag{49}$$

The ODE are:

$$\left(\frac{1}{\alpha} - \xi U\right) \frac{dN_i}{d\xi} - \xi N_i \frac{dU}{d\xi} - 3N_i U = 0 \ ,$$

$$\left(\frac{1}{\alpha} - \xi U\right) \frac{dU}{d\xi} - \xi \frac{dN_e}{d\xi} - 2N_e - U^2 = 0 \ , \qquad (50)$$

$$\xi^2 \frac{d^2 N_e}{d\xi^2} + 5\xi \frac{dN_e}{d\xi} + \left(4 - \frac{1}{\alpha^2}\right) N_i = 0 \ .$$

c) The PDE are:

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v_i)}{\partial x} = 0$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} = - \frac{\partial \phi}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = n_e - n_i \qquad (51)$$

$$n_e = e^{\phi}$$

The similarity variables are:

$$\xi = \frac{x + \frac{\mu}{\nu}}{t + \frac{\rho}{\nu}} - \frac{\delta}{\nu} \left[\ln\left(t + \frac{\rho}{\nu}\right) + \frac{\frac{\rho}{\nu}}{t + \frac{\rho}{\nu}} \right] ,$$

$$N_i(\xi) = \left(t + \frac{\rho}{\nu}\right)^2 n_i(x,t) \ ,$$

$$V_i(\xi) = v_i(x,t) - \frac{\delta}{\nu} \ln\left(1 + \frac{\rho}{\nu}\right) \ , \qquad (52)$$

$$N_e(\xi) = \left(t + \frac{\rho}{\nu}\right)^2 n_e(x,t) \ .$$

The ODE are:

$$-2N_i - \frac{\xi dN_i}{d\xi} + \frac{d(N_i V_i)}{d\xi} - \frac{\delta}{\nu}\frac{dN_i}{d\xi} = 0 \ ,$$

$$(V_i - \xi)\frac{dV_i}{d\xi} + \frac{1}{N_e}\frac{dN_e}{d\xi} - \frac{\delta}{\nu}\left(\frac{dV_i}{d\xi} - 1\right) = 0 \ , \qquad (53)$$

$$\frac{d}{d\xi}\left(\frac{1}{N_e}\frac{dN_e}{d\xi}\right) = N_e - N_i \ .$$

d) The PDE are:

$$\frac{\partial n}{\partial t} + \frac{\partial (n\nu)}{\partial x} = 0 \ ,$$

$$\frac{\partial \nu}{\partial t} + \nu \frac{\partial \nu}{\partial x} + \frac{V_A^2}{n_0}\frac{\partial n}{\partial x} = 0 \ . \qquad (54)$$

The similarity variables are:

$$\xi = \frac{x}{t^A} \ , \quad U = \frac{\nu}{t^{A-1}} \ , \quad N = \frac{n}{t^{2(A-1)}} \ , \qquad (55)$$

The ODE are:

$$-A\xi \frac{dN}{d\xi} + 2(A-1)N + \frac{d(NU)}{d\xi} = 0 \ ,$$

$$(A-1)U - A\xi \frac{dU}{d\xi} + U\frac{dU}{d\xi} + \frac{V_A^2}{n_0}\frac{dN}{d\xi} = 0 \ . \qquad (56)$$

F. Vlasov Model for Ion Acoustic Waves

The distribution function for ions in the absense of a magnetic field can be computed from the Vlasov equation

$$\frac{\partial f}{\partial \tau} + v\frac{\partial f}{\partial y} - \frac{e}{M}\frac{\partial \psi}{\partial y}\frac{\partial f}{\partial v} = 0 \qquad (57)$$

With a quasi-neutrality assumption, $n_e \sim n_i$ and with a Boltzmann approximation for electrons $n_e = n_0 \exp[e\psi/T_e]$, we write (57) as

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} - \frac{\partial f}{\partial U}\frac{d(\ln \int f dU)}{dx} = 0 . \qquad (58)$$

Standard dimensionless parameters are used in (58), namely

$$n = \frac{n_e}{n_0}, \quad t = \omega_{pi}\tau, \quad U = \frac{V}{\sqrt{T_{e/M}}}, \quad x = \frac{y\,\omega_p}{\sqrt{T_{e/M_i}}} .$$

Applying the linear group G (2) to (58), we find that the self similar variables are

$$\xi = \frac{x}{t} \quad \text{and} \quad \phi = \frac{f}{t^{\alpha/\gamma}} \qquad (59)$$

where ϕ satisfies

$$\frac{\alpha}{\gamma}\phi - \xi\frac{d\phi}{d\xi} + U\frac{d\phi}{d\xi} - \frac{d\phi}{dU}\frac{d(\ln\int \phi\, dU)}{d\xi} = 0 . \qquad (60)$$

Again, we have two possible boundary conditions that can be treated; (14) and (15). In a pioneering series of papers, Gurevich and Pitaevsky examined the solutions governed by boundary condition (14), i.e. $f(x = 0, t) = \text{constant}$. Their work has been summarized in a recent review paper (Gurevich and Pitaevsky (1975)). Experiments performed by Korn, Marshall and Schlesinger (1970) and others seem to confirm the predictions.

The second boundary condition (15), $\int f(x,t)dx = \text{constant}$ has received considerable attention from Jensen and his colleagues in their investigation of the Green's function for a linearized version of (58) (Anderson et al. (1971) Jensen et al. (1974) Christoffersen et al. (1974)). They found that the leading term in the series for the Green's function is the self-similar term $\frac{1}{t} h(\frac{x}{t})$ which is noted by setting $\alpha/\gamma = -1$ in (59). Equation (60) in this case is written

$$\frac{d[(U-\xi)\phi]}{d\xi} = \frac{d\phi}{dU} \frac{d[\ln \int \phi dU]}{d\xi} \tag{61}$$

or

$$\phi = \frac{1}{U-\xi} \frac{d\phi}{dU} \ln \int \phi dU \,\Big| .$$

They also experimentally confirmed their findings in a Q machine.

IV Conclusion

In this tutorial and review paper, we have summarized the technique of self-similar solution of partial differential equations and presented several examples where it has been applied to problems found in plasma physics.

The author wishes to acknowledge H. Hsuan, H. Shen, A. Hirose, and W. Ames who actively contributed to this work.

REFERENCES

Ahmadi, G., Hirose, A. and Lonngren, K. E. 1976, IEEE Trans. to be published.

Alexeff, I., Estabrook, K. and Widner, M. 1971, Phys. Fluids 14, 2355.

Allen, J. E. and Andrews, J. G. 1970, J. Plasma Phys. 4, 187.

Ames, W. F. 1965, Ind. Eng. Chem. Fund. 4, 72.

Ames, W. F. 1965, 1972, Nonlinear Partial Differential Equations in Engineering, (Academic Press).

Anderson, S. A., Christoffersen, G. B., Jensen, V. O., Michelsen, P. and Nielsen, P. 1971, Phys. Fluids 14, 990.

Barrenblatt, G. I. and Zel'dovich, Ya. B. 1972, Ann Rev. of Fluid Mech. 4, 285.

Berezen, Yu. A. and Karpman, V. 1964, Sov. Phys. JETP19, 1265.

Berryman, J. G. 1976, Phys. Fluids, to be published.

Bluman, G. W. and Cole, J. D. 1974, Similarity Methods for Differential Equations (Springer Verlag).

Boltzmann, L. 1894, Ann. Physik 53, 959.

Christoffersen, G. B., Jensen, V. O. and Michelson, P. 1974, Phys. Fluids 17, 390.

Crank, J. 1964, The Mathematics of Diffusion, Oxford University Press.

Drake, J. R. 1973, Phys. Fluids 16, 1554.

Eastlund, B. 1966, Phys. Fluids 9, 594.

Gagné, S. and Poussart, D. 1976, IEEE Trans. BME-23, 16.

Gautschi, W. 1964, Error Functions and Fresnel Integrals, in Handbook of Mathematical Functions, M. Abromowitz and I. A. Stegun, Eds., Nat. Bur. of Stand.

Gilding, B. H. and Peletier, L. 1976, J. Math. Anal. Appl. 55, 351.

Gintsburg, M. A. 1974, Sov. Phys. Dokl. 19, 216.

Gurevich, A. V. and Pitaevsky, L. P. 1975, Prog. Aerospace Sci. 16, 227.

Hirose, A., Alexeff, I., Jones, W. D., Kush, S. T. and Lonngren, K. E. 1970, Phys. Rev. Letters 25, 1563.

Hirose, A. and Alexeff, I. 1973, Phys. Fluids 16, 1087.

Hsuan, H. C. S., Lonngren, K. E. and Ames, W. F. 1974, J. Engr. Math. 8, 303.

Ikezi, H., Kiwamoto, Y., Lonngren, K. E., Burde, C. M. and Hsuan, H. C. S. 1973, Plasma Physics 15, 1141.

Jensen, V. O., Michelson, P. and Hsuan, H. C. S. 1974, Phys. Fluids 17, 2208.

Korn, P., Marshall, T. C. and Schlesinger, S. P. 1970, Phys. Fluids 13, 517.

Lie, S. 1881, Arch Math. (Kristiana) 6, 328.

Lonngren, K. E., Ames, W. F., Hirose, A. and Thomas, J. 1974, Phys. Fluids 17, 1919.

Lonngren, K. E., Hsuan H. C. S., Malik, N. R. and Shen, H. 1975, IEEE Trans. CAS-22, 882.

Lonngren, K. E. 1976, J. Appl. Phys. to be published.

Lonngren, K. E. and Hirose, A. 1976, Phys. Letters, to be published.

Michal, A. D. 1952, Proc. Nat. Acad. of Sci. 37, 623.

Moran, M. J. and Gaggioli, R. A. 1969, J. Engr. Math. 3, 151.

Morgan, A. J. A. 1952, Quart. J. of Math. 3, 250.

Ovsjannikov, L. V. 1962, Gruppovye Svoystva Differentsialny Uravneni,
Group Properties of Differential Equations (G. Bluman, transl. 1967)
Cal. Tech.

Peyrand, J. and Coste, J. 1976, Phys. Fluids 19, 388.

Phillip, J. R. 1960, Aust. J. Phys. 13, 1.

Sedov, L. I. 1959, Similarity and Dimensional Methods in Mechanics
(Academic Press).

Shen, H. and Ames, W. F. 1974, Phys. Letters 49A, 313.

Shen, H. and Lonngren, K. E. 1976a, J. Engr. Math. 10, 135.
Correction to be published, 11.

Shen, H. and Lonngren, K. E. 1976b, IEEE Trans. PS4, 144.

Shen, H. 1976, Ph.D. Thesis, University of Iowa, unpublished.

Shen, H. T. 1976, J. of Hyd. Div. Proc. ASCE HY6, 707.

Singh, R. 1967, J. of Hyd. Div. Proc. ASCE HY5, 43.

Swan, G. W. 1975, Math. Biosciences 25, 319.

Swan, G. W. 1976, Bull. of Math. Biology 38, 1.

Tuck, B. 1976, J. Phys. D Appl. Phys. 9, 1559.

Washimi, H. and Taniuti, T. 1966, Phys. Rev. Letters 24, 206.

Zhmudsky, A. A. 1975, Ukrainskii Fizicheskii Zhurnal 3, 492.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: 1698	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): SELF SIMILAR SOLUTION OF PLASMA EQUATIONS.		5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): K. E. Lonngren		8. CONTRACT OR GRANT NUMBER(s): DAAG29-75-C-0024; NSF-ENG-76-15645
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.		12. REPORT DATE: November 1976
		13. NUMBER OF PAGES: 27
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Technical summary rept.		15. SECURITY CLASS. (of this report): UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited. MRC-TSR-1698

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

30p.

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Heat Equation, Nonlinear Equations and Systems

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper describes the method of self similar solution of partial differential equations and reviews its application to several problems found in plasma physics.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 221 200

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)