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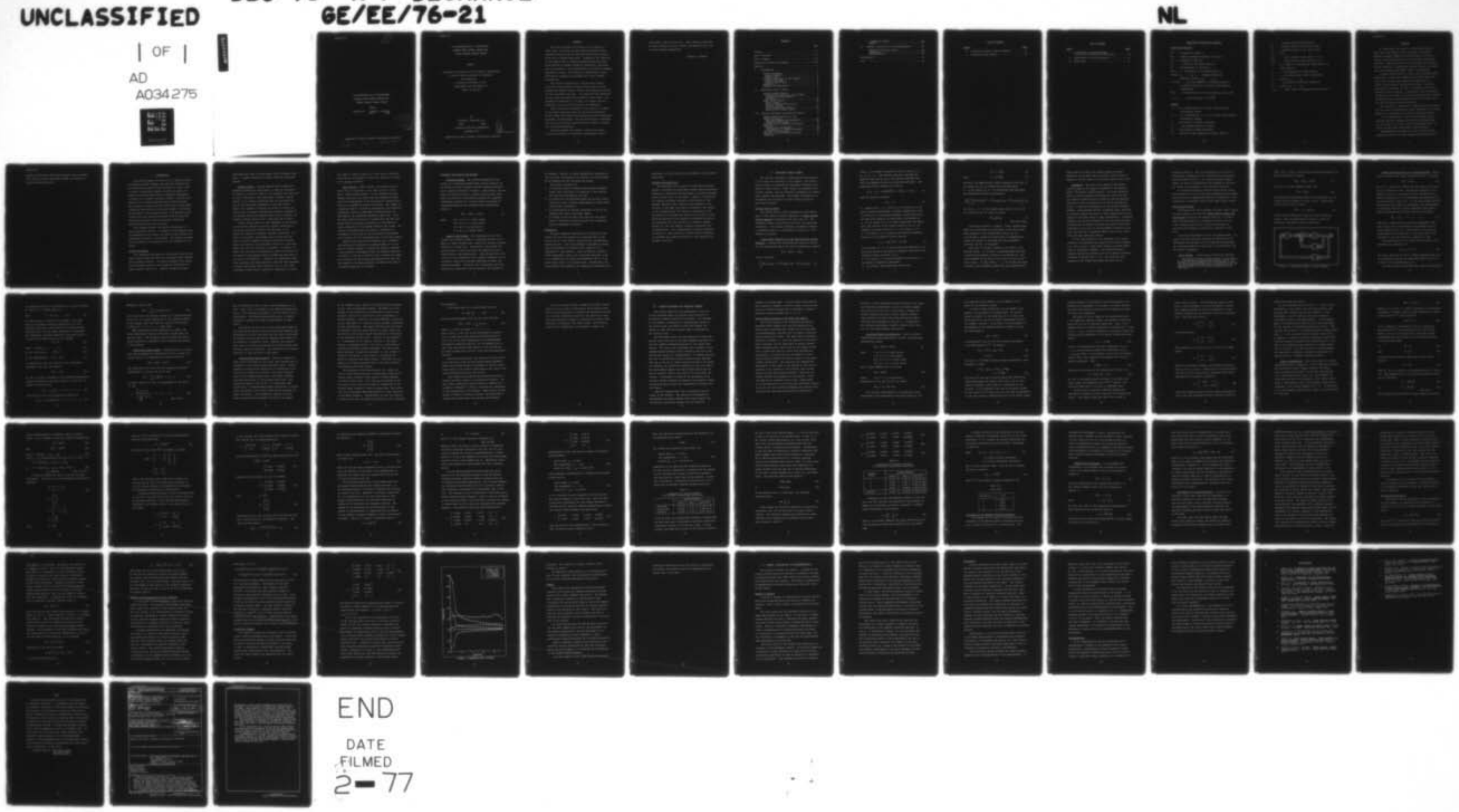

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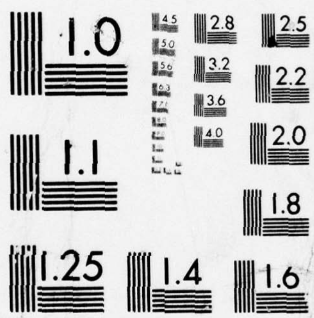
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AN INVESTIGATION OF A RELATIONSHIP

BETWEEN MODAL CONTROL THEORY AND

LINEAR OPTIMAL CONTROL THEORY

THESIS

GE/EE/76-21 Richard P. Dechance
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(See form 1473)

GE/EE/76-21

AN INVESTIGATION OF A RELATIONSHIP BETWEEN MODAL CONTROL THEORY AND LINEAR OPTIMAL CONTROL THEORY

THESIS

Presented to the Faculty of the School of Engineering
of the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
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Master of Science

by

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Preface

This report presents the results of my attempts to relate modal and linear optimal control theories so that weighting matrices could be specified which would provide a stable set of system eigenvalues. Ultimately, the objective of the research was to determine weighting matrices which would provide an optimal controller generating a specified set of eigenvalues. The investigation was limited to systems described by linear, time-invariant, deterministic matrix differential equations; and assumed full state feedback availability.

This project required that I become fairly deeply involved in the concepts of modal control theory; linear, steady-state optimal control theory; and generalized inverses of real and complex matrices. For supplying both background and real-time assistance with regards to these areas, I am indebted to Professors J.J. D'Azzo, C.H. Houpis, and D.G. Shankland of the Air Force Institute of Technology.

While the specific outcomes of this project in regards to the stated objectives were not entirely satisfactory, several areas with interesting possibilities were encountered, evaluated, and understood. The learning process I underwent during this project was sometimes trying, usually illuminating, and always beneficial.

The encouragement and support I received from many during this project are gratefully acknowledged; three in

particular I wish to single out: Capt. Thomas E. Moriarty, my thesis advisor; my wife, Gladys; and Edmund Hillary, for his ever-present watchfulness.

Richard P. Dechance

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Operational Notation and Symbols

Operational Notation

$\bar{x}^T$ . . . transpose of vector $\bar{x}$

$\bar{x}^*$ . . . complex conjugate transpose of vector $\bar{x}$

A^T . . . transpose of matrix A

A^{-1} . . . inverse of the square matrix A

$A^\dagger$ . . . pseudoinverse of the matrix A

det A . . . determinant of the square matrix A

diag$[\alpha_1, \alpha_2, \ldots, \alpha_n]$ . . . diagonal matrix with diagonal entries $\alpha_1, \alpha_2, \ldots, \alpha_n$

$M>0$, $M\geq 0$ . . . the real symmetric or Hermitian matrix M is positive definite or positive semi-definite, respectively

$\dot{\bar{x}}(t)$ . . . time derivative of the *time-varying vector*, $\bar{x}(t)$

$\| \bar{x} \|$ . . . norm of vector $\bar{x}$, $\triangleq [\bar{x}^* \bar{x}]^{\frac{1}{2}}$

Symbols

A . . . n x n plant matrix of a linear, time-invariant differential system

B . . . n x r input matrix, $r \leq n$, of a linear time-invariant differential system

F . . . gain matrix of optimal controller

I_i . . . unit identity matrix *of order* i

K . . . gain matrix of modal controller

$\underline{P}$ . . . solution to the steady-state Riccati equation

P_1 . . . terminal-state weighting matrix

R_1 . . . weighting matrix of the state vector

R_2 . . . weighting matrix of the input vector

t . . . time

$\overline{x}(t)$ . . . state variable, state vector (n x 1)

$\overline{z}(t)$ . . . input variable, input vector (r x 1)

U . . . n x n matrix of eigenvectors (modal matrix) of plant matrix A

$\overline{u}_i$ . . . i-th eigenvector of modal matrix U

λ_i . . . i-th characteristic value (eigenvalue)

Λ . . . $\mathrm{diag}[\lambda_1, \lambda_2, \ldots \lambda_n]$

ρ_i . . . i-th desired closed-loop eigenvalue

R . . . $\mathrm{diag}[\rho_1, \rho_2, \ldots \rho_n]$

$\overline{\xi}(t)$ . . . state vector of diagonalized plant matrix Λ

Abstract

An investigation was made of a relationship between modal and linear optimal control theories to determine whether the modal feedback gain matrix would be of help in finding the weighting matrices of the optimal steady-state criterion. Modal and optimal control theories are reviewed, and the concepts of steady-state optimal control, and single-input and multiple-input modal control are developed.

A general matrix solution, using a matrix pseudoinverse, determines a unique modal feedback gains matrix which provides a set of specified closed-loop eigenvalues for a linear, time-invariant, deterministic system. The modal gains matrix is used as an input to a modified form of the algebraic Riccati equation. A perturbation search technique is applied in an attempt to find the state and control weighting matrices which simultaneously satisfy the Riccati equation and the optimal control postulates.

The procedure is applied to a numerical control problem, with the results indicating the search technique is not fully effective in establishing the optimal weighting matrices.

It is concluded that a new and useful modal design technique has been developed utilizing the pseudoinverse of a real matrix, and that a valid relationship exists (in theory) between modal and optimal control theories.

Recommendations are made to pursue the modal design technique further; to further analyze the characteristics

determining Hermitian matrix definiteness; and to evaluate other types of search techniques capable of finding the optimal weighting matrices.

I. Introduction

This thesis presents the results of a research project to determine whether feedback gain matrices obtained from the application of modal control theories to linear, time-invariant, controllable systems can serve as inputs to a search for weighting matrices of a quadratic performance criterion for an optimal controller design. Initially the objective is to obtain weighting matrices which satisfy the postulates of optimal control theory and provide stable closed-loop controllers for realistic systems. Ultimately, the objective of the research project is to develop a relationship such that weighting matrices could be specified which would provide an optimal controller generating specified closed-loop pole locations.

This chapter presents the general philosophies of optimal and modal control, a statement of the specific problem addressed by this thesis, the scope of the problem, and assumptions pertinent to the problem. The chapter concludes with the sequence of presentation for the remainder of the report.

Control Philosophy

The underlying philosophy for both optimal control and modal control stems from the area of control theory commonly called "modern control." As D'Azzo and Houpis have stated, the state-space concept is the essential contribution of modern control [Ref 5:xvi]. However, optimal and modal techniques each treat the state-space from a different viewpoint. A general discussion of these differing viewpoints follows.

Optimal Control. Current optimal control deals with problems formulated in state-space, in that the response of the system is viewed in terms of the trajectory the state vector $\bar{x}(t)$, describes in the state-space. However, optimal control is not directly concerned with the variations over time of the individual components of the state vector. Rather, the "optimal" trajectory is chosen to be that trajectory which minimizes a performance criterion composed of the weighted states and weighted inputs (and in general, the weighted terminal states) over a specified time interval.

Optimal control design is thus dependent on the ability of the control engineer to specify suitable weightings on the various states and inputs: each variation in the weightings creating a different "optimal" controller. The task of the designer then becomes one of selecting from among an infinite number of optimal controllers one which causes the system to be controlled to behave in a specified manner over the time interval of interest. In general, insight into the choice of weightings is gained by analysis of the physical relationships within a system and experience on the part of the designer. A search of the applicable literature indicates that some analytical aids have been developed to assist the designer [Ref 5:Chap 15; 14; 17], but these techniques involve certain restrictive conditions on either the number of inputs allowed or, in the case of sequential input design, the sequence in which the inputs are applied, or are ad hoc procedures.

Modal Control. Modal control, like optimal control, deals with the trajectory of the state vector in state-space. Modal control however, is not concerned with performance criteria, but rather with the concepts of eigenvalues and eigenvectors of the plant matrix: concepts with which it is assumed the reader is familiar. The "shape" of a dynamical mode of a system is described by the associated eigenvector, while the time-domain characteristics are described by the associated eigenvalue [Ref 12:8]. Thus, the free (undriven) motion of a system which has been displaced from its equilibrium condition is described by a linear combination of the dynamic modes of the system. The stability characteristics and speed of convergence or divergence of the states of the system may be determined by an analysis of the "eigenproperties" of the plant matrix.

Modal control provides the designer with a method of pole-shifting to stabilize an unstable system (or in the case of a stable system, to improve the speed of response of the system to disturbances from its reference position). In general, it may be a difficult task to select the proper pole locations so that system specifications are met, but this aspect of the design problem lies beyond the scope of the modal design area of interest.

Statement and Scope of the Problem

Problem Statement. The problem addressed by this thesis is to determine whether modal control theories can be applied to obtain suitable starting weighting matrices for an optimal performance criterion. Specifically, can the modal gain matrix provide a suitable starting point from which to search for the weighting matrices of a quadratic performance criterion of a linear optimal controller of systems described by the following matrix differential equation:

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{z}(t) \tag{1}$$

where $\bar{x}(t)$ is an n x 1 state vector

$\bar{z}(t)$ is an r x 1 input vector

A is an n x n plant matrix

B is an n x r control matrix

Scope of the Problem. The investigation into this problem was limited to permit a reasonable analysis to be made of a small, defined area, rather than attempt a shallow analysis of a broad area. Specifically, the systems analyzed consist of those which could be modeled as linear, time-invariant, and deterministic. Thus the matrices A and B of Eq (1) are composed of real, constant elements; and no corruptive signals of sufficient magnitude to affect the system are present. Obviously, more complicated (and less restrictive) conditions could be placed on the systems to be examined. However, to permit mathematical tractability and facilitate interpretation of results, the first attempt on the problem area was restricted as stated.

The project consisted of:

1. A literature search, primarily using Defense Document Center (DDC) facilities, to determine if any previous work had been published in the area.
2. An analysis of the applicable modal and optimal control theories to gain a working understanding of these areas and how they interrelate.
3. An investigation of suitable techniques to obtain the feedback gain matrix from modal theory.
4. An attempt to use the modal gain matrices thus determined to establish weighting matrices for a quadratic optimal performance criterion.

Assumptions

The main assumption underlying all work done on this project was that the systems to be analyzed were of the form of Eq (1), with linear, time-invariant, deterministic characteristics. Full state feedback is assumed to circumvent the necessity of including state observers (e.g. Luenberger) in the analysis. Thus, no mention is made of output measurement vectors nor controlled variables, since it is assumed that all states are measurable and it is the state vector which is to be controlled. Additionally, the system models were assumed to be completely controllable and observable, and plant matrices were assumed to have distinct eigenvalues.

Sequence of Presentation

The remainder of this report is organized as follows. Chapter II presents the applicable control theory background necessary to understand the development of optimal and modal control theory given in this paper. Chapter III describes the research procedure followed in the analysis of the problem, starting with a general matrix approach to the solution of the modal gain matrix, followed by the concept of the "pseudoinverse" of a matrix, a discussion of the application of the modal gain matrix to the search for the weighting matrices of the quadratic performance criterion, and concluding with the results of a numerical example. Chapter IV, the last chapter, presents conclusions and recommendations. The recommendations primarily emphasize the need for a more efficient, powerful search technique to determine the weighting matrices of the quadratic performance criterion.

II. Applicable Control Theory

The theories necessary to understand the developments in this paper are presented in this chapter. The optimal control theory is developed and discussed, followed by the theory of modal control. Modal control is presented first as basic concept, then the necessary refinements for single-input systems are developed. The chapter concludes with a development and discussion of modal control applied to multi-input systems.

Optimal Control Theory

The optimal control theory necessary to the developments in this paper is stated completely in Linear Optimal Control Systems by Kwakernaak and Sivan [Ref 8]. The interested reader is referred to this text for additional information. The material necessary for present purposes is reproduced here as a restatement of Theorem 3.7, with changes in notation.

Steady-state Properties of the Time-invariant Optimal Regulator. Consider the time-invariant regulator problem for the system described by the matrix differential equation

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{z}(t) \qquad (1)$$

and the criterion

$$\int_{t_o}^{t_1} [\bar{x}^T(t)R_1\bar{x}(t) + \bar{z}^T(t)R_2\bar{z}(t)]dt + \bar{x}^T(t_1)P_1\bar{x}(t_1) \qquad (2)$$

(where t_1 is allowed to approach infinity), with $R_1 \geq 0$, $R_2 > 0$, $P_1 \geq 0$. (The notation used here indicates the symmetric matrices R_1 and P_1 are positive semi-definite, while the symmetric matrix R_2 is positive-definite.) The associated Riccati equation is given by

$$-\dot{P}(t) = R_1 - P(t)BR_2^{-1}B^TP(t) + A^TP(t) + P(t)A \qquad (3)$$

with the terminal condition

$$P(t_1) = P_1 \qquad (4)$$

a) Assume that $P_1 \geq 0$. Then as $t_1 \to \infty$ the solution of the Riccati equation approaches a constant steady-state value $\underline{P}$ if and only if the system possesses no poles that are at the same time unstable, uncontrollable, and observable.

b) If the system, Eq (1), is both *controllable and observable*, the solution of the Riccati equation, Eq (3), approaches the unique value $\underline{P}$ as $t_1 \to \infty$ for every $P_1 \geq 0$.

c) If $\underline{P}$ exists, it is a positive semi-definite symmetric solution of the algebraic Riccati equation

$$0 = R_1 - PBR_2{}^{-1}B^TP + PA + A^TP \qquad (5)$$

If the system, Eq (1), is controllable and observable, $\underline{P}$ is the unique positive semi-definite symmetric solution of the algebraic Riccati equation, Eq (5).

d) If $\underline{P}$ exists, it is positive-definite if and only if the system is completely observable.

e) If $\underline{P}$ exists, the steady-state control law

$$\bar{z}(t) = -F\bar{x}(t) \tag{6}$$

where
$$F = R_2^{-1}B^T\underline{P} \tag{7}$$

results in an asymptotically stable response if and only if the system, Eq (1), is controllable and observable.

f) If the system, Eq (1), is controllable and observable, the steady-state control law minimizes

$$\lim_{t_1 \to \infty} \left[\int_{t_o}^{t_1} [\bar{x}^T(t) R_1 \bar{x}(t) + \bar{z}^T(t) R_2 \bar{z}(t)] dt + \bar{x}^T(t_1) P_1 \bar{x}(t_1) \right] \tag{8}$$

for all $P_1 \geq 0$. For the steady-state control law, Eq (6), the criterion, Eq (8), takes the value

$$x^T(t_o)\underline{P}x(t_o) \tag{9}$$

[Ref 8:237-238].

Essentially Theorem 3.7 states: Given a system whose modes are controllable and observable, a full-state feedback according to the steady-state control, Eq (6), will provide a closed-loop system that is asymptotically stable and optimal in the sense that the performance criterion, Eq (8), is minimized.

Note that an "optimal" system is not "best" in any absolute sense. It is merely a system designed such that a specified performance criterion is minimized by a full-state feedback as the input to the plant for any initial state vector. In other words, ". . . we now have the means to devise linear feedback systems that are asymptotically stable and at the same time possess optimal transient properties in the sense that any nonzero initial state is reduced to the zero state in an optimal fashion [Ref 8:222]."

<u>Discussion</u>. The form of the quadratic performance criterion used in this paper is composed of two quadratic terms: $\bar{x}^T(t)R_1\bar{x}(t)$ and $\bar{z}^T(t)R_2\bar{z}(t)$. The state weighting matrix R_1 must be positive semi-definite, while the input (or control) weighting matrix R_2 must be positive-definite (and with no loss of generality, both can be assumed symmetric) to satisfy the postulates of the optimal regulator problem. Each weighting matrix, considered alone, specifies the relative weight each state or input combination has with respect to the other state or input combinations. Considered together, the relative magnitude of the two weighting matrices specifies the amount the states and inputs collectively contribute to the value of the performance criterion at any time, and thus act to establish upper limits to the values the states and inputs may take on.

In the usual approach to the synthesis of an optimal regulator, one specifies the values of the various elements in R_1 and R_2 using a "best guess" based on previous experience, or engineering judgment. The resulting Riccati equation may then be solved by any one of several currently available techniques [Ref 7; 16]. The resulting time response of the closed-loop system may then also be determined [Ref 16]. The time response may or may not be satisfactory, depending on the suitability of the weighting matrices chosen initially. An iterative process usually ensues, to obtain a satisfactory time response by adjusting the values of the elements in the weighting matrices.

An alternative method of control system design exists which allows one to be very specific regarding time responses, but which does not address the question of optimality. This alternative is discussed further under Modal Control theory.

Modal Control Theory

The material for the development of modal control theory is taken primarily from the text Modal Control: Theory and Applications by Porter and Crossley [Ref 12]. In particular, the development of a closed-form solution for the linear feedback gains matrix of both single-input and multiple-input systems is specifically credited to this text.

The entire development will not be presented here. Instead, the basic equations necessary to substantiate the development of this paper will be presented and discussed; the interested reader may then refer to the referenced text for further information.

Basic Concept.

In the words of Porter and Crossley:

> The central concept of modal control is very simple: It is merely that of generating the input vector of a system by linear feedback of the state vector in such a way that prescribed eigenvalues are associated with the dynamical modes of the resulting closed-loop system [Ref 12:2].

Thus, given a linear, time-invariant system described by the matrix differential equation

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{z}(t) \qquad (1)$$

the use of a linear feedback control law

$$\bar{z}(t) = K\bar{x}(t) \qquad (10)$$

can provide the prescribed closed-loop eigenvalues if K is chosen using the methods of modal control. Substitution of Eq (10) into Eq (1) yields

$$\dot{\bar{x}}(t) = (A + BK)\bar{x}(t) \qquad (11)$$

as the matrix differential equation of the closed-loop system. This is demonstrated in Figure 1, where the operations indicated are matrix operations.

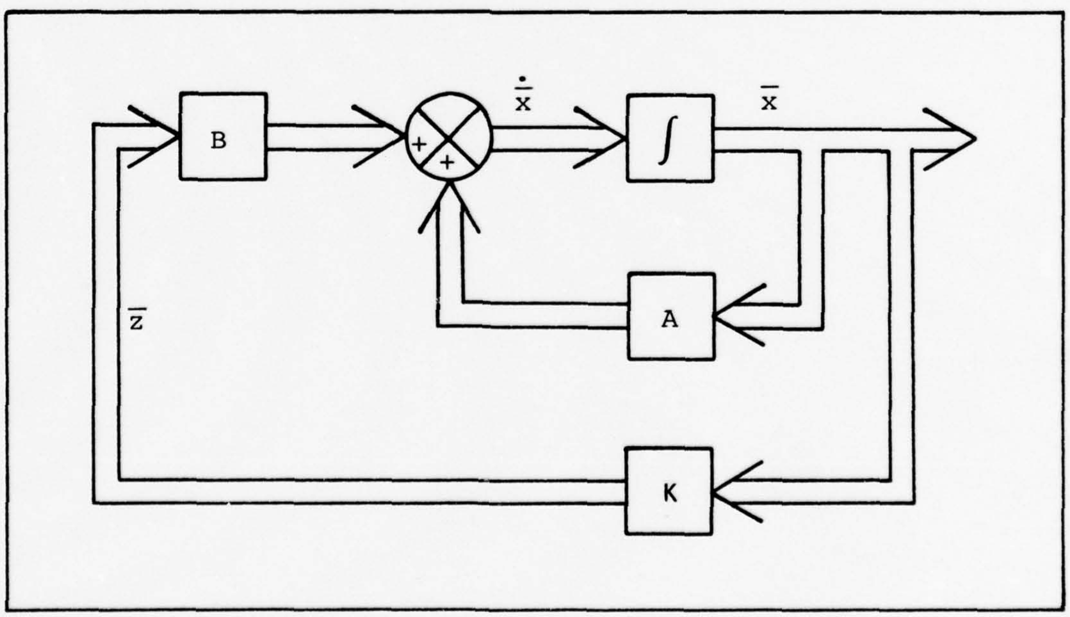

Figure 1. Closed-loop System of State Feedback

Dynamic Characteristics of an Undriven System. When no input is present, the system represented by Eq (1) becomes

$$\dot{\bar{x}}(t) = A\bar{x}(t) \tag{12}$$

The (n x 1) vector $\bar{x}(t)$ defines the free motion of the system as a function of time. The specific response of the system, Eq (12), to any nonzero initial condition can be found by examination of the eigenvalues and eigenvectors of the plant matrix A [Ref 12:5]. It has been assumed that the plant matrix has n distinct eigenvalues; the corresponding n eigenvectors must be linearly independent [Ref 9:108]. These eigenvectors may be found from the relationship

$$A\bar{u}_i = \lambda_i \bar{u}_i, \quad i = 1, 2, \ldots, n \tag{13}$$

Since the n eigenvectors are linearly independent, they form a basis for an n-dimensional "state space" for the system modeled by Eq (12). The motion of the system can be described by the trajectory of the state vector, $\bar{x}(t)$, generated in the state space as a function of time.

The eigenvalues of A are found from the roots of the characteristic equation of A

$$\det\,[\lambda I - A] = 0 \tag{14}$$

The roots, which may be real or complex conjugate pairs, of this polynomial equation are then substituted into Eq (13) to solve for the eigenvectors of A.

For the situation considered here, where the eigenvalues are assumed distinct, the eigenvectors of A may be arranged to form an (n x n) "modal matrix," U

where
$$U = [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n] \tag{15}$$

The modal matrix U, which will be complex if the plant matrix A has complex conjugate eigenvalues, is a non-singular, square matrix and therefore its inverse exists. The modal matrix has the useful property of "uncoupling" the differential equations embodied in the plant matrix A. In terms of the modal matrix, Eq (13) becomes

$$AU = U\Lambda \tag{16}$$

where $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ (17)

Eq (16) also implies $U^{-1}AU = \Lambda$ (18)

If the transformation $\bar{x}(t) = U\bar{\xi}(t)$ (19)

is substituted in Eq (12) along with the relationship expressed in Eq (18), one obtains

$$\dot{\bar{\xi}}(t) = \Lambda\bar{\xi}(t) \tag{20}$$

Eq (20) is a matrix statement of the uncoupled differential equations embodied in the plant matrix A. If Eq (20) is stated in scalar form as

$$\dot{\xi}_i(t) = \lambda_i \xi_i(t), \quad i = 1, 2, \ldots, n \tag{21}$$

the solution to the i-th differential equation is

$$\xi_i(t) = \xi_i(0)[\exp(\lambda_i t)], \quad i = 1, 2 \ldots, \quad n \tag{22}$$

Therefore, from Eq (19)

$$\bar{x}(t) = \sum_{i=1}^{n} \bar{u}_i \xi_i(0)[\exp(\lambda_i t)] \tag{23}$$

This states that the undriven response of a system described by Eq (12) is a linear combination of the "dynamical modes" of the system, where each mode is a function of the form $\bar{u}_i[\exp(\lambda_i t)]$, i = 1, 2, ..., n [Ref 12:8].

When considering forced responses of a system, two broad categories of forcing input arise: single-input and multiple-input. The necessary theories concerning each of these categories is discussed in the following two sections of this chapter.

Single-input Modal Control. Systems driven by a single input may be modeled as a modification of Eq (1) where the control matrix B is replaced by the control vector $\bar{b}$.

$$\dot{\bar{x}}(t) = A\bar{x}(t) + \bar{b}z(t) \tag{24}$$

The closed-loop form of Eq (24) will possess the desired eigenvalues if a control law of the form

$$z(t) = \sum_{j=1}^{m} K_j \bar{v}_j^T \bar{x}(t), \quad 1 \leq m \leq n \tag{25}$$

is chosen, where $\bar{v}_j$ is the j-th eigenvector of the matrix A^T, and

$$K_j = \frac{\prod_{k=1}^{m} (\rho_k - \lambda_j)}{p_j \prod_{\substack{k=1 \\ k \neq j}}^{m} (\lambda_k - \lambda_j)}, \quad j = 1, 2, \ldots, m \tag{26}$$

[Ref 12:70].

Note the presence of the p_j term in the denominator of Eq (26). This is the j-th element of the mode-controllability matrix, $U^{-1}\bar{b}$, and must be nonzero if the j-th mode is to be controllable. The mode-controllability matrix is further discussed in Chapter III under the development of modal theory.

Eigenvalues of any or all of the controllable modes of the plant may be shifted using just the single feedback loop determined by Eq (25), and since only one input is available no question of priorities arises regarding which modes will be shifted by a particular input. However, this question does arise in at least one application of modal control when multiple inputs are available. This problem will be further amplified in later sections of this paper.

Multiple-input Modal Control. The basic concepts of modal control developed in the single input approach may be extended to cover systems having multiple inputs. Several methods of achieving the desired set of closed-loop eigenvalues for a system are available, since many solutions exist to the underdetermined set of non-linear algebraic equations which arise in the multiple input case. Four methods discussed in Ref 12 are: 1) minimum-gain modal controllers, 2) prescribed-gain modal controllers, 3) dyadic modal controllers, and 4) multi-stage design procedure for modal controllers. The minimum-gain approach provides modal controllers which minimize the sum of the squares of the feedback gains, however a restriction exists regarding the class of applicable systems. The modes to be controlled in the system must be controllable by every input variable, which is too restrictive to be considered as a general approach. Similarly, the prescribed-gain controller is only applicable to a limited class of systems, eliminating it from consideration. The dyadic controller has the effect of changing a multiple input system to a single input form with each of the original inputs contributing a fixed amount of the equivalent single input. Implicit in this approach is a requirement for no inherent coupling in the system inputs, a requirement which is not met by a large class of interesting systems including airframes. The multi-stage design approach is generally applicable to many systems of interest, and further is well-suited to presenting the theory of multiple-input modal control. Therefore the only method to be further developed here is the multi-stage design procedure for modal control.

The technique of multi-stage design for a modal controller stems from the concept of sequentially applying the various inputs available, thus forming a system composed of several nested loops. Each loop may be designed to control part or all of any given subset of eigenvalues of the original plant (subject to certain constraints), thus giving rise to the previously mentioned question of input priorities in the design sequence. Disregarding, for now, the question of input priorities, the equations pertinent to this technique are presented.

If the input matrix B is partitioned such that

$$B = [\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_r] \tag{27}$$

then the system modeled by Eq (1) will have the form

$$\dot{\bar{x}}(t) = A\bar{x}(t) + \sum_{i=1}^{r} \bar{b}_i z_i(t) \tag{28}$$

where z_i is the i-th input, i = 1, 2, ..., r. Each of the z_i in Eq (28) can be determined by treating it as a single input and requiring specified shifts of any or all of the plant eigenvalues to be accomplished. The specific equations used will not be repeated here; a very clear and explicit application of this process can be found in Adams [Ref 1:27-31], along with the appropriate equations.

As was mentioned before, the question of input priorities arises when using the multi-stage design procedure. Porter and Crossley suggest a procedure of shifting eigenvalues based on the relative magnitudes of the terms in the mode-controllability matrix [Ref 12:92].

This chapter has presented the pertinent theories of optimal and modal control in an abbreviated fashion. The basic theory underlying optimal control is embodied in a statement of Theorem 3.7 [Ref 8]; followed by a discussion of the significance of the theorem. Modal control was presented in terms of the basic concepts, then amplified to include both single-input and multiple-input systems.

Since this thesis project intends to use modal control theory only as a means of determining gains matrices which serve as starting points to establish the weighting matrices of an optimal performance criterion, the question of input priorities should be avoided or eliminated to prevent later questions of optimality from being raised. The method by which this is accomplished is developed in Chapter III.

III. Research Procedure and Numerical Results

This chapter deals with the application of the previously developed modal and optimal control theories to the specific problem stated for this thesis. The approach in this chapter is essentially a chronological record of the various steps undertaken to solve the problem, the results of each, and the rationale used in proceeding to the following step.

The development began with the objective of using the general matrix solution of the modal feedback gain matrix to find the quadratic weighting matrices of the optimal controller. This led to the requirement for a method to solve underdetermined algebraic matrix equations and thus to the generalized inverse (or "pseudoinverse") of complex matrices. The lack of satisfactory results using the above approach then led to the use of "implicit-gain" matrices in an effort to circumvent the limitations uncovered in the early steps. The relationship between the implicit-gain matrix and the Riccati equation (used in the solution of the optimal controller) is developed, followed by the development of an iterative search technique to determine the quadratic weighting matrices which satisfy the requirements of optimal control theory.

Numerical examples were used in evaluating various stages in the research. The theoretical development is interspersed with these examples where they occurred and the specific conclusions reached from the numerical examples are stated there. Various sources were used for these numerical examples; since the intent of this project was not to develop problems but to determine a method of analyzing existing systems [Ref 4; 11, 15].

General Matrix Solution of the Modal Gain Matrix

The initial survey of the techniques available to determine modal feedback gain matrices emphasized one factor: When a multiple-input system was to be analyzed; an underdetermined, nonlinear set of algebraic equations must be solved [Ref 12:86]. The modal design techniques mentioned in Chapter II, with the exception of the multi-stage technique, are not applicable to general systems. These techniques have restrictions on the degree of control a system must have on a given mode (minimum-gain controllers), or restrictions on the degree of interaction existing between the various inputs to the system (dyadic controllers). When beginning the mathematical development of this paper it was decided to use a matrix approach to the solution of the multiple input system rather than the multi-stage technique. There were two primary reasons for proceeding in this manner. First, the question of optimality which arose, raised doubts as to the feasibility of the multi-loop procedure; specifically the fact that different feedback gain matrices were generated for each input and pole-shift combination.

Secondly, a matrix approach was better suited to the numerical solutions contemplated, in that the modal feedback gain matrix was to be only a first step in finding the optimal weighting matrices, and the computer program which existed to do multi-loop modal design was complicated and time-consuming to use [Ref 1]. Accordingly, the following development yielded a general matrix solution for the modal feedback gain matrix.

Applicable Theory and Equations. Given the matrix differential equation describing a linear, time-invariant, deterministic system

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{z}(t) \tag{1}$$

where A is a $(n \times n)$ plant matrix

B is a $(n \times r)$ input matrix

$\bar{x}(t)$ is a $(n \times 1)$ state vector

$\bar{z}(t)$ is a $(r \times 1)$ input vector

and a linear feedback law of the form

$$\bar{z}(t) = K\bar{x}(t) \tag{10}$$

where K ix a $(r \times n)$ real matrix.

Substitution of Eq (10) into Eq (1) yields

$$\dot{\bar{x}}(t) = (A + BK)\ \bar{x}(t) \tag{11}$$

The original time-response of the open-loop plant, determined by the eigenvalues of the plant matrix A, are thus modified by the addition of the feedback of the states of the system as inputs.

In order to deal effectively with the modes of the system, it is necessary to uncouple them. As was discussed in Chapter II, the modal matrix, U, of the plant matrix does this, and can be readily determined by existing computer routines. As was done in the development of the dynamics of the undriven system in Chapter II, when the transformation

$$\bar{x}(t) = U\bar{\xi}(t) \tag{19}$$

is substituted in Eq (11) and the resulting expression is pre-multiplied by U^{-1}, one obtains

$$\dot{\bar{\xi}}(t) = U^{-1}(A + BK)\ U\bar{\xi}(t) \tag{29}$$

$$= R\ \bar{\xi}(t) \tag{30}$$

The term $R = U^{-1}(A+BK)U$ requires further examination. When expanded it yields

$$\begin{aligned} U^{-1}(A + BK)U &= U^{-1}AU + U^{-1}BKU \\ &= \Lambda + U^{-1}BKU \end{aligned} \tag{31}$$

The term $U^{-1}B$ has been called by Porter and Crossley the "mode-controllability" matrix [Ref 12:45], and is a key factor in whether a given mode of a system may be controlled. If all elements of any row of the matrix formed by the product of U^{-1} and B are zero, the mode corresponding to that row cannot be controlled by any of the system inputs.

Although complete controllability has been assumed in the mathematical development in this paper, in actual practice the mode-controllability matrix must be evaluated to avoid violating this assumption.

The matrix R can be viewed as the diagonalized plant of an augmented, undriven system; the dynamics of which are specified as a set of desired eigenvalues. Thus, if the diagonal matrix of desired eigenvalues is denoted as $R = \text{diag}[\rho_1, \rho_2, \ldots, \rho_n]$, a matrix expression is obtained relating the existing plant eigenvalues to the desired closed-loop eigenvalues

$$R = \Lambda + U^{-1}BKU \tag{32}$$

Since it was assumed that the plant eigenvalues $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ and the desired closed-loop eigenvalues $[\rho_1, \rho_2, \ldots, \rho_n]$ are known, it was possible to solve for the unknown term, $U^{-1}BKU$, which generated the apparent shift of the system eigenvalues. Thus

$$U^{-1}BKU = R - \Lambda \tag{33}$$

which was then solved for the feedback gain matrix K as

$$K = (U^{-1}B)^{-1}(R - \Lambda)U^{-1} \tag{34}$$

Eq (34) is the general matrix solution of the modal gain matrix under the assumption that $(U^{-1}B)^{-1}$ exists. The situation where this is not true is addressed later.

At this point, a specific numerical example was employed to validate the matrix solution of the modal gain matrix. The problem chosen was taken from a paper by

Porter [Ref 11:17-21]. This problem was chosen at this point in the development to aid in validating a preliminary computer program for matrix eigenanalysis, as the problem presented all the necessary matrices in numerical form. Given the plant matrix

$$A = \begin{bmatrix} -1.0 & 1.0 \\ 0.5 & -1.5 \end{bmatrix} \tag{35}$$

and control matrix

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix} \tag{36}$$

The eigenvalues of A were λ_1=-0.5, λ_2=-2.0 with a modal matrix

$$U = \begin{bmatrix} 1.0 & 1.0 \\ 0.5 & -1.0 \end{bmatrix} \tag{37}$$

Based on the problem statement, the desired eigenvalues were ρ_1=-10.0, ρ_2=-15.0. Substitution of the known matrices into Eq (34), performing the indicated matrix multiplications and inversions resulted in

$$K = \begin{bmatrix} -10.67 & 2.33 \\ 2.33 & -23.67 \end{bmatrix} \tag{38}$$

This result agreed exactly with the values determined by Porter and thus validated the matrix approach concept to finding the modal gain matrix.

Eq (34) contained the term $(U^{-1}B)^{-1}$. In the initial development of this report, the form used by Porter and Crossley [Ref 12:66-70, 92-93] was followed. In that development, the $U^{-1}B$ term was called the "mode-controllability" matrix, discussed earlier. In the present paper, the general matrix solution led to the requirement for the inverse of the mode-controllability matrix. This required the inverse of a rectangular matrix (pseudoinverse) when $r < n$, and in general the rectangular matrix could be complex. A search of existing literature revealed that routines existed to determine the pseudoinverse of a real matrix, but apparently none existed for a complex matrix [Ref 6; 7:16-17]. Thus the next step in the development was undertaken to determine a method of generating the pseudoinverse of a complex, rectangular matrix.

Complex Pseudoinverse. Since the concept of a matrix pseudoinverse may be unfamiliar, a brief discussion follows. The pseudoinverse (more formally known as the "Moore-Penrose generalized inverse") is the best approximation (in a least-square-error sense) to a "true inverse" for those matrices which do not possess a true inverse [Ref 2: 44-45]. Specifically, only a non-singular, square matrix possesses a true inverse [denoted as $(\cdot)^{-1}$]. If the matrix H is a non-singular, square matrix, there is a unique matrix H^{-1} such that

$$HH^{-1} = H^{-1}H = I \tag{39}$$

However, if H is any r x n matrix (singular or non-singular, rectangular or square), there is some conditional or generalized inverse $H^{(-1)}$ which satisfied

$$HH^{(-1)}H = H \tag{40}$$

If H is singular or rectangular, there are infinitely many matrices $H^{(-1)}$ which satisfy Eq (40) [Ref 13:132]. However, there is one unique matrix which most closely approximates the true inverse of a non-invertible matrix in the following sense:

If $$H\bar{p} = \bar{q} \tag{41}$$

then $$\hat{p} = H^{\dagger}\bar{q} \tag{42}$$

is the minimum norm vector among those vectors which minimize

$$|| \bar{q} - H\bar{p} || \tag{43}$$

where $H^{\dagger}$ (an n x r matrix) is the pseudoinverse of H [Ref 2: 19]. The pseudoinverse of an arbitrary rectangular matrix may be found from

$$H^{\dagger} = (H^TH)^{\dagger}H^T \tag{44}$$

or $$H^{\dagger} = H^T(HH^T)^{\dagger} \tag{45}$$

[Ref 2:25].

The expressions H^TH and HH^T are always symmetric matrices, and the pseudoinverse of a symmetric matrix is easily found. Since a symmetric matrix may always be diagonalized

$$HH^T = TDT^{-1} \tag{46}$$

then

$$(HH^T)^\dagger = TD^\dagger T^{-1} \tag{47}$$

where $D = \text{diag}[d_1, \ldots, d_i, \ldots, d_n]$ (48)

and $d_1, \ldots, d_n$ are determined from roots of $|dI - HH^T| = 0$

$$D^\dagger = \text{diag}[1/d_1, \ldots, 1/d_i, \ldots, 1/d_n]$$

or $$D^\dagger \equiv \text{diag}[1/d_1, \ldots, \underset{\substack{\uparrow \\ \text{(ii element)}}}{0}, \ldots, 1/d_n] \text{ if } d_i = 0 \tag{49}$$

[Ref 2:22-23].

A numerical example may serve to clarify the above development. Given the set of underdetermined simultaneous equations

$$\begin{aligned} 3u + 2v + w &= 16 \\ u + 4v - w &= 6 \end{aligned} \tag{50}$$

$$\bar{p} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$H = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

$$\bar{q} = \begin{bmatrix} 16 \\ 6 \end{bmatrix}$$

Find $$\hat{p} = H^\dagger \bar{q} \tag{42}$$

Using Eq (45) to determine $H^\dagger$ with the diagonalization procedure mentioned above

$$H^\dagger = H^T(HH^T)^\dagger \tag{45}$$

Evaluation of the term $(HH^T)^\dagger$ proceeded as follows

$$(HH^T) = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 10 \\ 10 & 18 \end{bmatrix}$$

(Note: The resultant matrix $[HH^T]$ was a square, non-singular matrix, thus its pseudoinverse was equal to its true inverse; however the diagonalization procedure was followed to demonstrate its application.)

A straight-forward approach to matrix diagonalization is the use of the modal matrix and its inverse; this was the approach used here. Using standard eigenanalysis techniques, the diagonal matrix, D, and modal matrix, T, were found.

$$D = \begin{bmatrix} 26.198 & 0.0 \\ 0.0 & 5.802 \end{bmatrix}$$

$$T = \begin{bmatrix} 1.0 & 1.0 \\ 1.2198 & -0.8198 \end{bmatrix}$$

In this problem, the D matrix had no zero diagonal entries, thus from Eq (49) its pseudoinverse was

$$D^{\dagger} = \begin{bmatrix} 1/26.198 & 0.0 \\ 0.0 & 1/5.802 \end{bmatrix} = \begin{bmatrix} 0.0382 & 0.0 \\ 0.0 & 0.1724 \end{bmatrix}$$

and the pseudoinverse of $[HH^T]$ was then found from Eq (47).

$$[HH^T]^{\dagger} = TD^{\dagger}T^{-1}$$

$$= \begin{bmatrix} 0.1184 & -0.0658 \\ -0.0658 & 0.0921 \end{bmatrix} \tag{47}$$

Substituting this result into Eq (45) yielded

$$H^{\dagger} = \begin{bmatrix} 0.2895 & -0.1053 \\ -0.0263 & 0.2369 \\ 0.1842 & -0.1579 \end{bmatrix} \tag{51}$$

Thus $\hat{p} = H^{\dagger}\overline{q}$

$$\hat{p} = \begin{bmatrix} 4.0 \\ 1.0 \\ 2.0 \end{bmatrix} \tag{52}$$

Substitution of Eq (52) into Eqs (50) verified that this was a valid solution to the system of equations. The norm of $\hat{p}$ was found as

$$||\hat{p}|| = \sqrt{4^2 + 1^2 + 2^2} = 4.58 \tag{53}$$

In generating this numerical example, an arbitrary $\bar{p}$ vector was chosen as

$$\bar{p} = \begin{bmatrix} 1.0 \\ 3.0 \\ 7.0 \end{bmatrix} \tag{54}$$

which clearly satisfies Eqs (50). The norm of this vector is found to be

$$|| \bar{p} || = 7.68 \tag{55}$$

While this was far from an exhaustive test, it did not disprove the validity of the theory of the pseudoinverse in providing the unique vector of minimum norm from the set of vectors satisfying Eqs (50.

The particular application of the pseudoinverse for this paper required dealing with complex matrices. As was mentioned earlier, no source of computational routines was found to do this. Therefore a discussion ensued with Dr. D.G. Shankland of the Air Force Institute of Technology, Wright-Patterson Air Force Base regarding the feasibility of developing the necessary routines. The outcome of the discussion was the application of the concept of pseudoinverses to complex matrices by employing the complex conjugate transpose in place of the matrix transpose. Thus, for a complex, rectangular matrix H

$$H^{\dagger} = (H^{*}H)^{\dagger}H^{*} \tag{56}$$

or $$H^{\dagger} = H^{*}(HH^{*})^{\dagger} \qquad (57)$$

where H^{*} is the complex conjugate transpose of H.

[Ref 10:408].

Numerical tests run using a routine written to accomplish the matrix operations indicated in Eqs (56) and (57) partially validated the routine by pseudoinverting a complex matrix H = Re M+j0 and comparing the results with the pseudoinversion of Re M using LPSDOR [Ref 6]. Additional verification was obtained by noting product $HH^{\dagger}$ was an identity matrix.

Once the above procedure had been coded and incorporated into the developing program to find the optimum weighting matrices, specific numerical problems were run to evaluate the program capabilities. One of these problems was concerned with the design for an F-4 aircraft lateral-axis controller, discussed by Van Dierendonck [Ref 15:272-273]. The problem as given incorporated a sixth-order plant, which included actuator dynamics. Since it was not necessary to deal with the poles of the actuators, the plant was reduced to fourth-order with two inputs by eliminating the actuator dynamics. The resulting plant and input matrices were

$$A = \begin{bmatrix} -1.7680 & 0.4125 & -14.52 & 0.0 \\ -0.0007 & -0.3831 & 6.038 & 0.0 \\ 0.0016 & -0.9975 & -0.155 & 0.0586 \\ 1.0000 & 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad (58)$$

$$B = \begin{bmatrix} 2.031 & 8.952 \\ -3.398 & -0.3075 \\ 0.028 & -0.0036 \\ 0.0 & 0.0 \end{bmatrix} \qquad (59)$$

Eigenanalysis of the plant matrix yielded the following system dynamics:

$$\begin{aligned} &\text{Spiral mode, } \lambda_1 = -0.0156 \\ &\text{Roll subsidence, } \lambda_2 = -1.85 \\ &\text{Dutch roll mode, } \lambda_{3,4} = -0.219 \pm j2.48 \end{aligned} \qquad (60)$$

From aerodynamic considerations, a desired set of system eigenvalues was

$$\begin{aligned} &\text{Spiral mode, } \rho_1 = -0.015 \\ &\text{Roll subsidence, } \rho_2 = -3.5 \\ &\text{Dutch roll, } \rho_{3,4} = -1.0 \pm j2.29 \end{aligned} \qquad (61)$$

so as to improve aircraft response to roll deviations and increase damping of the Dutch roll. When the above values were incorporated into the program designed to evaluate Eq (34) using the complex pseudoinverse routines, the resulting modal feedback gain matrix was

$$K = \begin{bmatrix} -0.0105 & -0.2374 & -0.1299 & 0.0141 \\ 0.1468 & 0.1411 & 0.2677 & -0.0069 \end{bmatrix} \qquad (62)$$

Note the need for the pseudoinverse in this case due to the $U^{-1}B$ matrix having dimensions (4x2).

When this matrix was substituted into the expression for the augmented plant matrix

$$[A+BK] \tag{63}$$

the closed-loop eigenvalues were found to be

$$\begin{aligned} &\text{Spiral mode, } \lambda_1 = -0.0156 \\ &\text{Roll subsidence, } \lambda_2 = -3.12 \\ &\text{Dutch roll, } \lambda_{3,4} = -0.609 \pm j2.35 \end{aligned} \tag{64}$$

Comparison of the open-loop and closed-loop dynamics, Eq (60) and (64) respectively, indicated that the pseudoinverse technique was capable of providing reasonable gain matrices for a pure modal design approach, in addition to providing a satisfactory starting point in the search for the optimal quadratic weighting matrices.

Table I

A Comparison of System Dynamics
(F-4 Aircraft Lateral-Axis Controller)

	System Dynamics		
	Spiral	Roll	Dutch Roll
Open-loop	-0.0156	-1.85	$-0.219 \pm j2.48$
Closed-loop	-0.0156	-3.12	$-0.609 \pm j2.35$
Desired	-0.0150	-3.50	$-1.00 \pm j2.29$

As can be seen, the pseudoinverse approach to determining the modal gain matrix succeeded in providing closed-loop eigenvalues "closer" to the desired values. In each instance the modes were shifted in the correct direction, but not by the entire desired amount. It is felt that this is due to the nature of the pseudoinverse, in that it is the closest unique approximation to a true inverse available, with the approximation becoming exact when the original matrix is square and non-singular. Subsequently, it was felt that the possibility existed to determine the modal feedback gain matrix more closely for a specified set of desired eigenvalues by an iterative approach: treating the augmented plant matrix, A+BK, as a basic plant having unsatisfactory eigenvalues and re-entering the design sequence with the same set of desired eigenvalues. In this way, a second feedback gain matrix, K_2, was determined. The resulting system would appear as

$$(A+BK_1)+BK_2 \tag{65}$$

or

$$A+B(K_1+K_2) \tag{66}$$

In the general case of q iterations, the resulting system would be

$$A+B \sum_{i=1}^{q} K_i \tag{67}$$

This concept was evaluated numerically through three iterations (q=3) for the previous F-4 lateral control problem. The resulting feedback gain matrices are shown, followed by the closed-loop eigenvalues obtained from each iteration, Table II.

$$K_1 = \begin{bmatrix} -0.0105 & -0.2374 & -0.1299 & 0.0141 \\ 0.1468 & 0.1411 & 0.2677 & -0.0069 \end{bmatrix} \quad (62)$$

$$K_2 = \begin{bmatrix} -0.0035 & -0.1002 & 0.0384 & 0.0056 \\ 0.0453 & 0.0293 & 0.1494 & -0.0016 \end{bmatrix} \quad (68)$$

$$K_3 = \begin{bmatrix} -0.0006 & -0.0484 & 0.0344 & 0.0027 \\ 0.0048 & -0.0131 & 0.0332 & 0.0007 \end{bmatrix} \quad (69)$$

Table II

Eigenvalues of Iterative Analysis
(F-4 Aircraft Lateral-Axis Controller)

Iteration Cycle	System Dynamics		
	Spiral	Roll	Dutch Roll
0	-0.016	-1.85	$-0.219 \pm j2.48$
1	-0.016	-3.06	$-0.663 \pm j2.298$
2	-0.016	-3.45	$-0.831 \pm j2.288$
3	-0.016	-3.496	$-0.915 \pm j2.287$
Desired	-0.015	-3.500	$-1.000 \pm j2.290$

Examination of the corresponding elements in Eqs (62), (68), and (69) indicates the magnitude of each element is smaller for succeeding iterations, indicating a tendency toward convergence of the sequence

$$K = \lim_{q \to \infty} \sum_{i=1}^{q} K_i \quad (70)$$

where K is the modal feedback gain matrix which would provide the desired closed-loop eigenvalues for the system A+BK.

A further indicator of the usefulness of this procedure is shown by a comparison of the norm of the vector difference, $\| \bar{\sigma} \|$, between the desired eigenvalues and the system eigenvalues for each iteration, illustrated in Table III,

where $$\bar{\sigma}_j = \bar{\rho}_i - \bar{\gamma}_{ij}, \quad i=1,2,\ldots,n \tag{71}$$

$\bar{\rho}_i$: desired system eigenvalues

$\bar{\gamma}_{ij}$: actual closed-loop eigenvalues

and j is the iteration index. Since $\bar{\sigma}_j$ may be complex, the norm is given by

$$\| \bar{\sigma} \| = \sqrt{\bar{\sigma}^* \bar{\sigma}} \tag{72}$$

where $\bar{\sigma}^*$ is the complex conjugate transpose of $\bar{\sigma}_j$.

Table III

Vector Norm

Iteration, j	$\| \bar{\sigma}_j \|$
0	2.004
1	0.676
2	0.244
3	0.120

Determination of Quadratic Weighting Matrices

This segment of the report is concerned with the procedures to determine quadratic weighting matrices, using a modal gain matrix as a starting point. The mathematical development is given, incorporating the optimal linear feedback law and the matrix Riccati equation to solve explicitly for the state weighting matrix R_1, given the feedback gain matrix and a positive-definite control weighting matrix, R_2, assumed as the identity matrix. This development is followed by a discussion of the perturbation search technique developed to find the weighting matrices.

Mathematical Development. In the mathematical development, it was assumed that the modal gain matrix K used in the modal feedback control law developed in Chapter II

$$\overline{z}(t) = K\,\overline{x}\,(t) \tag{10}$$

could be substituted for the optimal gain matrix F in the optimal feedback control law also developed in Chapter II

$$\overline{z}(t) = -F\,\overline{x}\,(t) \tag{6}$$

where
$$F = R_2^{-1}\,B^T\,\underline{P} \tag{7}$$

Eqs (10), (6), and (7) were combined and manipulated to solve explicitly for the Riccati matrix $\underline{P}$

$$\underline{P} = -(B^T)^{\dagger} R_2 K \tag{73}$$

(Note that at this point the pseudoinverse was again needed, this time for a real matrix.)

Once the Riccati matrix corresponding to the modal gain matrix had been found, it and the assumed R_2 matrix were substituted into the algebraic Riccati equation, Eq (5), which could then be solved explicitly for R_1, the desired state weighting matrix, as

$$R_1 = \underline{P}BR_2^{-1}B^T\underline{P} - A^T\underline{P} - \underline{P}A \tag{74}$$

Implicit in the preceeding development was the requirement that $\underline{P}$ and R_1 must be symmetric and positive semi-definite, as developed in Chapter II. From a mathematical matrix viewpoint, this did not appear explicitly in the equations, but these requirements must be considered in developing a numerical search technique to find an acceptable set of quadratic weighting matrices. The methods by which these requirements were accommodated are included in the next section of this report.

Development of a Search Technique. A recap of the progress to this stage indicated the development of a practical capability to find suitable modal feedback gain matrices using a matrix solution, and the theoretical development of the relationships to determine the quadratic weighting matrices using the modal gain matrix as the input.

Initially, when the modal design phase had been proven using numerical inputs, the search for quadratic weighting matrices was attempted using an assumed input weighting matrix, $R_2 = I_r$. With this approach, the Riccati matrix $\underline{P}$ was found using Eq (73), then this result was substituted into Eq (74) to determine the corresponding state weighting matrix R_1. In this approach, the Riccati matrix found from Eq (73) was generally non-symmetric. Before substituting $\underline{P}$ into Eq (74) it was therefore necessary to convert the non-symmetric form to an equivalent symmetric form by replacing all sets of elements p_{ij} and p_{ji} by the average value $(p_{ij}+p_{ji})/2$ [Ref 5:462]. This procedure was also followed after determining the corresponding R_1 matrix from Eq (74).

Early results of this technique failed to provide usable results, in that either the Riccati matrix or the state weighting matrix would not be positive semi-definite. Therefore a rudimentary manual perturbation scheme was tried, to determine the changes in the $\underline{P}$ and R_1 matrices when elements of the R_2 matrix were varied. This approach seemed to indicate a method of obtaining positive semi-definite $\underline{P}$ and R_1 matrices, providing the proper elements of R_2 were varied. The manual procedure used was very time-consuming, but appeared to be a promising approach. Therefore a perturbation scheme was devised which could be implemented on a digital computer to vary sequentially each of the elements of the R_2 matrix and evaluate the effect of the shift on the $\underline{P}$ and R_1 matrices. The evaluation consisted of determining the effect of the R_2 element perturbations on the eigenvalues of R_1 and $\underline{P}$, since the determination of positive semi-definiteness could be accomplished readily by an examination of the eigenvalues [Ref 3:341]. After evaluation of the most effective element of R_2 (determined to be that element which caused the largest magnitude change in the most negative eigenvalue of either R_1 or $\underline{P}$), that element was then changed by a small, variable amount. The sign of the change was chosen to cause the most negative eigenvalue to become less negative. The sequence of the search was intended to ultimately force the most negative eigenvalue to zero, thereby causing both $\underline{P}$ and R_1 to be at least positive semi-definite, with R_2 remaining positive-definite.

A discussion of the outcome of the perturbation search technique will be postponed, to permit discussion of a concurrent development called the "implicit-gain matrix" approach.

The Implicit-Gain Matrix

Concurrent with the development of the perturbation search procedure, an examination was conducted on a variation of Eq (73), the explicit solution of the Riccati matrix.

$$\underline{P} = -(B^T)^{-1} R_2 K \tag{73}$$

The term $(B^T)^{-1}$ carried the implication that the control matrix B was square and non-singular for the inverse of its transpose to be defined. Obviously, this would not, in general, be valid. Thus, the pseudoinverse concept again arose, to be applied to a real rectangular matrix. At first, it was felt this should pose no significant problem since the early results using the complex pseudoinverse were encouraging. Problems occurring in determining suitable $\underline{P}$ and R_1 matrices for various numerical systems raised doubts as to the validity of the real pseudoinverse approach, and a way was sought to circumvent this difficulty. A reappraisal of Eq (33) indicated that an alternative solution could be found as

$$BK = U(R-\Lambda)U^{-1} \tag{75}$$

where the term BK (the "implicit-gain matrix") is a product matrix of the n x r control matrix B, and the r x n feedback gain matrix K. This eliminated the need for a complex pseudoinverse (since the feedback gain matrix was not explicitly needed) as the following development demonstrated. If the negative of the modal feedback gain matrix were substituted for the optimal feedback gain in Eq (7), and the result pre-multiplied by -B, one obtained

$$BK = -B\, R_2^{-1}\, B^T\, \underline{P} \tag{76}$$

Equating Eqs (75) and (76) yielded

$$-B\, R_2^{-1}\, B^T\, \underline{P} = U(R - \Lambda)U^{-1} \tag{77}$$

or, solving explicitly for $\underline{P}$

$$\underline{P} = -(BR_2^{-1} B^T)^{\dagger} U(R - \Lambda)U^{-1} \qquad (78)$$

[Note again the use of the pseudoinverse in Eq (78)]. Eq (78) does not explicitly involve the modal feedback gain matrix, and the right-hand side contains only one variable, R_2^{-1}, for any given system. From this point on in the development of the search for the quadratic weighting matrices, Eq (78) was used in lieu of Eq (73) to determine the Riccati matrix.

Outcome of the Perturbation Search Technique

At this point in the development, the entire approach seemed promising. The procedure had evolved to the point where inputs were necessary to two equations [Eqs (74) and (78)] central to the determination of quadratic weighting matrices incorporating a modal feedback gain matrix. Initial numerical attempts generated negative results to the search due to apparent conflicts in achieving simultaneous positive semi-definite $\underline{P}$ and R_1 matrices.

A further modification was undertaken to eliminate the search for a positive semi-definite Riccati matrix and concentrate strictly on finding a positive semi-definite state weighting matrix by combining Eqs (74) and (78), eliminating all explicit reference to the Riccati matrix. The resulting expression directly related the state and input weighting matrices through a combination of the algebraic Riccati equation, Eq (74), the optimal feedback control law, Eq (6), and the modal gain matrix, Eq (34).

$$R_1=[(BR_2^{-1}B^T)^{\dagger}U(R-\Lambda)U^{-1}](BR_2^{-1}B^T)[BR_2^{-1}B^T)^{\dagger}U(R-\Lambda)U^{-1}]$$
$$+A^T[(BR_2^{-1}B^T)^{\dagger}U(R-\Lambda)U^{-1}]+[(BR_2^{-1}B^T)^{\dagger}U(R-\Lambda)U^{-1}]A \qquad (79)$$

The perturbation search technique was applied to Eq (79) in a two-stage procedure. The elements of R_2^{-1} were sequentially perturbed by a small amount (typically $1.0\text{x}10^{-4}$) to determine the most effective element in forcing the most negative eigenvalue of R_1 to zero. This procedure incorporated a test of the positive-definiteness of R_2^{-1} at each step to avoid violating another constraint of optimal control theory. The most effective element of R_2^{-1} was then shifted by a reasonable amount (typically $1.0\text{x}10^{-4}$ to $5.0\text{x}10^{-4}$) so as to force the most negative eigenvalue of R_1 less negative. Unfortunately, this search procedure did not yield desired results when applied to two specific numerical systems, one of which is discussed here.

A Numerical Example

As a specific system for evaluation, the F-4 lateral-control problem discussed earlier was chosen. It was felt that this system was typical of those which one might wish to analyze using the procedure developed in this paper, in that it represented a controllable, multiple-input system whose time-responses were unsatisfactory. The plant and input matrices are reproduced here for convenience.

$$A = \begin{bmatrix} -1.7680 & 0.4125 & -14.52 & 0.0 \\ -0.0007 & -0.3831 & 6.038 & 0.0 \\ 0.0016 & -0.9975 & -0.155 & 0.0586 \\ 1.0000 & 0.0 & 0.0 & 0.0 \end{bmatrix} \tag{58}$$

$$B = \begin{bmatrix} 2.031 & 8.952 \\ -3.398 & -0.3075 \\ 0.028 & -0.0036 \\ 0.0 & 0.0 \end{bmatrix} \tag{59}$$

The desired closed-loop eigenvalues were chosen, as before, to provide a faster response to roll perturbations and improve Dutch roll damping: $\rho_1=-0.015$, $\rho_2=-3.5$, $\rho_{3,4}=-1.0\pm j2.29$.

Due to the requirement for eigenanalysis of the R_1 and R_2^{-1} matrices after each shift in the most effective element of R_2^{-1}, the program developed to implement the search for quadratic weighting matrices used a considerable amount of computational time on the CDC 6600 computer. A run involving three thousand iterations of the R_2^{-1} element shifts required on the average 280 seconds of central processor time. A sequence of twenty runs, each involving three thousand iterations, produced the eigenvalue shift pattern illustrated in Figure 2. This demonstrated the trend shown during runs with both numerical systems, and emphasized an inherent weakness in the present search

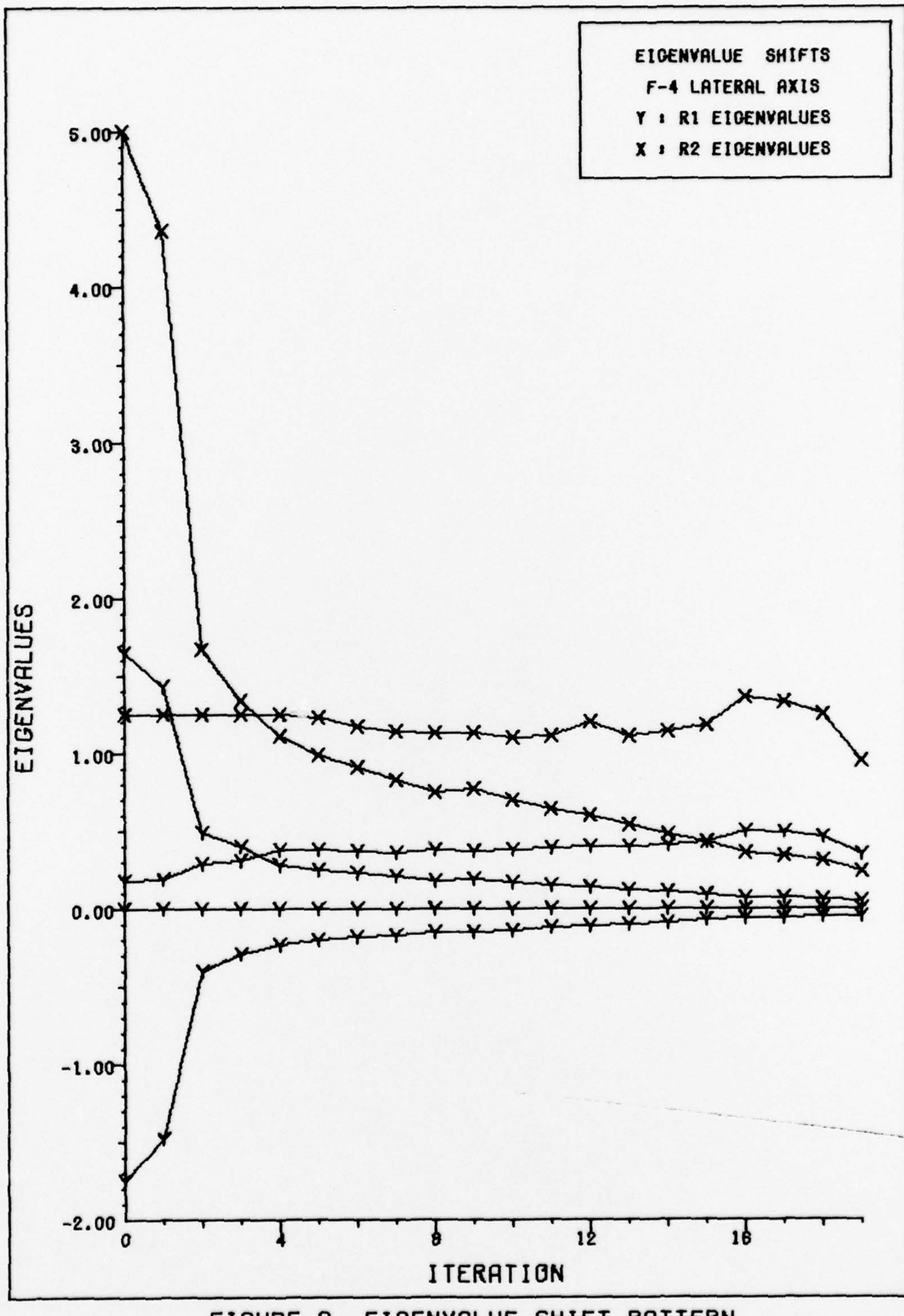

FIGURE 2. EIGENVALUE SHIFT PATTERN

technique: the inability to obtain a positive semi-definite R_1 matrix.

The numerical system analysis was terminated due to lack of time remaining, and the apparent ineffectiveness of the search technique developed to this point.

Summary

This chapter has presented the entire development involved in the search for quadratic weighting matrices derived from a modal feedback gain matrix. The development, essentially presented in chronological form, illustrated the various steps taken to solve sub-problems encountered in seeking the solution to the main problem stated at the beginning of this report. The results of each step were presented, and an attempt was made to describe the rationale involved in proceeding to the next step of the problem.

The areas covered were the general matrix solution for the modal feedback gain matrix, the development of the complex pseudoinverse, and the development of an iterative perturbation search technique to obtain satisfactory weighting matrices for an optimal controller.

Specific numerical examples were presented to substantiate various capabilities as they were developed, and in the final segment, to illustrate the shortcomings of the present search procedure.

The final chapter of this report deals with overall conclusions reached during the time spent on researching the problem, and presents specific recommendations for further work in this area.

IV. Summary, Conclusions, and Recommendations

This chapter concludes this report. A summary of the results obtained during the research is presented, followed by conclusions drawn from these results and certain observations noted during the research. The chapter concludes with recommendations for further investigation along the lines of this report.

Summary of Results

A review of Chapter III indicated four specific results which were determined during the period of research and analysis. Each of these results is presented and discussed below.

The first result was the successful solution to the modal design aspect of the thesis. In this area, a pseudoinverse approach was used to determine the modal feedback gain matrix which would provide a closed-loop set of eigenvalues. Used as a "one-shot" procedure, this method provides a closed-loop set of eigenvalues which are closer to a desired set of eigenvalues than the basic plant, but the "closeness" is dependent on how well the pseudoinverse used approximates a true inverse.

The second result of the research was essentially an outgrowth of the preceeding results: an iterative capability to determine a modal feedback gain matrix which provides closed-loop eigenvalues as close as desired to a specified set of eigenvalues. This procedure involved the repeated use of the pseudoinverse of a real matrix (the control matrix B) to establish a modal feedback gain matrix which is a summation of the individual gain matrices determined for each iteration. A heuristic analysis of this approach using the F-4 lateral axis controller indicated a tendency toward rapid convergence of the system eigenvalues to the desired set of eigenvalues (specifically, three iterations provided closed-loop eigenvalues within 10% of desired).

A third result, one which ultimately proved of minimal value to the project however, was the development of an algorithm to compute the pseudoinverse of an arbitrary complex matrix. While subsequent developments eliminated the need for the complex pseudoinverse, at the point in the research where it was developed it satisfied a definite requirement, and aided in identifying an unpromising avenue of research.

The fourth, and final, result to be noted was the development of a search procedure to identify and shift the eigenvalues of the quadratic weighting matrices, R_1 and R_2. As developed in Chapter III, this procedure was capable of identifying the elements of R_2 which had the most significant effect on the eigenvalues of R_1 and, on an iterative basis, modified this element of the input weighting matrix, R_2. Based on the results of two numerical examples investigated, this search technique has had limited success in modifying the weighting matrices.

Conclusions

Certain conclusions have been drawn, based on the four specific results stated above and observations made during the course of the research, as to the validity of the proposed method of solution to the initial problem of this thesis and the significance of the solutions proposed to the sub-problems described in the report. The original statement of the thesis problem was: Can the modal feedback gain matrix provide a suitable starting point from which to search for quadratic weighting matrices? Based on the results of this report, it is felt the answer is a qualified affirmative. The search technique developed for this report was essentially ineffective in obtaining the quadratic weighting matrices which would satisfy the postulates of optimal control, but the theory developed in Chapter III would indicate that the solution is feasible. Specific results obtained during the research substantiated this conclusion, and thus one might conclude that a more powerful search technique could aid in obtaining the desired results.

Additionally, it is felt the general matrix solution to the modal design problem (incorporating the iterative refinement to the feedback gain matrix) is a valid contribution to modal analysis and design techniques in that it provides an alternative solution to the designer.

The alternative design procedure for modal synthesis appears to be a new application of the pseudoinverse. It should be noted that, due to the uniqueness of the pseudoinverse, it is felt the resultant feedback gain matrix, Eq (70), is also unique. It is also believed that the least-square-error characteristic of the pseudoinverse provides a feedback gain matrix with desirable properties as regards the magnitude of the feedback gains, however this belief has not been investigated.

A final conclusion is concerned with the problems encountered during the search phase of the project. It is felt that the initial search, in which an attempt was made to obtain both the Riccati matrix, $\underline{P}$, and the state weighting matrix, R_1, positive semi-definite simultaneously, encountered difficulties due to the dimensionality of the problem. That is, with both $\underline{P}$ and R_1 n x n matrices, n(n + 1) undetermined elements existed; to be specified by m(m + 1)/2 elements of the m x m R_2 matrix. Thus, any search technique may have been inadequate. In retrospect, one must feel that further examination of the question of dimensionality was definitely in order, but was not accomplished due to time constraints.

Recommendations

It is recommended that further investigation be accomplished in regards to the findings of this report. Specifically, investigation of various methods of determining the quadratic weighting matrices, with the view of finding a functional search technique, is a necessity if a fully successful conclusion to the primary problem is to be reached. Additionally, the closed-loop systems generated by the iterative, pseudoinverse modal design should be investigated further--primarily in comparison with both optimal design results and other forms of modal designs--with respect to time response, variations of the state vector from nominal, and required input levels.

The questions of an analytic proof of the convergence of Eq (70) must also rank high as an area of interest, not specifically for solutions to the primary question of this report, but rather as a fundamental basis for a strong modal synthesis procedure.

As a final recommendation, a much deeper look at the characteristics and conditions pertaining to positive-definite and positive semi-definite matrices could indicate a new tack to be followed in regards to the search procedures and requirements. Specifically, the possibility of determining definiteness of a Hermitian matrix by investigation of the properties of the vector whose outer product produces the Hermitian matrix could provide useful results.

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An investigation was made of a relationship between modal and linear optimal control theories to determine whether the modal feedback gain matrix would be of help in finding the weighting matrices of the optimal steady-state criterion. Modal and optimal control theories are reviewed, and the concepts of steady-state optimal control, and single-input and multiple-input modal control are developed.

A general matrix solution, using a matrix pseudoinverse,

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) determines a unique modal feedback gains matrix which provides a set of specified closed-loop eigenvalues for a linear, time-invariant, deterministic system. The modal gains matrix is used as an input to a modified form of the algebraic Riccati equation. A perturbation search technique is applied in an attempt to find the state and control weighting matrices which simultaneously satisfy the Riccati equation and the optimal control postulates.

The procedure is applied to a numerical control problem, with the results indicating the search technique is not fully effective in establishing the optimal weighting matrices.

It is concluded that a new and useful modal design technique has been developed utilizing the pseudoinverse of a real matrix, and that a valid relationship exists (in theory) between modal and optimal control theories.

Recommendations are made to pursue the modal design technique further; to further analyze the characteristics determining Hermitian matrix definiteness; and to evaluate other types of search techniques capable of finding the optimal weighting matrices.