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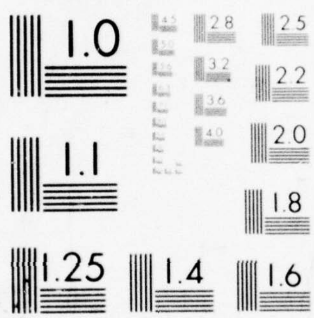

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See Form 1473

August 1976

(Received July 26, 1976)

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UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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Jean-Pierre Aubin, Arrigo Cellina and John Nohel*

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ABSTRACT

We prove existence of "monotone trajectories" for a class of discrete and continuous systems sufficiently general to include problems of some interest in economic and biological theory. We prove existence of critical points which are Pareto minima. We study stability properties of Pareto minima.

AMS (MOS) Subject Classification: 34D99

Key Words: dynamical system, stability, critical points, monotone trajectories, Pareto minima, Lyapunov function

Work Unit Number 1 (Applied Analysis)

*Sponsored in part by ARO Grant DAHC04-74-G0012, and NSF MCS75-21868.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

MONOTONE TRAJECTORIES OF MULTIVALUED DYNAMICAL SYSTEMS

Jean-Pierre Aubin, Arrigo Cellina and John Nohel *

Introduction

In "microsystems" one or several decision makers control the evolution of the state of a dynamical system in accordance with the rules of optimization or game theory. We define "macrosystems" to be dynamical systems which are not controlled (in the technical sense of control theory), but for which the state evolves by improving the values of several criteria. In our study of macrosystems we do not look for optimal trajectories, but rather we seek sufficient conditions guaranteeing the existence of "monotone trajectories" for a class of dynamical systems, sufficiently general to include problems of some interest in economic and biological theory.

It is reasonable to describe the evolution of such macrosystems by a dynamical system of the following type. Let U be a topological vector space, and let X be a convex, compact subset of U. For $i = 1, 2, \ldots, n$ let $f_i : X \to \mathbb{R}$ be n given functions, called loss functions; the values of the functions f_i should decrease as the state of the system evolves. Let S be a multivalued mapping from X into U. We seek a

*Sponsored in part by ARO Grant DAHC04-74-60012, and NSF MCS75-21868.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

function $x : t \to x(t) \in X$ such that

(*) $$x'(t) \in -S(x(t)) \quad (0 \le t < \infty;\ ' = d/dt)$$

where $x(0) = x^0$ is a given point, and such that

(**) each function : $t \to f_i(x(t))$ $(i = 1, \ldots, n;\ 0 \le t < \infty)$

is nonincreasing. Relation (*) states that if at time t the state of the system is $x(t)$, then the admissible velocities $x'(t)$ are constrained to lie in the set $-S(x(t))$ (this permits uncertainty to be included in the model). Relations (**) require that the evolution of the state of the system does not increase ("improves") the values of the n loss functions f_i.

We summarize our results as follows. In Section 1, we establish, under reasonable assumptions linking the correspondence S and the loss functions f_i, *the existence of monotone trajectories* for the discretized implicit system associated with (*), (**): find a sequence $t \to x^t \in X$ such that

(***) $$\frac{x^{t+1} - x^t}{k} \in -S(x^{t+1}) \quad (0 \le t < +\infty,\ k > 0)$$

where x^0 is given and such that (**) is satisfied. We prove in Section 2 that the piecewise linear functions which interpolate the solution of the discrete system (***), (**) converge to a solution of the system (*), (**). In Section 3, we show that the same assumptions imply that for all $\bar{x} \in X$ there exists a critical point $\bar{\bar{x}}$ of S (i.e., $0 \in -S(\bar{\bar{x}})$) such that $f_i(\bar{\bar{x}}) \le f_i(\bar{x})$ $(i = 1, \ldots, n)$. In particular, this shows that if $\bar{x}$ is a <u>Pareto</u> <u>minimum</u> (i.e., there is no $y \in X$ such that $f_i(y) \le f_i(\bar{x})$, $i = 1, \ldots, n$, with $f_j(y) < f_j(\bar{x})$ for at least one j), then there exists a critical point $\bar{\bar{x}}$ of S with the property that $f_i(\bar{x}) = f_i(\bar{\bar{x}})$ for all $i = 1, \ldots, n$.

We also show that the subsets $Q = \{x \in X : f_i(x) = f_i(\bar{x}),\ i = 1, \ldots, n\}$, where $\bar{x}$ is a Pareto minimum, are weakly stable in the following sense: For any neighborhood M of Q there exists a neighborhood N of Q such that for every $x^0 \in N$ there exists a trajectory of (***), (**) passing through x^0 which lies in M for all $t \geq 0$. In Section 4 we introduce the concept of Lyapunov function f for the correspondence S which has the property that $f(x^t)$ does not increase whenever $\{x^t\}_t$ is a trajectory of the dynamical system. We obtain stability properties analogous to classical results for ordinary differential equations. In the last section, we extend the results of the first section to other discretized systems

$$(****) \qquad \frac{x^{t+1} - x^t}{k} \in -S(\theta x^{t+1} + (1 - \theta)x^t) \qquad (0 \leq \theta \leq 1)$$

where x^0 is given and where the condition (**) must be satisfied (when $\theta = 0$, we obtain the "explicit" scheme). The assumptions needed for the existence of a solution of (****), (**) are stronger than those needed for the implicit scheme ($\theta = 1$).

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1. Existence of monotone trajectories in the discrete case

We shall make the following assumption:

$$(1) \quad \begin{cases} X \text{ is a } \underline{\text{convex}}\ \underline{\text{compact}} \text{ subset of a Hausdorff} \\ \text{locally convex space } U . \end{cases}$$

We consider a correspondence S from X into U with <u>nonempty closed convex images</u> $S(x)$. Then the Hahn-Banach theorem implies that these images are characterized by their support functions

$$(2) \quad \sigma^{\#}(S(x);p) = \sup_{y \in S(x)} \langle p, y \rangle$$

where p ranges over the topological dual U^* of U. We shall say that S is <u>upper hemi-continuous</u> if, for any $p \in U^*$, $x \to \sigma^{\#}(S(x);p)$ is upper semi-continuous. Any upper semi-continuous correspondence is upper hemi-continuous, *the converse being true when* $S(x)$ is compact for each $x \in X$. We shall assume that

$$(3) \quad \begin{cases} S \text{ is an upper hemi-continuous correspondence from} \\ X \text{ into } U \text{ with nonempty closed convex images } S(x) . \end{cases}$$

We introduce n "loss functions" f_i which satisfy:

$$(4) \quad \begin{cases} \text{loss functions } f_i : X \to \mathbb{R} \text{ are convex and lower} \\ \text{semi-continuous } (i = 1, \ldots, n). \end{cases}$$

Let $F : X \to \mathbb{R}^n$ be the multiloss operator defined by $F(x) = \{f_1(x), \ldots, f_n(x)\}$. We now need an assumption linking the "state set" X, the correspondence S and the multiloss operator F.

Definition 1.

We shall say that S and F are "consistent" on X if

$$(5)\quad \begin{cases} \forall p \in U^*,\ \forall \lambda \in \mathbb{R}^n_+ \text{ and } \forall x \in X \text{ which minimizes} \\ y \to \langle \lambda, F(y)\rangle - \langle p, y\rangle \text{ on } X, \text{ then } \sigma^{\#}(S(x);p) \geq 0 . \end{cases}$$

Remarks on the consistency assumption

Before stating the theorem, it is worthwhile to describe equivalent statements of (5) and give some examples.

Let ψ_X be the indicator of X, defined by $\psi_X(x) = 0$ when $x \in X$ and $\psi_X(x) = +\infty$ when $x \notin X$. If g is a function from U into $]-\infty, +\infty]$, let $\partial g(x) = \{p \in U^*$ such that $g(x) - g(y) \leq \langle p, x - y\rangle$ for all $y \in U\}$ be the subdifferential of g at x. Recall that $\partial\psi_X(x)$ is the normal cone to X at x. If assumption (4) holds, $x \in X$ minimizes $y \to \langle \lambda, F(y)\rangle - \langle p, y\rangle$ on X if and only if $p \in \partial(\sum_{i=1}^n \lambda^i f_i + \psi_X)(x)$. Hence, if we set

$$(6)\quad C(\lambda, x) = \inf_{p \in \partial(\sum_{i=1}^n \lambda^i f_i + \psi_X)(x)} \sigma^{\#}(S(x);p)$$

then S and F are consistent on X if and only if

$$(7)\quad \forall \lambda \in \mathbb{R}^n_+,\ \forall x \in X,\ C(\lambda, x) \geq 0 .$$

In particular, if the interior $\mathring{X}$ of X is not empty and if $x \in \mathring{X}$, then $\partial\psi_X(x) = \{0\}$ and

$$(8)\quad \forall x \in \mathring{X},\ C(\lambda, x) = \inf_{p \in \partial(\sum_{i=1}^n \lambda^i f_i)(x)} \sigma^{\#}(S(x);p) .$$

Let us remark also that when the images $S(x)$ are convex and compact, we can write

(9) $$C(\lambda, x) = \sup_{y \in S(x)} \inf_{p \in \partial(\sum_{i=1}^{n} \lambda^i f_i + \psi_X)(x)} \langle p, y \rangle .$$

Assume for instance that the functions f_i are Gâteaux differentiable at each point of X. Then we can write

(10) $$C(\lambda, x) = \inf_{q \in \partial\psi_X(x)} \sigma^{\#}(S(x); q + \sum_{i=1}^{n} \lambda^i Df_i(x))$$

where $Df_i(x)$ denotes the gradient of f_i at x. If the correspondence S is single-valued, then

(11) $$C(\lambda, x) = \inf_{q \in \partial\psi_X(x)} \langle q + \sum_{i=1}^{n} \lambda^i Df_i(x), S(x) \rangle .$$

Now let us consider the case where K is a convex compact set of controls. Define

(12) $$S(x) = A(x) + B(K)$$

where A is a single valued map from X into U and where B is linear. Then

(13) $$\begin{cases} C(\lambda, x) = \inf_{q \in \partial\psi_X(x)} \sup_{v \in K} \langle q + \sum_{i=1}^{n} \lambda^i Df_i(x), A(x) + B(v) \rangle \\ \quad = \sum_{i=1}^{n} \lambda^i \langle Df_i(x), A(x) \rangle + \sigma^{\#}(K, B^*(\sum_{i=1}^{n} \lambda^i Df_i(x))) \\ \quad + \inf_{q \in \partial\psi_x(x)} \sup_{v \in K} \langle q, A(x) + Bv \rangle . \end{cases}$$ ∎

By a "monotone trajectory of the discrete dynamical system" we mean a sequence of elements $x^t \in X$ satisfying

$$\frac{x^{t+1} - x^t}{k} \in -S(x^{t+1}) \tag{14}$$

and

$$F(x^{t+1}) \leq F(x^t), \tag{15}$$

where x^0 is given, where $k > 0$, and where the inequalities (15) are interpreted in an obvious way. Our basic existence result is:

Theorem 1.

Let the assumptions (1), (3), (4) hold. If S and Γ are consistent, there exists a solution $\{x^t\}_{t \geq 0}$ of the discrete dynamical system (14), (15) for any initial condition $x^0 \in X$.

In the proof we will use a theorem of Ky Fan which states that if X is convex compact, if $\forall y \in X$, $x \to a(x, y)$ is lower semi-continuous and if $\forall x \in X$, $y \to a(x, y)$ is concave, there exists $\bar{x} \in X$ such that $\sup_{y \in X} a(\bar{x}, y) \leq \sup_{y \in X} a(y, y)$.

Proof.

We have to prove the existence of $x^{t+1} \in X$ satisfying (14) and (15) when x^t is given.

Assume that no solution exists. Then, for any $x \in X$, either $x \notin x^t - kS(x)$ or $F(x) - F(x^t)$ is not positive. In the first case, by the Hahn-Banach theorem, there exists $p \in U^*$ such that

$\langle p, x^t\rangle > \langle p, x\rangle + k\sigma^\#(S(x);p)$ (since $S(x)$ is closed and convex). In the second case, there exists $\lambda \in \mathbb{R}^n_+$ such that $\langle \lambda, F(x) - F(x^t)\rangle > 0$. Let us introduce the subsets

(16) $$V_p = \{x \in X \text{ such that } \langle p, x^t\rangle > \langle p, x\rangle + k\sigma^\#(S(x);p)\}$$

and

(17) $$V_\lambda = \{x \in X \text{ such that } \langle \lambda, F(x) - F(x^t)\rangle > 0\} .$$

Since the functions $x \to k\sigma^\#(S(x);p) + \langle p, x\rangle$ and $x \to -\langle \lambda, F(x) - F(y)\rangle$ are upper semi-continuous by assumptions (3) and (4), these subsets are <u>open</u>. The negation of the conclusion implies that X is covered by the open subsets V_p and V_λ. Since X is compact, we can extract a finite covering: $X \subset \bigcup_{i=1}^{k} V_{\lambda_i} \cup \bigcup_{j=1}^{p} V_{p_j}$.

Let $\{\alpha_i, \beta_j\}_{\substack{1 \le i \le k \\ 1 \le j \le \ell}}$ be a continuous partition of unity with respect to this finite covering. Let a be the function defined on $X \times X$ by

(18) $$a(x, y) = \sum_{i=1}^{k} \alpha_i(x)\langle \lambda_i, F(x) - F(y)\rangle - \sum_{j=1}^{\ell} \beta_j(x)\langle p_j, x - y\rangle .$$

Since the functions f_i are lower semi-continuous, we deduce that for any $y \in X$, the function $x \to a(x, y)$ is also lower semi-continuous. The functions $y \to a(x, y)$ are concave because the functions f_i are convex. Also, $a(y, y) = 0$ for all $y \in X$. Hence the assumptions of the above Ky Fan theorem are satisfied: thus, there exists $\bar{x} \in X$ such that

(19) $$a(\bar{x}, y) \le 0 \quad \forall y \in X .$$

Let us set $\bar{\lambda} = \sum_{i=1}^{k} \alpha_i(\bar{x})\lambda_i \in \mathbb{R}^n_+$ and $\bar{p} = \sum_{j=1}^{\ell} \beta_j(\bar{x})p_j \in U^*$. We can write (19) in the form

$$\langle \bar{\lambda}, F(\bar{x}) - F(y)\rangle - \langle \bar{p}, \bar{x} - y\rangle \leq 0 \quad \forall y \in X . \tag{20}$$

Therefore, by the consistency assumption of S and F on X,

$$\sigma^{\#}(S(\bar{x});\bar{p}) \geq 0 . \tag{21}$$

We obtain a contradiction by showing that $a(\bar{x}, x^t) > 0$. Indeed, $\alpha_i(\bar{x}) > 0$ for at least one index i or $\beta_j(\bar{x}) > 0$ for at least one index j. If $\alpha_i(\bar{x}) > 0$, then $\bar{x} \in V_{\lambda_i}$ and thus, $\langle \lambda_i, F(\bar{x}) - F(x^t)\rangle > 0$. Hence $\langle \bar{\lambda}, F(\bar{x}) - F(x^t)\rangle = \sum_{i=1}^{k} \alpha_i(\bar{x})\langle \lambda_i, F(\bar{x}) - F(x^t)\rangle > 0$. If $\beta_j(\bar{x}) > 0$, then $\bar{x} \in V_{p_j}$ and this implies that $-\langle p^j, \bar{x} - x^t\rangle > k\sigma^{\#}(S(\bar{x});p^j)$. Therefore using (17) and $\sum_{j=1}^{\ell} \beta_j(\bar{x}) = 1$, we obtain $-\langle \bar{p}, \bar{x} - x^t\rangle > k \sum_{j=1}^{\ell} \beta_j(\bar{x})\sigma^{\#}(S(\bar{x}), p^j)$ $\geq k\sigma^{\#}(S(\bar{x});\bar{p} \geq 0$. Hence $a(\bar{x}, x^t) = \langle \bar{\lambda}, F(\bar{x}) - F(x^t)\rangle - \langle \bar{p}, \bar{x} - x^t\rangle > 0$. ■

Remarks on the existence of trajectories

In the case where $F = 0$, the consistency assumption becomes the so-called "Ky Fan boundary condition". The Browder - Ky Fan theorem states that this assumption and assumptions (1) and (3) imply the existence of a solution $\bar{x} \in X$ of the multivalued equation $0 \in -S(\bar{x})$ (see Browder, Ky-Fan, Cornet [1] and [2] for instance).

Theorem 1 states that under the same assumptions, there exists a solution $\{x^t\}_{t>0}$ of the discrete dynamical system (14). ■

Generalization

Let us consider now n functions $\varphi_i : X \times X \to \mathbb{R}$ satisfying

$$(22)\quad \begin{cases} \text{i) } \forall y \in X,\ x \to \varphi_i(x,y) \text{ is lower semi-continuous} \\ \text{ii) } \forall x \in X,\ y \to \varphi_i(x,y) \text{ is concave} \\ \text{iii) } \sup_{y \in X} \varphi(y,y) \leq 0 . \end{cases}$$

We set

$$\phi(x,y) = \{\varphi_i(x,y)\} .$$

Definition 2.

We shall say that S and ϕ are "consistent" if $\forall p \in U^*$, $\forall \lambda \in \mathbb{R}^n_+$, for any $x \in X$ satisfying

$$(23)\quad \sup_{y \in X} (\langle \lambda, \phi(x,y)\rangle - \langle p, x - y\rangle) \leq 0$$

then $\sigma^{\#}(S(x);p) \geq 0$.

Theorem 1 bis.

Let us assume that properties (1), (3) and (22) hold and that S and ϕ are consistent. For any initial condition $x^0 \in X$, there exists a solution of the dynamical system (14) satisfying

$$(24)\quad \forall i = 1, \ldots, n,\ \varphi_i(x^{t+1};x^t) \leq 0 \quad \text{for all } t \geq 0 .$$

Proof.

The proof is exactly the same than the proof of Theorem 1 where we replace $F(x) - F(y)$ by $\phi(x,y)$. ∎

Example.

Let us consider n loss functions $f_i : U \to]-\infty, +\infty]$ such that

(25) $\quad \forall i = 1, \ldots, n, \; f_i$ is convex, finite and continuous on X.

Then f_i is differentiable from the right and the function φ_i defined by

$$\varphi_i(x, y) = -Df_i(x)(y - x)$$

satisfies properties (22) [where $Df_i(x)(z) = \inf_{\theta \geq 0} \frac{f_i(x + \theta z) - f_i(x)}{\theta}$ is the derivative from the right, which is continuous with respect to x and z].

Corollary (of Theorem 1 bis).

Let us assume that properties (1), (3) and (25) hold. Suppose also that S and F are consistent on X. For each initial condition $x^0 \in X$, there exists a solution $\{x^t\}_{t \geq 0}$ of the discrete system (14) satisfying

(26) $\quad \forall i = 1, \ldots, n, \; f_i(x^t) - f_i(x^{t+1}) \geq Df_i(x^{t+1})(x^t - x^{t+1}) \geq 0$. ∎

2. Existence of monotone trajectories in the continuous case

We shall deduce from Theorem 1 of Section 1 the existence of monotone trajectories of a multivalued dynamical system in the continuous case. A trajectory is a continuous function $x \in C(0, T; \mathbb{R}^\ell)$, whose derivative (in the sense of distributions) belongs to $L^1(0, T; \mathbb{R}^\ell)$, satisfying

$$(1) \quad \begin{cases} \text{i)} \ \forall t \in [0, T], \ x(t) \in X \ \text{ and } \ x(0) = x^0, \ \text{ where } \ x^0 \in X \ \text{ is given} \\ \text{ii)} \ \text{for almost all } \ t \in [0, T], \ \dfrac{dx}{dt}(t) \in -S(x(t)) \ ; \end{cases}$$

in addition, the trajectory x is monotone if

$$(2) \qquad \text{for all } \ i = 1, \ldots, n, \ t \to f_i(x(t)) \ \text{ is decreasing,}$$

where f_i are given loss functions.

Theorem 2.

Let us assume that

$$(3) \qquad X \ \text{ is a convex compact subset of } \ \mathbb{R}^\ell$$

that

$$(4) \qquad \begin{cases} S \ \text{ is an upper semi-continuous correspondence from } \ X \ \text{ into} \\ \mathbb{R}^\ell \ \text{ with nonempty convex compact images} \end{cases}$$

and that

$$(5) \qquad \text{the } \ n \ \text{ functions } \ f_i : X \to \mathbb{R} \ \text{ are convex and continuous .}$$

If the correspondence S and the multiloss operator F are consistent on X, then for any $T > 0$ there exists a monotone trajectory of the system (1), (2) on $[0, T]$.

Proof.

Let N be a positive integer. Consider the discrete dynamical system (14), (15) of Section 1 with $k = \frac{T}{N}$. With any monotone trajectory $\{x^t\}_{t \geq 0}$ of this discrete system we associate the piecewise-linear function $x_k(\tau)$ which interpolates the values x^t at the nodes $tk (1 \leq t \leq N)$: we have

$$(6) \qquad x_k(tk) = x^t \quad \text{and} \quad \frac{d}{dt} x_k(\tau) = \frac{x^{t+1} - x^t}{k} \quad \text{if} \quad \tau \in \,]kt, (k+1)t[\, .$$

Since X is compact and S is upper semi-continuous, $S(X) = \bigcup_{x \in X} S(x)$ is compact. Hence, there exists a constant $a > 0$, independent of k and T, such that

$$(7) \qquad \sup_{\tau \in [0,T]} \|x_k(\tau)\| \leq a \quad \text{and} \quad \sup_{\tau \in [0,T]} \left\|\frac{d}{dt} x_k(\tau)\right\| \leq a \quad \text{for all } k \, .$$

This implies that the functions x_k are uniformly Lipschitzian and thus, remain in a bounded equicontinuous subset of $C(0, T; \mathbb{R}^\ell)$ which is relatively compact according to Ascoli's theorem. Hence we can extract a subsequence (again denoted x_k) which converges uniformly to a function $x \in C(0, T; \mathbb{R}^\ell)$.

Since for all $\tau \in [0, T]$, we have

$$(8) \qquad \int_0^\tau \frac{d}{dt} x_k(\sigma) d\sigma = x_k(\tau) - x^0 \quad \text{converges to} \quad x(\tau) - x^0 \, ,$$

we deduce that the sequence $\frac{d}{dt} x_k$ converges <u>weakly</u> in $L^1(0, T; \mathbb{R}^\ell)$ to the derivative (in the sense of distributions) of x, which belongs to $L^1(0, T; \mathbb{R}^\ell)$.

Since the correspondence S is upper semi-continuous, we can associate to any ball $B(\varepsilon)$ of radius $\varepsilon > 0$ a ball $B(\eta)$ of radius η such that

$$S(y) \subset S(x(t)) + B(\varepsilon) \quad \text{when} \quad y \in x(t) + B(\eta) . \tag{9}$$

On the other hand, since x_k lies in an equicontinuous set and converges uniformly to x, there exist $\alpha > 0$ and $k_0 > 0$ such that, for any $k \leq k_0$ and $|\tau - t| \leq \alpha$, we have $\|x_k(\tau) - x(t)\| \leq \eta$. Therefore

$$S(x_k(\tau)) \subset S(x(t)) + B(\varepsilon) \quad \text{when} \quad k \leq k_0 \quad \text{and} \quad |\tau - t| \leq \alpha . \tag{10}$$

Since the sequence $\frac{d}{dt} x_k$ converges weakly to $\frac{d}{dt} x$ in $L^1(0, T; \mathbb{R}^\ell)$, there exists some sequence of convex combinations $\mu_\ell = \sum_{k=k_0}^{\ell} \alpha_\ell^k \frac{d}{dt} x_k$ which converges strongly to $\frac{d}{dt} x$ in $L^1(0, T; \mathbb{R}^\ell)$. Let $\tau \in [0, T]$. Since

$$\mu_\ell(\tau) = \sum_{k=k_0}^{\ell} \alpha_\ell^k \frac{d}{dt} x_k(\tau) = \sum_{k=k_0}^{\ell} \alpha_\ell^k \frac{x^{t_k+1} - x^{t_k}}{k} \quad \text{where} \quad |\tau - t_k k| \leq \alpha$$

and since $\frac{x^{t_k+1} - x^{t_k}}{k} \in -S(x_k(t_k k))$, for some integer t_k we deduce from (10) that for any $\ell \geq k_0$,

$$-\mu_\ell(\tau) \in \sum_{k=k_0}^{\ell} \alpha_\ell^k S(x_k(t_k k)) \subset \sum_{k=k_0}^{\ell} \alpha_\ell^k [S(x(\tau)) + B(\varepsilon)] \subset S(x(\tau)) + B(\varepsilon)$$

for the latter subset is convex. It is also closed. Therefore, since some subsequence of μ_ℓ converges almost everywhere to $\frac{d}{dt} x$, we deduce that $-\frac{dx}{dt}(t)$ belongs to $S(x(t)) + B(\varepsilon)$ for almost all t. Thus, by letting ε converge to 0, we deduce that $\frac{dx}{dt}(t) \in -S(x(t))$ for almost all t.

It remains to prove that the functions $t \to f_i(x(t))$ are decreasing.

Let $t < s$. Since x_k. lies in an equicontinuous subset, we can associate with any η a number α such that $\|x_k(\tau) - x(t)\| \leq \eta$ and $\|x_k(\sigma) - x(s)\| \leq \eta$ when $|\tau - t| \leq \alpha$, $|\sigma - s| \leq \alpha$ and $k \leq k_0$. Since the functions f_i are continuous, we can associate with any $\varepsilon > 0$ a number η such that $f_i(x(t)) \leq f_i(y) + \frac{\varepsilon}{2}$ and $f_i(z) \leq f_i(x(s)) + \frac{\varepsilon}{2}$ when $\|x(t) - y\| \leq \eta$ and $\|x(s) - z\| \leq \eta$. Therefore, for any $\varepsilon > 0$, we can find k_0 such that, for any $k \leq k_0$, there exist integers t_k and s_k satisfying $t \leq t_k k \leq s_k k \leq s$ and $|t - t_k k| \leq \alpha$, $|s - s_k k| \leq \alpha$. Since $x_k(s_k k) \leq x_k(t_k k)$, we deduce that $f_i(x(s)) \leq f_i(x(t)) + \varepsilon$ for all i. By letting ε go to 0, we deduce that $f_i(x(s)) \leq f_i(x(t))$. ∎

Remark.

We refer to papers by Antosiewicz-Cellina, Brezis, Castaing-Valadier, Clarke, Henry [1], [2], Valadier, for the study of multivalued ordinary differential equations (but without the subsidiary condition (2)). ∎

Generalization.

In Theorem 2 (continuous case), we can drop the convexity assumptions, since it is not necessary to approximate the would-be solution by exact solutions of the discrete scheme. We can extend the Nagumo theorem (see Nagumo) to the case of monotone trajectories of multivalued differential equations, by adapting the proof of Crandall [see Crandall]. We obtain

Theorem 2 bis.

Let us assume that

(3 bis) $\quad X$ is a compact subset of $\mathbb{R}^\ell$

(4 bis) $\begin{cases} S \text{ is an upper semi-continuous correspondence from } X \\ \text{into } \mathbb{R}^{\ell} \text{ with nonempty convex compact images} \end{cases}$

(5 bis) the multiloss operator F is continuous from X into $\mathbb{R}^n$

<u>and that</u> S <u>and</u> F <u>are consistent on</u> X <u>in the sense that</u>

(11) $$\forall x \in X, \liminf_{k \to 0+} \frac{A(x,k)}{k} = 0 \quad \textit{uniformly over } X$$

<u>where we set</u>

(12) $$A(x,k) = \inf_{u \in S(x)} \inf_{\substack{F(v) \le F(x) \\ v \in X}} \|x - ku - v\|.$$

<u>Then, for any</u> $T > 0$, <u>there exists a monotone trajectory of the system</u> (1), (2) <u>on</u> $[0, T]$.

<u>Proof</u>.

Indeed, if we set $\eta(k) = \frac{1}{2} \sup_{x \in X} \frac{A(x,k)}{k}$, equation (12) allows us to construct an approximate solution of the explicit discrete scheme, by associating with any $x^t \in X$ an element $x^{t+1} \in X$ satisfying $F(x^{t+1}) \le F(x^t)$ and an element $y^t \in S(x^t)$ such that

$$\left\| \frac{x^{t+1} - x^t}{k} + y^t \right\| \le \eta(k) \quad \text{for any } t,$$

i.e., such that, for any t,

(13) $$\begin{cases} \text{i)} \quad \frac{x^{t+1} - x^t}{k} \in -S(x^t) + \eta(k)B \\ \text{ii)} \quad F(x^{t+1}) \le F(x^t) \end{cases}$$

where B is the unit ball of $\mathbb{R}^n$. The convergence proof of Theorem 2 shows that the piecewise-linear function which interpolates the solution of (13) converges to a monotone trajectory of the system (1), (2). ∎

Remark.

We can prove that (11) follows from the following property

$$
(14) \quad \begin{cases} S \text{ is continuous and} \\ \forall x \in X,\ \forall p \in \mathbb{R}^{\ell},\ \text{ for any } v \in X \text{ which minimizes} \\ y \to \|x - p - y\| \text{ under the constraint } F(y) \le F(x), \\ \text{then } \sup\limits_{z \in S(x)} \langle x - p - v, z \rangle \ge 0 \end{cases}
$$

by adapting the proof of Lemma 1 of Crandall. ∎

3. Stability of Pareto minima

The solutions $\bar{x}$ of the multivalued equation $0 \in S(\bar{x})$ are called the "critical points" of the correspondence S (or "equilibria" of S). If $\bar{x}$ is such a critical point, the constant trajectories $\{\bar{x}, \bar{x}, \ldots, \bar{x}, \ldots\}$ are obviously solutions of the dynamical systems (14) of Section 1 and (1) of Section 2.

Let us recall that $\bar{x} \in X$ is called a "Pareto minimum" of F if there is no $y \in X$ such that $F(y) \leq F(\bar{x})$ and $F(y) \neq F(\bar{x})$. In other words, if we set

(1) $$P^{-}(\bar{x}) = \{y \in X \text{ such that } F(y) \leq F(\bar{x})\}$$

we see that <u>$\bar{x}$ is a Pareto minimum if and only if</u>

(2) $$P^{-}(\bar{x}) = F^{-1}F(\bar{x}) .$$

There exist points which are <u>both</u> critical points and Pareto minima under the assumptions of Theorem 1 of Section 1.

<u>Theorem 3.</u>

<u>Let us suppose that assumptions of Theorem 1 of Section 1 hold. For any $\bar{x} \in X$, the subset $P^{-}(\bar{x})$ contains a critical point $\bar{\bar{x}}$ of S. In particular, if $\bar{x} \in X$ is any Pareto minimum of F, there exists a critical point $\bar{\bar{x}}$ of S such that $F(\bar{x}) = F(\bar{\bar{x}})$.</u>

<u>Proof.</u>

The proof is analogous to that of Theorem 1 of Section 1. Assume that the theorem is false. Then, for any $x \in X$, either there exists

$\lambda \in \mathbb{R}^n_+$ such that $\langle\lambda, F(x) - F(\bar{x})\rangle > 0$ or $0 \notin S(x)$, and thus, there exists $p \in U^*$ such that $0 > \sigma^{\#}(S(x);p)$. Hence X can be covered by the union of the subsets $V_\lambda = \{x \in X$ such that $\langle\lambda, F(x) - F(\bar{x})\rangle > 0\}$ and $V_p = \{x \in X$ such that $0 > \sigma^{\#}(S(x);p)\}$, which are open due to assumptions (3) and (4) of Section 1. Since X is compact, we can extract a finite covering and choose a continuous partition of unity subordinate to this covering.

We introduce the function a defined by (18) of Section 1. The assumptions of the Ky Fan theorem are satisfied: there exists $\bar{\bar{x}} \in X$ such that $\sup_{y \in X} a(\bar{\bar{x}}, y) \leq 0$, i.e., such that

$$\langle\bar{\lambda}, F(\bar{\bar{x}}) - F(y)\rangle - \langle\bar{p}, \bar{\bar{x}} - y\rangle \leq 0 \quad \forall y \in X. \tag{3}$$

The consistency assumption implies that $\sigma^{\#}(S(\bar{\bar{x}});\bar{p}) \geq 0$. Now, if $\beta_j(\bar{\bar{x}}) > 0$ for at least one index j, then $\bar{\bar{x}} \in V_{p_j}$ and thus, $\sigma^{\#}(S(\bar{\bar{x}});p^j) < 0$. Hence $\sigma^{\#}(S(\bar{\bar{x}}),\bar{p}) \leq \sum_{j=1}^{\ell} \beta_j(\bar{\bar{x}})\sigma^{\#}(S(\bar{\bar{x}});p^j) < 0$. This is impossible. Hence $\beta_j(\bar{\bar{x}}) = 0$ for all indices j. Therefore, $\alpha_i(\bar{\bar{x}}) > 0$ for at least one index i: then $\bar{\bar{x}} \in V_{\lambda_i}$ and thus, $\langle\lambda_i, F(\bar{\bar{x}}) - F(\bar{x})\rangle > 0$. Since $\bar{\lambda} \neq 0$ and $\bar{p} = 0$ in this case, we deduce that $\langle\bar{\lambda}, F(\bar{\bar{x}}) - F(\bar{x})\rangle > 0$ and, from (3), that $\langle\bar{\lambda}, F(\bar{\bar{x}}) - F(\bar{x})\rangle \leq 0$. Again, we obtain a contradiction.

Hence $\rho^-(\bar{x})$ contains a critical point of S.

Furthermore, if $\bar{x}$ is a Pareto minimum, $\rho^-(\bar{x}) = F^{-1}F(\bar{x})$ contains a critical point $\bar{\bar{x}}$. ∎

Let us consider the discrete and continuous dynamical systems (14) of Section 1 and (1) of Section 2.

Definition 3.

We shall say that a subset Q is "weakly stable" for the discrete system (14) of Section 1 (resp. for the continuous system (1) of Section 2) if for any neighborhood M of Q, there exists a neighborhood N of Q such that, for any initial value $x^0 \in N$, there exists a trajectory starting at x^0 of the discrete system (14) of Section 1 (resp. of the continuous system (1) of Section 2) lying in M.

Theorem 3.

Let us suppose that assumptions of Theorem 1 of Section 1 (resp. of Theorem 2 of Section 2) are satisfied. Let us assume also that

(4) the functions f_i are continuous on the set of Pareto minima .

Then, for any Pareto minimum $\bar{x} \in X$, the subsets $\rho^-(\bar{x})$ are weakly stable for the discrete system (resp. the continuous system).

Proof.

The proof is analogous to the proof of Theorem 3-8 of Maschler-Peleg. Let M be an open neighborhood of $\rho^-(\bar{x})$. Hence $K = \{x \in X$ such that $x \notin M\}$ is compact. For any $y \in K$, there exists at least one index i such that $f_i(y) > f_i(\bar{x}) + \varepsilon(y)$ where $\varepsilon(y) > 0$. Since the function f_i is lower semi-continuous, the subsets $B(y) = \{y \in X$ such that $f_i(y) > f_i(\bar{x}) + \varepsilon(y)\}$ are open; they form a covering of K, from which we can extract a finite covering $K \subset \bigcup_{k=1}^{m} B(y_k)$.

Let $\varepsilon = \min_{k=1,\ldots,m} \varepsilon(y_k) > 0$ and $N = \{y \in X$ such that $f_i(y) \leq f_i(\bar{x}) + \varepsilon$ for all $i = 1, \ldots, n\}$. Hence $N \subset M$. We also know that if we choose any $x^0 \in N$, then $f_i(x^0) \leq f_i(\bar{x}) + \varepsilon$ for all i. Let us consider the case of a discrete dynamical system: there exists a solution $\{x^t\}_t$ such that, $i = 1, \ldots, n$, $f_i(x^t) \leq f_i(x^0) \leq f_i(\bar{x}) + \varepsilon$ for all t, i.e., such that $x^t \in N \subset M$ for all t. Now, since the functions f_i are continuous on the set of Pareto minima, the set N is a neighborhood of $\rho^-(\bar{x}) = F_\bullet^{-1}F(\bar{x})$. Hence $\rho^-(\bar{x})$ is weakly stable. In the case of a continuous system, Theorem 2 implies the existence of a trajectory such that $f_i(x(t)) \leq f_i(x^0) \leq f_i(\bar{x}) + \varepsilon$ for all i. We deduce in the same way that $\rho^-(\bar{x})$ is weakly stable. ∎

4. Lyapunov functions

Let us consider a function $f : U \to]-\infty, +\infty]$ satisfying

(1) $\quad f$ is convex and finite and continuous on X.

It has a derivative $Df(x)(z)$ from the right at each point $x \in X$. Let us set

(2) $$B_f(x) = \inf_{z \in S(x)} Df(x)(z) .$$

Definition 4.

We shall say that f is a Lyapunov function for the correspondence S if it satisfies (1) and

(3) $$\forall x \in X, \quad B_f(x) \geq 0 .$$

With this definition we obtain the following result.

Proposition 1.

Let $\{x^t\}_{t \geq 0}$ be any solution of the discrete dynamical system (14) of Section 1. Let f be a Lyapunov function for S. Then, for all t

(4) $$f(x^{t+1}) - f(x^t) \leq -kB_f(x^{t+1}) \leq 0 .$$

Proof.

Since f is convex, we have

(5) $$\frac{1}{k}(f(x^{t+1}) - f(x^t)) \leq Df(x^{t+1})(-\frac{x^{t+1} - x^t}{k}) .$$

Since $-\frac{x^{t+1} - x^t}{k} \in S(x^{t+1})$, inequalities (2) and (5) imply

(6) $$f(x^{t+1}) - f(x^t) \leq -kB_f(x^{t+1}) .$$

Hence (4) follows from (6) and (3). ■

Remark.

If f is a Lyapunov function for S, then the sequences $t \to f(x^t)$ decrease for any trajectory of the discrete dynamical system (14) of Section 1. If $f_1, \ldots, f_n$ are n Lyapunov functions for S, then the subsets $\rho^-(x)$ are invariant; if x is a Pareto minimum, the subsets $\rho^-(x)$ are stable in the sense that for any neighborhood M of $\rho^-(x)$, there exists a neighborhood N of $\rho^-(x)$ such that, for any initial value $x^0 \in N$, all the trajectories starting at x^0 remain in M (the proof is analogous to that of Theorem 3). For related results, see the book of Brauer and Nohel, the papers of Champsaur, Champsaur-Dreze-Henry, Lassalle, Maschler-Peleg, Uzawa, Yoshizawa, etc.

Remark.

We can extend these results by assuming only that f is locally Lipschitz and by replacing the derivative from the right by the generalized directional derivative

$$\tilde{D}f(x)(z) = \limsup_{\substack{\theta \to 0+ \\ y \to x}} \frac{f(y + \theta z) - f(y)}{\theta} \tag{7}$$

introduced by F. H. Clarke. ■

Remark.

Since the inequality

$$\sum_{i=1}^{n} \lambda^i B_{f_i}(x) \le B_{\sum_{i=1}^{n} \lambda^i f_i}(x) \tag{8}$$

obviously holds when $\lambda \in \mathbb{R}^n_+$, then the functions $\sum_{i=1}^{n} \lambda^i f_i$ are Lyapunov functions for S if (and only if) the functions f_i are Lyapunov functions.

Remark.

We can also use Lyapunov functions to study asymptotic stability.

Let $f_1, \ldots, f_n$ be n Lyapunov functions. Let us assume that X is compact and that

$$(9)\quad \begin{cases} \text{whenever } x \text{ is not a Pareto minimum, there exist } \lambda \in \mathbb{R}^n_+, \\ \lambda \neq 0 \text{ and } \varepsilon > 0 \text{ such that } B_{\sum_{i=1}^n \lambda^i f_i}(y) \geq \varepsilon \text{ whenever} \\ f_i(y) \geq f_i(x) \text{ for all } i = 1, \ldots, n . \end{cases}$$

Then, for any solution $\{x^t\}$ of the discrete dynamical system (14) of Section 1 cluster points of $\{x^t\}$ are Pareto minima.

Proof.

Since the sequences $f_i(x^t)$ are bounded from below and decreasing, they converge to scalars c_i. Since X is compact, subsequences of x^t converge to elements $\bar{x}$ satisfying $f_i(\bar{x}) = \inf_{t \geq 0} f_i(x^t)$.

Let us assume that $\bar{x}$ is not a Pareto minimum: Assumption (9) implies that there exist $\lambda \in \mathbb{R}^n_+$, $\lambda \neq 0$ and $\varepsilon > 0$ such that $B_{\sum_{i=1}^n \lambda^i f_i}(x^t) > \varepsilon$ for all t. By Proposition 1, we obtain:

$$(10)\qquad \sum_{i=1}^n \lambda^i f_i(x^{t+1}) - \sum_{i=1}^n \lambda^i f_i(x^t) \leq -\varepsilon .$$

By adding these inequalities from $t = 0$ to $t = s - 1$, we deduce that $\sum_{i=1}^n \lambda^i f_i(x^s) \leq \sum_{i=1}^n \lambda^i f_i(x^0) - \varepsilon s$, which converges to $-\infty$ when $s \to \infty$. This is impossible. ∎

Remark.

Let us recall that if f is convex and continuous, its subdifferential $\partial f(x)$ is not empty and that $Df(x)(z) = \sup_{p \in \partial f(x)} \langle p, z \rangle$. Hence we can write

$$B_f(x) = \inf_{z \in S(x)} \sup_{p \in \partial f(x)} \langle p, z \rangle = \sup_{p \in \partial f(x)} \inf_{z \in S(x)} \langle p, z \rangle \tag{11}$$

(since $S(x)$ is closed and convex and $\partial f(x)$ is compact and convex). It is worthwhile to compare this formula with formula (9) of Section 1. ■

Examples of Lyapunov functions.

Let U be a Hilbert space.

Let us consider the correspondence $S = \partial g$ where g is a continuous convex function defined on an open convex subset. Then the functions $f_1 = g$ and $f_2(y) = \frac{1}{2} \|y - \bar{x}\|^2$ where $\bar{x}$ minimizes g are Lyapunov functions of $S = \partial g$.

Indeed, let $z \in \partial g(x)$. Since $Dg(x)(z) = \sup_{p \in \partial g(x)} \langle p, z \rangle$, we deduce that $\|z\|^2 \leq Dg(x)(z)$. Hence

$$B_g(x) \geq \inf_{z \in \partial g(x)} \|z\|^2 \geq 0$$

and g is a Lyapunov function for ∂g.

Let us consider now the case where $f(y) = \frac{1}{2} \|y - \bar{x}\|^2$. Since $Df(y)(z) = \langle y - \bar{x}, z \rangle$, we deduce that

$$B_f(x) = \inf_{z \in \partial g(x)} \langle x - \bar{x}, z \rangle = - \sup_{z \in \partial g(x)} \langle \bar{x} - x, z \rangle = -Dg(x)(\bar{x} - x) .$$

Since g is convex, we know that

$$Dg(x)(\bar{x} - x) \leq g(\bar{x}) - g(x) .$$

Hence

$$B_f(x) \geq g(x) - g(\overline{x}) \geq 0 \ ,$$

and f is a Lyapunov function for ∂g when $g(\overline{x}) = \inf_{x \in X} g(x)$. ■

More generally, let us assume that $\overline{x}$ satisfies

(12) $$\sigma^{\#}(S(x), \overline{x} - x) \leq 0 \quad \text{for all} \quad x \in X \ .$$

Hence the function f defined by $f(y) = \frac{1}{2}\|y - \overline{x}\|^2$ is a Lyapunov function for S. This kind of condition is used in mathematical economics (see Arrow-Hahn for instance). ■

The problem arises whether there exists a solution $\overline{x}$ of (12). Such a solution exists, not only in the case of $S = \partial g$, but in the more general case when, for example, S is a monotone correspondence:

<u>Let</u> S <u>be a monotone correspondence and</u> $\overline{x} \in X$ <u>a solution of the variational inequalities for</u> S <u>in the sense that</u>

$$\exists \overline{p} \in S(\overline{x}) \quad \text{such that} \quad \langle \overline{p}, \overline{x} - y \rangle \leq 0 \quad \forall y \in X \ .$$

<u>Then the function</u> $f(y) = \frac{1}{2}\|y - \overline{x}\|^2$ <u>is a Lyapunov function for</u> S.

Indeed, since S is monotone, we obtain for any $q \in S(y)$, $\langle q, \overline{x} - y\rangle \leq \langle \overline{p}, \overline{x} - y\rangle \leq 0$. Hence $\sigma^{\#}(S(y), \overline{x} - y) \leq 0$ for all $y \in X$. Recall that a solution $\overline{x}$ of the variational inequalities exist when X is convex compact and S is a monotone correspondence with nonempty convex compact values whose restriction to all finite dimensional spaces is upper hemi-continuous (see Brezis for instance). ■

Proposition 1 can be extended to the case of a continuous dynamical system (1) of Section 2. For that purpose, we need

Lemma 1.

Let f be a function satisfying (1). Let x be a differentiable function from $[0, T]$ into X. Then the Dini derivative $\left(\frac{d}{dt}\right)_+ [f(x(t))] = \limsup_{h \to 0+} \frac{f[x(t+h)] - f(x(t))}{h}$ satisfies for almost all t the inequality

$$(13) \qquad \left(\frac{d}{dt}\right)_+ f[x(t)] + Df(x(t))\left(-\frac{dx}{dt}\right) \leq 0 \,.$$

Proof.

Indeed, the convexity of f implies

$$(14) \qquad Df(x(t+h))\left(\frac{x(t) - x(t+h)}{h}\right) \leq \frac{f(x(t)) - f(x(t+h))}{h} \,.$$

Since $Df(y)(z) = \inf_{\theta \in [0,1]} \frac{f(y + \theta z) - f(y)}{\theta}$ is the infinimum of the continuous function $\{y, z, \theta\} \to \frac{f(y + \theta z) - f(y)}{\theta}$ when θ ranges over the compact set $[0,1]$, we deduce that $\{y, z\} \to Df(y)(z)$ is continuous. Hence we obtain (13) by letting h converge to 0 in (14). ∎

Proposition 2.

Let $x(\cdot)$ be any solution of the continuous dynamical system (1) of Section 2. Let f be a Lyapunov function for S. Then, for almost all t

$$(15) \qquad \left(\frac{d}{dt}\right)_+ f[x(t)] \leq -B_f(x(t)) \leq 0 \,.$$

Proof.

Since $-\frac{dx}{dt}(t) \in S(x(t))$ for almost all t, we deduce (15) from (2), (3) and (13).

5. Other discrete dynamical systems

The discrete dynamical system (14) of Section 1 is a particular example of a family of discrete dynamical systems.

$$x^{t+1} - x^t \in -kS(\theta x^{t+1} + (1-\theta)x^t) \tag{1}$$

where $\theta \in [0,1]$. For $\theta = 1$, we obtain the "implicit" discrete system (14) of Section 1. When $\theta = 0$, it is the so-called <u>explicit</u> dynamical system:

$$x^{t+1} - x^t \in -kS(x^t) \; . \tag{2}$$

For $\theta = 1/2$, we obtain the Crank-Nicholson system. In order to prove the existence of monotone trajectories of the system (1), we need a consistency assumption between S and Γ which depends upon θ:

Definition 5.

We shall say that S and Γ are "θ-consistent" on X if $\forall p \in U^*$, $\forall \lambda \in \mathbb{R}^n_+$, for any $x \in X$ which minimizes $y \to \langle \lambda, \Gamma(y)\rangle - \langle p, y\rangle$, we have

$$\langle \lambda, \Gamma(x)\rangle - \langle p, x\rangle \leq \inf_{y \in X} \left[\langle \lambda, \Gamma(y)\rangle - \langle p, y\rangle + k\sigma^{\#}(S(\theta x + (1-\theta)y; p)\right] . \tag{3}$$

Remark.

It is clear that the θ-consistency implies the consistency.

Theorem 4.

Let us assume that properties (1), (3) and (4) of Section 1 hold. If S and Γ are θ-consistent, there exists a solution $\{x^t\}_t$ of the discrete dynamical system (1) satisfying

$$\Gamma(x^{t+1}) \leq \Gamma(x^t) \quad \text{for all} \quad t \geq 0 \tag{4}$$

for any initial condition $x^0 \in X$.

Proof.

The proof is analogous to the proof of Theorem 1 and so we only sketch it. Assume that no solution exists. Then for any $x \in X$, either, there exists $p \in U^*$ such that $x \in V_p = \{y \in X$ such that $\langle p, x^t \rangle > \langle p, y \rangle + k\sigma^{\#}(S(\theta y + (1-\theta)x^t);p)\}$ or there exists $\lambda \in \mathbb{R}^n_+$ such that $x \in V_\lambda = \{y \in X$ such that $\langle \lambda, F(y) - F(x^t) \rangle > 0\}$ or both. Since X is compact, it can be covered by a finite union of these open subsets V_{λ_i} $(i = 1, \ldots, k)$ and V_{p_j} $(j = 1, \ldots, \ell)$. We introduce a continuous partition of unity $\{\alpha_i, \beta_j\}_{\substack{i=1,\ldots,k \\ j=1,\ldots,\ell}}$ subordinate to this covering and the function a defined by

$$a(x, y) = \sum_{i=1}^{k} \alpha_i(x)\langle \lambda_i, F(x) - F(y) \rangle - \sum_{j=1}^{\ell} \beta_j(x)\langle p_j, x - y \rangle . \tag{5}$$

The assumptions of the Ky Fan theorem are satisfied: hence there exists $\bar{x}$ such that $\sup_{y \in X} a(\bar{x}, y) \leq 0$. If we set $\bar{\lambda} = \sum_{i=1}^{k} \alpha_i(\bar{x})\lambda_i$ and $\bar{p} = \sum_{j=1}^{\ell} \beta_j(\bar{x})p_j$, we have

$$\langle \bar{\lambda}, F(\bar{x}) - F(y) \rangle + \langle \bar{p}, \bar{x} - y \rangle \leq 0 \ \forall y \in X . \tag{6}$$

The θ-consistency implies that (3) holds. We shall contradict it. Indeed, there exists i such that $\alpha_i(\bar{x}) > 0$ or there exists j such that $\beta_j(\bar{x}) > 0$. If $\alpha_i(\bar{x}) > 0$, then $\bar{x} \in V_{\lambda_i}$ and $\langle \lambda_i, F(\bar{x}) - F(x^t) \rangle > 0$. Hence $\langle \bar{\lambda}, F(\bar{x}) \rangle > \langle \bar{\lambda}, F(x^t) \rangle$. If $\beta_j(\bar{x}) > 0$, then $\bar{x} \in V_{p_j}$ and $\langle p^j, x^t \rangle > \langle p^j, \bar{x} \rangle + k\sigma^{\#}(S(\theta\bar{x} + (1-\theta)x^t);p^j)$. Hence $-\langle \bar{p}, \bar{x} \rangle > k\sigma^{\#}(S(\theta\bar{x} + (1-\theta)x^t);\bar{p}) - \langle \bar{p}, x^t \rangle$.

Therefore,

$$(7)\quad \begin{cases} \langle\bar{\lambda}, F(x^t)\rangle - \langle\bar{p}, x^t\rangle + k\sigma^{\#}(S(\theta\bar{x} + (1-\theta)x^t);\bar{p}) < \\ \langle\bar{\lambda}, F(\bar{x})\rangle - \langle\bar{p}, \bar{x}\rangle = \inf_{y\in X} [\langle\bar{\lambda}, F(y)\rangle - \langle\bar{p}, y\rangle] . \end{cases} \quad \blacksquare$$

Remark.

In the case where $\theta = 0$, we obtain the following result.

Proposition 3.

Let us assume that (1), (3) and (4) of Section 1 hold. For any $t \geq 0$, there exists a solution x^{t+1} of

$$(8)\qquad x^{t+1} - x^t \in -kS(x^t) \quad\text{and}\quad F(x^{t+1}) \leq F(x^t)$$

if and only if

$$(9)\quad \begin{cases} \forall\lambda \in \mathbb{R}^n_+,\ \forall p \in U^*,\ \inf_{y\in X} (\langle\lambda, F(y)\rangle - \langle p, y\rangle) \\ \leq \langle\lambda, F(x^t)\rangle - \langle p, x^t\rangle + k\sigma^{\#}(S(x^t);p) . \end{cases}$$

Proof.

Indeed, if x^{t+1} satisfies (8), we deduce that $\langle\lambda, F(x^{t+1})\rangle \leq \langle\lambda, F(x^t)\rangle$ and that $-\langle p, x^{t+1}\rangle \leq -\langle p, x^t\rangle + k\sigma^{\#}(S(x^t);p)$ for any $\lambda \in \mathbb{R}^n_+$ and $p \in U^*$. Hence (8) implies (9).

Conversely, the proof of Theorem 4 with $\theta = 0$ shows that the non-existence of a solution of x^{t+1} of (1) implies (7) with $\theta = 0$, which contradicts (9). $\blacksquare$

Remark.

In the case where $\theta = 0$, we do not need to assume that S is upper hemi-continuous in Proposition 3 (since the subsets $V_p = \{y \in X$ such that $\langle p, x^t\rangle > \langle p, y\rangle + k\sigma^{\#}(S(x^t);p)\}$ are always open).

The stability results of the implicit discrete system remain true for the other discrete systems. $\blacksquare$

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
1671 (14) MRC-TSR-1671

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER
(9) Technical

4. TITLE (and Subtitle)
(6) MONOTONE TRAJECTORIES OF MULTIVALUED DYNAMICAL SYSTEMS,

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
(10) Jean-Pierre Aubin, Arrigo Cellina and John Nohel

8. CONTRACT OR GRANT NUMBER(s)
(15) DAHC04-74-G-0012, DAAG29-75-C-0024, MCS75-21868
NSF

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
(12) 36p.

11. CONTROLLING OFFICE NAME AND ADDRESS
See Item 18 below.

12. REPORT DATE
(11) Aug [redacted] 76

13. NUMBER OF PAGES
33

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
dynamical system
stability
critical points
monotone trajectories
Pareto minima
Lyapunov function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We prove existence of "monotone trajectories" for a class of discrete and continuous systems sufficiently general to include problems of some interest in economic and biological theory. We prove existence of critical points which are Pareto minima. We study stability properties of Pareto minima.

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