AD-A034 398 FRANK J SEILER RESEARCH LAB UNITED STATES AIR FORCE --ETC F/G 12/1
OPTIMAL AND SUBOPTIMAL RESULTS IN FULL AND REDUCED ORDER LINEAR--ETC(U)
DEC 76 R B ASHER, C S SIMS

UNCLASSIFIED FJSRL-TR-76-0022 NL

1 OF 1

AD
A034398

END

DATE
FILMED
2-77

12

FRANK J. SEILER RESEARCH LABORATORY

SRL-TR-76-0022

DECEMBER 1976

OPTIMAL AND SUBOPTIMAL RESULTS IN FULL AND REDUCED ORDER LINEAR FILTERING

SCIENTIFIC

PROJECT 2304

AIR FORCE SYSTEMS COMMAND

UNITED STATES AIR FORCE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (14) FJSRL-TR-76-0022 AD A-	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) OPTIMAL AND SUBOPTIMAL RESULTS IN FULL AND REDUCED ORDER LINEAR FILTERING.		5. TYPE OF REPORT & PERIOD COVERED (9) Scientific rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Robert B. Asher, Craig S. Sims		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Frank J. Seiler Research Laboratory (AFSC) USAF Academy, Colorado 80840		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) DRS 61102F (17) F1 2304-F1-57
11. CONTROLLING OFFICE NAME AND ADDRESS Frank J. Seiler Research Laboratory (AFSC) USAF Academy, Colorado 80840		12. REPORT DATE (11) December 1976
		13. NUMBER OF PAGES 23
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 25p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Kalman Filtering	Suboptimal Estimation
Reduced Order Filtering	Linear Filtering
Linear Estimation	Suboptimal Filtering

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper considers the synthesis of linear reduced order filters and the synthesis of linear full order filters with minimum complexity. The objective of a reduced order filter is to estimate a linear transformation of the state vector with a filter of lower dimension. This type of filter occurs frequently in applications. Several cases are studied. In a number of cases it is shown that singular arcs exist. In instances where certain filter parameters are not subject to optimization, it is shown that the

over

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

319 920 LB

20. ABSTRACT

remaining parameters can be optimized with a relatively simple procedure. Closed form solutions for a number of cases have been obtained.

OPTIMAL AND SUBOPTIMAL RESULTS IN FULL AND REDUCED ORDER LINEAR FILTERING

†Craig S. Sims
School of Electrical Engineering
Oklahoma State University
Stillwater, Oklahoma 74074

and

††Robert B. Asher
Frank J. Seiler Research Laboratory
USAF Academy, Colorado 80840

† Associate Professor of Electrical Engineering. This research was accomplished during participation in the USAF-ASEE Summer Faculty Program at the Frank J. Seiler Research Laboratory.

†† Research Associate in the Applied Mathematics Division.

ABSTRACT

This paper considers the synthesis of linear reduced order filters and the synthesis of linear full order filters with minimum complexity. The objective of a reduced order filter is to estimate a linear transformation of the state vector with a filter of lower dimension. This type of filter occurs frequently in applications. Several cases are studied. In a number of cases it is shown that singular arcs exist. In instances where certain filter parameters are not subject to optimization, it is shown that the remaining parameters can be optimized with a relatively simple procedure. Closed form solutions for a number of cases have been obtained.

I. INTRODUCTION

The problem of optimal linear filtering has been examined from many viewpoints, including treatment within a control theoretic framework [1]. In this paper we wish to address the problem of estimating a linear transformation of a state vector using a filter which may be of reduced order relative to the dimension of the state vector. Control methodologies are used to optimize the parameters of the filter.

The motivation for this paper is based on the fact that often one is only interested in estimating a portion of a state vector. In a high accuracy pointing and tracking system for aircraft to satellite tracking [2], for example, one might be primarily interested in estimating certain physical variables such as angular rate of the line of sight and the pointing misalignment errors. The state vector might be very large, however, containing many states associated with detailed models of the error producing mechanisms in inertial instruments. If one were to design a full order filter, the complexity would be intolerable from the viewpoint of a flight computer implementation. Such difficulties have long been recognized and, consequently, there has been considerable research in this area [3-9]. Within a deterministic framework, Luenberger has shown that one may achieve considerable reduction of complexity in the state reconstruction process when only a linear combination of the state variables is required [10]. It would seem that a similar reduction in complexity should occur when the problem is stochastic. Indeed, it will become clear that this paper has some relationship to observer theory.

Although the primary thrust of this paper is towards the reduction of filter complexity through order reduction, it will be shown that it is even possible to reduce filter complexity in the full order case. This happens because for certain classes of problems the Kalman filter is not the only solution [1]. Other estimators can result in the same value for the performance measure.

The paper is divided into five sections. Section II casts the estimation process into a deterministic control problem which then can be solved via use of the matrix minimum principle. Section III contains the development of the problem solution. The conditions for unbiasedness are developed and the necessary conditions for optimality are given. Several cases where exact analytical solutions to the two point boundary value problem may be found are presented.

The first special case is that of terminal time estimation. The importance of this case may be found in orbit determination and in geodetic model determination in which one is interested in the optimal estimate at only one time instant. The solution for this case yields a singularity which may be taken advantage of by choosing the simplest filter configuration. By choosing part of the structure a priori, one needs to solve a simple linear two point boundary value problem which is easily accomplished. The second and third special cases are that of an interval estimation problem. The optimal reduced order solution is given in closed form in these cases for various assumptions on the transformation (whether time invariant or time varying) of the full order states to the reduced state space.

Section IV considers the solution to the two point boundary value problem when part of the structure is chosen a priori. A Riccati solution is given. This solution may be done off line and the gains either stored or approximated. Section V considers examples of the results.

The ability to obtain simple structures for filtering allows one to implement the least complex filter. This will yield a vast savings in on line computation or hardware implementation.

II. PROBLEM STATEMENT

The system of interest is a Markov process satisfying a linear stochastic differential equation

$$\dot{x}(t) = A(t)x(t) + w(t). \tag{1}$$

The observation model is also linear

$$y(t) = C(t)x(t) + v(t). \tag{2}$$

The state vector x is of dimension n and the observation vector y is of dimension m. It is assumed that the initial value of the state vector, $x(t_0)$, is uncorrelated with the plant noise, $w(t)$, and the measurement noise, $v(t)$. The noise processes are zero mean white processes, uncorrelated with each other, having covariance matrices

$$\begin{aligned} E\{w(t)w^T(\tau)\} &= Q(t)\delta(t-\tau) \\ E\{v(t)v^T(\tau)\} &= R(t)\delta(t-\tau). \end{aligned} \tag{3}$$

The initial state of the system has mean and variance

$$\begin{aligned} E\{x(t_0)\} &= \mu_0 \\ \mathrm{Var}\{x(t_0)\} &= P_0. \end{aligned} \tag{4}$$

The problem is to estimate a linear transformation of the state vector

$$z(t) = N(t)x(t) \tag{5}$$

where z is of dimension $\ell \leq n$, and presumably the $\ell \times n$ matrix, $N(t)$, is selected to indicate the part of the system which is of primary concern. As an example, if $N(t) = C(t)$, then we are interested in the noise free output as indicated by (2). The filter to be optimized is of the form

$$\dot{\hat{z}}(t) = F(t)\hat{z}(t) + K(t)y(t) + g(t) \tag{6}$$

where $\hat{z}$ is of dimension ℓ. The error, $e(t)$, is defined as

$$e(t) = z(t) - \hat{z}(t). \tag{7}$$

The deterministic vector, $g(t)$, and the filter initial condition $\hat{z}(t_0)$, are to be determined so that the filter is unbiased, i.e.,

$$E\{e(t)\} = 0 \qquad \forall t \geq t_0. \tag{8}$$

Note that we do not require conditional unbiasedness which is a much stronger requirement. The matrices $F(t)$ and $K(t)$ are then to be determined so that the performance measure

$$J = E\left\{\int_{t_0}^{t_f} e^T(t)U(t)e(t)dt + e^T(t_f)Se(t_f)\right\} \tag{9}$$

is a minimum. It is assumed that the weighting matrix S is positive definite. The weighting matrix $U(t)$ is critical to the problem formulation, as we obtain different results depending upon whether this matrix is positive definite or zero. It can make sense to have $U(t)$ be zero, when the only time of real importance is the final time such as in orbit determination. Indeed when $U(t)$ is zero and N is a constant transformation, it will be seen that performance does not depend on $F(t)$.

III. PROBLEM SOLUTION

We shall first examine the requirements for unbiasedness. By direct substitution from (1), (2), (5), and (6), the error equation is obtained

$$\dot{e} = [\dot{N}+NA-FN-KC]x + Fe + Nw - Kv - g. \tag{10}$$

If $g(t)$ is selected as

$$g = [\dot{N}+NA-FN-KC]\mu \tag{11}$$

where $\mu(t)$ is the mean value of the system state,

$$\dot{\mu}(t) = A(t)\mu(t) \quad ; \quad \mu(t_0) = \mu_0$$

then it is clear that taking expectation of (10) gives a homogeneous result

$$\frac{d}{dt}E\{e(t)\} = F(t)E\{e(t)\}. \tag{12}$$

Thus if $E\{e(t_0)\} = 0$, equation (8) is satisfied. This is true if

$$\hat{z}(t_0) = N(t_0)\mu_0 \tag{13}$$

since then

$$E\{e(t_0)\} = E\{N(t_0)x(t_0) - N(t_0)\mu_0\} = 0 \tag{14}$$

and the desired result follows. Hence (11) is the expression needed for g(t) and (13) specifies the filter initial condition. Equation (10) can be rewritten as

$$\dot{e} = [\dot{N}+NA-FN-KC]\bar{x} + Fe + Nw - Kv \tag{15}$$

where $\bar{x} = x-\mu$ satisfies the linear equation

$$\dot{\bar{x}} = A\bar{x} + w. \tag{16}$$

Certainly $\bar{x}$ is a zero mean process having initial variance P_0. It is convenient to work with the second moment matrix associated with (15) and (16) in formulating the optimal control problem which will give the requirements for F and K.

The performance measure (9) can be written as

$$J = tr\left\{\int_{t_0}^{t_f} U(t)P_{ee}(t)dt + SP_{ee}(t_f)\right\} \tag{17}$$

where

$$P_{ee}(t) = E\left\{e(t)e^T(t)\right\}. \tag{18}$$

From (15), P_{ee} satisfies

$$\begin{aligned} \dot{P}_{ee} &= [\dot{N}+NA-FN-KC]P_{\bar{x}e} + FP_{ee} + P_{ee}F^T \\ &\quad + P_{e\bar{x}}[\dot{N}+NA-FN-KC]^T + NQN^T + KRK^T \end{aligned} \tag{19}$$

where

$$P_{e\bar{x}}(t) = E\left\{e(t)\,\bar{x}(t)^T\right\} = P_{\bar{x}e}(t)^T. \tag{20}$$

Using (15) and (16), it is easy to show that

$$\dot{P}_{\bar{x}e} = AP_{\bar{x}e} + P_{\bar{x}\bar{x}}[\dot{N}+NA-FN-KC]^T + P_{\bar{x}e}F^T + QN^T \tag{21}$$

where

$$P_{\bar{x}\bar{x}}(t) = E\left\{\bar{x}(t)\bar{x}^T(t)\right\} \tag{22}$$

satisfies the equation

$$\dot{P}_{\bar{x}\bar{x}} = AP_{\bar{x}\bar{x}} + P_{\bar{x}\bar{x}}A^T + Q. \tag{23}$$

The term $P_{e\bar{x}}$ satisfies the transpose of (21). Initial conditions for (19), (21), and (23) are

$$
\begin{aligned}
P_{\overline{xx}}(t_0) &= P_0 \\
P_{\overline{x}e}^T(t_0) = P_{e\overline{x}}(t_0) &= N(t_0)P_0 \qquad (24) \\
P_{ee}(t_0) &= N(t_0)P_0N^T(t_0).
\end{aligned}
$$

The matrix $P_{\overline{xx}}(t)$ may be regarded as a known quantity since it is the solution to (23), i.e.,

$$P_{\overline{xx}}(t) = \phi(t,t_0)P_0\phi^T(t,t_0) + \int_{t_0}^{t} \phi(t,\tau)Q(\tau)\phi^T(t,\tau)dt \qquad (25)$$

where

$$\phi(t,t) = I$$

and

$$\dot{\phi}(t,\tau) = A(t)\phi(t,\tau). \qquad (26)$$

The optimization problem may then be stated in a deterministic way, so that the matrix minimum principle is applicable. The problem is to minimize (17) by properly selecting F and K, subject to the constraints imposed by (19) and (21). To preserve symmetry we include the transpose of (21) in the Hamiltonian also.

The Hamiltonian for this problem is

$$H = tr\left\{UP_{ee} + \dot{P}_{ee}\Lambda_{ee}^T + \dot{P}_{\overline{x}e}\Lambda_{\overline{x}e}^T + \dot{P}_{e\overline{x}}\Lambda_{e\overline{x}}^T\right\}. \qquad (27)$$

The equations for the Lagrange multiplier matrices are

$$\dot{\Lambda}_{ee} = -\frac{\partial H}{\partial P_{ee}} = -\left\{U + F^T\Lambda_{ee} + \Lambda_{ee}F\right\} \qquad (28)$$

$$\dot{\Lambda}_{\overline{x}e} = -\frac{\partial H}{\partial P_{\overline{x}e}} = -\left\{A^T\Lambda_{\overline{x}e} + \Lambda_{\overline{x}e}F + [\dot{N}+NA-FN-KC]^T\Lambda_{ee}\right\} \qquad (29)$$

with terminal conditions

$$\Lambda_{ee}(t_f) = S \qquad (30)$$

and

$$\Lambda_{\overline{x}e}(t_f) = 0. \qquad (31)$$

The matrix $\Lambda_{e\overline{x}}$ is just the transpose of $\Lambda_{\overline{x}e}$. Setting the gradient of H with respect to K equal to zero gives a necessary condition for the optimum gain

$$K = \left[P_{e\overline{x}} + \Lambda_{ee}^{-1}\Lambda_{e\overline{x}}P_{\overline{xx}}\right]C^TR^{-1} \qquad (32)$$

where R and Λ_{ee} have been assumed to be nonsingular. The matrix F presents a difficulty since it appears linearly in the Hamiltonian. It is convenient to proceed as in [14] by examining that part of the Hamiltonian which depends explicitly on F, defined as

$$H^* = tr\{F\Theta + \Theta^T F^T\} \tag{33}$$

where

$$\Theta = \left[P_{ee} - NP_{\bar{x}e}\right]\Lambda_{ee} + \left[P_{e\bar{x}} - NP_{\bar{x}\bar{x}}\right]\Lambda_{\bar{x}e}. \tag{34}$$

Clearly what we have called Θ is just the transpose of the gradient matrix, $\partial H/\partial F$. From the initial conditions (24), it may be seen that $\Theta(t_0)$ is zero. It is necessary to examine the time derivatives of Θ to determine the possible existence of a singular arc. Since different requirements will be obtained, it is convenient to examine Θ under different conditions. It is required that $\Theta(t)$ equal zero for all $t\varepsilon[t_0,t_f]$ if a singular arc is to exist.

3.1 Case 1

In this first case, it is assumed that U(t) is zero. Then it follows that

$$\dot{\Theta} = F\Theta + \Theta F + K\left[RK^T\Lambda_{ee} - CP_{\bar{x}\bar{x}}\Lambda_{\bar{x}e} - CP_{\bar{x}e}\Lambda_{ee}\right]. \tag{35}$$

The bracketed term in (35) is zero whenever K(t) is selected optimally, according to (32). Therefore (35) becomes

$$\dot{\Theta} = F\Theta + \Theta F \tag{36}$$

and since $\Theta(t_0)$ is zero, (36) assures us that $\Theta(t)$ remains zero for all $t\varepsilon[t_0,t_f]$, regardless of our choice of F. The implications of this result are significant. In the full order case (N=I), as was pointed out in [1], the Kalman filter is just one solution, i.e., one of the optimal linear filters. This means that for final time estimation problems, one should examine other realizations of the optimum linear filter. It will be seen that such realizations can be easier to implement than the Kalman filter. We have demonstrated here that for reduced order filters also the solution is not unique. One can pick F for some other reason than minimizing J, such as to reduce sensitivity, to satisfy a constraint, or to minimize some performance measure involving F. Consider adding a nonnegative term

$$J_1 = \int_{t_0}^{t_f} \rho[F(t)]dt \qquad (37)$$

to the performance measure J given in (17). Since $\Theta=0$, it is seen that a necessary condition for F to be optimal is that

$$\frac{\partial\rho}{\partial F} = 0 \qquad \forall t\varepsilon[t_0,t_f]. \qquad (38)$$

For example, it may be advantageous to use a measure of F as

$$\rho[F(t)] = tr\{F(t)^T F(t)\} \qquad (39)$$

which weights the magnitude of the elements of F. This particular choice of ρ yields an F identically equal to zero. This clearly is the simplest solution for F.

An alternative method of choosing the gain K may proceed as follows. This alternative method shows from another approach that, indeed, the results contained herein are reasonable. Let Φ_{KF} be the transition matrix for the optimal Kalman filter. Obviously, this is the transition matrix associated with $(A-K_0C)$ where K_0 is the optimal Kalman gain. If one considered the solution of the Kalman filter at the final time, assuming zero initial conditions, i.e.,

$$\begin{aligned} \hat{z}_0(t_f) &= N(t_f)\hat{x}_0(t_f) \\ &= \int_{t_0}^{t_f} N(t_f)\Phi_{KF}(t_f,\tau)K_0(\tau)y(\tau)d\tau, \end{aligned} \qquad (40)$$

where $\hat{x}_0$ is the optimal Kalman estimate, and the solution for the filter developed herein, i.e.,

$$\hat{z}(t_f) = \int_{t_0}^{t_f} \Phi_F(t_f,\tau)K(\tau)y(\tau)d\tau \qquad (41)$$

where Φ_F is the transition matrix associated with F, then it may be noticed that it is sufficient for optimality that $\hat{z}(t_f)$ to be equal to $\hat{z}_0(t_f)$, and this holds if

$$N(t_f)\Phi_{KF}(t_f,\tau)K_0(\tau) = \Phi_F(t_f,\tau)K(\tau), \qquad \forall\tau\varepsilon[t_0,t_f]. \qquad (42)$$

Since F is arbitrary for the singular solution or is zero for the solution of the problem with (39) in the Hamiltonian, F may be chosen as zero. If this is the case then the solution for the reduced order filter gain is

$$K(t) = N(t_f)\Phi_{KF}(t_f,t)K_0(t) \tag{43}$$

where the transition matrix $\Phi_{KF}(t_f,t)$ may be calculated from

$$\frac{d\,\Phi_{KF}(t_f,t)}{dt} = -\,\Phi_{KF}(t_f,t)(A-K_0C) \tag{44}$$

which is integrated backwards from $\Phi_{KF}(t_f,t_f) = I$. The solution for $K(t)$ may be accomplished offline and either stored or approximated.

Probably the most advantageous property of the singularity with respect to F is computational. If F is picked a priori, then one only needs to solve a linear two-point boundary-value problem (TPBVP), which is easy to do using linear systems theory. Alternatively one can use a procedure which leads to a Riccati equation type of solution. These methods are examined in a later section.

3.2 Case 2

In this case N is constant but U is assumed to be positive definite. This leads to an additional term in (36), i.e.,

$$\dot{\Theta} = F\Theta + \Theta F - \left(P_{ee} - NP_{\bar{x}e}\right)U. \tag{45}$$

Hence it is required that

$$P_{ee}(t) = NP_{\bar{x}e}(t) \qquad \forall t\varepsilon[t_0,t_f] \tag{46}$$

since U is positive definite. Clearly this cannot be satisfied for arbitrary F. Thus the existence of the singular arc depends on the choice of F in this case. Defining Ω,

$$\Omega \triangleq P_{ee} - NP_{\bar{x}e} \tag{47}$$

it is seen that

$$\dot{\Omega} = F\Omega + \Omega F^T + \left(P_{e\bar{x}} - NP_{\bar{x}\bar{x}}\right)[NA-FN-KC] - K[RK^T-CP_{\bar{x}e}]. \tag{48}$$

It is required that $\Omega(t)$ equal zero for all t in the interval of interest. From (24) it is apparent that $\Omega(t_0)$ is zero. In order to develop the equation

$$\dot{\Omega} = F\Omega + \Omega F^T \tag{49}$$

so that $\Omega(t)$ is zero, it is required that

$$NA - FN - KC = 0. \tag{50}$$

This leads to the result from (29) that

$$\Lambda_{e\bar{x}}(t) = 0 \tag{51}$$

so that the gain is evaluated as

$$K(t) = P_{e\bar{x}}(t)C^T(t)R^{-1}(t). \tag{52}$$

Clearly (52) implies that the last term in (48) is zero, and we have the desired result. Thus if F can be chosen so that (50) is satisfied, a singular arc exists, provided K is selected according to (52). In the full order case, when $N = I$, the proper choice of F is

$$F = A - KC. \tag{53}$$

The TPBVP is then solved by the Kalman filter, a result which is easily verified. In the reduced order case, (50) is simply the observer constraint equation, [10]. If the randomness were removed from the problem, (50) is required so that $\hat{z}(t) = Nx(t)$ for $t \geq t_0$ if there is no driving vector, g. Equation (46) has an interesting interpretation. It is easy to see that it is just the orthogonality requirement in a reduced space,

$$E\{\hat{z}(t)e^T(t)\} = 0. \tag{54}$$

It must be made clear that it is not always possible to satisfy (50) and (52) simultaneously, and consequently it is not always possible to have a singular arc. In such cases the problem should probably be reformulated with bounds on F, or a suboptimal solution accepted.

A necessary and sufficient condition that there be a solution F for (50) is that

$$(NA-KC)N^{\dagger}N = NA - KC \qquad \forall t \varepsilon [t_0, t_f] \tag{55}$$

where $N^{\dagger}$ is a pseudo inverse of the matrix N. In this case the solution for F is

$$F = [NA-KC]N^{\dagger} + \Gamma[I-NN^{\dagger}] \tag{56}$$

where Γ is an arbitrary $\ell \times \ell$ matrix [12]. In particular if NN^T is non-singular, then

$$F = [NA-KC]N^T[NN^T]^{-1} \tag{57}$$

is a solution. When the F of (57) can be found, the K which is the solution to (52) can be evaluated by solving the equation

$$\dot{P}_{e\bar{x}} = P_{e\bar{x}}A^T + \left[NA-P_{e\bar{x}}C^TR^{-1}C\right]N^T\left[NN^T\right]^{-1}P_{e\bar{x}} + NQ \tag{58}$$

from initial condition (24), and substituting the result in (52). Note that there are only $\ell \times n$ elements in $P_{e\bar{x}}$ which may be far less than if we were to solve the problem by estimating x first, and then use the result, $\hat{z} = N\hat{x}$. Doing it that way would require solving a Riccati equation with $n(n+1)/2$ distinct elements.

3.3 Case 3

In this case, N is not constant. By some fortunate cancellation, (45) is still applicable but (48) is not. Equation (48) is replaced by

$$\dot{\Omega} = F\Omega + \Omega F^T + \left(P_{e\bar{x}} - NP_{x\bar{x}}\right)[\dot{N}+NA-FN-KC] - K[RK^T-CP_{\bar{x}e}]. \quad (59)$$

It is clear that $\Omega(t)$ will be zero if

$$\dot{N} + NA - FN - KC = 0. \quad (60)$$

Then (51) and (52) and (54) are still true. To be able to solve (60) for F, it is necessary and sufficient that

$$(\dot{N}+NA-KC)N^{\dagger}N = [\dot{N}+NA-KC]\forall t\varepsilon[t_0,t_f]. \quad (61)$$

The solution for F is

$$F = [\dot{N}+NA-KC]N^{\dagger} + \Gamma[I-NN^{\dagger}]. \quad (62)$$

Again if NN^T is nonsingular

$$F = [\dot{N}+NA-KC]N^T[NN^T]^{-1}. \quad (63)$$

The gain can then be found by solving the equation

$$\dot{P}_{e\bar{x}} = P_{e\bar{x}}A^T + \left[\dot{N}+NA-P_{e\bar{x}}C^TR^{-1}C\right]N^T\left[NN^T\right]^{-1}P_{e\bar{x}} + NQ \quad (64)$$

from the proper initial condition and substituting the result in (52).

IV. SOLUTION FOR SPECIFIC F

In Case 1, it was found that the singular arc did not specify F. In Cases 2 and 3, it is very possible that no F can be found to maintain the singular arc. Therefore it is appropriate to investigate the solution to the TPBVP when F is specified a priori. It is observed that $\Lambda_{ee}(t)$ can be precomputed in this case, and may be regarded as a known quantity.

$$\Lambda_{ee}(t) = \psi(t,t_f)S\psi^T(t,t_f) + \int_t^{t_f} \psi(t,\tau)U(\tau)\psi^T(t,\tau)d\tau \quad (65)$$

where

$$\dot{\psi} = -F^T\psi$$

$$\psi(t,t) = I.$$

Substituting the expression for the gain (32), in (22) and (29) results in a linear TPBVP since $P_{\overline{xx}}(t)$ and $\Lambda_{ee}(t)$ are known. Equation (22) becomes

$$\dot{P}_{\overline{xe}} = AP_{\overline{xe}} + P_{\overline{xx}}\left[\dot{N}+NA-FN\right]^T + P_{\overline{xe}}F^T + QN^T - P_{\overline{xx}}C^TR^{-1}C\left[P_{\overline{xe}}+P_{\overline{xx}}\Lambda_{\overline{xe}}\Lambda_{ee}^{-1}\right]. \tag{66}$$

Equation (29) is rewritten as

$$\dot{\Lambda}_{\overline{xe}} = -A^T\Lambda_{\overline{xe}} - \Lambda_{\overline{xe}}F - \left[\dot{N}+NA-FN\right]^T\Lambda_{ee} + C^TR^{-1}CP_{\overline{xe}} + C^TR^{-1}CP_{\overline{xx}}\Lambda_{\overline{xe}}\Lambda_{ee}^{-1}. \tag{67}$$

Equations (66) and (67) are linear in the unknowns $P_{\overline{xe}}$ and $\Lambda_{\overline{xe}}$, so there is no difficulty other than possibly the high dimensionality of the problem to contend with. If Λ_{ee} is a scalar matrix, however, things work out particularly well. Suppose F, U, and S are scalar matrices, i.e.,

$$\begin{aligned} F &= fI \\ U &= uI \\ S &= sI. \end{aligned} \tag{68}$$

Then obviously Λ_{ee} is a scalar matrix also. We can thus reposition Λ_{ee} and F in (66) and (67) obtaining

$$\dot{P}_{\overline{xe}} = \left[A+F-P_{\overline{xx}}C^TR^{-1}C\right]P_{\overline{xe}} - \Lambda_{ee}^{-1}P_{\overline{xx}}C^TR^{-1}CP_{\overline{xx}}\Lambda_{\overline{xe}} + P_{\overline{xx}}\left[\dot{N}+NA-FN\right]^T + QN^T \tag{69}$$

and

$$\dot{\Lambda}_{\overline{xe}} = -\left[A^T+F-\Lambda_{ee}^{-1}C^TR^{-1}CP_{\overline{xx}}\right]\Lambda_{\overline{xe}} + C^TR^{-1}CP_{\overline{xe}} - \left[\dot{N}+NA-FN\right]^T\Lambda_{ee}. \tag{70}$$

Note that upon repositioning, the dimension of I in (68) will change in general. It is convenient to make the following definitions

$$\begin{aligned} G_{11} &\triangleq A + F^T - P_{\overline{xx}}C^TR^{-1}C \\ G_{12} &\triangleq -\Lambda_{ee}^{-1}P_{\overline{xx}}C^TR^{-1}CP_{\overline{xx}} \\ G_{21} &\triangleq C^TR^{-1}C \\ G_{22} &\triangleq -[A^T+F-\Lambda_{ee}^{-1}C^TR^{-1}CP_{\overline{xx}}] \\ D_1 &\triangleq P_{\overline{xx}}[\dot{N}+NA-FN]^T + QN^T \\ D_2 &\triangleq -[\dot{N}+NA-FN]^T\Lambda_{ee}. \end{aligned} \tag{71}$$

Then we have

$$\begin{bmatrix} \dot{P}_{\bar{x}e} \\ \hline \dot{\Lambda}_{\bar{x}e} \end{bmatrix} = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \begin{bmatrix} P_{\bar{x}e} \\ \hline \Lambda_{\bar{x}e} \end{bmatrix} + \begin{bmatrix} D_1 \\ \hline D_2 \end{bmatrix} \tag{72}$$

or

$$\begin{bmatrix} \dot{P}_{\bar{x}e} \\ \hline \dot{\Lambda}_{\bar{x}e} \end{bmatrix} = G \begin{bmatrix} P_{\bar{x}e} \\ \hline \Lambda_{\bar{x}e} \end{bmatrix} + D. \tag{73}$$

The boundary condition for $P_{\bar{x}e}$ is at t_0 and for $\Lambda_{\bar{x}e}$ is at t_f. If Φ is the transition matrix associated with G, i.e.,

$$\dot{\Phi} = \left[\begin{array}{c|c} \dot{\Phi}_{11} & \dot{\Phi}_{12} \\ \hline \dot{\Phi}_{21} & \dot{\Phi}_{22} \end{array}\right] = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \left[\begin{array}{c|c} \Phi_{11} & \Phi_{12} \\ \hline \Phi_{21} & \Phi_{22} \end{array}\right] \tag{74}$$

and $\Phi(t,t) = I$ then

$$\begin{aligned} P_{\bar{x}e}(t) &= \Phi_{11}(t,t_0)P_{\bar{x}e}(t_0) + \Phi_{12}(t,t_0)\Lambda_{\bar{x}e}(t_0) \\ &\quad + \int_{t_0}^{t} \Big[\Phi_{11}(t,\tau)D_1(\tau) + \Phi_{12}(t,\tau)D_2(\tau)\Big] d\tau. \end{aligned} \tag{75}$$

Also

$$\begin{aligned} 0 = \Lambda_{\bar{x}e}(t_f) &= \Phi_{21}(t_f, t_0)P_{\bar{x}e}(t_0) + \Phi_{22}(t_f,t_0)\Lambda_{\bar{x}e}(t_0) \\ &\quad + \int_{t_0}^{t_f} \Big[\Phi_{21}(t_f,\tau)D_1(\tau) + \Phi_{22}(t_f,\tau)D_2(\tau)\Big] d\tau. \end{aligned} \tag{76}$$

One can solve (76) for $\Lambda_{\bar{x}e}(t_0)$, substitute the result in (75) and solve (75) for $P_{\bar{x}e}(t)$ in terms of $P_{\bar{x}e}(t_0)$. Similarly, one can solve for $\Lambda_{\bar{x}e}(t)$ using

$$\begin{aligned} \Lambda_{\bar{x}e}(t) &= \Phi_{21}(t,t_0)P_{\bar{x}e}(t_0) + \Phi_{22}(t,t_0)\Lambda_{\bar{x}e}(t_0) \\ &\quad + \int_{t_0}^{t} \Big[\Phi_{21}(t,\tau)D_1(\tau) + \Phi_{22}(t,\tau)D_2(\tau)\Big] d\tau. \end{aligned} \tag{77}$$

The results can then be substituted in (32) to get the optimal gain. Under certain conditions the above procedure is a good approach. Sometimes it is convenient to use an approach which leads to a Riccati equation.

4.1 A Riccati Solution

Here we assume that Λ_{xe} can be written as

$$\Lambda_{\bar{x}e} = MP_{\bar{x}e} + \beta. \tag{78}$$

Then

$$\dot{\Lambda}_{\bar{x}e} = [\dot{M} + MG_{11} + MG_{12}M]P_{\bar{x}e} + MD_1 + \dot{\beta} + MG_{12}\beta$$

and

$$\dot{\Lambda}_{\bar{x}e} = [G_{21} + G_{22}M]P_{\bar{x}e} + G_{22}\beta + D_2. \tag{79}$$

If (78) and (79) are to hold for arbitrary $P_{\bar{x}e}$, then

$$\dot{M} + MG_{11} + MG_{12}M = G_{21} + G_{22}M \tag{80}$$

and

$$\dot{\beta} + MD_1 + MG_{12}\beta = G_{22}\beta + D_2. \tag{81}$$

Since $\Lambda_{\bar{x}e}(t_f) = 0$, the above equations are solved backwards in time from conditions

$$M(t_f) = 0 \tag{82}$$

and

$$\beta(t_f) = 0. \tag{83}$$

Solving the Riccati equation (80) and the equation for β must be done off line, as in a linear regulator problem, since the result is to be obtained by backward integration. The forward equation for $P_{\bar{x}e}$ is

$$\dot{P}_{\bar{x}e} = (G_{11} + G_{12}M)P_{\bar{x}e} + G_{12}\beta + D_1 \tag{84}$$

which is solved from the appropriate initial condition either on line or off line depending on computer requirements. The gain to be used in the filter is then

$$K = \left[P_{\bar{x}e}^T + \Lambda_{ee}^{-1}\left(MP_{\bar{x}e}+\beta\right)^T P_{\bar{x}\bar{x}}\right]C^T R^{-1}. \tag{85}$$

The results presented in this section are suboptimal in general, and optimal in the situation referred to as Case 1.

V. EXAMPLES

The first example considers estimating the output of a generalized Wiener process, i.e., the noise is Gaussian, A equals 0, N = C, and the problem is in the category indicated as Case 3. It is assumed that a solution to (60) exists and that CC^T is nonsingular. The optimal estimate of $z(t) = C(t)x(t)$ is obtained using the filter

$$\dot{\hat{z}}(t) = \dot{C}(t)C^T(t)\left[C(t)C^T(t)\right]^{-1}\hat{z}(t) + K(t)[y(t)-\hat{z}(t)]$$

with initial condition

$$\hat{z}(t_0) = C(t_0)\mu_0$$

and gain

$$K(t) = P_{e\bar{x}}(t)C^T(t)R^{-1}(t)$$

where

$$\dot{P}_{e\bar{x}} = \dot{C}C^T\left[CC^T\right]^{-1}P_{e\bar{x}} + CQ - P_{e\bar{x}}C^TR^{-1}P_{e\bar{x}}$$

with initial condition

$$P_{e\bar{x}}(t_0) = C(t_0)P_0.$$

Alternatively, since $CP_{\bar{x}e} = P_{ee}$, K may be evaluated as

$$K(t) = P_{ee}(t)\,R^{-1}(t)$$

where

$$\dot{P}_{ee} = \left[\dot{C}C^T(CC^T)^{-1}-K\right]P_{ee} + P_{ee}\left[\dot{C}C^T(CC^T)^{-1}-K\right]^T + CQC^T + KRK^T.$$

The above is appealing since P_{ee} has fewer elements to calculate than P_{xe}.

The next example is of the category referred to as Case 1 where a full order filter is used. It is a scalar case. The problem is to get a best estimate of x at $t = T$ where

$$\dot{x} = 0$$
$$y(t) = x(t) + v(t).$$

Initially at $t = 0$ x has mean zero and variance 1, and v(t) is zero mean white noise with unity covariance parameter. If F is selected as zero the filter is particularly simple with a constant gain

$$\dot{\hat{z}}(t) = \dot{\hat{x}}(t) = \frac{1}{1+T}\,y(t).$$

The initial condition is $\hat{x}(0) = 0$. The mean square error is

$$P(t) = 1 - \frac{(1+2T)t + t^2}{(1+T)^2}.$$

Evaluated at $t = T$, the above gives

$$P(T) = \frac{1}{1+T}$$

which is exactly what one would get using a Kalman filter. The filter is simpler than a Kalman filter, but the mean square error is larger than the Kalman filter except at $t = T$ where the mean square error equals that of the Kalman filter. This is illustrated in Figure 1 with $T = 1$. Note that the filter we have developed in this example is of the same structure as the MAP receiver for estimating a constant as presented in Van Trees [13].

The third example illustrates both Cases 1 and 2. In this example the system is given by a second order dynamic model representing a vehicle with random thrust, i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

where x_1 represents the position of the vehicle and x_2 the velocity and w is a zero mean white noise thrust with covariance q. The vehicle position is observed via

$$y = x_1 + r$$

where r is zero mean white measurement noise with covariance r. A reduced order filter is to be designed in order to estimate the position of the vehicle. The realizations for the Kalman filter and the filter developed here are shown in Figures 2 and 3. We have selected $F = 0$. The estimation performance index used is equation (17). The values for q and r are 10. and 1. respectively. The final time was chosen as one second. The first case is that of final time estimation ($U = 0$ and $S = 1$) and the second case corresponds to $U = 100.$ and $S = 1$. Figure 4 shows the mean square value of the first case. The Kalman results are not shown as the two results are nearly coincident. A measure of deviation from the Kalman results of

$$J^* = \int_{t_0}^{t_f} \left(P_{ee} - P_{KF}\right) dt$$

where P_{KF} is the Kalman filter mean square error yields $J^* = 0.0645$ Figure 5 shows the mean square value of the second case. Again, the results are too close to plot separately. The measure of deviation yields $J^* = 0.0052$.

VI. REMARKS AND CONCLUSIONS

We have presented new results in optimal and suboptimal filtering. These results are very attractive for implementation purposes particularly for problems where the Kalman filter is of too great a dimension to be practical. The results herein are in closed form and, thus, it is not necessary to solve a difficult two point boundary value problem. Some relatively simple realizations may be obtained via off line computation.

One of the limitations of the results is that there may be problems whereby a solution for F as in (50) or (60) may not exist. Also, one cannot conclude that the filter is conditionally unbiased when F is specified arbitrarily.

However, the results can be used in a number of problems to decrease the on-line computational burden with the filter structures described in this paper, and this is exceedingly important in many applications. This paper yields a method for synthesis of reduced order filters as well as a class of full order filters of simpler structure.

REFERENCES

1. J. S. Lee, "Optimal Linear Filter and its Relation to Singular Control," Proceedings of the 1970 JACC, June 22-26, 1970, Atlanta, Georgia, pp. 313-319.

2. R. B. Asher, R. A. K. Mitchell and P. S. Maybeck, "Filtering for Precision Pointing and Tracking with Application for Aircraft to Satellite Tracking," Frank J. Seiler Research Laboratory Report SRL-TR-75-0019, November 1975. Submitted to the IEEE Transactions on Aerospace and Electronic Systems.

3. R. B. Asher and R. M. Reeves, "Performance Evaluation of Suboptimal Filters," IEEE Transactions on Aerospace and Electronic Systems, May 1975, pp. 400-405.

4. R. B. Asher, K. D. Herring and J. C. Ryles, "Bias, Variance, and Estimation Error in Reduced Order Filters," Automatica, November 1976.

5. T. R. Damiani, "A General Partial State Estimator," Proceedings of the 14th Midwest Symp. Circuit Theory, Denver, Colorado, May 1971.

6. C. E. Hutchinson and J. A. D'Appolito, "Minimum Variance Reduced State Filters," Proceedings of the IEEE Conf. Decision and Control, New Orleans, December, 1972.

7. C. S. Sims, "An Algorithm for Estimating a Portion of a State Vector," IEEE Trans. Automatic Control, August 1974, pp. 391-393.

8. C. T. Leondes and L. M. Novak, "Optimal Minimal-Order Observers for Discrete-Time Systems - A Unified Theory," Automatica, Vol. 8, pp. 379-387.

9. M. Aoki and J. R. Huddle, "Estimation of the State Vector of a Linear Stochastic System with a Constrained Estimator," IEEE Trans. Automatic Control, Vol. AC-12, August 1967, pp. 432-433.

10. D. G. Luenberger, "Observers for Multivariable Systems," IEEE Trans. Automatic Control, Vol. AC-11, April 1966, pp. 190-197.

11. C. S. Sims and J. L. Melsa, "A Fixed Configuration Approach to the Stochastic Linear Regulator Problem," Proceedings of the 1970 JACC, June 22-26, 1970, Atlanta, Georgia, pp. 706-712.

12. T. L. Boullion and P. L. O'Dell, Generalized Inverse Matrices, Wiley-Interscience, New York, 1971, pp. 4-5.

13. H. L. Van Trees, Detection, Estimation, and Modulation Theory, Part 1, John Wiley and Sons, Inc., New York, 1968, pp. 271-273.

Fig. 1

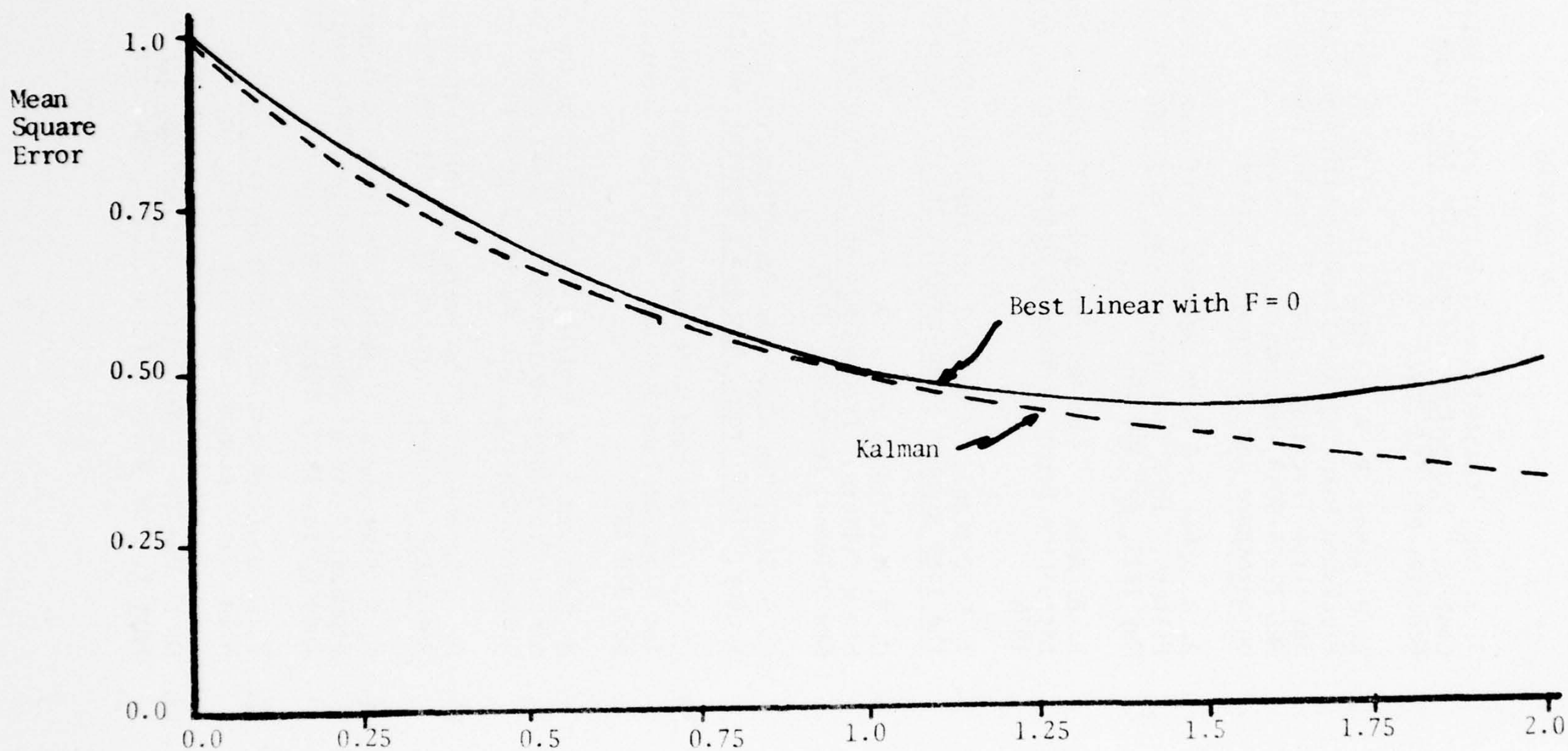

Figure 1 COMPARISON OF MEAN SQUARE ERROR

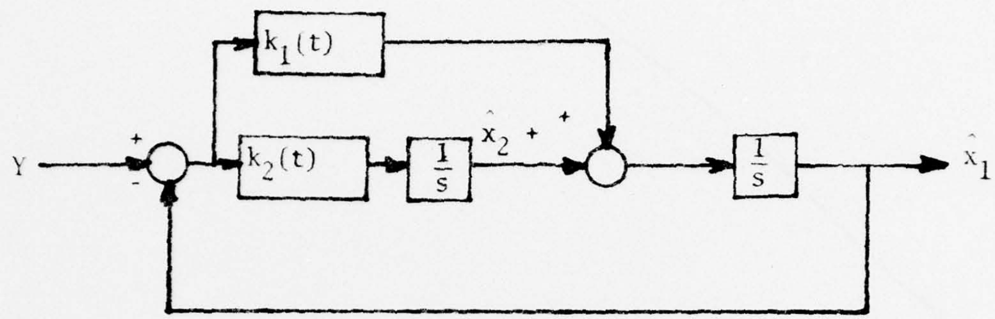

Figure 2 KALMAN FILTER REALIZATION

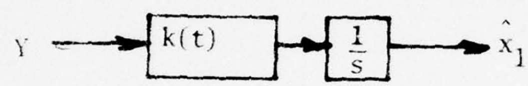

Figure 3 SIMPLIFIED TERMINAL TIME REALIZATION

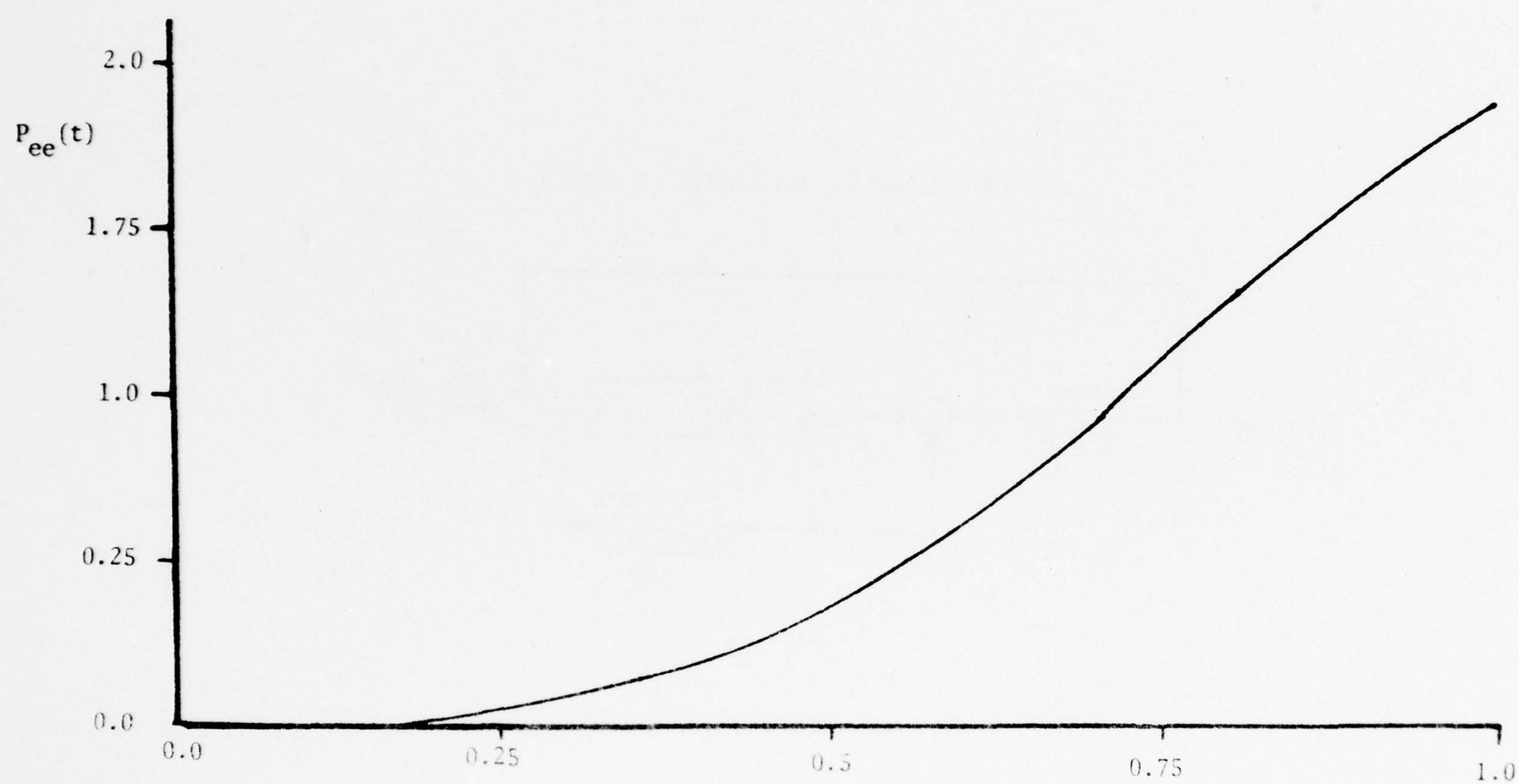

Figure 4 MEAN SQUARE ERROR VS. TIME, U = 0.0

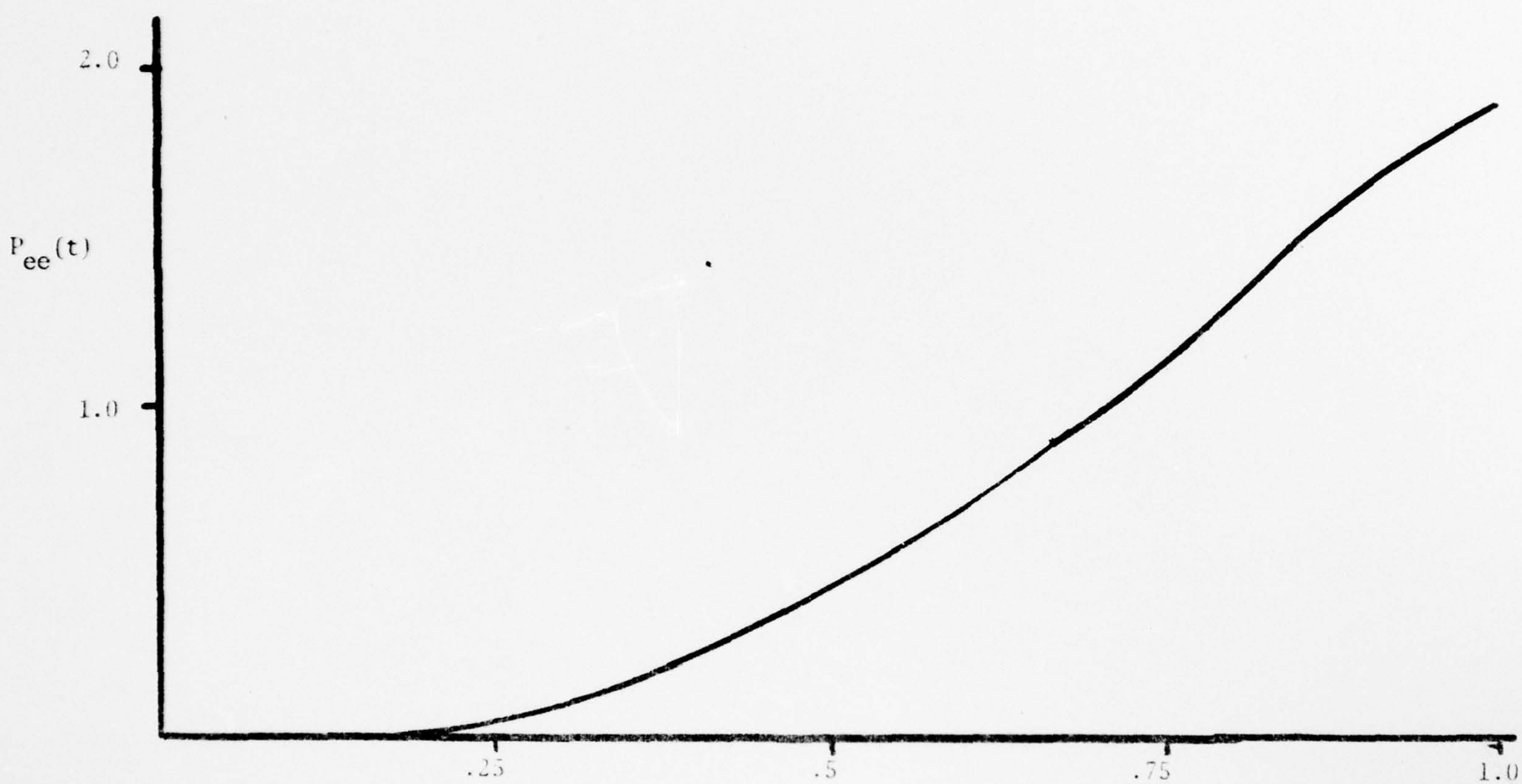

Figure 5 MEAN SQUARE ERROR VS. TIME, U = 100.0