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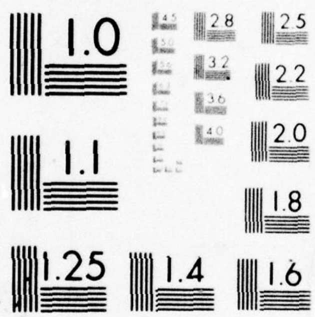

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AN ALGORITHM BASED ON THE EQUIVALENCE OF VECTOR AND SCALAR LABELS IN SIMPLICIAL APPROXIMATION,

By

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AN ALGORITHM BASED

ON THE EQUIVALENCE OF VECTOR AND SCALAR LABELS

IN SIMPLICIAL APPROXIMATION

by

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ABSTRACT

A scalar labeling is presented with the property that the complementary path follows the homotopy path tracked by the usual vector labeling. This produces an algorithm which determines the path without pivoting on a linear system and without the extra dimension introduced in the "sandwich" approach. The results should therefore lead to computational savings.

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1. INTRODUCTION

The theory of simplicial approximation has been applied by numerous authors to problems involving the computation of fixed points and the equivalent problem of finding solutions to a system of nonlinear equations [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]. In most of the previous works in this field two apparently different approaches can be recognized. These two approaches pivot upon the same triangulation (of, for example, $R^n \times [0, 1]$) but they differ in that one uses a "scalar labeling" and the other a "vector labeling." The scalar labeling was initially presented by Scarf [15] for the problem of computing fixed points on the unit simplex, and, by simple extension, for a map $f: C \to C$, C compact in R^n. Scalar labelings were then presented by Fisher and Gould [5a] for the nonlinear complementarity problem and by Fisher, Gould and Tolle [6], [7], and Gould and Tolle [12] for the general problem of solving $f(x) = 0$, $f: R^n \to R^n$. The so-called "vector labelings" were developed most notably by Eaves [2], [3], [4], and Merrill [13], both for the fixed point problem and the problem of solving $f(x) = 0$ on bounded and unbounded regions. Theorems regarding the existence of homotopy paths were shown in Garcia [9], and Charnes, Garcia and Lemke [1]. Merrill [13] and Eaves [5] independently solved the restart problem for vector labelings. The problem was solved by Fisher, Gould, and Tolle [7] for scalar labelings.

In this paper we shall for convenience present the entire discussion in the framework of seeking a solution, say x^*, to the system $f(x) = 0$. Although the scalar and vector approaches appear to be different, it is known that in a purely formal way any scalar labeling ℓ can be encompassed within the vector framework as follows. Let v^i be any vertex in the triangulation. Associated with this vertex, define the column

$$A^i = \begin{cases} \begin{bmatrix} e^j \\ 1 \end{bmatrix} & \text{if } \ell(v^i) = j \qquad (1 \leq j \leq n) \\ \begin{bmatrix} -e \\ 1 \end{bmatrix} & \text{if } \ell(v^i) = n + 1 \end{cases}$$

where e^j is the $j^{\underline{th}}$ unit vector in R^n and e is the n-vector of ones. The scalar path generated by ℓ will be followed by using the vector labels A^i in conjunction with the linear system (the matrix A has columns A^i)

$$A\lambda = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda \geq 0 \quad (0 \in R^n).$$

Though this transition is purely formal and not of computational interest it prompts the converse question as to whether or not vector labelings can be expressed in the scalar framework. For example, the "sandwich" vector procedure of Merrill [13] assigns a label A^i to each vertex v^i of $R^n \times [0, 1]$ as follows.

$$A^i = \begin{cases} \begin{bmatrix} v^i \\ 1 \end{bmatrix} & \text{if } v^i \in R^n \times \{0\} \\ \begin{bmatrix} f(v^i) \\ 1 \end{bmatrix} & \text{if } v^i \in R^n \times \{1\} \end{cases}$$

Using these labels, the vector approach to solving $f(x) = 0$ pivots on the triangulation through successive basic feasible solutions of the system

$$\sum_{I_0} \lambda_i \begin{bmatrix} v^i \\ 1 \end{bmatrix} + \sum_{I_1} \lambda_i \begin{bmatrix} f(v^i) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (0 \in R^n)$$

where I_0 indices vertices in $R^n \times \{0\}$ and I_1 indices vertices in $R^n \times \{1\}$. That is, the algorithm proceeds via complementary pivoting to generate a sequence of simplices $F_0, S_0, F_1, S_1, F_2, S_2, \ldots,$ where F_j and F_{j+1} are facets (n-simplices) of the n+1-simplex S_j. The move from F_j to S_j is made by joining a single vertex to the $n+1$ vertices of F_j. The rule for selecting the vertex to bring in is given by the triangulation formula. The move from S_j to F_{j+1} is made by dropping a vertex from S_j. The drop-out rule is provided via the linear system by pivoting from the basic feasible solution associated with F_j to the one associated with F_{j+1}. It is known (Charnes, Garcia and Lemke [1]) that when the simplices of the triangulation are suitably small the path generated by the above vector labeling is (in the limit) the homotopy path $H(x, t) = 0$ where $H(x, t) \equiv tf(x) + (1 - t)x$, $t \in [0, 1]$, $x \in R^n$. Without loss of generality we have chosen the initial point to be $0 \in R^n$. Thus $H(0, 0) = 0$ and if the algorithm converges to $(x^*, 1)$ we have $H(x^*, 1) = 0 = f(x^*)$.

The principal result of this paper is that a scalar labeling and a modified pivoting procedure have been discovered for generating precisely the same homotopy path. In this sense the vector labeling can be cast into the scalar framework. The result is of computational significance, for it eliminates both the extra "sandwich" dimension and the need to pivot on the linear system.

2. THE ALGORITHM AND THE LABELING

Assume that $f\colon R^n \to R^n$ is continuous, $f_i(0) \neq 0$, $i = 1,\ldots, n$, $f(x^*) = 0$, and x^* is interior to some orthant. Each orthant of R^n is triangulated (in general, orthants are determined by a starting point ω rather than the origin). There is no need to add an extra dimension by triangulating $R^n \times [0, 1]$. The algorithm begins in some orthant denoted E^J at the origin with an n-simplex (n + 1 vertices) containing the labels 1 through n + 1. A simplex with labels 1 through n + 1 is termed an (n + 1)-complete (or a completely labeled) simplex. Complementary pivoting with scalar labels produces a path in E^J of n-simplices with labels 1 through n. A simplex with labels 1 through n is termed an n-complete (or a (n + 1)-almost complete simplex). We note that although an (n + 1)-complete simplex must be an n-simplex, an n-complete simplex may be an n-simplex or an (n - 1)-simplex. The simplicial path which we generate is described by the following possibilities.

(P1) The path consists of an unbounded sequence of n-complete n-simplices in E^J.

(P2) The path terminates at an (n + 1)-complete simplex which is near x^* (interior to E^J).

(P3) The path terminates at an n-complete (n - 1)-simplex in the boundary of E^J.

In case (P3) a modified pivoting technique will generate a sequence of boundary (n - 1)-simplices in such a way that either

(P4) (a) the boundary path is unbounded, or

(P4) (b) the path reverts back into the orthant E^J or it breaks through the boundary into a new orthant $E^{\hat{J}}$.

If (P4) (b) occurs, then one of the possibilities (P1), (P2), or (P3) will again occur, with E^J replaced by $E^{\hat{J}}$.

It will be shown that when the triangulation grid is suitably small the above described path follows the same homotopy path which is followed by the vector labeling.

A new scalar labeling is introduced for our purpose. At each step of the algorithm the current simplex either intersects the interior of an orthant or lies in a boundary of an orthant. We shall term the "current orthant" to be either the orthant whose interior the current simplex intersects or, in case the current simplex is in a boundary, the current orthant is the one whose interior was last visited.

Let E^J denote the current orthant and suppose $x \in E^J$. If $x_i = 0$ for some i, define

$$\frac{f_i(x)}{x_i} = \begin{cases} +\infty \text{ if } f_i(x) \geq 0 \text{ and } y_i \geq 0 \text{ for all } y \text{ in } E^J \text{ or if } f_i(x) < 0 \text{ and } y_i \leq 0 \text{ for all } y \text{ in } E^J \\ -\infty \text{ if } f_i(x) \geq 0 \text{ and } y_i \leq 0 \text{ for all } y \text{ in } E^J \text{ or if } f_i(x) < 0 \text{ and } y_i \geq 0 \text{ for all } y \text{ in } E^J \end{cases}$$

Now, for any $x \in E^J$ our labeling (depending on the orthant E^J) is specified as follows:

(1)
$$\ell_J(x) = \begin{cases} n + 1 \text{ if } x_i f_i(x) \geq 0, \text{ all } i \\ \text{smallest } i \text{ s.t. } \dfrac{f_i(x)}{x_i} \leq \dfrac{f_j(x)}{x_j} \text{ all } j, \text{ otherwise} \end{cases}$$

3. THE INITIAL SIMPLEX

We now identify a starting n-simplex which is a unique $(n + 1)$-complete simplex in a neighborhood of the origin. The space R^n is triangulated in such a way that the interior of any n-simplex will not intersect any hyperplane $x_i = 0$ for any i. Thus each n-simplex is entirely in an orthant of R^n.

As stated earlier it is assumed that $f_i(0) \neq 0$, $i = 1, \ldots, n$. Let D denote the $n \times n$ matrix whose diagonal terms are given by $d_{ii} = -\text{sgn } f_i(0)$ and whose off-diagonal terms d_{ij} $(j \neq i)$ are zero. Let e^i be the ith unit vector in R^n and e be the n-vector of ones. Finally, let $\varepsilon > 0$ be the grid size and triangulate in such a way as to include the simplex $\{v^0, v^1, \ldots, v^n\}$ where

$$v^0 = 0$$

$$v^i = \varepsilon D(e - e^i), \quad i = 1, \ldots, n .$$

Thus, for $i = 1, \ldots, n$, $v^i_i = 0$ and, for $j \neq i$, $v^i_j = -\varepsilon \text{ sgn } f_j(0)$. This n-simplex will be the starting simplex, denoted σ^n_0 and it will be shown that this is a unique $(n + 1)$-complete simplex in some neighborhood of the origin. Let E^J denote the orthant containing σ^n_0. Note that if x is in the interior of E^J then $x_j f_j(0) < 0$ for all j.

<u>Lemma 1</u>. Let $N(0, \delta)$ be a neighborhood of the origin for which $f_i(x) \neq 0$, all i, for every x in $N(0, \delta)$. Then for $\varepsilon > 0$ sufficiently small the simplex σ^n_0 is a unique $(n + 1)$-complete simplex contained in $N(0, \delta)$.

<u>Proof</u>. It is immediate from the labeling rules (1) that $\ell_J(v^0) = n + 1$. Let ε be chosen so small that σ^n_0 is contained in $N(0, \delta)$. Note that for any x in $E^J \cap N(0, \delta)$ such that $x_j \neq 0$ it must be the case that x_j and $f_j(x)$ have opposite signs. Pick an index $i \in \{1, \ldots, n\}$. Recall $v^i_i = 0$ and $v^i_j \neq 0$, $j \neq i$. Hence,

$$\frac{f_i(v^i)}{v^i_i} = -\infty, \frac{f_j(v^i)}{v^i_j} > -\infty, \text{ all } j \neq i$$

and therefore $\ell_J(v^i) = i$. This shows that σ^n_0 is $(n+1)$-complete. Now let σ^n be any other $(n+1)$-complete simplex in $N(0, \delta)$. If the interior of σ^n is in some orthant other than E^J, then for each x in the interior of σ^n it must be that x_j and $f_j(x)$ have the same sign for some j, and hence this label would be excluded from σ^n. Consequently σ^n and σ^n_0 must be in the same orthant, E^J. Furthermore, the origin (v^0) must be a vertex of σ^n. Otherwise, if v is the vertex of σ^n labeled $n+1$, then, from the labeling rule (1), $v_j f_j(v) \geq 0$, all j. If some $v_i \neq 0$, then, since $f_i(v)$ is also nonzero, it follows that both have the same sign which contradicts the fact that for any x in $E^J \cap N(0, \delta)$ it must be that if $x_i \neq 0$ then x_i and $f_i(x)$ have opposite signs. Since v^0 (the origin) is in σ^n, there is a coordinate i such that for each of the other vertices the ith coordinate is nonzero and some other coordinate is zero (see Figure 1). Hence, if $\sigma^n \neq \sigma^n_0$ the label i is excluded from σ^n, a contradiction. #

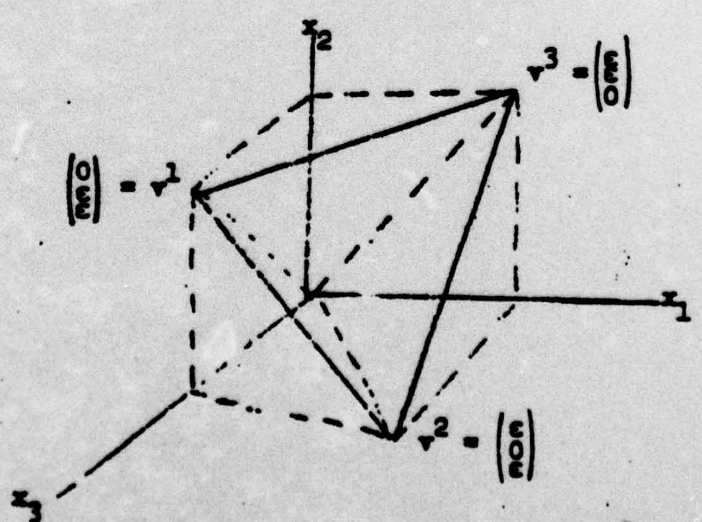

Figure 1: The Starting Simplex and Triangulation in R^3 (for $f(0) < 0$)

4. THE PATH IN E^J.

Starting with the $(n+1)$-complete simplex σ_0^n in the orthant E^J, the algorithm generates an $(n+1)$-almost complementary path, i.e., a sequence of distinct n-simplices $\{\sigma_0^n, \sigma_1^n, \sigma_2^n, \ldots\}$ where

(i) each σ_j^n is in E^J

(ii) each σ_j^n is n-complete, for $j > 0$

(iii) $\sigma_j^n \cap \sigma_{j+1}^n$ is an n-complete $(n-1)$-simplex, all j.

Either the path of simplices is unbounded or else it stops with

(a) either an $(n+1)$-complete simplex σ_T^n, or

(b) an n-complete $(n-1)$-simplex on some boundary $x_i = 0$ on E^J.

We now show that in the limit this path of simplices in E^J coincides with the homotopy path $H(x, t) = 0$ where $H(x, t) \equiv tf(x) + (1-t)x$, $t \in [0, 1]$, $x \in R^n$. Note that $H(x, 0) = 0$ if and only if $x = 0$, and if $t \neq 0$ then $H(x, t) = 0$ if and only if $f(x) = \lambda x$ for some $\lambda \leq 0$. Thus, for $x \neq 0$ the homotopy path is described by the known condition [1]

$$f(x) = \lambda x, \quad \text{some } \lambda \leq 0 .$$

<u>Theorem 1.</u> Let ε_k be a sequence of grid sizes such that $\varepsilon_k \to 0$ and let C denote the connected closed set which is the limiting curve of the above generated path of simplices in E^J. Then, for each $x \in C$, $f(x) = \lambda x$ for some $\lambda \leq 0$.

<u>Proof</u>. Suppose $x \in C$ and $x_i \neq 0$, $x_j \neq 0$. Then since each simplex is n-complete, and by continuity, it must be that $f_i(x)/x_i = f_j(x)/x_j \leq 0$. Thus there is a $\lambda \leq 0$ such that $f_k(x) = \lambda x_k$ for each $x_k \neq 0$. Suppose $x_i = 0$. Then in each neighborhood of x there is a point y in E^J with label i and hence at which $f_i(y)/y_i \leq 0$. In each neighborhood of x there is also a point $z \in E^J$ with label $j \neq i$. Thus $f_j(z)/z_j \leq f_i(z)/z_i$. Since z_i is near zero, and the possible values of z_j can be bounded away from zero, the last inequality means that $f_i(z)/z_i \geq 0$. Since y and z are both in E^J, it must be the case that y_i and z_i have the same sign. Hence $f_i(y)$ and $f_i(z)$ have opposite signs. It follows that $f_i(x) = 0$ and hence x_i satisfies $f_i(x) = \lambda x_i$ for any λ. #

Note: By reasoning analogous to that in the proof of Theorem 1 it can be seen that any point in a "small" (n + 1)-complete terminal simplex is an approximate zero of f.

5. THE ANALYSIS WHEN THE PATH TERMINATES IN THE BOUNDARY OF E^J

For simplicity of exposition, we shall make the nondegeneracy assumption that except for the origin the homotopy path $\{x: f(x) = \lambda x, \lambda \leq 0\}$ intersects any boundary $x_i = 0$ only at points x such that $x_j \neq 0$, $j \neq i$.

Now assume that the simplicial path terminates with an n-complete (n - 1)-simplex in the boundary of E^J. Under the nondegeneracy assumption there will be a $K > 0$ such that if the grid ε is sufficiently small then for every x in the simplex $|x_j| \geq K$, all $j \neq i$. A modified pivoting procedure will now be described for use on $x_i = 0$ and it will be shown that this path in the boundary of E^J remains arbitrarily close to the homotopy path.

Suppose σ^{n-1} is an n-complete simplex in that boundary of E^J for which $x_i = 0$. Let $\sigma^{n-1} = \{v^1, \ldots, v^n\}$, $\ell(v^j) = j$, $v_i^j = 0$, all j, and for each j we have, by nondegeneracy, $|v_m^j| \geq K$ all $m \neq i$ and some $K > 0$. The $(n-1)$-simplex σ^{n-1} is a facet of exactly two n-simplices. One of these, σ^n, is in E^J. The other, $\hat{\sigma}^n$, is in the orthant $E^{\hat{J}}$ on the "opposite side" of the boundary $x_i = 0$. Let $\ell_{\hat{J}}$ denote the labeling function on the orthant $E^{\hat{J}}$, where the rules for $\ell_{\hat{J}}$ are also given by (1). Points on the boundary $x_i = 0$ will have different labels depending on whether the labeling function ℓ_J or $\ell_{\hat{J}}$ is employed. This is because for such points $f_i(x)/x_i$ will be $+\infty$ on one side of $x_i = 0$ and $-\infty$ on the other side, depending on the sign of f_i at x.

<u>Lemma 2</u>. Suppose v is in the boundary $x_i = 0$, where $\ell_J(v) \neq n + 1$

(i) $\ell_J(v) = i \iff \ell_{\hat{J}}(v) \neq i$.

(ii) If $\ell_J(v) = i$ then $\ell_{\hat{J}}(v) =$ smallest $j \neq i$ such that $f_j(v)/v_j \leq f_k(v)/v_k$, all $k \neq i$.

<u>Proof</u>. Observe that

$$\ell_J(v) = i \iff \frac{f_i(v)}{v_i} = -\infty \text{ on } E^J \iff \frac{f_i(v)}{v_i} = +\infty \text{ on } E^{\hat{J}} \iff \ell_{\hat{J}}(v) \neq i$$

Hence, (i) holds, and (ii) follows immediately. #

Now let σ^{n-1} be an <u>arbitrary</u> $(n-1)$-simplex on the plane $x_i = 0$ and let σ^n and $\hat{\sigma}^n$ be the n-simplices in E^J and $E^{\hat{J}}$ respectively such that $\sigma^{n-1} = \sigma^n \cap \hat{\sigma}^n$.

Lemma 3.

(i) σ^{n-1} is an n-complete facet of σ^n (with respect to ℓ_J) iff $\ell_J(\sigma^{n-1}) = \{1, \ldots, n\}$.

(ii) σ^{n-1} is an n-complete facet of $\hat{\sigma}^n$ (with respect to $\ell_{\hat{J}}$) iff $\ell_J(\bar{v}) \neq i$ for a unique $\bar{v}$ of σ^{n-1} and $\ell_{\hat{J}}(\sigma^{n-1} - \{\bar{v}\}) = \{1, \ldots, n\} - \{i\}$.

Proof. Part (i) is definitional. For part (ii), σ^{n-1} is n-complete (with respect to $\ell_{\hat{J}}$ iff $\ell_{\hat{J}}(\sigma^{n-1}) = \{1, \ldots, n\}$. The conclusion follows from (i) in Lemma 2. #

Definition 1: Let σ be a simplex on $x_i = 0$. This simplex is termed jointly n-complete (jnc) if

(i) $\ell_J(v) = i$ for some vertex v of σ.

(ii) $\ell_J(v) \neq i$ for some vertex v of σ.

(iii) $\ell_J(\sigma) \cup \ell_{\hat{J}}(\sigma) = \{1, \ldots, n\}$.

Note that only an $(n - 2)$-simplex or an $(n - 1)$-simplex on $x_i = 0$ can be jnc. Also, by Lemma 2, ℓ_J can be replaced by $\ell_{\hat{J}}$ in (i) and (ii) of the above definition.

Lemma 4. Let σ^{n-1} be a simplex on $x_i = 0$ such that σ^{n-1} is n-complete with respect to either ℓ_J or $\ell_{\hat{J}}$. Then σ^{n-1} has a unique jnc facet σ^{n-2}.

Proof. Suppose $\sigma^{n-1} = \{v^1, \ldots, v^n\}$. Suppose $\ell_J(\sigma^{n-1}) = \{1, \ldots, n\}$ and let $\ell_J(v^j) = j$, all j. Suppose $\ell_{\hat{J}}(v^i) = k$. By Lemma 2, $k \neq i$. Then $\sigma^{n-1} - \{v^k\}$ satisfies (i), (ii), and (iii) of Definition 1 and hence is jnc. Let $\sigma = \sigma^{n-1} - \{v^j\}$, $j \neq i, k$ denote any other facet of σ^{n-1}. Then $j \notin \ell_J(\sigma) \cup \ell_{\hat{J}}(\sigma)$

and hence $\sigma^{n-1} - \{v^j\}$ is not jnc. Also, $\sigma^{n-1} - \{v^i\}$ is not jnc because it contains no vertex v with $\ell_J(v) = i$. Hence $\sigma^{n-1} - \{v^k\}$ is the <u>only</u> jnc facet of σ^{n-1}. The same argument applies if $\ell_{\hat{J}}(\sigma^{n-1}) = \{1, \ldots, n\}$. #

<u>Lemma 5</u>. Let σ^{n-1} be a jnc simplex on $x_i = 0$ but not n-complete with respect to either ℓ_J or $\ell_{\hat{J}}$. Then σ^{n-1} has exactly two jnc facets.

<u>Proof</u>. Let $\sigma^{n-1} = \{v^1, \ldots, v^n\}$. Since σ^{n-1} is jnc we can assume

A1. $\ell_J(v^i) = i$, $\ell_{\hat{J}}(v^i) = k \neq i$.

A2. for some $s \neq i$, $\ell_J(v^s) = s$ (s could equal k)

A3. $\{\ell_J(v^j), \ell_{\hat{J}}(v^j)\} = \{j, i\}$, for $j \neq i$.

Case (i). $\ell_J(v^k) = i$. Note that $v^k \in \sigma^{n-1} - \{v^i\}$ and $v^s \in \sigma^{n-1} - \{v^i\}$. Thus parts (i) and (ii) of Definition 1 are satisfied. From A3 plus the assumption that $\ell_J(v^k) = i$ it follows that $\sigma^{n-1} - \{v^i\}$ is jnc.

Case (ii). $\ell_J(v^k) = k \neq i$. There is a vertex $\tilde{v} \in \sigma^{n-1} - \{v^i, v^k\}$ such that $\ell_J(\tilde{v}) = i$. Otherwise $\ell_J(\sigma^{n-1}) = \{1, \ldots, n\}$ which is not possible by the hypothesis of the lemma. Since $\tilde{v} \in \sigma^{n-1} - \{v^i\}$ and $v^k \in \sigma^{n-1} - \{v^i\}$, it follows again that $\sigma^{n-1} - \{v^i\}$ is jnc.

Now consider the facet $\sigma^{n-1} - \{v^k\}$.

Case (i). $\ell_J(v^k) = i$. Note that $v^i \in \sigma^{n-1} - \{v^k\}$ and $v^s \in \sigma^{n-1} - \{v^k\}$. From A3 and the fact that $\ell_J(v^i) = i$ it follows that $\sigma^{n-1} - \{v^k\}$ is jnc.

Case (ii). $\ell_J(v^k) = k \neq i$. There is a vertex $\tilde{v} \in \sigma^{n-1} - \{v^i, v^k\}$ such that $\ell_J(\tilde{v}) \neq i$. Otherwise $\ell_{\hat{J}}(\sigma^{n-1}) = \{1, \ldots, n\}$, which is not possible by the hypothesis of the lemma. Since $\tilde{v} \varepsilon \sigma^{n-1} - \{v^k\}$ and $v^i \varepsilon \sigma^{n-1} - \{v^k\}$, it follows that $\sigma^{n-1} - \{v^k\}$ is jnc. Note that these are the only two jnc facets of σ^{n-1}. #

We now show that in the limit the simplicial path in the boundary of E^J coincides with the homotopy path.

<u>Theorem 2</u>. Let x be a point of convergence of a sequence of jnc simplices on $x_i = 0$. Then $f(x) = \lambda x$ for some $\lambda \leq 0$.

<u>Proof</u>. Let x be a point of convergence of a sequence of jnc simplices σ^{n-2} on $x_i = 0$. By (i) in Definition 1,

$$f_i(x)/x_i = -\infty \text{ on } E^J$$

By (ii) of Definition 1,

$$f_i(x)/x_i = +\infty \text{ on } E^J$$

Hence $f_i(x) = 0$. By (iii) of Definition 1,

$$f_j(x)/x_j \leq f_k(x)/x_k \text{ for every } k \neq i,\ j \neq i.$$

Hence $f(x) = \lambda x$ for some $\lambda \leq 0$. #

This completes our description of the modified pivoting in the boundary of E^J. Starting from σ_0^{n-1}, the first n-complete facet on $x_i = 0$, we generate a sequence of distinct $(n - 1)$-simplices on $x_i = 0$. The sequence is of the form

$$\{\sigma_0^{n-1}, \sigma_1^{n-1}, \sigma_2^{n-1}, \ldots\}$$

where $\sigma_j^{n-1} \cap \sigma_{j+1}^{n-1}$ are jnc for all j, until we obtain a terminal facet σ_τ^{n-1} which is again an n-complete facet of E^J or $E^{\hat{J}}$ (in which case we either pivot "back" into E^J or "through" the boundary into $E^{\hat{J}}$). Note that because of the assumption that a solution x^* is interior to some orthant, for a small grid size we cannot encounter a vertex labeled $n + 1$. Also, by the nondegeneracy assumption, we cannot intersect the boundary of the plane $x_i = 0$ (for a small grid size). Noncycling is assured as σ_0^{n-1} and σ_τ^{n-1} have only one jnc facet whereas $\sigma_1^{n-1}, \sigma_2^{n-1}, \ldots, \sigma_{\tau-1}^{n-1}$ each have exactly two such facets.

In summary, we have the following rules for pivoting in the boundary of E^J on which $x_i = 0$:

1. First $(n - 1)$-simplex, σ^{n-1}, is n-complete. Let $\ell_J(v^i) = i$, $\ell_{\hat{J}}(v^i) = k$. Remove the vertex z with $\ell_J(z) = k$ to obtain $\sigma^{n-2} = \sigma^{n-1} - \{z\}$, where σ^{n-2} is jnc.

2. Given any σ^{n-2} which is jnc, join a vertex $\bar{v}$ by the triangulation rule to obtain a new $\sigma^{n-1} = \sigma^{n-2} \cup \{\bar{v}\}$.

a. If σ^{n-1} is complete with respect to ℓ_J, return into E^J by adding an $(n + 1)^{\underline{st}}$ vertex in the interior of E^J and forming a σ^n in E^J.

b. If σ^{n-1} is complete with respect to $\ell_{\hat{J}}(\ell_J(\tilde{v}) \neq i$ for a unique $\tilde{v}$ and $\ell_{\hat{J}}(\sigma^{n-1} - \{\tilde{v}\}) = \{1, \ldots, n\} - \{i\})$ go through $x_i = 0$ into $E^{\hat{J}}$ by adding an $(n + 1)^{\underline{st}}$ vertex in interior of $E^{\hat{J}}$ and forming a $\hat{\sigma}^n$ in $E^{\hat{J}}$

c. Otherwise, either: case (i) $\ell_J(\bar{v}) = i$, $\ell_{\hat{J}}(\bar{v}) = p$ or case (ii) $\ell_J(\bar{v}) = p \neq i$. In either case, remove the vertex $z \neq \bar{v}$ such that

(a) $\ell_J(z) = p$, or (b) $\ell_J(z) = i$, $\ell_{\hat{J}}(z) = p$. Since σ^{n-2} is jnc either (a) or (b) will occur, but not both.

6. GEOMETRIC ILLUSTRATIONS

Prior to this work it has been quite difficult to illustrate the course of the homotopy path. This state of affairs is now considerably improved. As an illustration, Figure 2 shows the new scalar labeling for the functions

$$f_1(x, y) = x - y - 10$$

$$f_2(x, y) = 2x - y - 5 \quad .$$

Note that the path does not converge. If the function f were to be replaced by

$$g(x) = J_f^{-1}(0) \cdot f(x) = \begin{pmatrix} x - 5 \\ y - 5 \end{pmatrix}$$

then a representation similar to Figure 3 would be obtained and the homotopy path would converge. This idea of using the Jacobian transformation for scalar labelings was first presented by Fisher, Gould and Tolle [7] and was later studied by Wolsey [17] and Todd [16]. The present work underscores the importance of this transformation for the new scalar labeling presented herein and hence for the vector labeling as well.

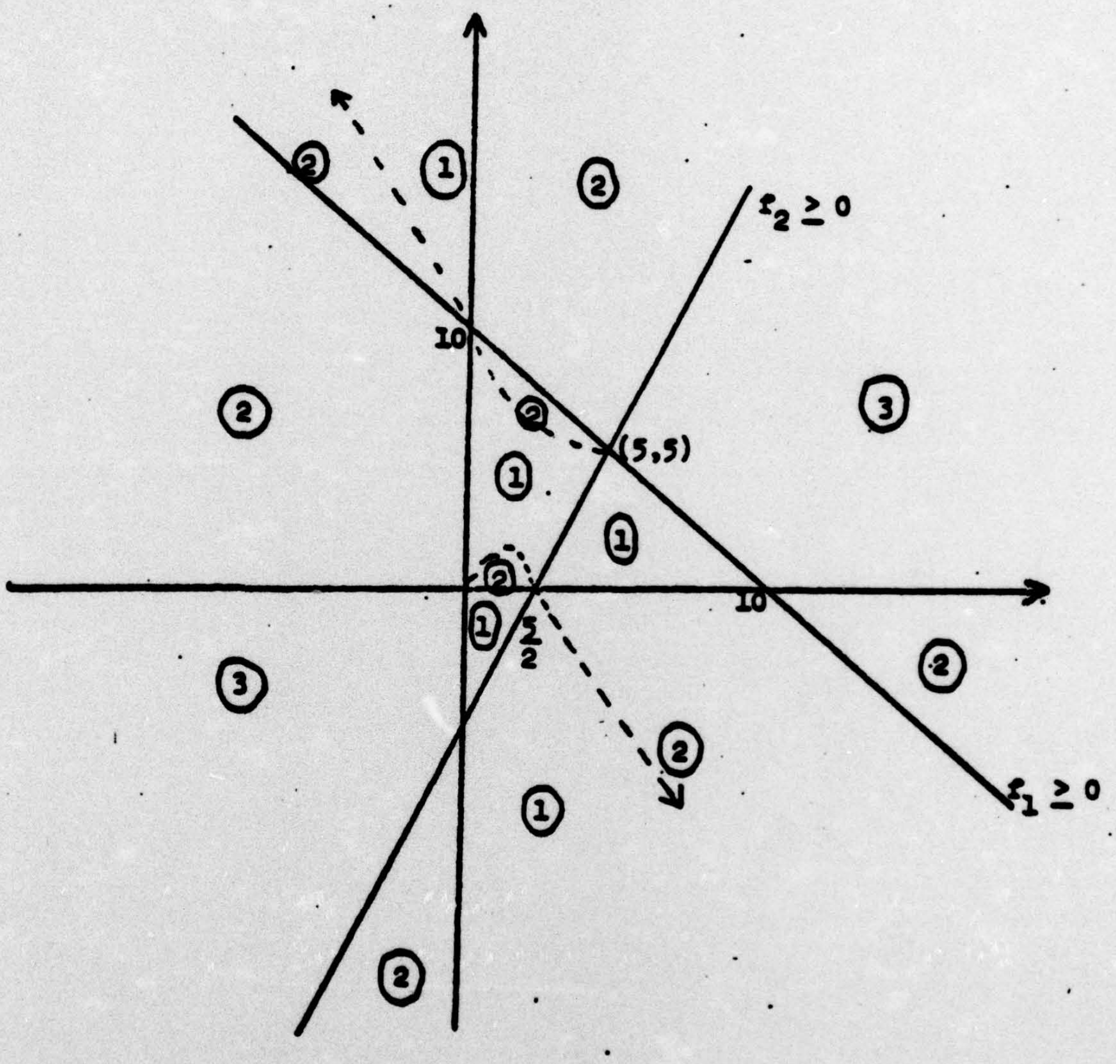

Figure 2: Scalar Labeling and H path for f

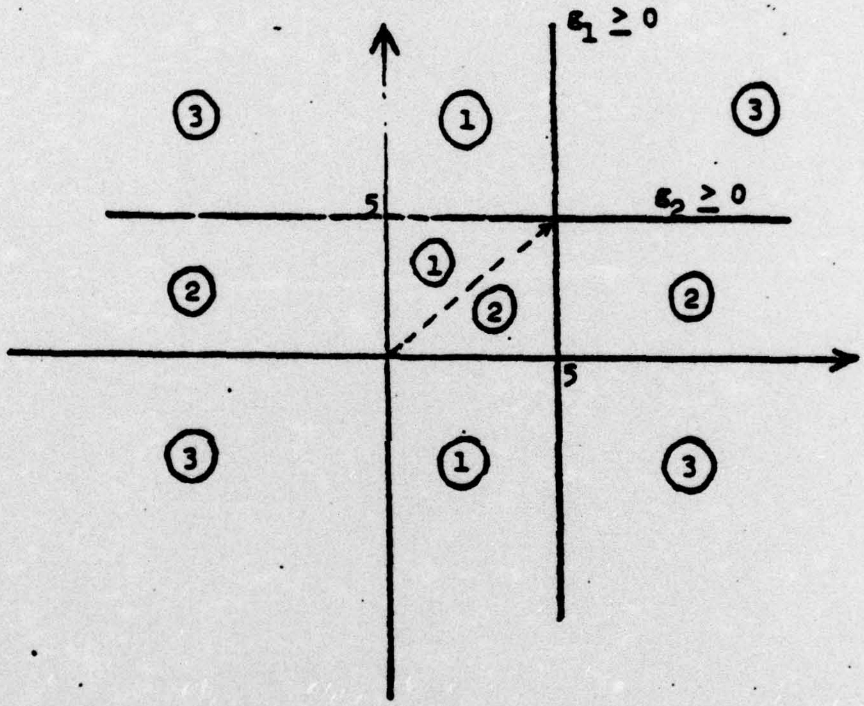

Figure 3: Scalar Labeling and H path for g

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