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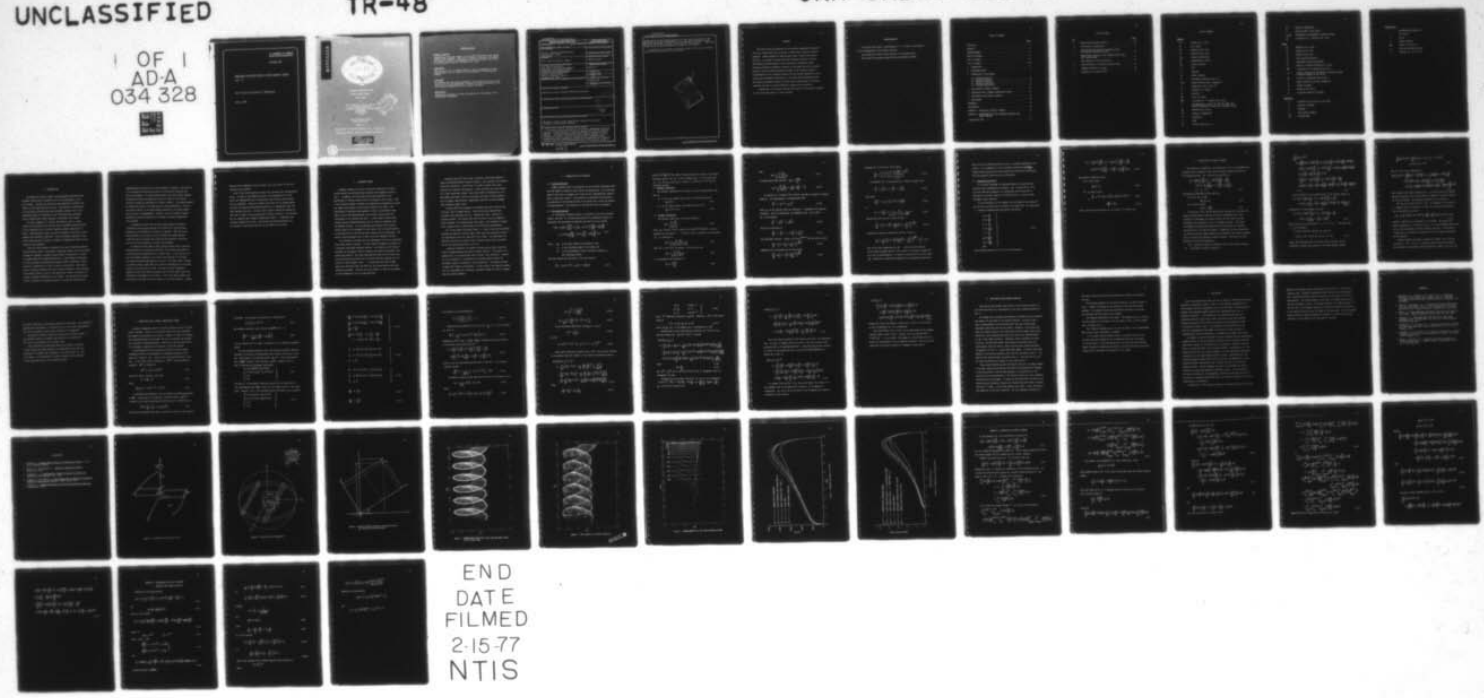

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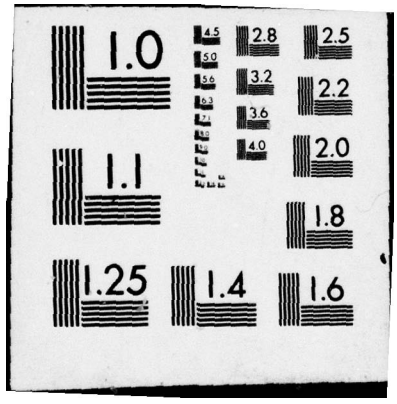
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NONLINEAR HELICOPTER ROTOR LIFTING SURFACE THEORY

PART III

WEST VIRGINIA UNIVERSITY, MORGANTOWN

APRIL 1976

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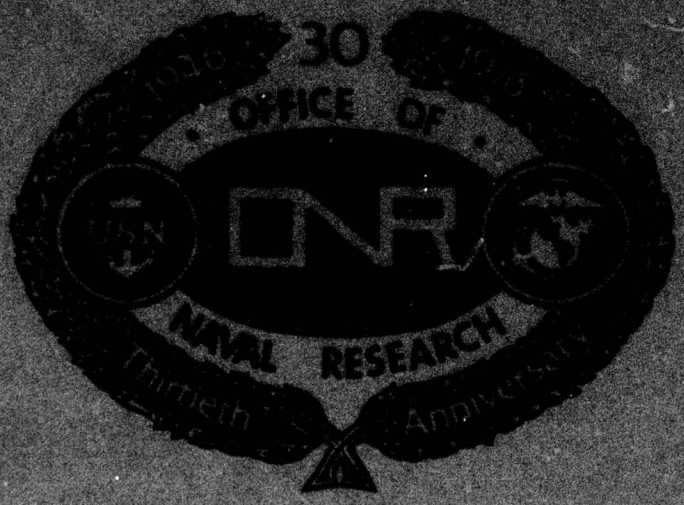

NONLINEAR HELICOPTER ROTOR LIFTING SURFACE THEORY

FINAL REPORT

by

Hsi F. Chou and Jerome B. Fanucci
West Virginia University
Department of Aerospace Engineering
Morgantown
West Virginia
26506

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Green's function approach for the subsonic compressible potential flow over a helicopter rotor in hover or steady state vertical climb was examined. Several methods of approach were taken to derive the Green's function. An attempt to derive the Green's function directly from the differential equation written in the non-inertial coordinate system (rotating Cartesian coordinates attached to the blade) was not successful. However, since the Green's function was known in the inertial system, a

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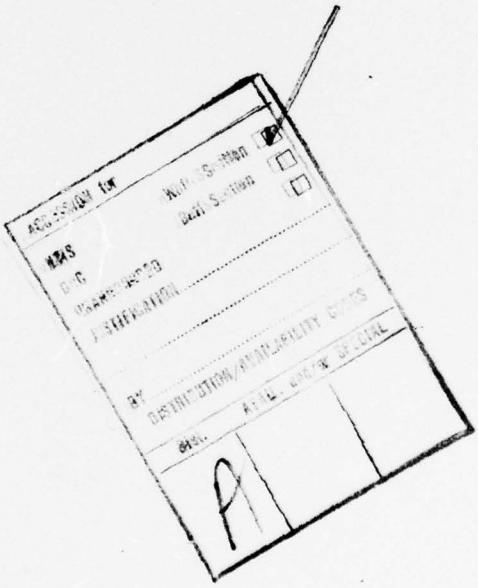
transformation of the integral equation for the velocity potential in this system was transformed successfully to the blade fixed coordinate system. The resulting integral equation involves surface derivatives of which precludes the use of previous numerical schemes for its solution.

A second part of the report includes the results of the second iteration for the force-free wake of a rotor in hover.



ABSTRACT

The Green's function approach for the subsonic compressible potential flow over a helicopter rotor in hover or steady state vertical climb was examined. Several methods of approach were taken to derive the Green's function. An attempt to derive the Green's function directly from the differential equation written in the non-inertial coordinate system (rotating Cartesian coordinates attached to the blade) was not successful. However, since the Greens function was known in the inertial system, a transformation of the integral equation for the velocity potential in this system was transformed successfully to the blade fixed coordinate system. The resulting integral equation involves surface derivatives of φ which precludes the use of previous numerical schemes for its solution.

A second part of the report includes the results of the second iteration for the force-free wake of a rotor in hover.

ACKNOWLEDGMENTS

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LIST OF FIGURES

LIST OF SYMBOLS

English

A	defined in Eq. (A-2)
a	sonic speed
B	defined in Eq. (A-5)
C	defined in Eq. (A-8)
C_p	pressure coefficient
E	defined in Eq. (5-26)
G	Green's function
i	$\sqrt{-1}$
J	Jacobian
J_n	Bessel function
L	an operator defined by Eq. (4-1)
m, n	dummy variables used in Appendix B
n	magnitude of unit vector $\hat{n}$
P	defined in Eq. (B-8A)
p	pressure
R	rotor tip radius
$\mathcal{R}$	a function of r defined in Eq. (B-2)
r	the magnitude of vector $\vec{r}$, $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$; radial coordinate of cylindrical coordinate system
r_α	defined by Eq. (5-23)
S	surface of integration
T	temperature
t	time
U	velocity used by Ref. 7

∇	volume of integration
V	velocity used in this report
x, y, z	coordinates of rectangular coordinate system
Z	a function of z defined by Eq. (B-2)

Greek

α	defined by Eq. (5-20)
β	defined by Eq. (3-8)
γ	specific heat ratio
ϕ	total velocity potential
φ	perturbation velocity potential
ρ	density; a variable defined by Eq. (B-8)
Θ	a function of θ defined by Eq. (B-2)
θ	angular coordinate of cylindrical coordinate system; a variable defined by Eq. (5-21)
Σ	surface of integration used in Ref. 7
σ	dummy variable for surface integration
μ	doublet strength
λ	defined by Eq. (B-8)
ω	rotational speed of the blades

Subscripts

1	coordinates at any point in the field
0	reservoir condition
λ	climbing
∞	free stream condition
TE	trailing edge

Superscripts

$'$	nondimensional quantities
$\hat{}$	unit vector
$\rightarrow$	vector
$*$	adjoint operator
MK	symbol used in Ref. 7
$-$	inertial coordinate system
θ	evaluated at $\bar{x}_1 = \bar{x}^I - \theta$

1. INTRODUCTION

The demand for heavily loaded, high performance helicopters has prompted a great deal of research work on the aerodynamics of the rotor blades. Lifting line theory has been employed to indicate the effects of the distorted wake vortices. Various ways of prescribing or locating the distorted wake were developed, however, due to the tremendous amount of calculation, very few studies have been made using lifting surface theory applied to helicopter rotor blades.

The major drawback of a lifting line model is that it cannot predict the variation of the induced flow in chordwise direction which is very significant for a heavily loaded helicopter, especially in hover. Lifting surface theory can give the details of the flow in both spanwise and chordwise directions. This is extremely important in new blade design because the method is capable of showing the effects of any change in blade geometry.

There are three methods of approach that can be used when applying lifting surface theory. One method of approach predicts the local surface loadings by assuming a loading function which is expressed in a series of assumed modes with unknown coefficients. These unknown coefficients are then obtained by satisfying the normal velocity conditions, either directly or indirectly at a set of points on the surface. The second method of approach predicts the local surface loadings by first assuming an influence function, such as sink and source, vortex, or doublets, of unknown strengths distributed on the surface. By satisfying the normal velocity condition, an integral equation is formed and then solved by

approximating the surface with a finite number of elements. The solution of the influence distribution provides the velocities, pressures and hence the loadings. The third method predicts the local surface loading by directly solving the governing equation in terms of the velocity potential through the application of Green's theorem. An integral equation is formed and solved in the same way as in the second method. In fact, this method results in the same integral equation as the second method when the flow is incompressible. However, the use of Green's theorem generalizes the second method and is capable of solving compressible flow problems.

It should be noticed that the first method gives the pressure coefficient without going through the calculation of velocities and pressures, as is required in the latter two methods. When the actual wetted surface of the blade is used to satisfy the normal boundary conditions and to calculate the velocities and pressures, these data can be used in the analysis of viscous flow problems.

An attempt has been made to apply lifting surface theory using the second approach to an arbitrary helicopter rotor blade system which is in axial motion including hover as a special case. A test case was run with the elements assumed on the actual wetted surface of the blade. Two reports have been published on this work. The first report[1] assumed a classical helical rigid wake with a doublet distribution on the wetted surface of the blade and the wake. The second report[2] attempted to compute the force free wake by starting with the rigid wake results and iterating on the geometry of the wake. A force free wake requires that each point on the wake is free to convect at its local velocity. However,

because of the complexity of this problem, only the results of the first iteration were included.

Both of the above referenced reports are concerned with incompressible flows. In order to investigate the effect of compressibility, in this report the compressible case of helicopter rotor lifting surface theory will be considered. Since the influence functions for an incompressible case will not satisfy the governing equation of a compressible case, the third approach was used. In other words, Green's theorem was applied in this case in order to get an appropriate influence function. The analysis will be compared with a general compressible theory. In addition, the iteration scheme used in Ref. 2 was further pursued and modified, and the results of the second iteration are included in this report.

2. LITERATURE REVIEW

Extensive reviews, as well as listing of references on lifting surface theory and helicopter wake analysis were presented in Refs. 1 and 2. In summary, very few reports have been published on the application of lifting surface theory to helicopter rotor blades. Most of the previous studies circumvent the complicated calculations by using the camber plane instead of the wetted surface of the blade. The description of helicopter wake geometry, either the prescribed rigid wake or force free distorted wake, has had some success under the limitations of the lifting line model. In most cases the wake sheet concept were not maintained wherein the tip vortex is separated from the inboard vortices. In the compressible helicopter rotor lifting surface theory to be considered in this report, the true wetted surface on the blade will be used and the sheet concept will be maintained. The Green's theorem approach will be used to get the influence function.

The literature concerned with the compressible flow over a helicopter rotor blade system is sparse. In 1969, Sopher[3] presented an analysis on a hovering, nonlifting thickness problem. The solution of the velocity potential in the general acoustic equation was taken from Garrick's non-steady wing theory.[4] With some approximations made for the steady case, the velocities and pressures were calculated and then compared with the results of blade element theory. It illustrated that blade element theory was inaccurate near the blade tip and became worse as the blade speed was increased. This was the first attempt to check the dependence of three-dimensional flow on compressibility.

Caradonna and Isom[5] also solved a hovering, nonlifting problem by using the mixed-difference relaxation method. Both subsonic and transonic flows were considered. Scale factors in terms of aspect ratio were introduced to simplify the equations. These equations were then solved by a finite difference scheme. The results show the importance of tip Mach number and aspect ratio on the growth and extent of shock waves in the tip region, and indicate a significant reduction in shock strength with decreasing aspect ratio.

A second paper[6] was presented by Caradonna and Isom in 1975 on the transonic case in forward motion. Scale factors were employed and Successive Over Relaxation (S.O.R.) schemes were used. This was an unsteady problem. The results show that the flow in the tip region is mostly unsteady in the decelerating flow region. The influence of aspect ratio, advance ratio and Mach number on this process was discussed.

All of the papers cited above (Refs. 4, 5, and 6) deal with nonlifting problems and these are the only major works that have attempted to solve the compressible flow over helicopter rotor blades. The approaches are different from what is ususally called lifting surface theory and wakes are not included because there is no lift.

A general formulation of subsonic lifting surface theory using the Green's theorem approach was given by Morino and Kuo.[7] The theory is so general that it can include any type of motion, even rotational. However, it is very difficult to visualize the rotational problem unless the coordinate system is attached to the rotating body. No attempt was made to apply this theory to helicopters. In this report, this general equation will be transformed into rotational coordinate systems in order to compare with the present analysis.

3. FORMULATION OF THE PROBLEM

3.1 Problem Definition

Steady potential flow is considered over an arbitrary helicopter rotor with any number of blades in axial motion including hover. The rotational speed of the rotor is assumed to be large so that compressibility effects must be taken into account. The problem is to determine the velocities and pressures on the wetted surfaces of the blade and to study the effects of compressibility. The influence of the wake will also be included.

3.2 Governing Equation

If a Cartesian coordinate system is attached to one of the rotating blades, (see Fig. 1), the governing equation for linearized subsonic or supersonic flow was shown by Caradonna and Isom[5] to be as follows:

$$\left[a_\infty^2 - (\omega y)^2\right]\frac{\partial^2\varphi}{\partial x^2} + \left[a_\infty^2 - (\omega x)^2\right]\frac{\partial^2\varphi}{\partial y^2} + a_\infty^2\frac{\partial^2\varphi}{\partial z^2}$$

$$+ 2\omega^2 xy\frac{\partial^2\varphi}{\partial x\partial y} + \omega^2 x\frac{\partial\varphi}{\partial x} + \omega^2 y\frac{\partial\varphi}{\partial y} = 0 \qquad (3\text{-}1)$$

where a_∞ is the sonic speed of the medium at rest,

ω is the rotational speed of the blade, and

φ is the perturbation velocity potential, defined in the following equation.

The total velocity at any point in the flow field is

$$\vec{V} = -(\omega\hat{z}\times\vec{r}) + V_\lambda\hat{z} + \nabla_{\vec{r}}\varphi \qquad (3\text{-}2)$$

where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$, $\nabla_{\vec{r}}$ means taking gradient with respect to the xyz coordinate system, and V_λ is the climbing speed. For the hover case $V_\lambda = 0$. $\vec{V}$ is the velocity as viewed by an observer in the rotating coordinate system.

3.3 Boundary Conditions

The boundary conditions are the same as in the incompressible case. That is,

(1) At very far regions where there is no induced velocity

$$\nabla_{\vec{r}}\, \varphi = 0. \tag{3-3}$$

(2) On the blade surface, the normal velocity is 0

$$\vec{V}_n = \vec{V} \cdot \hat{n} = 0. \tag{3-4}$$

3.4 Pressure Coefficient

The pressure coefficient is usually defined by

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} \tag{3-5}$$

where V_∞ from Eq. (3-2) is equal to $\left| -(\omega \hat{z} \times \vec{r}) + V_\lambda \hat{z} \right|$, which is dependent on the radius r. In order to use a constant as the reference, C_p is redefined as

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty (\omega^2 R^2 + V_\lambda^2)}, \tag{3-6}$$

where R is the rotor tip radius. For the hover case

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty (\omega R)^2}. \tag{3-7}$$

If the parameter β is defined as

$$\beta = \frac{\omega R}{a_\infty}, \tag{3-8}$$

then

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty a_\infty^2 \beta^2} \tag{3-9}$$

By using perfect gas relations $a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}$,

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\gamma p_\infty \beta^2} = \frac{2}{\gamma \beta^2}\left(\frac{p}{p_\infty} - 1\right) \tag{3-10}$$

To express p in terms of the velocity, one has to consult the energy equation. For compressible or incompressible flow

$$\frac{V^2}{2} + c_p T = c_p T_o^2 \tag{3-11}$$

where c_p is the specific heat and subscript o designates the reservoir condition. With the expressions $c_p = \gamma R/(\gamma - 1)$ and $a^2 = \gamma R T$, Eq. (3-11) becomes

$$\frac{V^2}{2} + \frac{a^2}{\gamma - 1} = \frac{a_o^2}{\gamma - 1} \tag{3-12}$$

This can be rewritten as

$$\frac{a_o^2}{a^2} = \frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}\left(\frac{V}{a}\right)^2 \tag{3-13}$$

The isentropic relation $p/p_o = (T/T_o)^{\gamma/(\gamma-1)}$ may be used to obtain

$$\frac{p_o}{p} = \left[1 + \frac{\gamma - 1}{2}\left(\frac{V}{a}\right)^2\right]^{\frac{\gamma}{\gamma-1}} \tag{3-14}$$

Similarly for the pressure in the free stream

$$\frac{p_o}{p_\infty} = \left[1 + \frac{\gamma - 1}{2}\left(\frac{V_\infty}{a_\infty}\right)^2\right]^{\frac{\gamma}{\gamma-1}} \tag{3-15}$$

Dividing Eq. (3-15) by Eq. (3-14) given

$$\frac{p}{p_\infty} = \left[\frac{2+(\gamma-1)\left(\frac{V_\infty}{a_\infty}\right)^2}{2+(\gamma-1)\left(\frac{V}{a}\right)^2}\right]^{\frac{\gamma}{\gamma-1}} \tag{3-16}$$

To eliminate a^2, one may write Eq. (3-12) in another form

$$\frac{V^2}{2}+\frac{a^2}{\gamma-1} = \frac{V_\infty^2}{2}+\frac{a_\infty^2}{\gamma-1} \tag{3-17}$$

This gives

$$\frac{a^2}{\gamma-1} = \frac{1}{2}\left(V_\infty^2 - V^2\right)+\frac{a_\infty^2}{\gamma-1}, \tag{3-18}$$

or

$$(\gamma-1)\frac{V^2}{a^2} = \frac{V^2}{\frac{1}{2}\left(V_\infty^2-V^2\right)+\frac{a_\infty^2}{\gamma-1}} \tag{3-19}$$

Substituting this into Eq. (3-16) one gets

$$\frac{p}{p_\infty} = \left[1+\frac{\gamma-1}{2}\left(\frac{V_\infty}{a_\infty}\right)^2\left(1-\frac{V^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}, \tag{3-20}$$

therefore the pressure coefficient from Eq. (3-10) is

$$C_p = \frac{2}{\gamma\beta^2}\left\{\left[1+\frac{\gamma-1}{2}\left(\frac{V_\infty}{a_\infty}\right)^2\left(1-\frac{V^2}{V_\infty^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1\right\}. \tag{3-21}$$

This is the exact expression for C_p. Since the local velocity can be easily obtained from Eq. (3-2), there is no need to expand this out to get an approximation. It should be noted that for a rotor blade V_∞ usually has a significant magnitude in the spanwise direction as

well as in the chordwise direction, and is a variable depending on the radius. It is dangerous to use the familiar expression $2(\partial\varphi/\partial x)/V_\infty$ for regular wings especially near the inboard tip because the contribution from the spanwise direction will be overlooked.

3.5 Nondimensionalization

It is always convenient to nondimensionalize the equations and introduce some nondimensional parameters. β as given by Eq. (3-8) is a very important parameter for compressible rotary flow and is equivalent to the Mach number in rectilinear flow. It will be used throughout these equations.

In the following, all the lengths will be divided by tip radius R and all the velocities will be divided by tip speed ωR. Let superscript $'$ designate nondimensional quantities, and define

$$\left.\begin{aligned} x' &= \frac{x}{R} \\ y' &= \frac{y}{R} \\ z' &= \frac{z}{R} \\ \varphi' &= \frac{\varphi}{\omega R^2} \\ n' &= \frac{n}{R} \\ V_\infty' &= \frac{V_\infty}{\omega R} \\ V_n' &= \frac{V_n}{\omega R} \\ &\text{etc.} \end{aligned}\right\} \qquad (3\text{-}22)$$

The governing equation, Eq. (3-1) can be rewritten as

$$\left[1-(\beta y')^2\right]\frac{\partial^2\varphi'}{\partial x'^2}+\left[1-(\beta x')^2\right]\frac{\partial^2\varphi'}{\partial y'^2}+\frac{\partial^2\varphi'}{\partial z'^2}$$

$$+\beta^2\left(2x'y'\frac{\partial^2\varphi'}{\partial x'\partial y'}+x'\frac{\partial\varphi'}{\partial x'}+y'\frac{\partial\varphi'}{\partial y'}\right)=0 \qquad (3\text{-}23)$$

The boundary conditions become

(1) in free stream

$$\nabla_{\vec{r}'}\,\varphi'=0 \qquad (3\text{-}24)$$

(2) on blade surface

$$\vec{V}_n'=\vec{V}'\cdot\hat{n}'R=\left(\vec{V}_\infty'+\nabla_{\vec{r}'}\varphi'\right)\cdot\hat{n}'R=0 \qquad (3\text{-}25)$$

or

$$-\frac{\partial\varphi'}{\partial n'}=V_{\infty n}' \qquad (3\text{-}25a)$$

Other quantities and equations can be written in a similar way.

4. APPLICATION OF GREEN'S THEOREM

For the sake of convenience, the superscript "′" in the nondimensional equations will be dropped in this section, without being confused with the dimensional ones. Eq. (3-23) may be rewritten as

$$\left[1-(\beta y)^2\right]\frac{\partial^2\varphi}{\partial x^2}+\left[1-(\beta x)^2\right]\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$
$$+\beta^2\left(2xy\frac{\partial^2\varphi}{\partial x\partial y}+x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}\right)$$
$$=L\varphi=0. \tag{4-1}$$

The boundary conditions now have the form

(1) in the free stream

$$\nabla_r\varphi=0, \tag{4-2}$$

(2) on blade surface

$$V_{\infty_n}=-\frac{\partial\varphi}{\partial n}. \tag{4-3}$$

To solve Eq. (4-1), one may multiply the equation by a function G, which is called Green's function and will be determined later, and integrate it throughout the volume in which the flow field is being considered, then transform it into an integral equation. The volume of integration contains the region from the body surface to infinity, therefore this is an exterior problem. Integration by parts was applied to each term and the volume integral was transformed into surface integral. This is actually the approach of Green's theorem. The details of this derivation are given in Appendix A.

Fig. 2 shows the volume of integration V and the surface boundary S. The result of Green's theorem is

$$\iiint_V G L\varphi \, dV$$

$$= -\iint_S \Big[\Big\{ G\left[1-(\beta y)^2\right]\frac{\partial\varphi}{\partial x} - \left[1-(\beta x)^2\right]\frac{\partial G}{\partial x}\varphi + 2\beta^2 G x y \frac{\partial\varphi}{\partial y} + \beta^2 G x \varphi \Big\}\hat{x}$$

$$+ \Big\{ G\left[1-(\beta x)^2\right]\frac{\partial\varphi}{\partial y} - \left[1-(\beta x)^2\right]\frac{\partial G}{\partial y}\varphi - 2\beta^2\left(Gy + xy\frac{\partial G}{\partial x}\right) + \beta^2 G y \varphi \Big\}\hat{y}$$

$$+ \Big\{ G\frac{\partial\varphi}{\partial z} - \frac{\partial G}{\partial z}\varphi \Big\}\hat{z} \Big] \cdot \hat{n} \, dS$$

$$+ \iiint_V \Big\{ \left[1-(\beta y)^2\right]\frac{\partial^2 G}{\partial x^2} + \left[1-(\beta x)^2\right]\frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}$$

$$+ 2\beta^2\left(G + y\frac{\partial G}{\partial y} + x\frac{\partial G}{\partial x} + xy\frac{\partial^2 G}{\partial x \partial y}\right) - \beta^2\left(x\frac{\partial G}{\partial x} + G\right) - \beta^2\left(y\frac{\partial G}{\partial y} + G\right) \Big\}\varphi \, dV$$

$$= 0. \qquad (4\text{-}4)$$

Let the operator in the curled bracket of the volume integral be

$$L^* = \left[1-(\beta y)^2\right]\frac{\partial^2}{\partial x^2} + \left[1-(\beta x)^2\right]\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$+ 2\beta^2\left(1 + y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial x \partial y}\right) - \beta^2\left(x\frac{\partial}{\partial x} + 1\right) - \beta^2\left(y\frac{\partial}{\partial y} + 1\right) \qquad (4\text{-}5)$$

Incidentally, by rearranging, one finds

$$L^* = \left[1-(\beta y)^2\right]\frac{\partial^2}{\partial x^2} + \left[1-(\beta x)^2\right]\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$+ \beta^2\left(2xy\frac{\partial^2}{\partial x \partial y} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \qquad (4\text{-}6)$$

which turns out to be the same as L, i.e., $L^* = L$. This shows that L is self-adjoint.

If one can find the function G such that

$$L^* G = L G = \delta(x - x_1, y - y_1, z - z_1), \qquad (4\text{-}7)$$

where $\delta(x-x_1, y-y_1, z-z_1)$ is the Dirac delta function, then by definition of Dirac delta function, the volume integral becomes

$$\iiint_V \varphi L^* G \, dV = \iiint_V \varphi \, \delta(x-x_1, y-y_1, z-z_1) \, dV$$
$$= \varphi(x_1, y_1, z_1) \qquad (4\text{-}8)$$

Now. Eq. (4-4) can be rewritten as

$$\varphi(x_1,y_1,z_1) = -\iint_S \Big[\Big\{ G\big[1-(\beta y)^2\big]\frac{\partial \varphi}{\partial x} - \big[1-(\beta y)^2\big]\frac{\partial G}{\partial x}\varphi + 2\beta^2 G x y \frac{\partial \varphi}{\partial y}$$
$$+ \beta^2 G x \varphi \Big\} \hat{x} + \Big\{ G\big[1-(\beta x)^2\big]\frac{\partial \varphi}{\partial y} - \big[1-(\beta x)^2\big]\frac{\partial G}{\partial y}\varphi$$
$$- 2\beta^2 \Big(Gy + xy\frac{\partial G}{\partial x}\Big)\varphi + \beta^2 G y \varphi \Big\} \hat{y} + \Big\{ G\frac{\partial \varphi}{\partial z} - \frac{\partial G}{\partial z}\varphi \Big\} \hat{z} \Big] \cdot \hat{n} \, dS \qquad (4\text{-}9)$$

This includes the surface integral only. It could be solved by approximating the surface with a finite number of elements if Green's function G can be obtained. The remaining question is how to find G .

G has to be solved from Eq. (4-7). The symmetry of the operator L suggests that it is better to do this in cyclindrical coordinates. After transformation one gets

$$L G = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial G}{\partial r}\Big) + (1-\beta^2 r^2)\frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} + \frac{\partial^2 G}{\partial z^2} = \delta(r-r_1, \theta-\theta_1, z-z_1) \qquad (4\text{-}10)$$

Consider $LG=0$ first and separate the variables. The details are shown in Appendix B. It turns out that L is of the standard Bessel type and G may be expressed in terms of Bessel functions. Unfortunately, it still has to be written in integral form and no compact expressions can be achieved.

Several attempts were made to expand the integral form in series such as β , β^2 , or $z-z_1$, however all these series are divergent. The technique of perturbation was also applied in light of the fact that

the Green's function for the Laplace equation is well known, but singularity problems were encountered. The details of all of the above investigation will not be presented in this report because of its complexity.

The search for Green's function has been one of the main tasks of this project, however, a workable form could not be obtained and Eq. (4-9) is not solvable in its present form.

Some word should be mentioned finally with regard to notation. In this section the dummy variables of integration do not have the subscript "$_1$", whereas the coordinates of the field point do. This is a result of the derivation of Green's theorem, and is opposite to the convention used in the following section.

5. COMPARISON WITH A GENERAL COMPRESSIBLE THEORY

Unsteady compressible flow was treated by Marino and Kuo[7] by using Green's function. This is a very general theory that will accomodate any complex configuration in any motion. However, in order to visualize the rotational motion of the blades, one has to transform the equation into rotational coordinate systems. This will be done in this section in an attempt to compare with the results in the last section.

The perturbation velocity potential in Marino and Kuo's paper (designated as MK in the following) is different from that in this report by a constant U_∞, which is the free stream velocity of the body. If the total velocity potential is ϕ^{MK}, the perturbation potential φ^{MK} is defined by

$$\phi^{MK} = U_\infty (x + \varphi^{MK}) \tag{5-1}$$

while this report, from Eq. (3-2), uses

$$\phi = \varphi_\infty + \varphi \tag{5-2}$$

where

$$\nabla_{\vec{r}}\, \varphi_\infty = -(\omega \hat{z} \times \vec{r}) + V_\infty \hat{z} \tag{5-3}$$

Clarifying this difference, one can rewrite the governing equation in MK, with respect to an inertial coordinate system $\bar{x}\bar{y}\bar{z}\bar{t}$ attached to the body and moving with velocity U_∞ in x-direction, as

$$\nabla^2 \varphi = \frac{1}{a_\infty^2} \left(\frac{\partial}{\partial \bar{t}} + U_\infty \frac{\partial}{\partial \bar{x}} \right)^2 \varphi , \tag{5-4}$$

where all the notations have been, and will be, written in this report's

convention. The surface of integration is represented by

$$S(\bar{x}, \bar{y}, \bar{z}, \bar{t}) = 0 \tag{5-5}$$

The boundary condition can be written as $DS/D\bar{t} = 0$, or

$$\frac{\partial \varphi}{\partial \bar{n}} = -\frac{1}{|\nabla_r S|}\left(\frac{\partial S}{\partial \bar{t}} + U_\infty \frac{\partial S}{\partial \bar{x}}\right), \tag{5-6}$$

All the overhead bars (—) mean the quantity in the inertial coordinate system.

For the hovering helicopter case, the inertial coordinate system is attached to the body of the helicopter with z-axis at the center of the hub. Since there is no advancing speed, $U_\infty = 0$, the transformation from $\bar{x}\bar{y}\bar{z}\bar{t}$ system to the rotational $xyzt$ systems (refer to Fig. 3) then has the following relationship:

$$\left.\begin{aligned} x &= \bar{x}\cos(\omega\bar{t}) + \bar{y}\sin(\omega\bar{t}) \\ y &= -\bar{x}\sin(\omega\bar{t}) + \bar{y}\cos(\omega\bar{t}) \\ z &= \hat{z} \\ t &= \bar{t} \end{aligned}\right\} \tag{5-7}$$

The time t in rotational coordinate systems will be considered for the transformation and $\partial/\partial t$ terms will be dropped later for the steady case. Using Eq. (5-7), the following relevant relations can be obtained:

$$\left.\begin{aligned} \bar{x} &= x\cos(\omega t) - y\sin(\omega t) \\ \bar{y} &= x\sin(\omega t) + y\cos(\omega t) \\ \bar{z} &= z \\ \bar{t} &= t \end{aligned}\right\} \tag{5-8}$$

$$\left.\begin{aligned}
\frac{\partial}{\partial \bar{x}} &= \cos(\omega t)\frac{\partial}{\partial x} - \sin(\omega t)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \bar{y}} &= \sin(\omega t)\frac{\partial}{\partial x} + \cos(\omega t)\frac{\partial}{\partial y} \\
\frac{\partial}{\partial \bar{z}} &= \frac{\partial}{\partial z} \\
\frac{\partial}{\partial \bar{t}} &= \omega y \frac{\partial}{\partial x} - \omega x \frac{\partial}{\partial y} + \frac{\partial}{\partial t} \\
&= -(\omega \hat{z} \times \vec{r}) \cdot \nabla_{\vec{r}} + \frac{\partial}{\partial t}
\end{aligned}\right\} \quad (5\text{-}9)$$

$$\left.\begin{aligned}
\hat{x} &= \hat{\bar{x}} \cos(\omega \bar{t}) + \hat{\bar{y}} \sin(\omega \bar{t}) \\
\hat{y} &= -\hat{\bar{x}} \sin(\omega \bar{t}) + \hat{\bar{y}} \cos(\omega \bar{t}) \\
\hat{z} &= \hat{\bar{z}}
\end{aligned}\right\} \quad (5\text{-}10)$$

$$\left.\begin{aligned}
\hat{\bar{x}} &= \hat{x} \cos(\omega t) - \hat{y} \sin(\omega t) \\
\hat{\bar{y}} &= \hat{x} \sin(\omega t) + \hat{y} \cos(\omega t) \\
\hat{\bar{z}} &= \hat{z}
\end{aligned}\right\} \quad (5\text{-}11)$$

$$\nabla_{\vec{\bar{r}}} = \nabla_{\vec{r}} \quad (5\text{-}12)$$

$$\nabla^2_{\vec{\bar{r}}} = \nabla^2_{\vec{r}} \quad (5\text{-}13)$$

The Jacobian of the transformation

$$J = \frac{\partial(x, y, z, t)}{\partial(\bar{x}, \bar{y}, \bar{z}, \bar{t})} = 1 \tag{5-14}$$

Using the above relations, with $U_\infty = 0$ and $\frac{\partial}{\partial \bar{t}} = 0$, one can rewrite Eq. (5-4) as

$$\nabla_{\vec{r}}^2 \varphi - \frac{1}{a_\infty^2}\left[-(\omega \hat{z} \times \vec{r}) \cdot \nabla_{\vec{r}}\right]^2 \varphi = 0. \tag{5-15}$$

Rembering $(\omega \hat{z} \times \vec{r}) \cdot \nabla_{\vec{r}} = \omega y \frac{\partial}{\partial x} - \omega x \frac{\partial}{\partial y}$ and expanding the square brackets in the above, this now has the form

$$\left[1 - \left(\frac{\omega y}{a_\infty}\right)^2\right]\frac{\partial^2 \varphi}{\partial x^2} + \left[1 - \left(\frac{\omega x}{a_\infty}\right)^2\right]\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$
$$+ \left(\frac{\omega}{a_\infty}\right)^2\left(2xy\frac{\partial^2 \varphi}{\partial x \partial y} + x\frac{\partial \varphi}{\partial x} + y\frac{\partial \varphi}{\partial y}\right) = 0, \tag{5-16}$$

which is just the governing equation given in Section 3. The boundary condition becomes

$$\frac{\partial \varphi}{\partial n} = -\frac{1}{|\nabla_{\vec{r}} S|}\left[-(\omega \hat{z} \times \vec{r})\right] \cdot \nabla S. \tag{5-17}$$

The Green's function in the inertial system for the unsteady case is

$$G = -\frac{1}{4\pi r_\alpha}\,\delta(\bar{t}_1 - \bar{t} + \theta) \tag{5-18}$$

where

$$r_\alpha = \left\{(\bar{x} - \bar{x}_1)^2 + \alpha^2\left[(\bar{y} - \bar{y}_1)^2 + (\bar{z} - \bar{z}_1)^2\right]\right\}^{1/2} \tag{5-19}$$

$$\alpha = \sqrt{1 - \left(\frac{U_\infty}{a_\infty}\right)^2} \tag{5-20}$$

$$\theta = \frac{1}{a_\infty \alpha^2}\left[\frac{U_\infty}{a_\infty}(\bar{x}_1 - \bar{x}) + r_\alpha\right] \tag{5-21}$$

In the rotational system now, with $U_\infty = 0$, $\alpha = 1$,

$$G = -\frac{1}{4\pi r_\alpha} \tag{5-22}$$

but now

$$r_\alpha = \left[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2\right]^{1/2} \tag{5-23}$$

When Green's theorem is applied to Eq. (5-4), the integral equation in the general form with respect to the inertial coordinate system is

$$\begin{aligned} 4\pi E\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{t}) &= -\iint_{\Sigma^\theta}\left[\nabla_{\bar{r}_1} S \cdot \nabla_1 \varphi - \frac{1}{a_\infty^2}\frac{dS}{d\bar{t}_1}\frac{d\varphi}{d\bar{t}_1}\right]^\theta \frac{1}{r_\alpha}\frac{d\sigma^\theta}{|\nabla_{\bar{r}_1} S^\theta|} \\ &+ \iint_{\Sigma^\theta}\left[\nabla_{\bar{r}_1} S \cdot \nabla_1\left(\frac{1}{r_\alpha}\right) - \frac{1}{a_\infty^2}\frac{dS}{d\bar{t}_1}\frac{d}{d\bar{t}_1}\left(\frac{1}{r_\alpha}\right)\right]^\theta [\varphi]^\theta \frac{d\sigma^\theta}{|\nabla_{\bar{r}_1} S^\theta|} \\ &- \frac{\partial}{\partial \bar{t}}\iint_{\Sigma^\theta}\left[\nabla_{\bar{r}_1} S \cdot \nabla_1 \theta - \frac{1}{a_\infty^2}\frac{dS}{d\bar{t}_1}\left(1 + U_\infty \frac{\partial \theta}{\partial \bar{x}_1}\right)\right]^\theta [\varphi]^\theta \frac{d\sigma^\theta}{|\nabla_{\bar{r}_1} S^\theta|} \end{aligned} \tag{5-24}$$

where

$$\frac{d}{d\bar{t}_1} = \frac{\partial}{\partial \bar{t}_1} + U_\infty \frac{\partial}{\partial \bar{x}_1} \tag{5-25}$$

$$\left.\begin{aligned} E &= 0 && \text{inside } \sigma \\ E &= \tfrac{1}{2} && \text{on } \sigma \\ E &= 1 && \text{outside } \sigma \end{aligned}\right\} \qquad (5\text{-}26)$$

and $[\ ]^{\theta}$ indicates evaluation at $\bar{t}_1 = \bar{t} - \theta$. Similarly, Σ^{θ} is the surface given by

$$S^{\theta} = S(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{t} - \theta) \qquad (5\text{-}27)$$

where $(\bar{x}_1, \bar{y}_1, \bar{z}_1)$ is the dummy point of integration on Σ^{θ}.

Substituting in the transformation relations, Eq. (5-24) in the inertial system with $U_\infty = 0$ and $\partial/\partial\bar{t} = 0$ becomes

$$\begin{aligned} &4\pi E \varphi(x,y,z) \\ &= -\iint_S \left\{ \nabla_{\vec{r}_1} S \cdot \nabla_{\vec{r}_1} \varphi - \frac{1}{a_\infty^2}\left[(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1} S\right]\left[(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1} \varphi\right] \frac{1}{r_\alpha} \frac{dS_1}{|\nabla_{\vec{r}_1} S|} \right. \\ &+ \iint_S \left\{ \nabla_{\vec{r}_1} S \cdot \nabla_{\vec{r}_1}\left(\frac{1}{r_\alpha}\right) + \frac{1}{a_\infty^2}\left[(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1} S\right]\left[(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1}\left(\frac{1}{r_\alpha}\right)\right][\varphi] \frac{dS_1}{|\nabla_{\vec{r}_1} S|} \right. \\ &+ (\omega \hat{z} \times \vec{r})\cdot \nabla_{\vec{r}} \iint_S \left[\nabla_{\vec{r}_1} S \cdot \nabla_{\vec{r}_1} \theta + \frac{1}{a_\infty^2}(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1} S \right][\varphi] \frac{1}{r_\alpha} \frac{dS_1}{|\nabla_{\vec{r}_1} S|} \end{aligned} \qquad (5\text{-}28)$$

where

$$\theta = \frac{r_\alpha}{a_\infty} \qquad (5\text{-}29)$$

and σ^{θ} or Σ^{θ} now is just the surface given by $S(x,y,z) = 0$, which is independent of time.

Substituting the boundary condition, Eq. (5-17), and remembering $(\omega \hat{z}_1 \times \vec{r}_1)\cdot \nabla_{\vec{r}_1} = \omega y_1 \frac{\partial}{\partial x_1} - \omega x_1 \frac{\partial}{\partial y_1}$ and $\frac{\nabla_{\vec{r}_1} S}{|\nabla_{\vec{r}_1} S|} \cdot \nabla_{\vec{r}_1} \varphi = \frac{\partial \varphi}{\partial n_1}$. Eq. (5-28) can be rewritten as

$$4\pi E\varphi(x,y,z)$$

$$= -\iint_S \left[\frac{\partial\varphi}{\partial n_1} + \frac{1}{a_\infty^2}\frac{\partial\varphi}{\partial n_1}\left(\omega y_1\frac{\partial\varphi}{\partial x_1} - \omega x_1\frac{\partial\varphi}{\partial y_1}\right)\right]\frac{1}{r_\alpha}\,dS_1$$

$$+\iint_S \left[\frac{\partial}{\partial n_1}\left(\frac{1}{r_\alpha}\right) - \frac{1}{a_\infty^2}\frac{\partial\varphi}{\partial n_1}\left(\omega y_1\frac{\partial}{\partial x_1} - \omega x_1\frac{\partial}{\partial y_1}\right)\left(\frac{1}{r_\alpha}\right)\right]\varphi\,dS_1$$

$$+\left(\omega y\frac{\partial}{\partial x} - \omega x\frac{\partial}{\partial y}\right)\iint_S \left(\frac{\partial\theta}{\partial n_1} - \frac{1}{a_\infty^2}\frac{\partial\varphi}{\partial n_1}\right)\frac{\varphi}{r_\alpha}\,dS_1 \qquad (5\text{-}30)$$

All of the above equations in this section are written in dimensional form. In order to compare with the results in the last section, these equations have to be nondimensionalized. Using the same notation defined in Section 3, and noting that $dS = dx\,dy = R^2\,dx'dy' = R^2 dS'$ one can rewrite Eq. (5-30) as

$$4\pi E\varphi'(x',y',z')$$

$$= -\iint_{S'} \left[\frac{\partial\varphi'}{\partial n_1'} + \beta^2\frac{\partial\varphi'}{\partial n_1'}\left(y_1'\frac{\partial\varphi'}{\partial x_1'} - x_1'\frac{\partial\varphi'}{\partial y'}\right)\right]\frac{1}{r_\alpha'}\,dS_1'$$

$$+\iint_{S'} \left[\frac{\partial}{\partial n_1'}\left(\frac{1}{r_\alpha'}\right) - \beta^2\frac{\partial\varphi'}{\partial n'}\left(y_1'\frac{\partial}{\partial x_1'} - x_1'\frac{\partial}{\partial y_1'}\right)\left(\frac{1}{r_\alpha'}\right)\right]\varphi'\,dS_1'$$

$$+\left(y'\frac{\partial}{\partial x'} - x'\frac{\partial}{\partial y'}\right)\iint_{S'} \left[\frac{\partial}{\partial n_1'}\left(\beta r_\alpha'\right) - \beta^2\frac{\partial\varphi'}{\partial n_1'}\right]\frac{\varphi'}{r_\alpha'}\,dS_1' \qquad (5\text{-}31)$$

To compare this with Eq. (4-9), one should ignore the primes ("′") and remember that the variables with subscript "1" are dummies of integration. Eq. (4-9) can be rewritten in the following form using the convention of this section

$$\varphi'(x_1', y_1', z)$$

$$= \iint_{S'} \Big\{ G \frac{\partial \varphi'}{\partial n_1'} - G\beta^2 \Big(y_1'^2 \frac{\partial \varphi'}{\partial x_1'} \hat{x}_1' + x_1'^2 \frac{\partial \varphi'}{\partial y_1'} \hat{y}_1' \Big) \cdot \hat{n}_1'$$

$$- \varphi' \frac{\partial G}{\partial n_1'} - \varphi' \beta^2 \Big(y_1'^2 \frac{\partial G}{\partial x_1'} \hat{x}_1 + x_1'^2 \frac{\partial G}{\partial y_1'} \hat{y}_1 \Big) \cdot \hat{n}_1'$$

$$+ \beta^2 \Big[G \Big(x_1' y_1' \frac{\partial \varphi'}{\partial y_1'} + \varphi' x_1' \Big) \hat{x}_1' - \varphi' \Big(x_1' y_1' \frac{\partial G}{\partial x_1'} + G y_1' \Big) \hat{y}_1' \Big] \cdot \hat{n}_1' \Big\} dS_1' \qquad (5\text{-}32)$$

Though this looks very similar in form as Eq. (5-31), it is not easy to get an expression for G by comparison.

The terms $\partial\varphi'/\partial n_1'$ in Eq. (5-31) can be substituted by the boundary condition shown in Eq. (3-25a). However, the term ($y_1' \partial\varphi'/\partial x_1' - x_1' \partial\varphi'/\partial y_1'$) is not known. This makes Eq. (5-31) difficult to be solved by transforming it into an algebraic equation. This is also the reason why no numerical solution is attempted in this report.

6. SOME RESULTS FROM SECOND ITERATION

This section will present some results for the second iteration of the force-free wake and a continuation of the first iteration results of Ref. 2.

The damped first iteration wake geometry calculated in this reference was taken to be the basis of the second iteration. In order to get a better understanding of the wake distortion near the trailing edge, additional steps were used in this region. Moreover, it was found that since the control points on the radial lines were no longer linear after distortion, some irregularities would occur due to the differentiation of φ in the radial direction. Therefore, before starting the second iteration, the distorted radial lines should be projected onto each radial plane, which was done by using the projection routine discussed in Appendix B of Ref. 2, to make them linear again. The final input wake geometry of the second iteration looks like the one shown in Fig. 4. The sizes of the inserted steps are shown in Fig. 6. The step size for the far wake calculations was similar to that used in Ref. 2.

This second iteration process has been carried out, including finding a new wake, solving the equations and finally recalculating the loadings. In the calculation of the new wake geometry, the contribution to the velocity due to the derivative of the velocity potential in the streamwise direction was also included. It should be understood that the value of φ along any given streamline leaving the trailing edge would remain constant and equal to $\Delta\varphi_{TE}$ if the wake geometry was correct. This contribution was neglected in the first iteration. All the techniques developed for

the first iteration were used and therefore the program is essentially the same.

The resulting geometry of the second iteration is shown in Figs. 5 and 6. Instead of rolling up, the points near both tips were washed downward. The downward distortion increased the loading on the blade and therefore made the result more divergent.

The damping technique was used again. The geometry used to calculate the new loading is the average of the one shown in Fig. 5 and the damped first iteration, Fig. 4.

The $\Delta\mu_{TE}$ and loading curve, are shown on Figs. 7 and 8 respectively. The overall thrust coefficient is .005424.

It is obvious that the present result is caused by the fact that the wake distortion scheme was not working well near the outer edges of the wake sheet so that the expected tip roll-ups were not generated. Before carrying out additional iterations, the wake distortion calculation scheme must be thoroughly investigated in this region.

7. CONCLUSIONS

One of the objectives of this work was to adapt our lifting surface theory programs, as described in Refs. 1 and 2, to the subsonic compressible use. One procedure that was attempted was the use of the Green's theorem approach whereby the governing integral equation was transformed into a set of algebraic equations. It was hoped that this approach would lead to an analytical and exact form of the Green's function so that the resulting integral equation could be solved by applying the same numerical scheme as developed in Refs. 1 and 2. The analysis was not restricted to any particular compressible flow region on the blade so that the effect of compressibility would be accounted for throughout the complete flow field.

The attempt to derive the Green's function directly from the differential equation written in the non-inertial coordinate system (rotating Cartesian coordinates attached to the blade) was not successful. However, since the Green's function was known in the inertial system, a transformation of the integral equation for the velocity potential in this system was transformed successfully to the blade fixed coordinate system. Investigation of the resulting integral equation (Eq. (5-31)) shows that the appearance of terms involving surface derivatives of φ precludes the use of our previous numerical scheme. Further attempts were made to investigate the Green's function in a rotating cylindrical coordinate system. It was found that the Green's function for the appropriate differential equation would be in the form of an integral which would result in a numerical scheme that was too complicated to be practical.

A second part of this study was related to improving the force free wake

geometry by extending further the procedures used in Ref. 2 for the first iteration wake. Extensive calculations were carried out in attempting to find a convergent procedure which would result in the correct force free wake geometry. The results appear to show that the inboard portion of the wake was convergent, however, the outboard portion of the wake diverged. It was felt that this divergent tendency was caused by inaccurate calculations of the induced velocities near the edges of the wake sheet. Further attempts should be made at improving numerical procedures to compute these induced velocities in the region of the free edges of the wake.

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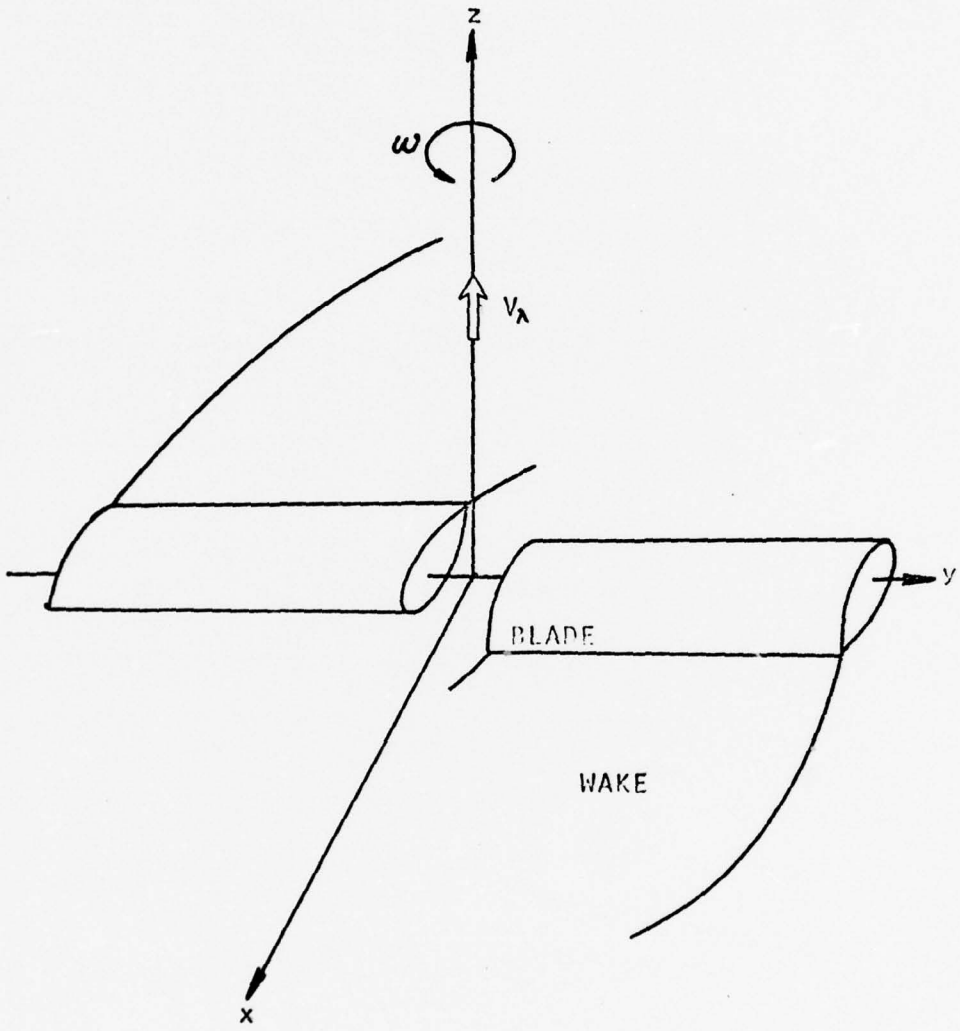

Figure 1 Blade-Fixed Coordinate System

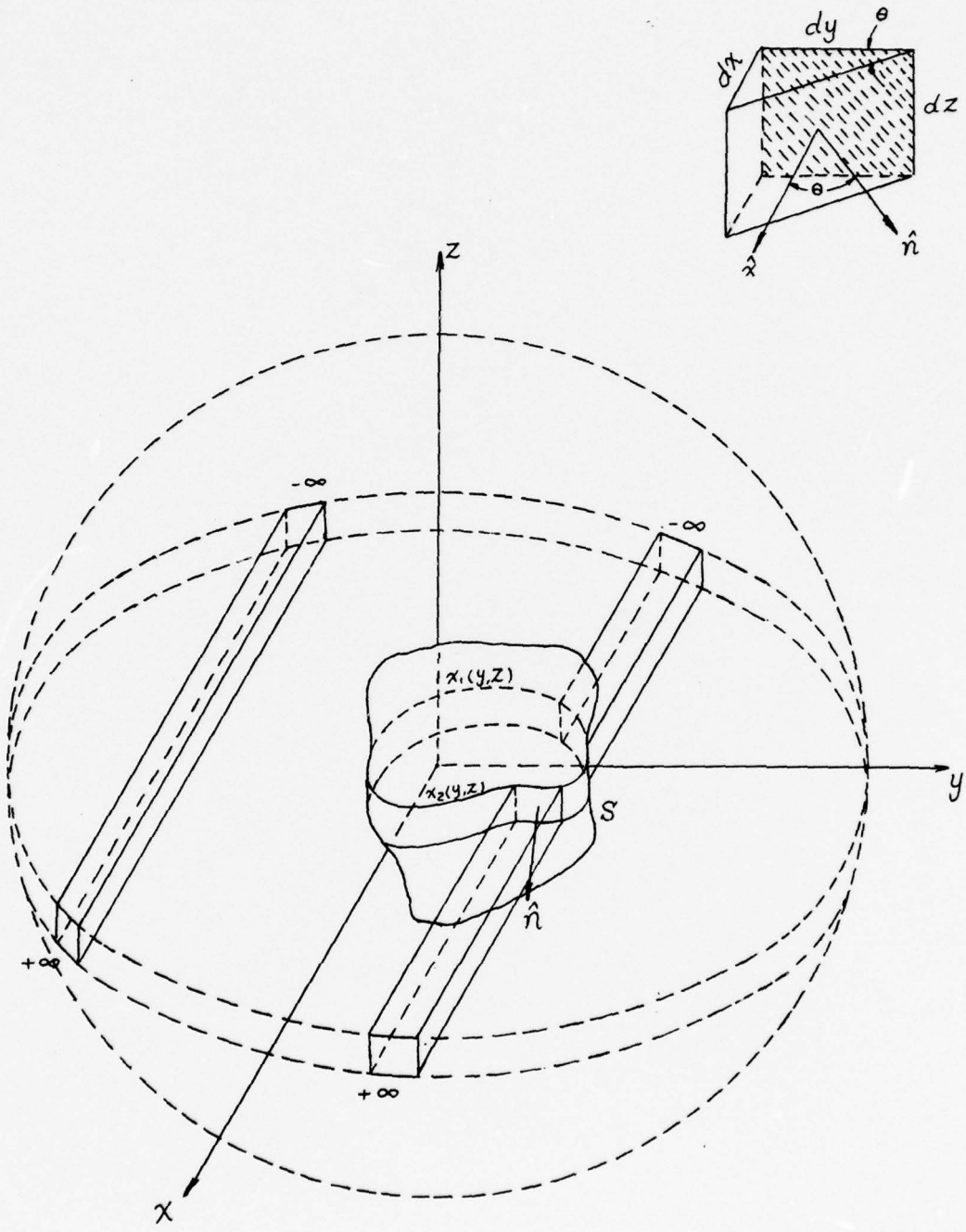

Figure 2 Flow Field for Integration

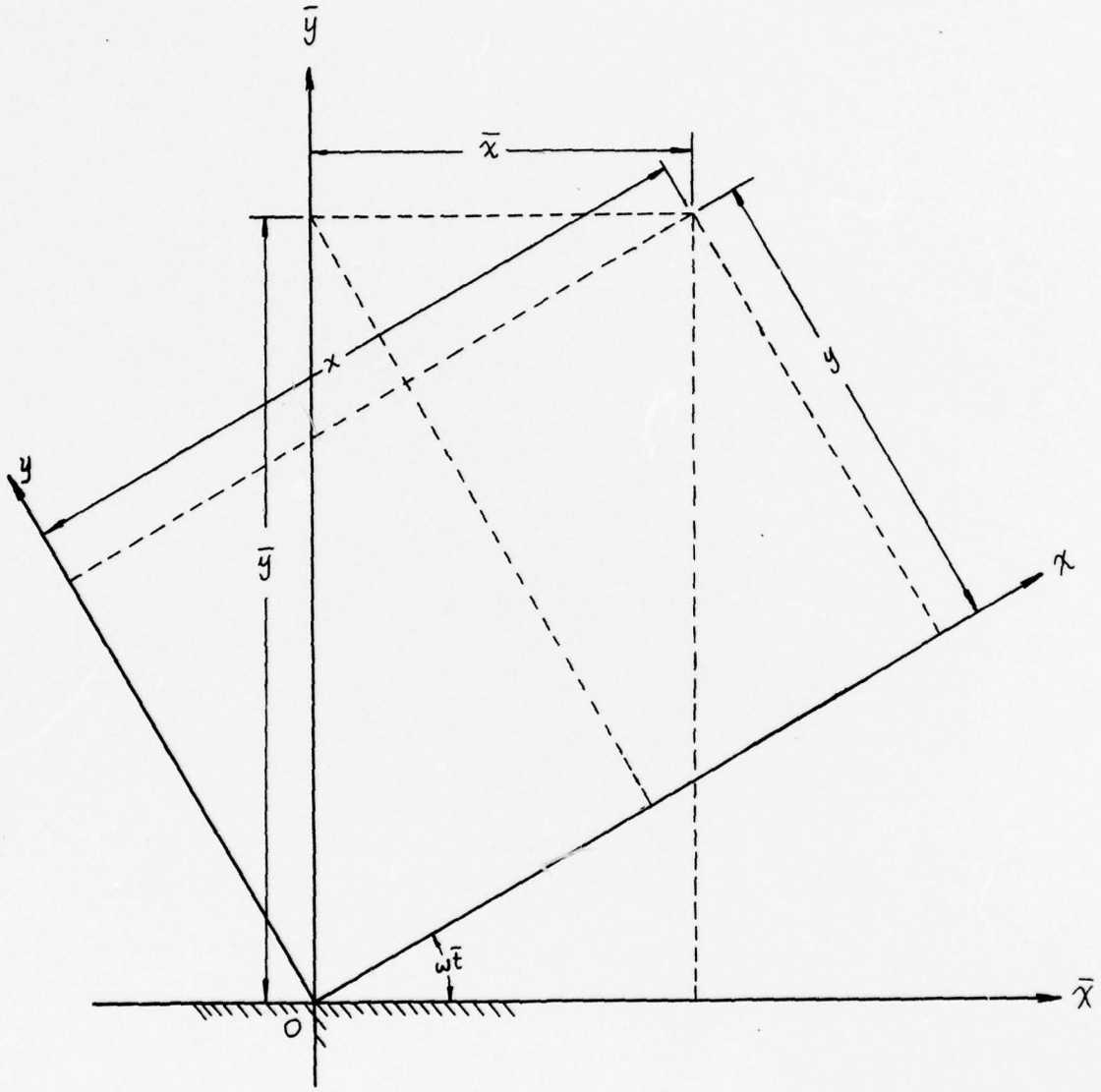

Figure 3 Relation between Rotation Coordinate System and Inertial Coordinate System

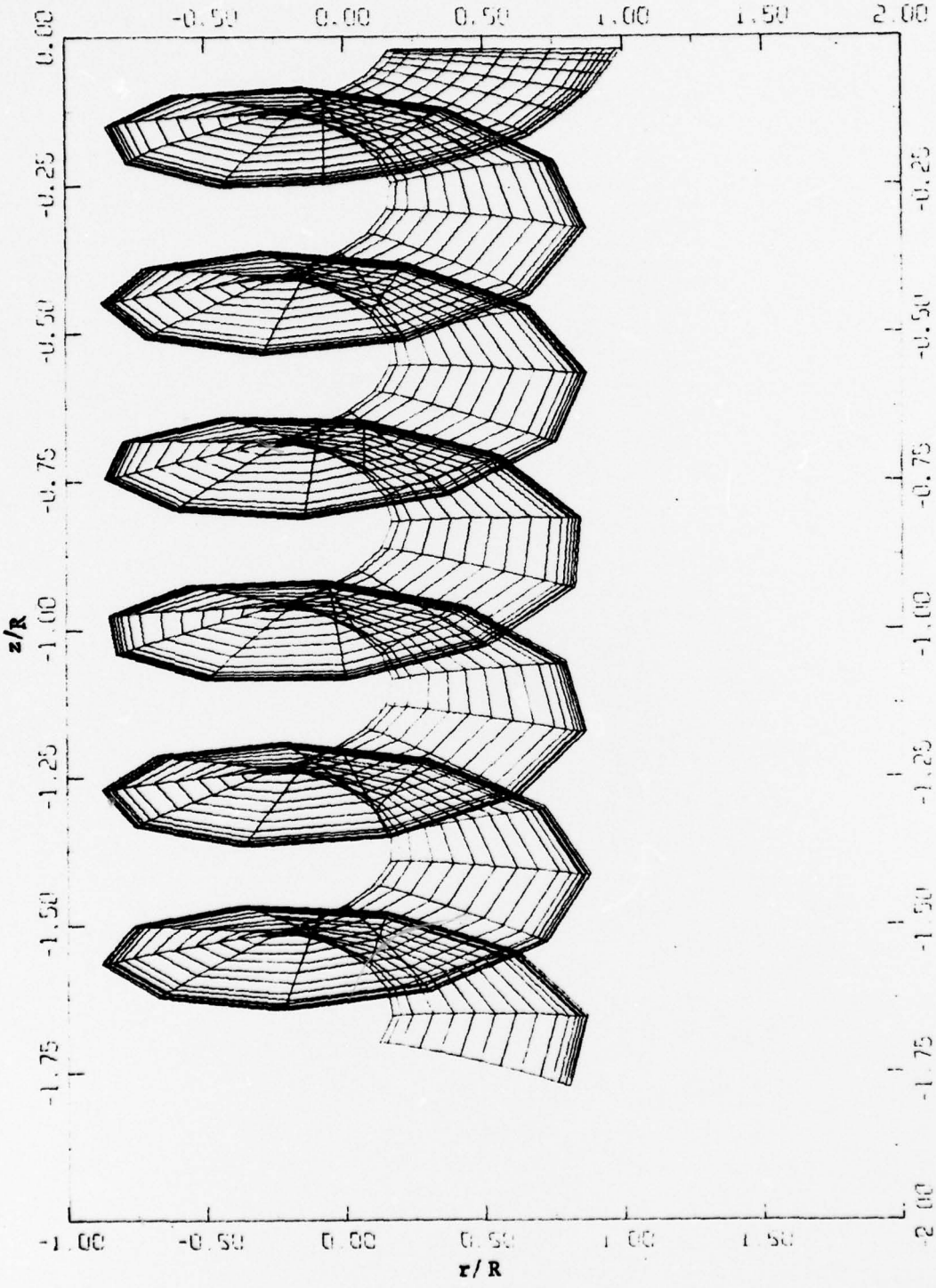

Figure 4 **Damped First Distortion with Inserted Small Steps Near Trailing Edge**

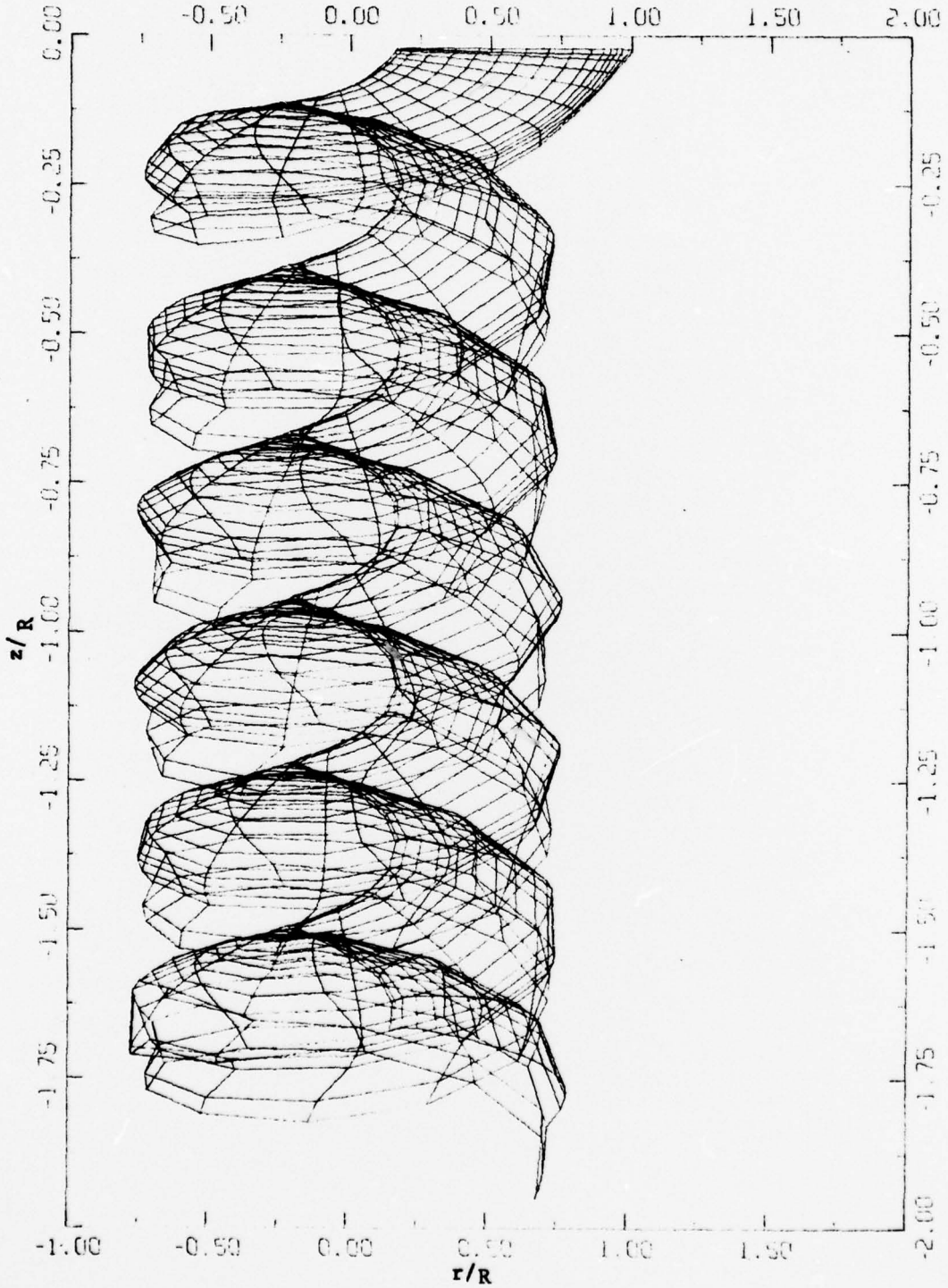

Figure 5 Wake Geometry of Second Iteration

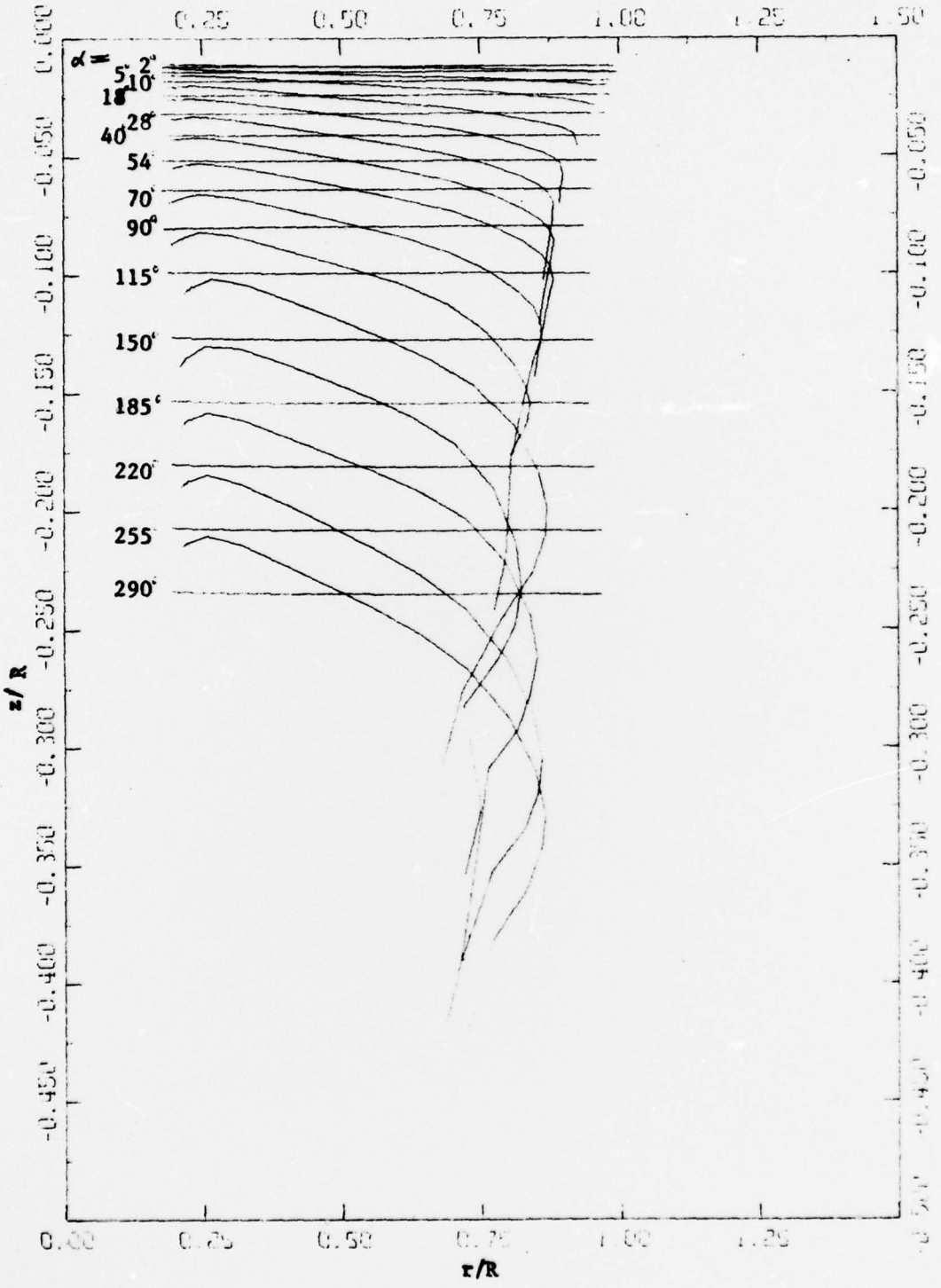

Figure 6 Cross-Sections of the Second Distorted Wake

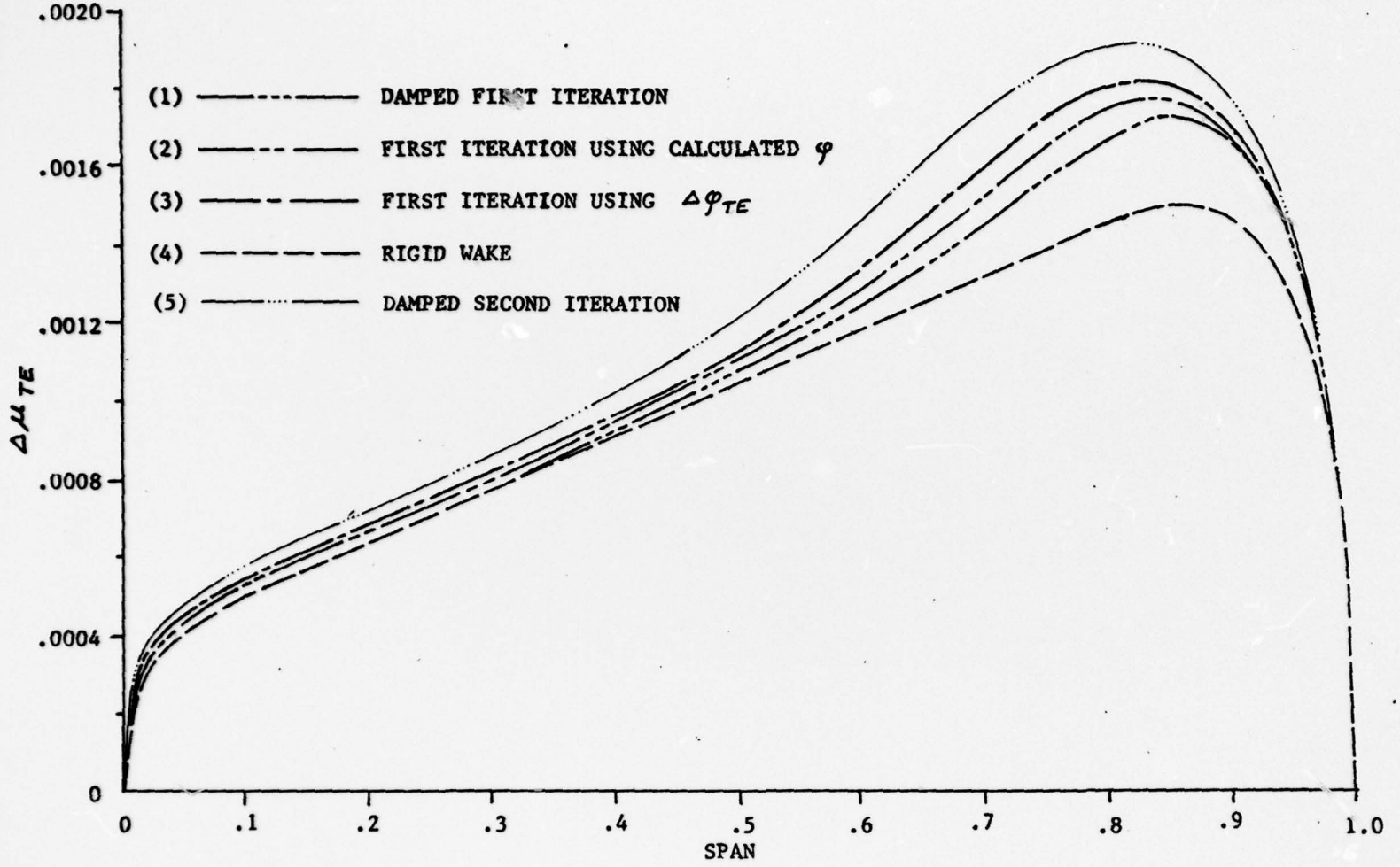

Figure 7 Spanwise Variation of $\Delta\mu_{TE}$

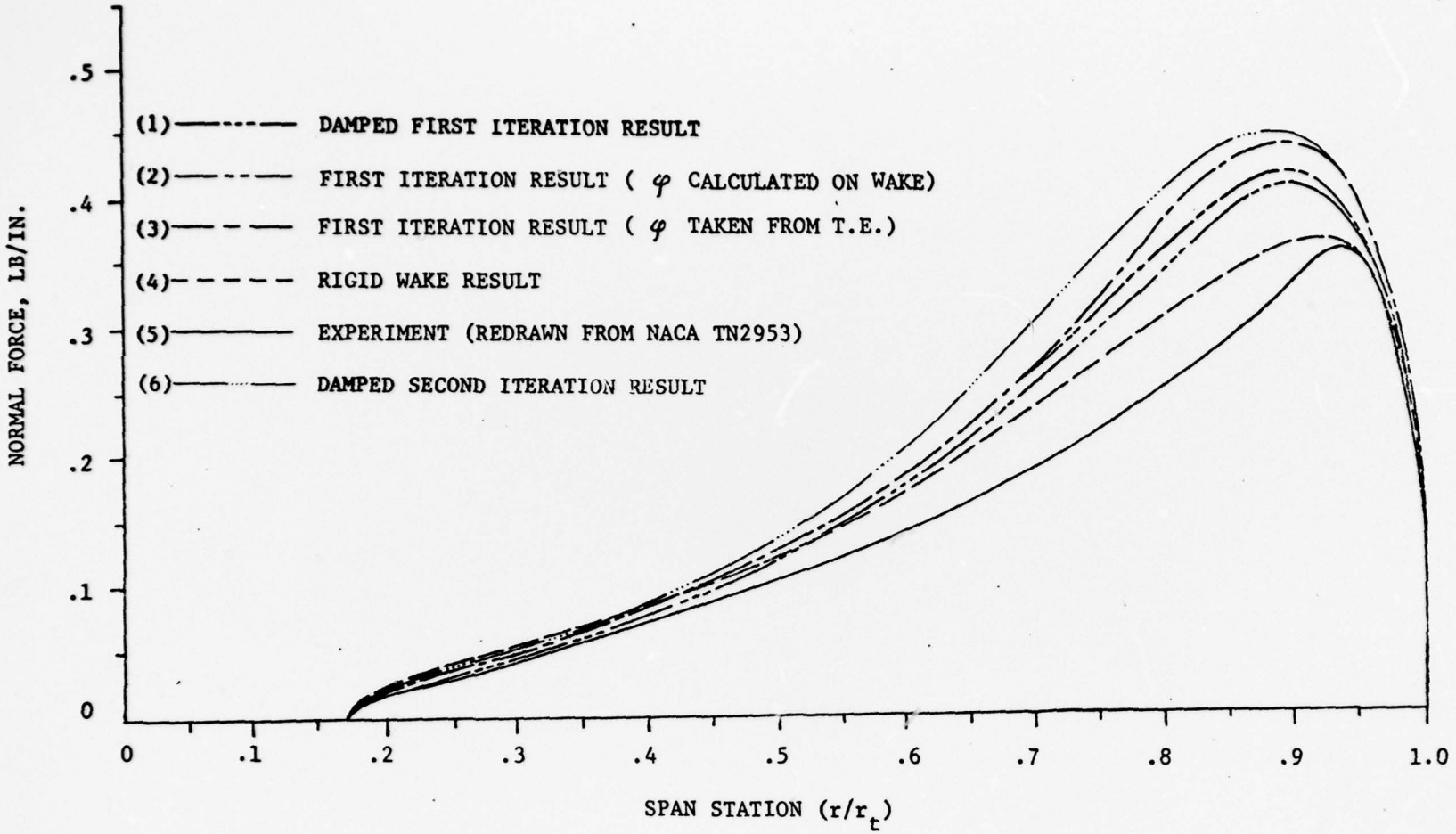

Figure 8 Comparison of Loading Curves

APPENDIX A DERIVATION OF GREEN'S THEOREM

In this appendix, Eq. (4-1) multiplied by function G

$$G\left[1-(\beta y)^2\right]\frac{\partial^2\varphi}{\partial x^2}+G\left[1-(\beta x)^2\right]\frac{\partial^2\varphi}{\partial y^2}+G\frac{\partial^2\varphi}{\partial z^2}$$

$$+\beta^2 G\left(2xy\frac{\partial^2\varphi}{\partial x\partial y}+x\frac{\partial\varphi}{\partial x}+y\frac{\partial\varphi}{\partial y}\right)=0 \qquad \text{(A-1)}$$

will be integrated throughout on volume V, and by using integration by parts the volume integral will be transferred into surface integral.

Consider each term one by one. For the first term, let

$$\iiint_V G\left[1-(\beta y)^2\right]\frac{\partial^2\varphi}{\partial x^2}\,dV=\iiint_V GA\frac{\partial^2\varphi}{\partial x^2}\,dx\,dy\,dz \qquad \text{(A-2)}$$

Referring to Fig. 2, consider the integration in the x-direction first. For a strip that does not pass through the internal boundary surface S, the limit is from $-\infty$ to ∞, therefore for x-direction

$$\begin{aligned}\int_{-\infty}^{\infty}GA\frac{\partial^2\varphi}{\partial x^2}\,dx&=(GA)\frac{\partial\varphi}{\partial x}\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial(GA)}{\partial x}\frac{\partial\varphi}{\partial x}\,dx\\&=0-\int_{-\infty}^{\infty}\frac{\partial(GA)}{\partial x}\frac{\partial\varphi}{\partial x}\,dx\\&=-\frac{\partial(GA)}{\partial x}\varphi\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial^2(GA)}{\partial x^2}\varphi\,dx\\&=\int_{-\infty}^{\infty}\frac{\partial^2(GA)}{\partial x^2}\varphi\,dx\end{aligned} \qquad \text{(A-3)}$$

For a strip that passes through S, it is split into two parts:

$$\left(\int_{-\infty}^{x_1(y,z)}+\int_{x_2(y,z)}^{\infty}\right)GA\frac{\partial^2\varphi}{\partial x^2}\,dx$$

$$=\left[GA\frac{\partial\varphi}{\partial x}\Big|_{-\infty}^{x_1(y,z)}-\int_{-\infty}^{x_1(y,z)}\frac{\partial(GA)}{\partial x}\frac{\partial\varphi}{\partial x}\,dx\right]+\left[GA\frac{\partial\varphi}{\partial x}\Big|_{x_2(y,z)}^{\infty}-\int_{x_2(y,z)}^{\infty}\frac{\partial(GA)}{\partial x}\frac{\partial\varphi}{\partial x}dx\right]$$

$$= -GA\frac{\partial\varphi}{\partial x}\Big|_{x_1(y,z)}^{x_2(y,z)} - \left[\left(\int_{-\infty}^{x_1(y,z)} + \int_{x_2(y,z)}^{\infty}\right)\frac{\partial(GA)}{\partial x}\frac{\partial\varphi}{\partial x}\,dx\right]$$

$$= -GA\frac{\partial\varphi}{\partial x}\Big|_{x_1(y,z)}^{x_2(y,z)} - \left[\frac{\partial(GA)}{\partial x}\varphi\Big|_{-\infty}^{x_1(y,z)} - \int_{-\infty}^{x_1(y,z)}\frac{\partial^2(GA)}{\partial x^2}\varphi\,dx\right]$$

$$- \left[\frac{\partial(GA)}{\partial x}\varphi\Big|_{x_2(y,z)}^{\infty} - \int_{x_2(y,z)}^{\infty}\frac{\partial^2(GA)}{\partial x^2}\varphi\,dx\right]$$

$$= \left[-GA\frac{\partial\varphi}{\partial x} + \frac{\partial(GA)}{\partial x}\varphi\right]\Big|_{x_1(y,z)}^{x_2(y,z)} + \left(\int_{-\infty}^{x_1(y,z)} + \int_{x_2(y,z)}^{\infty}\right)\frac{\partial^2(GA)}{\partial x^2}\varphi\,dx \qquad \text{(A-4)}$$

Now consider the integration in y- and z-directions. Since

$$dy\,dz = \hat{x}\cdot\hat{n}\,ds$$

the bracketed terms in Eq. (A-4), after putting back into the volume integral, become

$$\iint_S \left[-GA\frac{\partial\varphi}{\partial x} + \frac{\partial(GA)}{\partial x}\varphi\right]\hat{x}\cdot\hat{n}\,ds$$

The other terms in Eq. (A-4) together with the term in Eq. (A-3) gives, for the total volume V,

$$\iiint_V \frac{\partial^2(GA)}{\partial x^2}\varphi\,dV.$$

therefore

$$\iiint_V GA\frac{\partial^2\varphi}{\partial x^2}\,dx\,dy\,dz = \iint_S \left[-GA\frac{\partial\varphi}{\partial x} + \frac{\partial(GA)}{\partial x}\varphi\right]\hat{x}\cdot\hat{n}\,ds + \iiint_V \frac{\partial^2(GA)}{\partial x^2}\varphi\,dV \qquad \text{(A-5)}$$

By substitution of Eq. (A-2)

$$\iiint_V G\left[1-(\beta y)^2\right]\frac{\partial^2\varphi}{\partial x^2}\,dV$$

$$=\iint_S\left\{-G\left[1-(\beta y)^2\right]\frac{\partial\varphi}{\partial x}+\frac{\partial G\left[1-(\beta y)^2\right]}{\partial x}\varphi\right\}\hat{x}\cdot\hat{n}\,dS$$

$$+\iiint_V\frac{\partial^2 G\left[1-(\beta y)^2\right]}{\partial x^2}\,dV$$

$$=\iint_S\left\{-G\left[1-(\beta y)^2\right]\frac{\partial\varphi}{\partial x}+\left[1-(\beta y)^2\right]\frac{\partial G}{\partial x}\varphi\right\}\hat{x}\cdot\hat{n}\,dS$$

$$+\iiint_V\left[1-(\beta y)^2\right]\frac{\partial^2 G}{\partial x^2}\varphi\,dV \qquad \text{(A-6)}$$

Similarly,

$$\iiint_V G\left[1-(\beta x)^2\right]\frac{\partial^2\varphi}{\partial y^2}\,dV=\iiint_V GB\,\frac{\partial^2\varphi}{\partial y^2}\,dV$$

$$=\iint_S\left[-GB\frac{\partial\varphi}{\partial y}+\frac{\partial(GB)}{\partial y}\varphi\right]\hat{y}\cdot\hat{n}\,dS+\iiint_V\frac{\partial^2(GB)}{\partial y^2}\varphi\,dV$$

$$=\iint_S\left\{-G\left[1-(\beta x)^2\right]\frac{\partial\varphi}{\partial y}+\left[1-(\beta x)^2\right]\frac{\partial G}{\partial y}\varphi\right\}\hat{y}\cdot\hat{n}\,dS$$

$$+\iiint_V\left[1-(\beta x)^2\right]\frac{\partial^2 G}{\partial y^2}\varphi\,dV \qquad \text{(A-7)}$$

and

$$\iiint_V G\,\frac{\partial^2\varphi}{\partial z^2}\,dV=\iint_S\left[-G\frac{\partial\varphi}{\partial z}+\frac{\partial G}{\partial z}\varphi\right]\hat{z}\cdot\hat{n}\,dS+\iiint_V\frac{\partial^2 G}{\partial z^2}\varphi\,dV \qquad \text{(A-8)}$$

For

$$\iiint_V G\cdot 2xy\,\frac{\partial^2\varphi}{\partial x\partial y}\,dV=\iiint_V GC\,\frac{\partial^2\varphi}{\partial x\partial y}\,dx\,dy\,dz,$$

in x- and y-direction it could be either

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} GC\frac{\partial^2\varphi}{\partial x\partial y}\,dx\,dy=\int_{-\infty}^{\infty}\left\{(GC)\frac{\partial\varphi}{\partial y}\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dx\right\}dy$$

$$=0-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dy\,dx$$

$$=-\int_{-\infty}^{\infty}\left[\frac{\partial(GC)}{\partial x}\varphi\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial(GC)}{\partial x\partial y}\varphi\,dy\,dx\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2(GC)}{\partial x\partial y}\varphi\,dy\,dx \qquad \text{(A-9)}$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} GC\frac{\partial^2\varphi}{\partial x\partial y}\,dx\,dy=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{x_1(y,z)}+\int_{x_2(y,z)}^{\infty}\right)GC\frac{\partial^2\varphi}{\partial x\partial y}\,dx\,dy$$

$$=\int_{-\infty}^{\infty}\left\{(GC)\frac{\partial\varphi}{\partial y}\Big|_{-\infty}^{x_1(y,z)}-\int_{-\infty}^{x_1(y,z)}\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dx\right.$$

$$\left.+(GC)\frac{\partial\varphi}{\partial y}\Big|_{x_2(y,z)}^{\infty}-\int_{x_2(y,z)}^{\infty}\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dx\right\}dy$$

$$=\left(\int_{-\infty}^{y_3(z)}+\int_{y_4(z)}^{\infty}\right)\left[-(GC)\frac{\partial\varphi}{\partial y}\right]\Big|_{x_1(y,z)}^{x_2(y,z)}dy-\int_{-\infty}^{x_3(z)}\left(\int_{-\infty}^{y_1(z,x)}+\int_{y_2(z,x)}^{\infty}\right)\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dy\,dx$$

$$-\int_{x_4(z)}^{\infty}\left(\int_{-\infty}^{y_1(z,x)}+\int_{y_2(z,x)}^{\infty}\right)\frac{\partial(GC)}{\partial x}\frac{\partial\varphi}{\partial y}\,dy\,dx$$

$$=\left(\int_{-\infty}^{y_3(z)}+\int_{y_4(z)}^{\infty}\right)\left[-(GC)\frac{\partial\varphi}{\partial y}\right]\Big|_{x_1(y,z)}^{x_2(y,z)}dy+\left(\int_{-\infty}^{x_3(z)}+\int_{x_4(z)}^{\infty}\right)\left[\frac{\partial(GC)}{\partial x}\varphi\right]\Big|_{y_1(z,x)}^{y_2(z,x)}dx$$

$$+\int_{-\infty}^{x_3(z)}\left(\int_{-\infty}^{y_1(z,x)}+\int_{y_2(z,x)}^{\infty}\right)\frac{\partial^2(GC)}{\partial x\partial y}\varphi\,dy\,dx$$

$$+\int_{x_4(z)}^{\infty}\left(\int_{-\infty}^{y_1(z,x)}+\int_{y_2(z,x)}^{\infty}\right)\frac{\partial^2(GC)}{\partial x\partial y}\varphi\,dy\,dx \qquad \text{(A-10)}$$

Together with the integration in z-direction by noting

$$dy\,dz = \hat{x}\cdot\hat{n}\,ds$$

$$dx\,dz = \hat{y}\cdot\hat{n}\,ds$$

one has

$$\iiint_V GC\frac{\partial^2\varphi}{\partial x\partial y}dV = \iint_S\left[GC\frac{\partial\varphi}{\partial y}\hat{x} + \frac{\partial(GC)}{\partial x}\varphi\hat{y}\right]\cdot\hat{n}\,ds + \iiint_V\frac{\partial^2(GC)}{\partial x\partial y}\varphi\,dV$$

$$-\iint_S\left[-G(2xy)\frac{\partial\varphi}{\partial y}\hat{x} + \frac{\partial(G\cdot 2xy)}{\partial x}\varphi\hat{y}\right]\cdot\hat{n}\,ds + \iiint_V\frac{\partial^2(2xyG)}{\partial x\partial y}\varphi\,dV$$

$$= \iint_S\left[-2xyG\frac{\partial\varphi}{\partial y}\hat{x} + 2\left(Gy + xy\frac{\partial G}{\partial x}\right)\varphi\hat{y}\right]\cdot\hat{n}\,ds$$

$$+\iiint_V 2\left[G + y\frac{\partial G}{\partial y} + x\frac{\partial G}{\partial x} + xy\frac{\partial^2 G}{\partial x\partial y}\right]\varphi\,dV. \qquad \text{(A-11)}$$

Also

$$\iiint_V Gx\frac{\partial\varphi}{\partial x}dV = \iint_S\left[-Gx\varphi\right]\hat{x}\cdot\hat{n}\,ds - \iiint_V\left(\frac{\partial G}{\partial x}x + G\right)\varphi\,dV \qquad \text{(A-12)}$$

$$\iiint_V Gy\frac{\partial\varphi}{\partial y}dV = \iint_S\left[-Gy\varphi\right]\hat{y}\cdot\hat{n}\,ds - \iiint_V\left(\frac{\partial G}{\partial y}y + G\right)\varphi\,dV \qquad \text{(A-13)}$$

Putting all these together into Eq. (A-1) one has

$$\iiint_V GL\varphi\,dV = 0$$

$$= -\iint_S\Big[\left\{G\left[1-(\beta y)^2\right]\frac{\partial\varphi}{\partial x} - \left[1-(\beta y)^2\right]\frac{\partial G}{\partial x}\varphi + 2\beta^2 Gxy\frac{\partial\varphi}{\partial y} + \beta^2 Gx\varphi\right\}\hat{x}$$

$$+\left\{G\left[1-(\beta x)^2\right]\frac{\partial\varphi}{\partial y}-\left[1-(\beta x)^2\right]\frac{\partial G}{\partial y}\varphi-2\beta^2(Gy+xy\frac{\partial G}{\partial x})+\beta^2Gy\varphi\right\}\hat{y}$$

$$+\left\{G\frac{\partial\varphi}{\partial z}-\frac{\partial G}{\partial z}\varphi\right\}\hat{z}\Big]\cdot\hat{n}\,dS$$

$$+\iiint_V\Big\{\Big[1-(\beta y)^2\Big]\frac{\partial^2 G}{\partial x^2}+\Big[1-(\beta x)^2\Big]\frac{\partial^2 G}{\partial y^2}+\frac{\partial^2 G}{\partial z^2}$$

$$+2\beta^2\left(G+y\frac{\partial G}{\partial y}+x\frac{\partial G}{\partial x}+xy\frac{\partial^2 G}{\partial x\partial y}\right)-\beta^2\left(\frac{\partial G}{\partial x}x+G\right)-\beta^2\left(\frac{\partial G}{\partial y}y+G\right)\Big\}\varphi\,dV$$

(A-14)

APPENDIX B TRANSFORMATION OF THE GOVERNING EQUATION INTO BESSEL EQUATION

Consider the following equation

$$LG = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \left(1-\beta^2 r^2\right)\frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} + \frac{\partial^2 G}{\partial z^2} = 0 \tag{B-1}$$

Let

$$G = R(r)\Theta(\theta)Z(z) \tag{B-2}$$

then Eq. (B-1) becomes

$$LG = \frac{1}{r}\frac{\partial}{\partial r}\left(r\Theta Z\frac{dR}{dr}\right) + \frac{1}{r^2}RZ\frac{d^2\Theta}{d\theta^2} - \beta^2 RZ\frac{d^2\Theta}{d\theta^2} + R\Theta\frac{d^2Z}{dz^2} \tag{B-3}$$

Again let

$$\Theta = e^{in\theta}, \qquad Z = e^{mz} \tag{B-4}$$

where $i=\sqrt{-1}$; then

$$\left.\begin{aligned}\frac{d^2\Theta}{d\theta^2} &= -n^2 e^{in\theta} = -n^2\Theta \\ \frac{d^2Z}{dz^2} &= m^2 e^{mz} = m^2 Z\end{aligned}\right\} \tag{B-5}$$

and

$$LG = \Theta Z\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - RZ\frac{n^2}{r^2}\Theta + \beta^2 n^2 RZ\Theta + R\Theta m^2 Z = 0 \tag{B-6}$$

Dividing through by $R\Theta Z$,

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\frac{n^2}{r^2}+\beta^2n^2+m^2=0 \qquad \text{(B-7)}$$

or

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\left[\left(m^2+\beta^2n^2\right)-\frac{n^2}{r^2}\right]R=0 \qquad \text{(B-7a)}$$

Letting

$$r=\frac{\rho}{\lambda}=\frac{\rho}{\sqrt{m^2+\beta^2n^2}},$$

and

$$R(r)=P(\rho), \qquad \text{(B-8)}$$

then

$$\frac{d}{dr}=\frac{d}{d\rho}\frac{d\rho}{dr}=\lambda\frac{d}{d\rho} \qquad \text{(B-9)}$$

Eq. (B-7a) becomes

$$\frac{\lambda}{\rho}\lambda\frac{d}{d\rho}\left(\frac{\rho}{\lambda}\cdot\lambda\frac{dP}{d\rho}\right)+\left[\lambda^2-\frac{n^2\lambda^2}{\rho^2}\right]P=0, \qquad \text{(B-10)}$$

or

$$\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\left[\rho-\frac{n^2}{\rho}\right]P=0. \qquad \text{(B-10a)}$$

This is the standard form of Bessel equation whose solution is

$$P=J_n(\rho)$$

where

$$J_n(\rho) = \left(\frac{\rho}{2}\right)^n E_n(\rho) = \left(\frac{\rho}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\rho}{2}\right)^{2k}}{k!(n+k)!}$$

Therefore, by substitution,

$$R(r) = J_n\left(\sqrt{m^2+\beta^2 n^2}\, r\right)$$

and

$$\Psi = J_n\left(\sqrt{m^2+\beta^2 n^2}\, r\right) e^{in\theta}\, e^{mz}$$