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THE USE OF SPHERICAL HARMONICS IN SUBOPTIMAL ESTIMATOR DESIGN (U)
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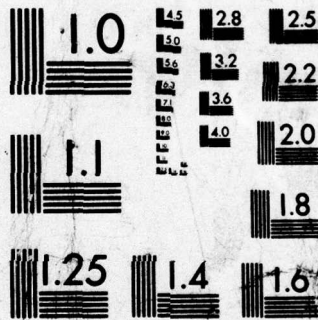
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Abstract

The state estimation problem for bilinear stochastic systems evolving on spheres is considered. The problem is motivated by some applications involving rotational processes in three dimensions. Then the theory of harmonic analysis on spheres is used to define assumed density approximations which result in implementable suboptimal estimators for the state of the bilinear system.

1. Introduction

Fourier series analysis has been applied in several recent studies [1]-[4] to estimation problems for stochastic processes evolving on the circle S^1 . Willsky [4] used Fourier series methods to define "assumed density" approximations for certain phase tracking and demodulation problems. In fact, a system designed using these techniques performed better than other estimators, including an optimal phase-lock loop.

In this paper we study bilinear systems evolving on spheres (the more general case of systems evolving on compact Lie groups or homogeneous spaces is discussed in [13]). The optimal estimator is in general infinite dimensional [7], and our approach to the design of suboptimal estimators is a generalization of that of Willsky [4], whose work is described in Section 3. The basic approach involves the definition of an "assumed density" form for the conditional density of the system state at time t given observations up to time t . These densities are defined via the techniques of harmonic analysis on spheres [5].

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[6] (which generalize the Fourier series on the Lie group S^1). Our method differs from most previous assumed density approximations in that our approximation is defined on the appropriate compact manifold (as opposed to Gaussian approximations, for example, which are defined on R^n [7]).

In Section 2 we discuss some general properties of stochastic bilinear systems and present an example involving satellite tracking. The use of harmonic analysis will be motivated by the phase tracking example of Willsky [4] in Section 3. In Section 4, we discuss the optimal estimation problem on the 2-sphere S^2 and the application of harmonic analysis to the design of suboptimal estimators.

2. Stochastic Bilinear Systems

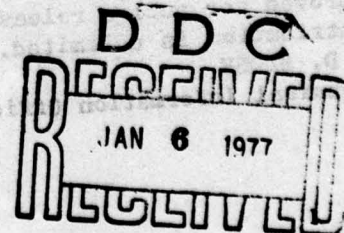
The basic stochastic bilinear system considered here is described by the Ito stochastic differential equation [4], [9], [10], [12], [13], [14]-[16], [17], [18].

$$dx(t) = \left\{ A_0 + \sum_{j=1}^N Q_{1j}(t) A_j \right\} dt + \sum_{i=1}^N A_i dw_i(t) x(t) \quad (1)$$

where x is an n -vector, A_j are $n \times n$ matrices, Q_{ij} is the (i,j) th element of Q , and w is a Brownian motion (Wiener) process with strength $Q(t)$ such that $E[w(t)w'(s)] = \int_0^{\min(t,s)} Q(\tau) d\tau$. Following the notation of [8]-[10], [11], we define $\mathfrak{L} = \{A_0, A_1, \dots, A_N\}_{LA}$ to be the Lie algebra containing these matrices. The corresponding connected matrix Lie group is denoted by $G = \{\exp \mathfrak{L}\}_G$.

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Then the solution $x(t)$ of (1) will evolve on the homogeneous space $G \cdot x(0)$ (i.e., $x(t) \in Gx(0)$ for all $t \geq 0$) in the mean-square sense and almost surely [15], [16].

Associated with the Ito equation (1) is a sequence of equations for the moments of the state $x(t)$, first derived by Brockett [9], [10], [14]. If $N(n, p)$ denotes the binomial coefficient $\binom{n+p-1}{p}$,

then given an n -vector x , we define $x^{[p]}$ to be the $N(n, p)$ -vector with components

$$\sqrt{\binom{p}{p_1} \binom{p-p_1}{p_2} \dots \binom{p-p_1-\dots-p_{n-1}}{p_n}} \cdot x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}; \sum_{i=1}^n p_i = p; p_i \geq 0 \quad (2)$$

(ordered lexicographically). These components are the monomials (or homogeneous polynomials) of degree p in x_1, \dots, x_n , the components of x ,

scaled so that $\|x\|^p = \|x^{[p]}\|$. Given an $m \times n$ matrix A , we denote by $A^{[p]}$ the unique matrix which verifies

$$y = Ax \Rightarrow y^{[p]} = A^{[p]} x^{[p]} \quad (3)$$

It is clear that if x satisfies the linear differential equation

$$\dot{x}(t) = Ax(t) \quad (4)$$

then $x^{[p]}$ also satisfies a linear differential equation

$$\dot{x}^{[p]}(t) = A^{[p]} x^{[p]}(t) \quad (5)$$

We regard this as the definition of $A^{[p]}$, which is the infinitesimal version of $A^{[p]}$. In fact, $A^{[p]}$ can be easily computed from A [17]. Brockett has shown that if x satisfies (1), then $x^{[p]}$ satisfies the Ito equation

$$dx^{[p]}(t) = \left[A_0^{[p]} + \frac{1}{2} \sum_{i,j=1}^N Q_{ij}(t) A_{i1}^{[p]} A_{j1}^{[p]} \right] x^{[p]}(t) dt + \sum_{i=1}^N A_{i1}^{[p]} x^{[p]}(t) dw_i(t) \quad (6)$$

As we shall see in the sequel, the sequence of moment equations is a valuable tool in the study of state estimation.

The observation model considered in this paper consists of linear observations of the state corrupted by additive white noise, or

$$dz(t) = H(t)x(t)dt + R^{\frac{1}{2}}(t)dv(t) \quad (7)$$

where z is a p -vector, $R > 0$, and v is a Wiener process. This bilinear system-linear observation model is useful in the study of certain practical problems, as illustrated by the following example.

Consider a satellite in circular orbit about some celestial body. Because of a variety of effects including anomalies in the gravitational field of the body, effects of the gravitational fields of nearby bodies, and the effects of solar pressure, the orbit of the satellite is perturbed. In this case, the position x of the satellite can be described by the simplified bilinear model [18]

$$dx(t) = \left\{ \sum_{i=1}^3 f_i(t) R_i + \frac{1}{2} \sum_{i,j=1}^3 Q_{ij}(t) R_i R_j \right\} dt + \sum_{i=1}^3 R_i dw_i(t) x(t) \quad (8)$$

where

$$R_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad R_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (9)$$

form a basis for $so(3)$, the Lie algebra of 3×3 skew-symmetric matrices. Also, f_i are the components of the nominal angular velocity and w_i are the components of a Wiener process with strength $Q(t)$. If $E[x'(0)x(0)] = 1$, then $E[x'(t)x(t)] = 1$ for all t ; thus x evolves on the 2-sphere S^2 (the same statement can be made almost surely [15], [16]). We note that the assumption in (8) that the perturbations in the angular velocity are white is a simplification [13], but the simplified model (8) can lead to simple but accurate on-line tracking schemes. If we are then given the noisy observations (7) of the satellite position, the problem is to estimate $x(t)$ given the past observations $z^t \triangleq \{z(s), 0 \leq s \leq t\}$.

3. A Phase Tracking Problem on S^1

We first discuss a phase tracking problem studied by Bucy, et al, [1] and Willsky [4], in which the phase θ and the observation z are described by

$$d\theta(t) = \omega_c dt + q^{\frac{1}{2}}(t)dw(t), \quad \theta(0) = \theta_0 \quad (10)$$

$$dz(t) = s^n \theta(t) dt + r^{\frac{1}{2}}(t) dv(t) \quad (11)$$

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where v and w are independent standard Brownian motion processes independent of the random initial phase θ_0 . We wish to estimate $\theta(t) \bmod 2\pi$ given z^t , and we take as our optimal estimation criterion the minimization of

$$E[(1 - \cos(\theta(t) - \tilde{\theta}(t)) | z(s), 0 \leq s \leq t] \quad (12)$$

where this notation denotes the conditional expectation given the σ -field $\sigma\{z(s), 0 \leq s \leq t\}$ generated by the observation process up to time t .

Noting that we are essentially tracking a point on the unit circle S^1 , we reformulate the problem in Cartesian coordinates. Let $x_1 = \sin \theta(t)$, $x_2 = \cos \theta(t)$. Then

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} -q(t)dt/2 & \omega_c dt + q^{1/2}(t)dw(t) \\ -(\omega_c dt + q^{1/2}(t)dw(t)) & -q(t)dt/2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (13)$$

$$dz(t) = x_1(t)dt + r^{1/2}(t)dv(t) \quad (14)$$

which are of the bilinear process - linear measurement type discussed in Section 2.

In Cartesian coordinates our estimation problem is to choose an estimate $(\tilde{x}_1(t), \tilde{x}_2(t))$ on the unit circle. If we use the constrained least squares criterion

$$J = \frac{1}{2} E[(x_1(t) - \tilde{x}_1(t))^2 + (x_2(t) - \tilde{x}_2(t))^2 | z(s), 0 \leq s \leq t] \quad (15)$$

subject to $\tilde{x}_1^2(t) + \tilde{x}_2^2(t) = 1$, or equivalently subject to $\tilde{x}_1(t) = \sin \tilde{\theta}(t)$, $\tilde{x}_2(t) = \cos \tilde{\theta}(t)$ our criterion reduces to the criterion (12) used in [1] and [4]. One can show [1], [4] that

$$(\tilde{x}_1(t), \tilde{x}_2(t)) = (\hat{x}_1(t|t), \hat{x}_2(t|t)) / \sqrt{\hat{x}_1^2(t|t) + \hat{x}_2^2(t|t)} \quad (16)$$

or

$$\tilde{\theta}(t) = \tan^{-1}(\hat{x}_1(t|t) / \hat{x}_2(t|t)) \quad (17)$$

where the conditional expectation is denoted by the equivalent notations

$$\hat{x}_1(t|t) \triangleq E[x_1(t) | z(s), 0 \leq s \leq t] \triangleq E^t[x(t)] \quad (18)$$

As discussed in [1] and [4], the optimal (constrained least-squares) filter is described as follows. The conditional probability density of θ given $\{z(s), 0 \leq s \leq t\}$ can be expanded in the Fourier series

$$p(\theta, t) = \sum_{n=-\infty}^{+\infty} c_n(t) e^{in\theta} \quad (19)$$

where

$$c_n(t) \triangleq \frac{1}{2\pi} E[e^{-in\theta(t)} | z(s), 0 \leq s \leq t] \triangleq b_n(t) - ia_n(t) \quad (20)$$

Then the optimal filter is given by

$$\begin{aligned} dc_n(t) = & -[in\omega_c + \frac{n^2}{2} q(t)] c_n(t) dt \\ & + \frac{c_{n-1}(t) - c_{n+1}(t)}{2i} + 2\pi c_n(t) \text{Im}(c_1(t)) \frac{dv(t)}{r(t)} \end{aligned} \quad (21)$$

$$\tilde{\theta}(t) = \tan^{-1}(a_1(t)/b_1(t)) \quad (22)$$

where $dv(t) = dz(t) + 2\pi \text{Im}(c_1(t)) dt$.

The structure of the optimal filter is that of an infinite bank of filters, but the equation for c_n is coupled only to the filters for c_1 , c_{n-1} , and c_{n+1} . This fact plays an important part in the following approximation. In order to construct a finite-dimensional suboptimal filter, we wish to approximate the conditional density (19) by a density determined by a finite set of parameters. Following Willsky [4], we assume that $p(\theta, t)$ is a folded normal density with mode $\eta(t)$ and "variance" $\gamma(t)$:

$$\begin{aligned} p(\theta, t) &= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{-n^2 \gamma(t)/2} e^{in(\theta - \eta(t))} \\ &= F(\theta; \eta(t), \gamma(t)) \end{aligned} \quad (23)$$

The folded normal density is the solution of the standard diffusion equation on the circle (i.e., it is the density for S^1 Brownian motion processes) and is as important a density on S^1 as the normal is on R^1 ; this point will be discussed in more detail in the next section. In this case, if c_1 has been computed and if $p(\theta, t)$ satisfies (23) then c_{N+1} can be computed (for any N) from the equation

$$c_{N+1} = (2\pi)^{(N+1)^2 - 1} |c_1|^{N(N+1)} c_1^{(N+1)} \quad (24)$$

Thus the bank of filters described by (21) can be truncated by approximating c_{N+1} by (24) and substituting this approximation into the equation for c_N . This was done for $N=1$ in [4], and the resulting Fourier coefficient filter (FCF) was compared to a phase-lock loop and to the Gustafson-Speyer "state-dependent noise filter" (SDNF) [19]. The FCF performed consistently better than the other systems, although the SDNF performance was quite close.

4. Estimation on S^2

This section is devoted to the study of the estimation problem for the system with the bilinear state equation (1) with the linear measurement (7), where the matrices $\{A_i; i=1,2,3\}$ are given by the skew-symmetric matrices $\{R_i; i=1,2,3\}$ of (9).

This system is motivated by the satellite tracking problem of Section 2; as discussed there, the state $x(t)$ evolves on the 2-sphere S^2 .

The estimation criterion which will be used for this problem is the constrained least-squares criterion, which is analogous to the criterion discussed in Section 3 for the phase estimation problem. That is, we seek $\hat{x}(t|t)$ which minimizes

$$J = E[(x(t) - \hat{x}(t|t))' (x(t) - \hat{x}(t|t) | z_2^t)] \quad (25)$$

subject to the S^2 constraint $\|\hat{x}(t|t)\|^2 =$

$\hat{x}(t|t)' \hat{x}(t|t) = 1$. It is easily shown [13] that the optimal estimate is

$$\hat{x}(t|t) = \hat{x}(t|t) / \|\hat{x}(t|t)\| \quad (26)$$

where $\hat{\cdot}$ denotes conditional expectation given the σ -field generated by the past observations (see (18)). Thus we must compute the conditional expectation of the state $x(t)$ given the past observations $z^t = \{z(s), 0 \leq s \leq t\}$.

The equations for computing the conditional expectation can be derived from the general nonlinear filtering equation [7, p. 184] and the moment equation (6). The resultant equations are

$$dE^t[x^{[p]}(t)] = [A_{0[p]} + \frac{1}{2} \sum_{i,j=1}^3 Q_{ij}(t) A_{i[p]} A_{j[p]}] E^t[x^{[p]}(t)] dt + [E^t[x^{[p]}(t) x'(t)] - E^t[x^{[p]}(t)] E^t[x'(t)] H'(t) R^{-1}(t) dv(t) \quad (27)$$

$$dv(t) = dz(t) - H(t) \hat{x}(t|t) dt. \quad (28)$$

The structure of these equations is quite similar to that of (21)—i.e., the estimator consists of an infinite bank of filters, and the filter for the p^{th} moment is coupled only to those for the first and $(p+1)^{\text{st}}$ moments. Therefore, we are led to the design of suboptimal estimators. Motivated by the success of Bucy and Willisky's phase tracking example evolving on S^1 , we would like to design suboptimal estimators for the S^2 system using similar techniques.

In our discussion of estimation on S^2 , we

will refer to a point on S^2 in terms of the Cartesian coordinates $x^A = (x_1, x_2, x_3)$ or the polar coordinates (θ, φ) (see the Appendix, in which harmonic analysis on S^2 is summarized). We first review the notions of Brownian motion and Gaussian densities on homogeneous spaces, which have received much attention in the literature (see [5], [21]). Yosida [21] proved that the density $p(x, t)$ of a Brownian motion process on a Riemannian homogeneous space M (such as S^2) with respect to the Riemannian measure is the fundamental solution of

$$\partial p(x, t) / \partial t - G^* p(x, t) = 0 \quad (29)$$

where G^* is the formal adjoint of a differential operator expressible in local coordinates as

$$G = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^n Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \quad (30)$$

with constant f and $Q = Q' \geq 0$. In particular, if G is the Laplace-Betrami operator (Laplacian) Δ_{g^2} on S^2 (see (A.4)) which is self-adjoint [5], the fundamental solution of

$$\partial p(\theta, \varphi, t) / \partial t - \gamma \Delta p(\theta, \varphi, t) = 0 \quad (31)$$

where $\gamma > 0$, is a Brownian motion on S^2 . According to [5], the fundamental solution of (31) is given by

$$p(\theta, \varphi, t; \theta_0, \varphi_0, t_0) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta_0, \varphi_0) \exp(-l(l+1)(t-t_0)\gamma) \quad (32)$$

where $\{Y_{lm}, -l \leq m \leq l\}$ are the spherical harmonics of degree l (defined in (A.2), (A.3)). The function $p(\theta, \varphi, t; \theta_0, \varphi_0, t_0)$ is the solution to (31) with initial condition equal to the singular distribution concentrated at $(\theta, \varphi) = (\theta_0, \varphi_0)$. Also,

Grenander [20] defines a Gaussian (normal) density to be the solution of (29) for some t .

The folded normal density $F(\theta; \eta, \gamma)$ used by Willisky as an assumed density approximation for the phase tracking problem is indeed a normal density on S^1 in the sense of Grenander [4]. Motivated by the success of Willisky's suboptimal filter, we will design suboptimal estimators for bilinear systems on S^2 by employing a normal assumed conditional density of the form

$$p(\theta, \varphi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\eta(t), \lambda(t)) \cdot \exp(-\ell(\ell+1)\gamma(t)) \quad (33)$$

where $\eta(t)$, $\lambda(t)$ and $\gamma(t)$ are parameters of the density which are to be estimated. In other words, the "generalized Fourier coefficients"

$$c_{\ell m}(t) = \int_0^{2\pi} \int_0^{\pi} Y_{\ell m}^*(\theta(t), \varphi(t)) p(\theta, \varphi, t) \sin \theta d\theta d\varphi = E[Y_{\ell m}^*(\theta(t), \varphi(t))] \quad (34)$$

are assumed to be

$$c_{\ell m}(t) = Y_{\ell m}^*(\eta(t), \lambda(t)) e^{-\ell(\ell+1)\gamma(t)} \quad (35)$$

In order to assume the existence of the conditional density, it is sufficient to assume a technical controllability condition [13].

In order to truncate the optimal estimator (27) after the N^{th} equation (for $\hat{x}^{[N]}(t|t)$) using the assumed density (33), we must compute $E^t[x^{[N]}(t)x'(t)]$, or equivalently, $\hat{x}^{[N+1]}(t|t)$, in terms of $\hat{x}^{[p]}(t|t)$, $p = 1, 2, \dots, N$. However, if $\hat{x}(t|t)$ is known, so are $c_{10}(t)$ and $c_{11}(t)$, and a simple computation yields

$$\gamma(t) = -\frac{1}{2} \log \left[\frac{4\pi}{3} (c_{10}^2(t) + 2|c_{11}(t)|^2) \right] \quad (36)$$

$$\cos \eta(t) = c_{10}(t) / [c_{10}^2(t) + 2|c_{11}(t)|^2]^{\frac{1}{2}} \quad (37)$$

$$\sin \eta(t) = \pm \sqrt{2} |c_{11}(t)| / [c_{10}^2(t) + 2|c_{11}(t)|^2]^{\frac{1}{2}} \quad (38)$$

If $c_{11}(t) = 0$, then the density is independent of $\lambda(t)$; otherwise,

$$\exp(2i\lambda(t)) = c_{11}^*(t) / c_{11}(t) \quad (39)$$

Then $\{c_{N+1, m}(t), m = -(N+1), \dots, N+1\}$ can be computed from

$$c_{N+1, m}(t) = Y_{N+1, m}^*(\eta(t), \lambda(t)) \exp(-(N+1)(N+2)\gamma(t)) = (-1)^m \sigma^{\frac{1}{2}} P_{N+1, m} \left(\frac{c_{10}(t)}{(c_{10}^2(t) + 2|c_{11}(t)|^2)^{\frac{1}{2}}} \right) \cdot \left(\frac{c_{11}(t)}{c_{11}^*(t)} \right)^{m/2} \left[\frac{4\pi}{3} (c_{10}^2(t) + 2|c_{11}(t)|^2) \right]^{\frac{1}{2} (N+1)(N+2)} \quad (40)$$

$$\text{where } \sigma = \frac{(N+1-m)! 2N+3}{(N+1+m)! 4\pi}$$

The decomposition of (A.5) of homogeneous polynomials of degree n (restricted to S^2) in terms of the spherical harmonics of degree $\leq n$ implies the existence of a nonsingular matrix P_n such that

$$P_n x^{[n]} = [Y_n(x), Y_{n-2}(x), \dots, Y_0(x)]' \quad (41)$$

where $Y_{\ell}(x)$ is the $(2\ell+1)$ -vector whose components are the spherical harmonics $\{Y_{\ell m}, -\ell \leq m \leq \ell\}$ of degree ℓ and δ is zero or one depending on whether n is even or odd. Specializing this result to the present approximation implies the existence of a nonsingular matrix P_{N+1} such that

$$P_{N+1} x^{[N+1]} = [Y_{N+1}(x), x^{[N-1]}] \quad (42)$$

Thus $\hat{x}^{[N+1]}(t|t)$ can be computed from $\{c_{N+1, m}^{[N+1]}(t|t), -(N+1) \leq m \leq N+1\}$ and $\hat{x}^{[N-1]}(t|t)$. The optimal estimator (27) is truncated by substituting this approximation for $\hat{x}^{[N+1]}(t|t)$ into the equation for $\hat{x}^{[N]}(t|t)$.

We note that one can show that

$$\sigma(t) \triangleq \sqrt{\|\hat{x}(t|t)\|^2} \leq 1 \quad (43)$$

and this quantity can be used as a measure of our confidence in our estimate. Specifically, if $\hat{x}(t|t)$ satisfies the assumed density (33),

$$\sigma(t) = \|\hat{x}(t|t)\| = e^{-\gamma(t)} \quad (44)$$

so $\gamma = 0$ (zero "variance") implies $\sigma = 1$, and $\gamma = \infty$ (infinite "variance") implies $\sigma = 0$ (see [4] for the S^1 analog).

Example: Suppose that we truncate the optimal S^2 estimator (27) after $N = 1$ —i.e., we approximate $\hat{x}^{[2]}(t|t)$ using the above approximation. The resulting suboptimal estimator is (for $Q(t) = I$)

$$d\hat{x}(t|t) = [A_0 + \frac{1}{2} \sum_{i=1}^N A_i^2] \hat{x}(t|t) dt + P(t) H'(t) R^{-1}(t) dz_2(t) - H(t) \hat{x}(t|t) dt \quad (45)$$

where the "covariance" matrix $P(t)$ is given by

$$P_{11}(t) = \hat{x}_1^2(t|t) \left(\frac{2}{3} \|\hat{x}(t|t)\| - 1 \right) - \frac{1}{3} (\hat{x}_j^2(t|t) + \hat{x}_k^2(t|t)) \|\hat{x}(t|t)\| + \frac{1}{3} \quad (46)$$

for $i \neq j$, $i \neq k$, $j \neq k$, and

$$P_{ij}(t) = \hat{x}_i(t|t) \hat{x}_j(t|t) (\|\hat{x}(t|t)\| - 1) \quad (47)$$

for $i \neq j$. Notice that, from (44), $\|\hat{x}(t|t)\| = 1$ implies that the "variance" $\gamma(t) = 0$; in fact, if $\|\hat{x}(t|t)\| = 1$, we see from (46)-(47) that the covariance matrix $P(t)$ is identically zero. Thus if $\|\hat{x}(t|t)\| = 1$, this first order suboptimal filter assumes that it has perfect knowledge of $x(t)$ and disregards the measurements.

5. Conclusion

In this paper we have considered the estimation problem for bilinear stochastic systems evolving on spheres. The techniques of harmonic analysis have been applied to the design of suboptimal estimators for such systems. The performance of these estimators is presently being investigated by means of computer simulation, and the results will be presented in a future paper.

Appendix - Harmonic Analysis on S^2

Any point (x_1, x_2, x_3) on the 2-sphere

$S^2 = \{x \in R^3 | x \cdot x = 1\}$ can be expressed in the polar coordinates (θ, φ) , where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, by defining

$$x_1 = \cos \theta; x_2 = \sin \theta \cos \varphi; x_3 = \sin \theta \sin \varphi. \quad (A.1)$$

Harmonic analysis on S^2 is studied in terms of the spherical harmonics [6], [22], [23]; the normalized spherical harmonics of degree l on S^2 are defined by [6]

$$Y_{lm}(\theta, \varphi) = (-1)^m \left[\frac{(l-m)!}{(l+m)!} \frac{(2l+1)}{4\pi} \right]^{\frac{1}{2}} P_{lm}(\cos \theta) e^{im\varphi} \quad (A.2)$$

$$Y_{l,-m}(\theta, \varphi) = (-1)^m Y_{lm}^*(\theta, \varphi) \quad (A.3)$$

for $l=0, 1, \dots$ and $m=0, 1, \dots, l$, where $P_{lm}(\cos \theta)$ are the associated Legendre functions and $*$ denotes complex conjugate.

Let \mathcal{S}_l denote the space of homogeneous polynomials of degree l on R^3 (i.e., $f(cx_1, cx_2, cx_3) = c^l f(x_1, x_2, x_3)$). Then the space H_l of spherical harmonics of degree l on S^2 can be characterized in the following equivalent ways:

(1) the restriction to S^2 of the subspace of R^3 which is orthogonal to the subspace $\{(x_1^2 + x_2^2 + x_3^2) \cdot f(x_1, \dots, x_3) | f \in \mathcal{S}_{l-2}\}$;

(2) the eigenspace of the Laplacian Δ_{S^2} on S^2 with eigenvalue $-l(l+1)$, where [9]

$$\Delta_{S^2} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (A.4)$$

Property (1) implies that each $f \in \mathcal{S}_l$ has a unique expansion

$$f = \sum_{j=0}^l \|x\|^{2j} f_{l-2j} \quad (A.5)$$

where $f_{l-2j} \in H_{l-2j}$ and $[t]$ is the largest integer $\leq t$ (Brockett [9] also discusses this point). One can show [22, p. 109] that the span of $\{H_l, l=1, 2, \dots\}$ is dense in the space of continuous functions on S^2 and in $L_p(S^2)$, $1 \leq p < \infty$.

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