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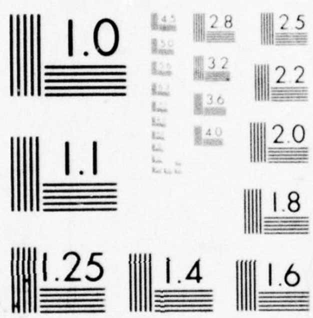

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ISOPARAMETRIC AND SUBPARAMETRIC VARIATIONAL PROBLEMS

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ABSTRACT

Let T be an operator defined on the subset U of a Banach space X determined by equality or inequality constraints for a finite number of functionals on X. We find necessary conditions that the solution of the minimization problem $\inf\{\|Tu\| : u \in U\}$ must satisfy. A case of particular interest is when T has values in L^∞. We analyze several examples in detail to show how the necessary conditions yield detailed information about the solution. We also introduce the notion of a t-point and show how the information becomes complete at such a point.

AMS (MOS) Subject Classifications: 49A25, 49B30, 46B10

Key Words: extremal problem, dual extremal problem, constrained minimization, calculus of variations, Frechét derivative of an operator, isoparametric and subparametric problems

Work Unit Number 6 (Spline Functions and Approximation Theory)

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ISOPARAMETRIC AND SUBPARAMETRIC VARIATIONAL PROBLEMS

Stephen D. Fisher

Introduction

Let T be an operator defined on a subset U of a real Banach space X with values in a real Banach space Y. We shall suppose that U is determined by a finite number of functional constraints:

$$U = \{x \in X : \ell_j(x) \le r_j, \ j = 0, \dots, N\}$$

or

$$U = \{x \in X : \ell_j(x) = r_j, \ j = 0, \dots, N\}$$

which we term the subparametric and isoparametric cases, respectively. In Section 1 we give a theorem on the existence of solutions of the variational problem

$$\inf\{ \|Tu\| : u \in U\} . \tag{0.1}$$

In Section 2 we find necessary conditions that any solution of (0.1) must satisfy. In Section 3 we show that the results of Section 2 can be applied to solve some non-linear equations of the form $Tu = 0$. In Section 4 we introduce the notion of a t-point and show how at a t-point we gain more complete information about the nature of solutions of (0.1). In Sections 5 and 6 we analyze in detail the problem of minimum curvature in L^∞, first for real-valued functions and then for complex analytic functions.

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Throughout the paper a case of particular interest is when $Y = L^{\infty}(\Omega, \mu)$ or some other dual space and indeed, this is a major point of difference between our analysis of (0.1) and that say of [5] since here the functional $f(x) = \|Tx\|$ to be minimized is not Fréchet differentiable.

§1. Existence

Let X and Y be real Banach spaces and let $U \subset X$. We distinguish two cases:

Case I: X and Y are real Banach spaces with separable dual spaces X^* and Y^*, respectively.

Case II: X and Y are each the dual of separable, non-reflexive Banach spaces W and Z, respectively.

The main novelty and interest in this paper is Case II but since the proofs work (with weaker hypotheses) for Case I as well, we include both.

The following is a simple sufficient condition for existence of solutions to (0.1).

__Theorem 1.1__. __Let__ T __be weakly (respectively, weak-*) sequentially continuous on__ U. __Suppose that__ U __is weakly (resp., weak-*) closed and that there is a bounded minimizing sequence for the problem__ (0.1). __Then there is a solution of__ (0.1).

__Proof__. The conclusion is immediate. If $\{u_n\}$ is a bounded minimizing sequence, then $\{u_n\}$ has a weak (resp. weak-*) cluster point $u \in U$. Since X^* is separable (resp. X is separable) we may assume that u_n converges weakly (resp. weak-*) to u. Hence, $Tu_n \to Tu$ either weakly or weak-* and thus $\alpha = \|Tu\|$ showing that u is a solution of (0.1).

__Corollary 1.2__. __Let__ $\ell_0, \ldots, \ell_N$ __be weakly (resp., weak-*) sequentially continuous functionals on__ X __and let__

(1.1) $$U = \{x \in X : \ell_j(x) \le r_j, \ j = 0, \ldots, N\}$$

or

(1.1)' $$U = \{x \in X : \ell_j(x) = r_j, \ j = 0, \ldots, N\} .$$

If T *is weakly (resp., weak-*) sequentially continuous and if there is a bounded minimizing sequence, then there is a solution of* (0.1).

Remark. We can clearly have U defined by some mixture of equality and inequality constraints; this case will not be mentioned specifically in the sequel. We may also drop the assumption of separability on W, Z by assuming that T is weak-* continuous.

§2. Necessary conditions for extremality

We assume in Theorem 2.1 that U is of the form (1.1) where $\ell_0, \ldots, \ell_N$ are Frechét differentiable functionals on X and that the Frechét derivatives of $\ell_0, \ldots, \ell_N$ are always linearly independent.

Theorem 2.1. Let u_0 be a solution of (0.1) and let J be those indices j for which $\ell_j(u_0) = r_j$. Let ℓ'_j be the Frechét derivative of ℓ_j at u_0 and set

$$V = \{v \in X : \ell'_j(v) \leq 0 \quad \text{for all} \quad j \in J\}$$

$$V_0 = \{v \in X : \ell'_j(v) = 0 \quad \text{for all} \quad j \in J\} .$$

Suppose there is a bounded linear operator L with range in Y whose domain contains V and for which

$$T(u_0 + \varepsilon v) = Tu_0 + \varepsilon Lv + o(\varepsilon), \quad v \in V, \ \varepsilon \to 0 . \tag{2.1}$$

Finally, suppose in Case I that LV_0 is closed in Y and in Case II that LV_0 is of finite codimension in Y and L is weak-* continuous. Let

$$V^* = \{\ell \in Y^* : \ell(Lv) \geq 0 \quad \text{for all} \quad v \in V\}$$

and

$${}^*V = \{m \in Z : m(Lv) \geq 0 \quad \text{for all} \quad v \in V\}$$

for Cases I and II, respectively. Then

$$\alpha = \inf\{\|Tu_0 + Lv\|_Y : v \in V\} \tag{2.2}$$

and

$$\alpha = \sup\{|\ell(Tu_0)| : \ell \in V^*, \ \|\ell\| \leq 1\} \tag{2.3}$$

or

$$\alpha = \sup\{|m(Tu_0)| : m \in {}^*V, \ \|m\| \leq 1\} \tag{2.4}$$

where (2.3) holds in Case I and (2.4) holds in Case II. In either Case I or II, the supremum is actually a maximum.

Proof. Let $x \in X$ satisfy $\ell_j'(x) < 0$ for $j \in J$ (or $\ell_j(x) = 0$ if ℓ_j is linear for some j). Then $u_0 + \varepsilon x \in U$ for all sufficiently small $\varepsilon > 0$ so that

$$\alpha \leq \|T(u_0 + \varepsilon x)\|_Y = \|Tu_0 + \varepsilon Lx\|_Y + o(\varepsilon) .$$

If $\|Tu_0 + Lx\| \leq \alpha - \delta$ for some $\delta > 0$, then

$$\alpha \leq \|Tu_0 + \varepsilon Lx\| + o(\varepsilon) \leq \varepsilon \|Tu_0 + Lx\| + (1-\varepsilon)\|Tx_0\| + o(\varepsilon)$$
$$\leq \alpha - \varepsilon\delta + o(\varepsilon) < \alpha$$

for ε small, a contradiction. Hence, $\|Tu_0 + Lx\|_Y \geq \alpha$ and so by continuity

$$\alpha = \inf\{\|Tu_0 + Lx\|_Y : x \in V\}$$

which is (2.2).

Next, note that (2.2) implies that the convex set $Tu_0 + LV$ is disjoint from the open ball in Y radius α. By the separation theorem [5; p. 133] there is a continuous linear functional ℓ of norm 1 with

(2.5) $$\sup\{\ell(y) : \|y\| < \alpha\} \leq \inf\{\ell(Tu_0 + Lv) : v \in V\}$$

which gives conclusion (2.3) in Case I. In Case II we note that LV_0 is closed ([4; p. 186]) and hence weak-* closed since L is weak-* continuous. Further, since $V_0 \subset V$, the convex set $(Tu_0 + LV)/LV_0$ in Y/LV_0 is disjoint from the open ball of radius α in the finite dimensional space Y/LV_0 and hence there is an element $m \in Z$ with $\|m\| = 1$, $m(Lv) = 0$ for all $v \in V_0$ and

$$\sup\{m(y) : \|[y]\|_{Y/LV_0} < \alpha\} \leq \inf\{m(Tu_0 + Lv) : v \in V\}$$

which gives conclusion (2.4) in Case II.

Corollary 2.2. *Let* u_0 *be a solution of* (0.1). *Then* $\alpha = \|Tu_0\|$ *is the distance in* Y *from* Tu_0 *to* LV_0; *if* $U = X$, *then* $\|Tu_0\|$ *is the distance in* Y *from* Tu_0 *to* $L(X)$.

Corollary 2.3. *Let Case* II *hold with* $Y = L^{\infty}(\Omega, \mu)$. *Then under the hypotheses of Theorem* 2.1 *there is a function* $h \in L^1$ *with* $\|h\|_1 = 1$ *and*

$$\text{(2.6)}\qquad \begin{array}{ll} \text{(i)} & 0 \leq \int_{\Omega} hLv, \quad \text{all } v \in V \\ \text{(ii)} & hTu_0 \geq 0 \quad \text{a.e. } \mu \\ \text{(iii)} & |Tu_0| = \alpha \quad \text{a.e. } \mu \text{ where } h \neq 0. \end{array}$$

Proof. Conclusions (ii) and (iii) follow from equality in Hölder's inequality.

The Isoparametric Case

The isoparametric case is almost the same as the sub-parametric one; we do only Case II. Existence is covered by Corollary 1.2; the necessary conditions for an extremal are given below.

Theorem 2.4. *Let* $\ell_0, \ldots, \ell_N$ *be Frechét differentiable functionals on* X *and let*

$$U = \{x \in X : \ell_j(x) = 0, \ j = 0, \ldots, N\}.$$

Let T *be an operator from* U *to* Y *and suppose* $u_0 \in U$ *satisfies*

$$\text{(2.7)} \qquad \alpha = \|Tu_0\| = \inf\{\|Tu\| : u \in U\}.$$

Let $\ell'_0, \ldots, \ell'_N$ *be the Frechét derivatives of* $\ell_0, \ldots, \ell_N$, *respectively, at* u_0 *and suppose that there is a bounded linear operator* L *with* range in Y whose domain contains a neighborhood of V_0:

$$V_0 = \{x \in X : \ell_j'(x) = 0,\ j = 0, \dots, N\}$$

and for which

$$T(u_0 + v) - Tu_0 - Lv = o(\|v\|),\ \|v\| \to 0 .$$

Suppose further that $\ell_0', \dots, \ell_N'$ are linearly independent, L is weak-* continuous, and LV_0 has finite codimension in Y. Then

$$\alpha = \inf\{\|Tu_0 + Lv\| : v \in V_0\} \tag{2.8}$$

and there is an element $z_0 \in Z$ with

(i) $\|z_0\| = 1$

(2.9) (ii) $\langle z_0, Lv\rangle = 0$, all $v \in V_0$

(iii) $\langle z_0, Tu_0\rangle = \alpha$.

Proof. We shall only show (2.8); the rest is as in Theorem 2.1. Let

$$X = X_1 \oplus V_0$$

where V_0 is the intersection of the null spaces of $\ell_0', \dots, \ell_N'$ and X_1 is spanned by the $N+1$ vectors $x_0, \dots, x_N$ with

$$\ell_j'(x_k) = \delta_{jk},\ j,k = 0, \dots, N .$$

Such vectors exist since $\ell_0', \dots, \ell_N'$ are linearly independent. Define $\Lambda : X \to \mathbb{R}^{N+1}$ by

$$\Lambda(x) = (\ell_0(x), \dots, \ell_N(x)) .$$

Then the Frechet derivative, Λ', of Λ at u_0 exists and is an isomorphism of X_1 onto $\mathbb{R}^{N+1}$. If $v \in V_0$, then the implicit function theorem (for Euclidean space) assures us that there are continuous functions

$g_0, \ldots, g_N$ defined in some neighborhood of $\varepsilon = 0$ such that $g_j(0) = 0$ and

$$\Lambda(u_0 + \varepsilon v + \sum_0^N g_j(\varepsilon)x_j) = 0 .$$

Let $y(\varepsilon) = \sum_0^N g_j(\varepsilon)x_j$; then

$$0 = \Lambda(u_0) + \varepsilon\Lambda'(v) + \Lambda'(y(\varepsilon)) + o(\varepsilon) + o(\|y(\varepsilon)\|) .$$

Since Λ' is invertible we find that $\|y(\varepsilon)\| = o(\varepsilon)$. Thus,

$$\alpha \le \|T(u_0 + \varepsilon v + y(\varepsilon))\| = \|Tu_0 + \varepsilon Lv\| + o(\varepsilon)$$

and the rest follows as in Theorem 2.1.

<u>Remark</u>. Consider an $\ell \in V^*$; then certainly

$$\ell(Lx) = 0 \quad \text{if} \quad \ell_j'(x) = 0 \quad \text{for} \quad j = 0, \ldots, N .$$

Hence,

$$\ell(Lx) = \sum_{j=0}^N c_j \ell_j'(x), \; x \in X, \; c_0, \ldots, c_N \in \mathbb{R} . \tag{2.10}$$

Similarly, for $m \in {}^*V$, we have

$$m(Lx) = \sum_{j=0}^N c_j \ell_j'(x), \; x \in X, \; c_0, \ldots, c_N \in \mathbb{R} . \tag{2.11}$$

Suppose that $Y = L^p(\Omega, \mu)$, $1 \le p \le \infty$; let p' be the conjugate exponent of p. Then, assuming that the hypotheses of Theorem 2.1 or 2.4 hold there is a function $h \in L^{p'}(\Omega, \mu)$ with

$$\int_\Omega h\, Lx\, d\mu = \sum_{j=1}^N c_j \ell_j'(x) .$$

Now suppose that $\ell'_0, \ldots, \ell'_n$ span the dual of kernel L; then $c_0 = \cdots = c_n = 0$. Finally, suppose that for $j = n+1, \ldots, N$, ℓ'_j has the form

$$\ell'_j(x) = \int_\Omega Q_{j-n} Lx \, d\mu \quad \text{when} \quad x \in \bigcap_{j=1}^{n} \ker \ell'_j$$

where $Q_1, \ldots, Q_{N-n}$ are some $L^{p'}$ functions. Since L is 1 - 1 on $\bigcap_{j=1}^{n} \ker \ell'_j$, we have $h \in \text{span}\{Q_1, \ldots, Q_{N-n}\}$ modulo functions orthogonal to the range of L. If L is known to be onto then h lies in the span of $Q_1, \ldots, Q_{N-n}$. In particular if $Q_1, \ldots, Q_{N-n}$ is a subset of a (weak) Chebyshev system of size $r \geq N - n$, then h has at most $r - 1$ zeros (sign changes) and, in particular, when $p = \infty$, $Tu_0 = \pm\alpha$ with $r - 1$ or fewer sign changes (strong sign changes.)

<u>Example</u>. Let g be a bounded smooth monotone decreasing function on $\mathbb{R}$ and let

$$Tu = u'' + g(u)$$

for $u \in W^{2,\infty}$. Let $\varphi_0, \ldots, \varphi_N$ be a Chebyshev system on $[0,1]$ and set

$$U = \{u \in W_0^{2,\infty} : \int_0^1 u\varphi_j = y_j, \ j = 0, \ldots, N\}$$

where $y_0, \ldots, y_N$ are prescribed numbers. Let u_0 be any solution of

$$\alpha = \inf\{\|Tu\|_\infty : u \in U\}$$

and let h be the L^1 function (assured by Theorem 2.4) with $hTu_0 \geq 0$. Now L is given by

$$Lv = v'' + Av$$

where $A(t) = g'(u_0(t))$ is negative so that L is $1-1$ and onto. Let $G(s,t)$ be the Green's function for L with homogeneous initial conditions. Then

$$y_j = \ell_j(u) = \int_0^1 u\varphi_j = \int_0^1 \{\int_0^1 G(s,t)(Lu)(s)ds\}\varphi_j(t)dt = \int_0^1 (Lu)(s)\psi_j(s)ds$$

where $\psi_j(s) = \int_0^1 G(s,t)\varphi_j(t)dt$. However, $\psi_0, \ldots, \psi_N$ form a Chebyshev system since $G(s,t)$ is totally positive and hence h has N or fewer zeros on $[0,1]$. Thus, for any u_0 with $\|Tu_0\| = \alpha$ we find that $u_0'' + g(u_0) = \pm\alpha$ with N or fewer sign changes.

§ 3. Solutions of some non-linear equations

There are certain happy cases when it is immediate that the operator L in Theorem 2.1 maps V_0 onto Y and in these cases we can directly conclude from Corollary 2.2 that the equation $Tu = 0$ has a solution, provided the minimization problem (0.1) has a solution. We illustrate this below with several examples.

Example 1. Let $F(t, x)$ be a C^1 function on $[0,1] \times \mathbb{R}^n$ and consider the initial value problem

$$(3.1) \qquad \begin{cases} y^{(n)}(t) + F(t, y(t), \ldots, y^{(n-1)}(t)) = 0, & 0 \leq t \leq 1 \\ y^{(\nu)}(0) = a_\nu, \ \nu = 0, \ldots, n-1 . & \end{cases}$$

The Frechét derivative of $Ty = y^{(n)} + F(\cdot, y, \ldots, y^{(n-1)})$ at u_0 is

$$(Lv)(t) = v^{(n)}(t) + \sum_1^n \frac{\partial F}{\partial x_j}(t, u_0(t), \ldots, u_0^{(n-1)}(t))v^{(j-1)}(t)$$

and hence L maps V_0, which in this case is the space of functions satisfying homogeneous initial conditions, onto L^p. It is easily verified that if either

(a) $|F(t,x)| \leq (1-\delta)|x| + M$, some $\delta, M > 0$

or

(b) $|F(t,x)| \leq C|x_n||\log x_n| + M$, some $C, M > 0$

then the minimization problem $\inf\{\|Tu\|_\infty : u \in H^{n,\infty}(0,1)\}$ has a solution and hence so does (3.1).

Example 2. Let $L_0 > 0$ be given and let g be a function in $L^p(0, L_0)$. We wish to show that there is a smooth curve $t \mapsto (x(t), y(t))$ of length L_0

or less whose curvature is g. That is, if $X = \{(x, y) : x, y \in W^{2,p}(0, L_0)\}$, then we wish to show that there is an element of X with

$$(3.2)\qquad \begin{cases} \text{(a)} & \int_0^{L_0} (\dot{x}^2 + \dot{y}^2)^{1/2} \le L_0 \\ \text{(b)} & (\dot{x}\ddot{y} - \ddot{x}\dot{y})(\dot{x}^2 + \dot{y}^2)^{-3/2} = g . \end{cases}$$

Let

$$T(x, y) = (\dot{x}\ddot{y} - \ddot{x}\dot{y})(\dot{x}^2 + \dot{y}^2)^{-3/2} - g, \quad (x, y) \in X$$

and

$$(3.3)\qquad \alpha = \inf\{ \|T(x, y)\|_p : (x, y) \in X\} .$$

If $\dot{x}^2 + \dot{y}^2 \equiv 1$, then we have $(T(x, y) + g)^2 = \ddot{x}^2 + \ddot{y}^2$ so that both $\ddot{x}$ and $\ddot{y}$ lie in a ball in L^p of radius no more than the norm of $T(x, y)$ plus the norm of g. Hence, a weak compactness argument shows there is a solution (x, y) of (3.3) which satisfies $\dot{x}^2 + \dot{y}^2 \equiv 1$ on $[0, L]$, $L \le L_0$. If $L < L_0$, extend x, y linearly on $[L, L_0]$. According to Corollary 2.2, α is the distance of $T(x, y)$ to LV_0 where

$$L(u, v) = -2\dot{u}\ddot{y} - \ddot{u}\dot{y} + 2\dot{v}\ddot{x} + \ddot{v}\dot{x}$$

is the Frechét derivative of T at (x, y) and V_0 is the null space of the Frechét derivative of $\ell_0(u, v)$ at (x, y):

$$V_0 = \{(u, v) : \int_0^{L_0} \dot{x}\dot{u} + \dot{y}\dot{v} = 0\} .$$

In deriving the formula for L we have used the fact that $k\dot{x} = \ddot{y}$ and $k\dot{y} = -\ddot{x}$ where k is the curvature of (x, y). For $w \in W_0^{2,p}$, define u, v by

$$\dot{u} = -\dot{y}\ddot{w}, \quad u(0) = 0$$

$$\dot{v} = \dot{x}\ddot{w}, \quad v(0) = 0 .$$

Then $(u, v) \in V_0$ and $L(u, v) = \ddot{w}$ so that LV_0 contains all L^p functions and hence $\alpha = 0$.

Example 3. Let g be a continuous monotone increasing function on $\mathbb{R}$. Consider the boundary-value problem

$$(3.4)\quad \begin{cases} \text{(i)} & u'(t) + g(u(t)) = f(t) \qquad 0 \le t \le 1, \ f \in L^1(0,1) \\ \text{(ii)} & u(0) = u(1) = a . \end{cases}$$

We shall show: (3.4) has a solution if and only if the number $b = \int_0^1 f$ lies in the range of g.

The necessity of this condition follows by integrating (3.4) (i) over $[0,1]$. If b lies on the boundary of the range of g then either $g(x) = b$ for all $x \ge x_0$ or $g(x) = b$ for all $x \le x_1$, say the former. Let $u_0'(t) = f(t) - b$ with $u_0(0) = u_0(1) = a$ and put $u_1(t) = u_0(t) + c$ where c is some large constant. Then $g(u_1(t)) = b$ for all t and $u_1' = u_0' = f - b = f - g(u_1)$. Hence, we may assume that b lies in the interior of the range of g. By subtracting b from both sides and by a translation we may assume that $b = 0$, $g(0) = 0$, and 0 is in the interior of the range of g.

Suppose $\{u_j\}$ is a sequence of functions in $W^{1,1}(0,1)$ and the functions $w_j = u_j' + g(u_j) - f$ lie in ball of L^∞ of radius C. Then we have

$$\frac{1}{2}u_j^2(x) \le \frac{1}{2}u_j^2(x) + \int_0^x g(u_j)u_j = \int_0^x (f + w_j)u_j \le (\|f\|_1 + C)\|u_j\|_\infty .$$

Hence, $\|u_j\|_\infty \le C'$ for all j. Thus, $\|g(u_j)\|_\infty \le C''$ for all j and we have

$$|u_j'(x)| \le |f(x)| + C + C'' \text{ for all } x \text{ and all } j .$$

Hence, $\{u_j\}$ is uniformly bounded and equicontinuous in $C[0,1]$ so that we may assume that $u_j \to u$ uniformly. We may assume that w_j converges weak-* in L^∞ (and hence weakly in L^1) to a function w. Thus, we get $u_j' \to u'$ weakly in L^1 and we find

$$u' + g(u) = f + w$$

$$u(0) = u(1) = a .$$

This shows that the following holds: if

$$\alpha = \inf\{\|u' + g(u) - f\|_\infty : u(0) = u(1) = a\}$$

then there is a $\tilde{u} \in W^{1,1}$ with $\alpha = \|\tilde{u}' + g(\tilde{u}) - f\|_\infty$. (That α is finite is easy: let u_0 solve $u_0' = f$, $u_0(0) = u_0(1) = a$; then $u_0' + g(u_0) - f = g(u_0) \in L^\infty$.)

If $V_0 = \{v \in W^{1,1} : v(0) = v(1) = 0\}$, then Corollary 2.2 asserts that α is the distance of $\tilde{u}' + g(\tilde{u}) - f$ to LV_0 where L is the Frechét derivative of $Tu = u' + g(u) - f$ given by

$$Lv = v' + Av$$

and

$$A(t) = g'(\tilde{u}(t)) \ge 0 .$$

However, it is obvious that L maps V_0 both 1 - 1 and onto the space of functions with 0 mean-value. Thus, $\alpha = 0$ and we are done.

§4. Tight constraints

Suppose that one of the functionals, say ℓ_0, has the property that α is not a constant function of r_0. Let us fix $r_1, \ldots, r_N$ and allow $r = r_0$ to vary and consider the function

$$(4.1) \qquad \alpha(r) = \inf\{\|Tu\| : u \in U(r)\}$$

where

$$(4.2) \qquad U(r) = \{x \in X : \ell_0(x) \leq r,\ \ell_j(x) \leq r_j,\ j = 1, \ldots, N\}$$

in the subparametric case or

$$(4.3) \qquad \beta(r) = \inf\{\|Tu\| : u \in U(r)\}$$

where

$$(4.4) \qquad U(r) = \{x \in X : \ell_0(x) = r,\ \ell_j(x) = r_j,\ j = 1, \ldots, N\}$$

in the isoparametric case. We will be interested in values of r at which α (or β) has a definite change. We define a point r_0 in the interior of the domain of a continuous function f to be a <u>t-point</u> if there is a $\Delta \neq 0$ and a sequence $\varepsilon_n \to 0$ with

$$(4.5) \qquad \begin{cases} \Delta\varepsilon_n > 0, & n = 1, 2, \ldots \\ \varepsilon_n^{-1}(f(r_0 + \varepsilon_n) - f(r_0)) \to \Delta . \end{cases}$$

(The terminology is derived from the fact that at a t-point for α the constant $\ell_0(x) \leq r_0$ becomes tight: $\ell_0(x) = r_0$.) We have the following elementary proposition which shows that any continuous non-constant function f has (many) t-points.

<u>Proposition 4.1. Let</u> f <u>be a non-constant continuous function on</u> $[a, b]$. <u>Then</u> f <u>has a t-point in</u> (a, b).

Proof. We may assume $f(b) \neq f(a)$. Suppose that for each $x \in [a, b)$ we have

$$\limsup_{h \to 0^+} h^{-1}(f(x + h) - f(x)) \leq 0 .$$

Let $\varepsilon > 0$ be given and set $F(x) = \varepsilon(x - a) - f(x) + f(a)$. If $t > x$ but is close to x, then $f(t) - f(x) \leq (\varepsilon/2)(t - x)$ so that

$$F(t) - F(x) = \varepsilon(t - x) - (f(t) - f(x)) > 0 .$$

Hence, F is increasing so that $F(b) > F(a) = 0$. This implies $f(b) \leq f(a) + \varepsilon(b - a)$. Let $\varepsilon \to 0$ and conclude $f(b) \leq f(a)$. Hence, if $f(b) > f(a)$, then there must be a point x_0 and a $\Delta > 0$ with

$$\limsup_{h \to 0^+} h^{-1}(f(x_0 + h) - f(x_0)) = \Delta .$$

If $f(b) < f(a)$, apply the reasoning to $g(x) = f(a + b - x)$ which satisfies $g(b) > g(a)$.

Theorem 4.2. Let r_0 be a t-point of α and let $x_0 \in U(r_0)$ satisfy $\|Tx_0\| = r_0$. Let L be the Frechét derivative of T at x_0, ℓ_0' the Frechét derivative of ℓ_0 at x_0; in Case II assume that ℓ_0' is weak-* continuous. Set

$$U_0 = \{x \in X : \ell_j'(x) = 0 \quad \text{for} \quad j = 1, \ldots, N\} .$$

Then there are elements $y^* \in Y^*$ and $z \in Z$ of norm 1 with

$$\text{(4.6)} \qquad \begin{array}{ll} \text{(i)} & y^*(Tx_0) = \|Tx_0\| \quad \text{and} \quad y^*(Lx) = c\ell_0'(x), \quad x \in U_0 \\ \text{(ii)} & z(Tx_0) = \|Tx_0\| \quad \text{and} \quad z(Lx) = c\ell_0'(x), \quad x \in U_0 \end{array}$$

for Cases I and II, respectively, where $c\Delta > 0$ and $|c| \geq |\Delta| \|Tx_0\|$.

Proof. We shall do the case when $\Delta > 0$; the case when $\Delta < 0$ is entirely similar. We first show that in U_0, the kernel of L lies in the kernel of ℓ_0'. If not, then there is an $x \in U_0$ with $Lx = 0$ but $\ell_0'(x) = 2$. Hence, $x_0 + \varepsilon x \in U(r_0 + \varepsilon)$ for ε small and so, in the case $\Delta > 0$,

$$\begin{aligned}\varepsilon \Delta \leq \alpha(r_0 + \varepsilon) - \alpha(r_0) &\leq \|T(x_0 + \varepsilon x)\| - \|Tx_0\| \\ &= \|Tx_0 + \varepsilon Lx\| + o(\varepsilon) - \|Tx_0\| \\ &= o(\varepsilon) ,\end{aligned}$$

a contradiction.

If $\ell_0'(x) > \delta$, then $x_0 + \varepsilon x \in U(r_0 + \varepsilon\delta)$ for ε small enough and hence we find for a sequence of $\varepsilon \to 0$

$$\begin{aligned}\delta\Delta &\leq \varepsilon^{-1}(\alpha(r_0 + \varepsilon\delta) - \alpha(r_0)) \\ &\leq \varepsilon^{-1}(\|Tx_0 + \varepsilon Lx\| - \|Tx_0\|) + o(1)\end{aligned}$$

Thus, $$\delta\Delta \leq \lim_{\varepsilon \to 0^+} \varepsilon^{-1}(\|Tx_0 + \varepsilon Lx\| - \|Tx_0\|) .$$

Write $y_0 = Tx_0$; we make use of the formula

$$\|y_0\| \lim_{\varepsilon \to 0^+} \varepsilon^{-1}(\|y_0 + \varepsilon Lx\| - \|y_0\|) = \max_{y^* \in C} y^*(Lx) , \tag{4.7}$$

where

$$C = \{y^* \in Y^* : \|y^*\| = 1, \ y^*(y_0) = \|y_0\|\} ;$$

see [1; Theorem V.9.5, p. 447]. Let $C_1 = \{c\ell_0' : c \geq \Delta \|y_0\|\}$ and $C_2 = \{L^*y^* : y^* \in C\}$. Then C_1 and C_2 are closed convex sets and C_2 is weak-* compact. If C_1 and C_2 are disjoint, then there is an $x_0 \in U_0$ which strictly separates C_1 and C_2:

$$c\ell_0'(x) \leq r < r + \delta \leq y^*(Lx_0), \ c \geq \Delta \|y_0\| \text{ and } y^* \in C.$$

Hence, $\ell_0'(x_0) \leq 0$. If $\ell_0'(x_0) = -\eta < 0$, then

$$-r \leq \eta \Delta \|y_0\| \leq \max_{y^* \in C} y^*(L(-x_0)) \leq -r - \delta ,$$

a contraction. If $\ell_0'(x_0) = 0$, then $r \geq 0$ and

$$y^*(L(-x_0)) \leq -r - \delta < 0, \quad \text{all } y^* \in C .$$

However, a continuity argument shows that

$$\max_{y^* \in C} y^*(L(-x_0)) \geq 0 ,$$

again a contradiction. Hence, there is a $c \geq \Delta$ and a $y^* \in C$ with

$$c\ell_0'(x) = y^*(Lx), \; x \in U_0 .$$

This is the proof for Case I; for Case II the proof is done in Y/LV_0 and hence the functionals come from elements of Z.

I would like to thank C. Micchelli for pointing out (4.7) to me and thus giving a proof of Theorem 4.2 which is appreciably shorter than my original proof.

An almost identical proof yields the following isoparametric version which we state only for Case II.

__Theorem 4.3__. __Let__ r_0 __be a t-point for the isoparametric problem__ (4.3) __and let__ $x_0 \in U(r_0)$ __satisfy__ $\|Tx_0\| = \beta(r_0)$. __Let__ L __and__ ℓ_0' __be the Frechét derivative of__ T __and__ ℓ_0, __respectively, at__ x_0 __and suppose__ ℓ_0' __is weak-* continuous__. __Then there is an element__ $z \in Z$ __of norm__ 1 __and a number__ c, $c\Delta > 0$ __and__ $|c| \geq |\Delta| \, \|y_0\|$ __with__

(4.8) $$z(Tx_0) = \|Tx_0\| \quad \text{and} \quad c\ell_0'(x) = z(Lx), \; x \in U_0 .$$

__Remark__. In the subparametric case $\alpha(r)$ is a monotone decreasing function of r and we always have $\beta(r) \geq \alpha(r)$. If we know that r_0 is a t-point for α, then clearly $\ell_0(u_0) = r_0$. If we assume that the remaining constraints are of the form $\ell_j(x) = r_j$, $j = 1, \ldots, N$, then we must have $\beta(r_0) = \alpha(r_0)$ and hence r_0 is a t-point for isoparametric problem as well as for the subparametric problem.

The following Proposition gives a simple condition which guarantees that α is continuous; it applies equally as well to β but we do not give the details for this case. We consider only the setting of Case II.

Proposition 4.4. (1) α is upper semi-continuous. (2) If T is weak-* continuous and if there is a constant C with $\|u_r\| \leq C$ for all r, $|r - r_0| < \delta$, where $\|Tu_r\| = \alpha(r)$, then α is continuous at r_0.

Proof. (1) Let $\alpha(r_0) = \|Tu_0\|$; let $v \in U_0$ satisfy $\ell_0'(v) = 2$. Then for ε small, either positive or negative, we have

$$\begin{aligned}\alpha(r_0 + \varepsilon) &\leq \|T(u_0 + \varepsilon v)\| \\ &\leq \|Tu_0 + \varepsilon Lv\| + o(\varepsilon) \\ &\leq \alpha(r_0) + \varepsilon\|Lv\| + o(\varepsilon) \\ &= \alpha(r_0) + 0(\varepsilon).\end{aligned}$$

(2) Since $\|u_r\| \leq C$, there is a subsequence $u_{r_n} \xrightarrow{*} u$. Clearly $u \in U(r)$ and thus

$$\alpha(r) \leq \|Tu\| \leq \underline{\lim}\,\|Tu_{r_n}\| = \underline{\lim}\,\alpha(r_n).$$

Example. Let $0 \leq x_1 < \cdots < x_N \leq 1$ be given points, let $y_1, \ldots, y_N$ be real numbers and let L_0 be a positive number. Consider the problem of finding a function f satisfying (1) $f(x_j) = y_j$ for $j = 1, \ldots, N$ and (2) the length of the curve $(x, f(x))$ is no more than (or is exactly) L_0 and (3) f'' is smallest in L^∞. Precisely, for the first case let

$$U = \{f \in W^{2,\infty}(0,1) : f(x_j) = y_j,\ j = 1, \ldots, N \text{ and } \int_0^1 (1+(f')^2)^{1/2} \le L_0\}$$

and

$$\alpha(L_0) = \inf\{\|f''\|_\infty : f \in U\}.$$

For the second case replace the inequality in the definition of U by an equality. Suppose L_0 is a t-point of α. Then the function h must satisfy

$$\int_0^1 hg'' = \lambda \int_0^1 f'g'(1 + (f')^2)^{-1/2}$$

for all $g \in W^{2,\infty}$ with $g(x_j) = 0$, $j = 1, \ldots, N$. Integration by parts shows that h is continuous on $[0,1]$, h is in $W^{2,\infty}$ on each segment (x_j, x_{j+1}) and h satisfies

$$h' = b_j - \lambda f'(1 + (f')^2)^{-1/2} \quad \text{on} \quad (x_j, x_{j+1}). \tag{4.13}$$

If h vanishes on some segment (a, b) in (x_j, x_{j+1}), then so does h' and so

$$b_j^2 = (f')^2(\lambda^2 - b_j^2).$$

If $b_j \neq 0$, then f' is constant on (a, b). If $b_j = 0$, then since $\lambda \neq 0$, we must have $f' = 0$ on (a, b). If h is positive or negative on some segment, then we already know that $f'' = \alpha$ or $-\alpha$ on that segment. Since

$$h'' = -\lambda f''(1 + (f')^2)^{-3/2}$$

we see that h'' is strictly bounded below (above) zero on any segment on which h is positive (negative). Hence, if h is say positive on $(a, b) \subset (x_j, x_{j+1})$ with $h(b) = 0$, then h is negative for $x \in (b, b + \varepsilon)$.

However, more is true. Since $f'' = \alpha$ when $h > 0$ and $f'' = -\alpha$ when $h < 0$ we can use (4.13) to conclude that h' is even about the point b and hence h is odd about b. Thus, if there are points $\xi_1 < \xi_2 < \xi_3$ in (x_j, x_{j+1}) with $h(\xi_k) = 0$, $h > 0$ in (ξ_1, ξ_2) and $h < 0$ in (ξ_2, ξ_3), then $\xi_3 - \xi_2 = \xi_2 - \xi_1$. Thus h can have only a finite number of sign changes in (x_j, x_{j+1}) and there is no segment in (x_j, x_{j+1}) on which h vanishes identically. Thus, the graph of f is composed of finitely many sections of parabolas and, possibly finitely many straight line segments. If a straight line occurs, then it must join successive points (x_j, y_j) and (x_{j+1}, y_{j+1}).

For this problem it is immediate that the hypotheses of Proposition 4.4 (a) are fulfilled and hence α is continuous. To show that α has t-points it suffices to show that α is not constant. Let L_0 be the length of the polygonal curve joining in succession the points $\{(x_j, y_j) : j = 1, \ldots, N\}$. Then for each $\delta > 0$ there is a function $g \in W^{2,\infty}(0,1)$, $g(x_j) = y_j$, the length of the curve $(x, g(x))$ is no more (or is exactly) $L_0 + \delta$ and $\alpha(L_0 + \delta) = \|g''\|$. Clearly as $\delta \downarrow 0$ we must have $\alpha(L_0 + \delta) \to \infty$. Consequently, α is not constant.

§5. Curvature problems in $\mathbb{R}^2$

Let us consider the following problem: Let $\mathcal{P} = \{p_1, \ldots, p_N\}$ be a set of distinct points in $\mathbb{R}^2$. We wish to pass a smooth curve $t \mapsto (x(t), y(t))$ through the set $\mathcal{P}$ whose curvature, measured in the L^∞ norm, is as small as possible. We must impose the constraint that the lengths of the competing curves are uniformly bounded for otherwise the infimum of the curvatures is zero (piece together arcs of very large circles) and, except in the trivial case when all the points lie on a single straight line, there is no smooth curve with zero curvature passing through all the points. Once, however, we do impose this length constraint then there will be a curve with minimal curvature. This is most easily seen by parametrizing a competing curve by arc-length. In this case the curvature formula

$$k(t) = (\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t))(\dot{x}^2(t) + \dot{y}^2(t))^{-3/2} \tag{5.1}$$

reduces to

$$k^2(t) = \ddot{x}^2(t) + \ddot{y}^2(t) \tag{5.2}$$

since $\dot{x}^2(t) + \dot{y}^2(t) \equiv 1$, $0 \le t \le L \le L_0$. Hence, a uniform bound on k produces a uniform bound on both $\|\ddot{x}\|_\infty$ and $\|\ddot{y}\|_\infty$ and since we have $\ddot{x}, \ddot{y} \in L^\infty(0, L_0)$ for a fixed L_0, we may apply a simple compactness argument to obtain a curve $(x(t), y(t))$, $0 \le t \le L$, which passes through all the points of $\mathcal{P}$ and for which $\|k\|_\infty$ is minimal. We shall prove the following result.

<u>Theorem 5.1</u>. <u>Let distinct points</u> $p_1, \ldots, p_N$ in $\mathbb{R}^2$ <u>be given and let</u> L <u>be a number large enough that there is some smooth curve of length</u> L

passing through all the points $p_1, \ldots, p_N$. _Then there is a_ C^1 _curve containing the points_ $p_1, \cdots, p_N$ _with length no more than_ L _and which consists of a finite number of arcs of circles of some fixed radius or straight line segments. This curve minimizes the sup norm of the curvature among all smooth curves of length not exceeding_ L _which pass through the given points_ $p_1, \ldots, p_N$.

Proof. The existence of a curve minimizing $\|k_f\|_\infty$ has been given above. We shall now determine the properties of such a curve. Fix points $0 = t_1 < t_2 < \cdots < t_N = L$ for which $(x(t_j), y(t_j)) = p_j$, $j = 1, \ldots, N$. Let $\ell_j(u, v) = (u(t_j), v(t_j))$ for $j = 1, \ldots, N$ and $u, v \in H^{2,\infty}(0, L)$ and let

$$\ell_0(u, v) = \int_0^L \sqrt{\dot{u}^2 + \dot{v}^2} \; ;$$

that is, ℓ_0 assigns to the curve (u, v) its length. With this notation we are now in the context of Theorem 2.1, Case II with $X = H^{2,\infty}(0, L) \oplus H^{2,\infty}(0, L)$, $Y = L^\infty$, and U consisting of all pairs (u, v) in X for which $\ell_0(u, v) \leq L_0$, $\ell_j(u, v) = p_j$ for $j = 1, \ldots, N$, and $T(u, v) =$ curvature of the curve $t \mapsto (u(t), v(t))$. Note that the solution (x, y) of the problem

$$\alpha = \inf\{ \|T(u, v)\|_\infty : (u, v) \in U\}$$

arrived at above has the property that $\dot{x}^2 + \dot{y}^2 \equiv 1$ so that if (u, v) is any element of X, then $T((x, y) + \varepsilon(u, v))$ is well-defined for small enough ε. Hence, the operator L referred to in Theorem 2.1 is just the Frechét derivative of T at (x, y):

(5.3) $\qquad L(u,v) = -\dot{v}\ddot{x} + \ddot{v}\dot{x} + \dot{u}\ddot{y} - \ddot{u}\dot{y} - 3k(\dot{x}\dot{u} + \dot{y}\dot{v}), \quad u, v \in H^{2,\infty}$

(5.3)' $\qquad = -2\dot{u}\ddot{y} - \ddot{u}\dot{y} + 2\dot{v}\ddot{x} + \ddot{v}\dot{x}$

where $k = T(x, y)$ and to derive (5.3)' we've made use of the fact that $k\dot{x} = \ddot{y}$ and $k\dot{y} = -\ddot{x}$. The next thing to note is that L maps X onto L^{∞}. To see that the equation $L(u,v) = g$, $g \in L^{\infty}$, has a solution we make the substitutions $\dot{u} = -\dot{y}w$ and $\dot{v} = \dot{x}w$ where $w \in H^{2,\infty}$ is to be found. The equation $L(u,v) = g$ then reduces to the equation $\ddot{w} = g$ which surely has a solution. Hence, we have shown the hypotheses of Theorem 2.1, Case II, are satisfied. Thus, according to Theorem 2.1 there is a function $h \in L^1$ with norm 1 and

(5.4) $$0 \leq \int h\, L(u,v) \quad \text{whenever} \quad 0 \geq \ell_0'(u,v)$$

and $u(t_j) = v(t_j) = 0$ for $j = 1, \ldots, N$. Further, h and k have the same sign and $|k| = \alpha$ a.e. where $h \neq 0$. A simple computation gives

$$\ell_0'(u,v) = \int (\dot{u}\dot{x} + \dot{v}\dot{y}) .$$

Taking $u \equiv 0$ and v to be in $C_0^{\infty}(I_j)$, $I_j = (t_j, t_{j+1})$, we integrate by parts and find that

$$0 \leq \int \ddot{v}(h\dot{x} + P) \quad \text{whenever} \quad 0 \leq \int \ddot{v}x$$

where $\dot{P} = h\ddot{x} + 3k\dot{y}h$. Hence, there is a non-negative scalar λ_j and real numbers a_j, A_j with

$$h\dot{x} + P = \lambda_j y + A_j + a_j t .$$

Likewise, taking $v \equiv 0$ we find

$$h\dot{y} + Q = \mu_j x + B_j + b_j t, \; \mu_j \leq 0$$

where $\dot{Q} = h\ddot{y} - 3k\dot{x}h$. Since $\dot{x}^2 + \dot{y}^2 \equiv 1$ we find h is differentiable on I_j and we obtain the two equations

(5.5) $$2\ddot{x}h + \dot{h}\dot{x} + 3k\dot{y}h = \lambda_j \dot{y} + a_j \quad \text{on } I_j$$

(5.6) $$2\ddot{y}h + \dot{h}\dot{y} - 3k\dot{x}h = \mu_j \dot{x} + b_j \quad \text{on } I_j .$$

Returning to the fundamental relation (5.4) we find

$$\sum_1^N \int_{I_j} (\dot{u}\dot{x} + \dot{v}\dot{y}) \le 0 \quad \text{implies} \quad \sum_1^N (\mu_j \int_{I_j} \dot{x}\dot{u} - \lambda_j \int_{I_j} \dot{y}\dot{v}) \ge 0 .$$

It follows that $\mu_j = -\lambda_j = \lambda$ for all j. (If $\dot{x}$ or $\dot{y}$ is constant on some I_j, this may involve changing a_j or b_j). *Hence, the equations* (5.5), (5.6) become

(5.7) $$2\ddot{x}h + \dot{h}\dot{x} + 3k\dot{y}h = -\lambda\dot{y} + a_j$$

(5.8) $$2\ddot{y}h + \dot{h}\dot{y} - 3k\dot{x}h = \lambda\dot{x} + b_j .$$

We also find from (5.4) that h is continuous on $[0, L]$ and $h(0) = h(L) = 0$.

Multiply (5.7) by $\dot{x}$, (5.8) by $\dot{y}$ and add the resulting equations to yield

(5.9) $$\dot{h} = a_j\dot{x} + b_j\dot{y} .$$

Multiply (5.7) by $\dot{y}$ and (5.8) by $-\dot{x}$ and again add the resulting equations to get

(5.10) $$\lambda + kh = a_j\dot{y} - b_j\dot{x} .$$

Applying a similar technique with $\ddot{x}$ and $\ddot{y}$ yields the two equations

(5.11) $$\lambda k + k^2 h = -a_j\ddot{x} - b_j\ddot{y}$$

and

(5.12) $$\dot{h}k = a_j\ddot{y} - b_j\ddot{x} .$$

Comparing (5.9) and (5.11) we find that

$$\ddot{h} + k^2 h = -\lambda k \quad on \ [0, L] - \{t_i\}_{i=1}^{m} . \tag{5.13}$$

Suppose first that L is a t-point of α. Then λ can not be zero. For referring to (5.4) we see that h would be orthogonal to LU_0 if $\lambda = 0$; but this can be specifically ruled out by Theorem 4.2. Hence, if (a, b) is an interval in (t_i, t_{i+1}) on which h vanishes identically, then k must also vanish identically. The same is true of any set of positive measure in some (t_i, t_{i+1}) on which $h = 0$. If h is positive (or negative) on some interval (a, b) in (t_i, t_{i+1}) then $k \equiv \alpha$ (or $k \equiv -\alpha$) on (a, b) and, further, (5.13) yields

$$h = -\lambda/k + A \cos kt + B \sin kt$$

on (a, b) which, together with the fact that $-\lambda/k$ has the same sign as h on (a, b) implies that either

$$(b - a)\alpha \geq \pi \quad \text{if} \quad h(a) = h(b) = 0 \tag{5.14}$$

or

(5.15) there are at most 2 arcs and 1 line segment between the successive points p_i and p_{i+1}.

Note that the arcs are all of the same radius, namely $R = 1/\alpha$.

If L is not a t-point of α, but is in the closure of the set of t-points, then a simple limiting argument shows the validity of Theorem 5.1. Finally, if L is not in the closure of the set of t points of α then L can be reduced without altering α to a point L' in the closure of the set of t points. (A simple argument shows that α has t-points all the way down to the minimum possible length.)

Note that the isoparametric problem here is basically no different than the subparametric problem for the simple reason that the curve can be extended linearly to increase its length but not change the norm of the curvature. Hence if

$$\beta(L) = \inf\{\|k_f\|_\infty : \ell_0(f) = L\}$$

then β is monotone decreasing and continuous. In fact, $\beta \equiv \alpha$ in this case. For we know that $\beta = \alpha$ on the closure of the set of t-points; if (a, b) is a maximal interval in the complement of the closure of the set t-points, then α is constant on (a, b) and $\beta(a) = \alpha(a) = \alpha(b) = \beta(b)$ so that $\beta = \alpha$ on (a, b) as well.

§6. Minimal curvature, the analytic case

Here we consider the following variant of the curvature problem of §5:

Let distinct points $p_1, \ldots, p_N$ *be given in the complex plane.* Find a function f holomorphic in the open unit disc $D = \{|z| < 1\}$ and smooth on the closure of D such that the image of the unit circle $T = \{|z| = 1\}$ under f contains the points $p_1, \ldots, p_N$ and the closed curve $f(T)$ has curvature as small as possible in the supremum norm.

The first major difference between this problem and the one considered in §5 is that existence is much less obvious. Here we can not reparametrize the curve by arc-length and still remain in the class of holomorphic functions. A second major difference is that, once a solution f has been found, we can only perturb it by holomorphic functions and hence we can not use C_0^∞ functions to obtain local information about f.

The existence question is handled in Proposition 6.1 (which uses a lemma) and Theorem 6.2. Necessary conditions that the solution must satisfy are derived in the material following Theorem 6.2. For reference we recall that the curvature of a holomorphic function f is given by

$$k_f(z) = |z|^{-1}|f'(z)|^{-1}\mathrm{Re}(1 + zf''(z)/f'(z)), \quad |z| \leq 1 \tag{6.1}$$

in terms of the derivative with respect to z and by

$$k_f(e^{i\theta}) = |\dot{f}(e^{i\theta})|^{-1} \,\mathrm{Im}(\ddot{f}(e^{i\theta})/\dot{f}(e^{i\theta})) \tag{6.2}$$

in terms of the derivative with respect to θ.

We recall that H^p is defined as the space of those functions h which are holomorphic on D and which satisfy

$$\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}|h(re^{it})|^p dt\right\}<\infty \quad \text{if} \quad 0<p<\infty$$

or

$$\sup_{|z|<1}|f(z)|<\infty \quad \text{if} \quad p=\infty.$$

We shall use a number of facts about H^p; good references are [2] and [3].

Lemma 6.1. (1) *If* g *is holomorphic in* D *and if the range of* g *lies in* $\{re^{i\theta}: -\alpha\le\theta\le\alpha\}$, *then* $g \in H^p$ *for all* $p<\pi/\alpha$; *if* $g(0)=1$, *then* $\|g\|_p \le C_p$ *where* C_p *is a constant independent of* g.

(2) *If* g *is in* H^1, $g\ne 0$ *in* D, *and if arg* g *has a continuous extension to* $\bar{D}$, *then* $g \in H^p$ *for all* $p<\infty$.

Proof. (1) The first part of (1) is well-known; see [2; #2, p. 13]. To prove the second assertion of (1), let ϕ be the conformal map of D onto the sector $\{z: |\arg z|<\alpha\}$ with $\phi(0)=1$; then $h=\phi^{-1}\circ g$ maps D into D and $h(0)=0$. Hence, $g=\phi\circ h$ is subordinate to ϕ so that $\|g\|_p \le \|\phi\|_p = C_p$; see [2; Theorem 1.7].

(2) Given $p<\infty$ choose points $\theta_0=0<\theta_1<\cdots<\theta_{m+1}=2\pi$ and an r_0, $0<r_0<1$, such that the variation of $\arg g(z)$ for $z\in D_j=\{re^{i\theta}: r_0\le r<1,\ \theta_j\le\theta\le\theta_{j+1}\}$ is no more than $\pi/2p$. Then $f\in H^p(D_j)$ by (1). Putting the finite number of regions $D_0,\ldots,D_m$ together we see that $f\in H^p$, as desired.

Theorem 6.2. *Let* S *consist of those holomorphic functions* h *in* D *which satisfy*

$$\text{(6.3)}\quad \begin{array}{ll} \text{(i)} & f'' \in H^p, \quad \text{some} \quad p < 1 \\ \text{(ii)} & |f'| \geq \nu = \nu(f) > 0 \quad \text{on} \quad D \quad \text{and arg} \quad f' \quad \text{is continuous on} \quad \overline{D} \\ \text{(iii)} & \int_0^{2\pi} |f'(e^{i\theta})| d\theta \leq L \\ \text{(iv)} & f(T) \ \underline{\text{contains}}\ \underline{\text{the}}\ \underline{\text{points}} \quad p_1, \ldots, p_N . \end{array}$$

Then S contains an element f_0 which satisfies

$$\text{(6.4)} \qquad \alpha = \|k_{f_0}\|_\infty = \inf\{\|k_f\|_\infty : f \in S\} .$$

Proof. We shall show that if $\{f_n\}$ is a sequence in S with

$$\lim \|k_{f_n}\|_\infty = \alpha$$

then there is a subsequence, again denoted by $\{f_n\}$, and a function f, with $f_n \to f$ uniformly, $f'_n \to f'$ in H^1, $|f'| \geq \nu > 0$ on $\overline{D}$ and $\|k_f\|_\infty \leq \alpha$. This will establish the conclusion.

Let $x_n + iy_n$ be the arc-length parametrization of the curve $f_n(T)$ so that

$$x_n(s) + iy_n(s) = f_n(e^{it})$$

where

$$s = s(t) = \frac{1}{2\pi} \int_0^t |f'(e^{i\theta})| d\theta .$$

Then

$$p_n(e^{i\theta}) = \dot{f}_n(e^{i\theta})/|\dot{f}_n(e^{i\theta})| = (d/ds)(x_n + iy_n)$$

and $(d^2/ds^2)(x_n + iy_n)$ lies in the ball of radius $1 + \alpha$ in L^∞. Hence, we may assume that $(d/ds)(x_n + iy_n)$ converges uniformly on $[0, L]$. Hence, $p_n \to p$ uniformly so that $\overline{p}_n \to \overline{p}$ uniformly. Now we may also assume that the measures $\{f'_n d\theta\}$ converge weak-* to the measure $g d\theta$ where $g \in H^1$ by the F. and M. Riesz theorem. Hence, $f'_n \to g$ uniformly on compact subsets on D and $\|g\|_1 \le \liminf \|f'_n\|_1$. Hence,

$$\int |f'_n| = \int f'_n \overline{p}_n \to \int g\overline{p} \le \|g\|_1 .$$

Thus, $\|f'_n\|_1 \to \|g\|_1$ so that $f'_n \to g$ in H^1 (see [6]). Consequently, we may write $g = f'$ where $f_n \to f$ uniformly and $f'_n \to f'$ in H^1.

Next, note that $\int f'\overline{p} = \int |f'|$ and hence $p = f'/|f'|$ a.e. If we normalize f_n so that $\arg(f'_n(1)) = 0$, then from the fact that $\{p_n\}$ is equicontinuous we see that the sequence $\{\arg f'_n\}$ is equicontinuous and uniformly bounded so that we may assume $\{\arg f'_n\}$ converges uniformly on T and hence on $\overline{D}$ to $\arg f'$ and thus $f' \in H^p$ for all $p < \infty$ by the lemma. Further, the second part of (1) of the lemma implies that the H^p norms of f'_n are uniformly bounded since $f'_n \to f'$ uniformly on compact subsets of D. (Indeed, $f'_n \to f'$ in H^p for all $p < \infty$). Hence, the functions

$$u_n = \mathrm{Re}(e^{i\theta} f''_n(e^{i\theta})/f'_n(e^{i\theta}))$$

lie in a fixed ball in L^p for each $p < \infty$ and thus so do their harmonic conjugates

$$v_n = \mathrm{Im}(e^{i\theta} f''_n(e^{i\theta})/f'_n(e^{i\theta})) .$$

Hence, $\{f''_n/f'_n\}$ lies in a fixed ball of H^p for each $p < \infty$ and so

$\{\log f'_n\}$ lies in a fixed ball of the disc algebra. Thus, $|f'_n|$ is uniformly bounded away from zero. We may assume that f''_n/f'_n converges weakly in H^p to f''/f' for each $p < \infty$. Thus, for $p < \infty$, $\|k_f\|_p \leq \liminf \|k_{f_n}\|_p \leq \liminf \|k_{f_n}\|_\infty \leq \alpha$ and so $\|k_f\|_\infty \leq \alpha$. This concludes the proof.

To cast the analytic curvature problem in the mold required by Theorem 2.1 we let f be a solution assured by Theorem 6.2 and let X consist of all those holomorphic functions h on D for which

$$\mathrm{Re}(z h''(z)/f'(z))$$

is bounded on D; we norm X by

$$\|h\|_X = \|\mathrm{Re}(z h''(z)/f'(z))\|_\infty + |h'(0)| + |h(0)| .$$

X is complete with this norm. It follows immediately that among all functions h in X which satisfy

(a) $p_1, \ldots, p_N$ lie in $h(T)$

(b) $\int_0^{2\pi} |h'(e^{i\theta})| d\theta \leq L$

(c) $|h'| \geq \nu = \nu(h) > 0$ on D

f has the smallest curvature. Note that since $|f'| \geq \delta$ on D the curvature of $f + \varepsilon h$ is defined and finite for each $h \in X$ as soon as ε is small enough. Thus the operator L in Theorem 2.1 is the Fréchet derivative of the curvature operator and is given by

$$Lg = -k\,\mathrm{Re}(g'/f') + (1/|f'|)\mathrm{Re}(z(f'g'' - f''g')(f')^{-2})$$

where $k = k_f$ the curvature of the solution f.

The operator L maps X into L^∞; we first show that L has closed range. Suppose that $\{g_k\}$ is a sequence of elements in X all of which are of distance 1 to the null space of L but for which $\|Lg_k\|_\infty \to 0$. By altering g_k by an element of the null space of L we may assume that $\|g_k\|_X \leq 2$. Hence,

$$\|\operatorname{Re}(z g_k''(z)/f'(z))\|_\infty \leq 2 .$$

This implies that $\|g_k''\|_p \leq A_p$ for all $p < \infty$; by extracting a subsequence we may assume that $g_k'' \to g''$ weakly in H^p and $g_k' \to g'$ and $g_k \to g$ uniformly on D. But $Lg_k \to 0$ uniformly so that $\operatorname{Re}(e^{i\theta} g_k''/f') \to k \operatorname{Re}(g'/f') + (1/|f'|)\operatorname{Re}(e^{i\theta} f''/(f')^2) = \operatorname{Re}(e^{i\theta} g''/f')$ uniformly. However, $g \in \ker(L)$ and so we've just established that $g_k \to g$ in X, a contradiction. Thus, the range of L is closed and since L is weak-* continuous, the range of L is weak-* closed.

It is easier to work with derivatives taken with respect to t rather than $z = e^{it}$. We shall write $\dot{g}(t)$ for the derivative of $g(e^{it})$ with respect to t. In this way we have

$$Lg = -k \operatorname{Re}(\dot{g}/\dot{f}) + (1/|\dot{f}|)\operatorname{Im}(\ddot{g}/\dot{f}) \tag{6.5}$$

and

$$\ell_0(g) = \int |\dot{g}| \tag{6.6}$$

and X consists of those holomorphic functions g for which $\operatorname{Im}(\ddot{g}/\dot{f})$ is bounded on $\overline{D}$.

Let points $0 \leq t_1 \leq \cdots \leq t_N \leq 2\pi$ be selected with $f(t_j) = p_j$ for $j = 1, \ldots, N$ and let ℓ_j be the continuous linear functional on X

given by

(6.7) $$\ell_j(g) = g(t_j), \quad j = 1, \ldots, N .$$

To apply Theorem 2.1 we shall investigate those (real) functions $h \in L^1_r$ for which

(6.8) $$0 \leq \int h\, Lg$$

whenever

(6.9) $$\text{(a)} \quad 0 = \ell_j(g), \qquad j = 1, \ldots, N$$

$$\text{(b)} \quad 0 \geq \ell'_0(g) .$$

In particular we shall first show that the set $\mathcal{P}$ those functions h for which equality holds in (6.8) whenever it holds in (6.9) (a) and (b) form a finite dimensional subspace of L^1_r and hence LV_0 has finite codimension in L^∞, as required by the hypotheses of Theorem 2.1, Case II. Integration by parts for smooth enough g yields the formula

$$\int h\, Lg = \mathrm{Im} \int \ddot{g} P + \mathrm{Im}[\dot{g}(2\pi) H(2\pi)]$$

where

(6.10) $$P = h(\dot{f}|\dot{f}|)^{-1} - H$$

and

(6.11) $$\dot{H} = -h(\dot{f}|\dot{f}|)^{-1}[(\ddot{f}/\dot{f}) + i\, \mathrm{Im}(\ddot{f}/\dot{f})]$$

$$H(0) = 0 .$$

We have

$$\ell'_0(g) = \mathrm{Re} \int \dot{g} s$$

where

$$s = |\dot{f}|/\dot{f} .$$

Define, for a function $h \in \mathcal{P}$, a linear functional ℓ on X by the rule

$$\ell(g) = \int \ddot{g}P + \dot{g}(2\pi)[H(2\pi)]$$

where P and H are related to h by (6.10), (6.11). We know that $\operatorname{Im} \ell(g) = 0$ if $\ell_j(g) = 0$ for $j = 1, \ldots, N$ and $\operatorname{Re} \ell'_0(g) = 0$. Replacing g by ig we see that $\ell(g) = 0$ when g lies in the intersection of the null spaces ℓ'_0 and ℓ_j, $j = 1, \ldots, N$. Hence, there are (complex) scalars $\lambda_0, \lambda_1, \ldots, \lambda_N$ such that

$$\ell = \sum_1^N \lambda_j \ell_j + \lambda_0 \ell'_0 \tag{6.12}$$

as linear functionals on X. I now claim that λ_0 is pure imaginary. To see this, note again that $\operatorname{Im} \ell(g) = 0$ whenever $\ell_j(g) = 0$ for $j = 1, \ldots, m$ and $\operatorname{Re} \ell'_0(g) = 0$. Hence,

$$0 = \operatorname{Im} \ell(g) = \operatorname{Im}(\lambda_0 \ell'_0(g)) = (\operatorname{Re} \lambda_0)(\operatorname{Im} \ell'_0(g)) .$$

However, it is trivial to show that there is a $g \in X$ with $\ell_j(g) = 0$ for $j = 1, \ldots, m$, and $\ell'_0(g) = i$. This implies that $\operatorname{Re} \lambda_0 = 0$, as claimed. Write $\lambda_0 = i\lambda$ where λ is real.

We have the following representations:

$$\ell'_0(g) = \int \ddot{g}F$$

where $\dot{F} = -s$ and

$$\ell_j(g) = \int_0^{2\pi} (t - t_j)_+ \ddot{g}(t)dt + (t_j - 2\pi)\dot{g}(2\pi) + g(2\pi) .$$

Hence, from (6.12) we obtain the following

$$\sum_1^N \lambda_j (t - t_j)_+ + i\lambda F = P + G \tag{6.13}$$

for some $G \in H^1$. We also find

$$\sum_1^N \lambda_j(t_j - 2\pi) = H(2\pi) \tag{6.14}$$

and

$$\sum_1^N \lambda_j = 0 . \tag{6.15}$$

However, it is (6.13) which will yield the information we want about h. From (6.13) we see that $h|\dot{f}|^{-1} - \dot{f}G$ lies in $W^{1,1}$ and, in particular, is bounded. Now $\dot{f}G = \sigma + i\tau$ is analytic and both τ and $\sigma + h|\dot{f}|^{-1}$ are real and in L^∞. Hence, σ lies in L^q for all $q < \infty$ (as the harmonic conjugate of the bounded function τ) and hence $h|\dot{f}|^{-1}$ lies in L^q, $q < \infty$. Thus, $h \in L^q$. This implies that $H \in W^{1,q}$ and once again (6.13) implies that $h|\dot{f}|^{-1} - \dot{f}G$ lies in $W^{1,q}$. Thus, $\tau \in W^{1,q}$ so that $\sigma \in W^{1,q}$ also. Finally, this implies that $h \in W^{1,q}$. Since $\wp$ is a closed subspace of L^1 all of whose elements are bounded (indeed, in $\mathrm{Lip}(1 - 1/q)$) we see immediately that $\wp$ is finite dimensional. Moreover, we can differentiate both sides of (6.13) to obtain

$$\sum_1^N \lambda_j \chi_j - i\lambda s = \dot{h}(\dot{f}|\dot{f}|)^{-1} - h(\dot{f}|\dot{f}|)^{-1}\overline{(\ddot{f}/\dot{f})} + \dot{G} \tag{6.16}$$

where χ_j is the characteristic function of $[t_j, 2\pi]$. Multiply (6.16) by $\dot{f}$ and separate real and imaginary parts to obtain the two equations

$$\mathrm{Re}(\chi\dot{f}) = \dot{h}|\dot{f}|^{-1} - h|\dot{f}|^{-1}\mathrm{Re}(\ddot{f}/\dot{f}) + \mathrm{Re}(\dot{f}\dot{G}) \tag{6.17}$$

and

(6.18) $$\operatorname{Im}(\chi\dot{f}) = hk + \lambda|\dot{f}| + \operatorname{Im}(\dot{f}\ddot{G})$$

where $\chi = \sum_1^N \lambda_j\chi_j$ and k is the curvature of f. Consequently we find

(6.19) $$h(t) = |\dot{f}(t)|(\operatorname{Re}\int_0^t (\chi - \dot{G})\dot{f} + c), \quad c \in \mathbb{R}$$

(6.20) $$kh = -\lambda|\dot{f}| + \operatorname{Im}(\chi - \dot{G})\dot{f}\ .$$

We next claim that $\dot{G} = 0$. To see this note that if h is positive on a segment $(a, b) \subset [t_j, t_{j+1}]$ then $k \equiv \alpha$ on (a, b) so that we can multiply the right side of (6.19) by α and equate this to the right side of (6.20). After dividing by $|\dot{f}|$ we differentiate with respect to θ. Since $(d/d\theta)(\dot{f}/|\dot{f}|) = i\dot{f}k$ we find that $\operatorname{Im}(\dddot{G}\dot{f}) = 0$ on every segment where h is positive or negative. If $h = 0$ a.e. on a set E of positive measure in some $[t_j, t_{j+1}]$, then (6.19) implies

$$\operatorname{Re}(\chi - \dot{G})\dot{f} = 0 \quad \text{a.e. on } E$$

and (6.20) yields, after dividing by $|\dot{f}|$ and differentiating,

$$\operatorname{Im}(\dddot{G}\dot{f}) = 0 \quad \text{a.e. on } E\ .$$

Hence, $\dddot{G}\dot{f}$ is a purely real constant. But $\dot{G}f$ has mean-value 0 so that $\dddot{G}f \equiv 0$ which implies $\ddot{G}$ and $\dot{G}$ are identically zero. Thus, we have

(6.19)' $$h(t) = |\dot{f}(t)|\{\operatorname{Re}\int_0^t \chi\dot{f} + c\}$$

(6.20)' $$kh(t) = -\lambda|\dot{f}| + \operatorname{Im}\chi\dot{f}\ .$$

Assume first that L is a t-point of α; then $\lambda \neq 0$ as in §5. Suppose that $h = 0$ on some set E of positive measure; then by (6.19)' and (6.20)' we find

(6.21) $$\chi \dot{f} = i\lambda |\dot{f}| \quad \text{a.e. on } E$$

which implies that $\arg \dot{f}$ is constant on E. Thus, $k = 0$ on E. Hence, if L is a t-point of α, then the curve $f(T)$ consists of arcs of circles of radius $R = 1/\alpha$ or straight line segments. We also note that formulas (6.19)' and (6.20)' show that if $[a, b]$ is a subset of $[t_j, t_{j+1}]$ and $h(a) = h(b)$ and $h > 0$ on (a, b), then $(b - a)\alpha \geq \pi$. Hence, conclusions (5.14) and (5.15) hold as well for the analytic case. Next, a limiting argument shows that if L is in the closure of the set of t-points, then there is a solution consisting of arcs of circles (of the same radius) or straight line segments. The general case is solved by decreasing L until a point in the closure of the set of t-points is reached. The final point to be touched on is the existence of critical points for α (or β); equivalently, we just have to show that α is not constant. A brief sketch goes like this. Let L_0 be the infimum of all the lengths of $f(T)$ where f is holomorphic in D, smooth on $\bar{D}$, $f' \neq 0$ on $\bar{D}$, and $p_1, \ldots, p_N \in f(T)$. Let $\{f_n\}$ be a sequence with $\int |f_n'| \to L_0$ and $\|k_{f_n}\|_\infty = \alpha(L_0 + \delta_n)$, $\delta_n \to 0$. If $\|k_{f_n}\|_\infty \leq C$ for all n, then as in Theorem 6.2 we would have $f_n \to f$ uniformly and $f_n' \to f'$ in H^1. Thus there would be a smooth curve f with minimal length passing through $p_1, \ldots, p_N$. But a simple variational argument shows that for this f

$$0 = \int g\,|\dot{f}|\dot{f}^{-1}k_f$$

for all holomorphic g with $g(t_j) = 0$. Thus, $\operatorname{Im}(\ddot{f}/\dot{f}) = |\dot{f}|k_f = 0$ a.e. which implies that f is constant, a contradiction.

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Let T be an operator defined on the subset U of a Banach space X determined by the equality or inequality constraints for a finite number of functionals on X. ~~We find~~ The necessary conditions that the solution of the minimization problem ~~$\inf\{\|Tu\| : u \in U\}$~~ must satisfy is determined. A case of particular interest is when T has values in $\mathcal{L}^{\infty}$ $L(\infty)$. ~~We analyze~~ several examples in detail are analyzed to show how the necessary conditions yield detailed information about the solution. ~~We also introduce~~ the notion of a t-point is introduced and ~~show~~ how the information becomes complete at such a point is shown.

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