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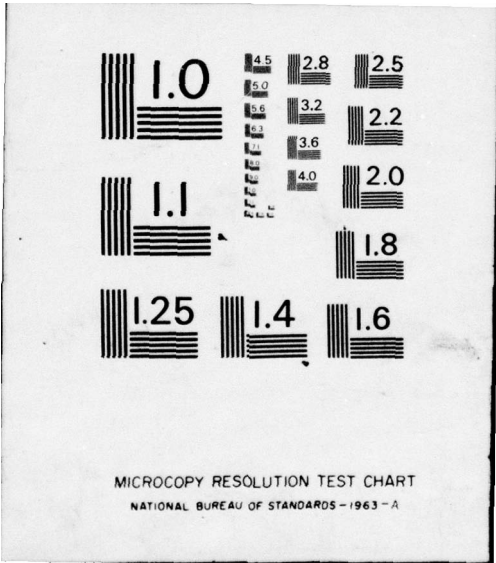UNCLASSIFIED   NUSC-TR-5439   NL

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An Alternate Approach to Optimum Bearing Estimation

Wolfgang K. Fischer

Special Projects Department

27 September 1976

NUSC

NAVAL UNDERWATER SYSTEMS CENTER

Newport, Rhode Island ● New London, Connecticut

PREFACE

This report was prepared under NUSC Project No. A-750-12, "Narrow Band Bearing Tracker," Principal Investigator, Dr. W. K. Fischer (Code 315), Navy Subproject No. ZR-0000101, sponsoring activity NAVSEA, Code 06H-1.

The Technical Reviewers for this report were Dr. G. C. Carter (Code 313) of NUSC and Professor P. Schultheiss of Yale University.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An alternate processing scheme for bearing estimation locates the peaks of the acoustic energy distribution in k-ω space and combines the estimates obtained at each temporal frequency under a least mean-square-error criterion. This processor is shown to be optimum in the sense that the variance of the resultant bearing estimate is equal to the Cramer-Rao lower bound under the same conditions assumed by MacDonald and Schultheiss. The relevance of this processing scheme for bearing estimation of impulsive signal spectra is briefly discussed in the context of recent sonar systems.

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TABLE OF CONTENTS

AN ALTERNATE APPROACH TO OPTIMUM
BEARING ESTIMATION

INTRODUCTION

The optimum passive bearing estimation problem is examined in this report from a "k-ω space" point of view. (The two-dimensional Fourier transform of the acoustic field in the space-time aperture generates the energy distribution in the k-ω domain.) Since the representation of the acoustic field in k-ω space has proved useful in a number of applications and since recent sonar systems essentially generate a sampled function of the energy distribution in k-ω space, this viewpoint is expected to provide additional insight into the processing required for optimum bearing estimation.

In particular, guided by an intermediate result of MacDonald and Schultheiss[1] and a result by Bouvier and Ianniello,[2] it is postulated that the maximum likelihood estimator (MLE) merely determines an estimate of the location of the primary peaks of the energy distribution in k-ω space and combines these estimates, through a linear combination, to arrive at an overall "best" estimate of the target bearing. This postulate is examined in this report, and the performance of such an estimator is compared with the Cramer-Rao lower bound under the same conditions assumed by MacDonald and Schultheiss.

PRELIMINARIES

Consider a finite, one-dimensional array to be described by an effective sensor density p(x), where p(x) includes the effect of sensor location and sensor shading and is considered to be zero for values of $|x| > L/2$, where L is the total length of the array. In practice, p(x) is usually discrete, but mathematically it may be treated as a continuous function of x multiplied by an appropriate sampling function. If h(x, t) represents the acoustic field at any point x and time t, then the output from a conventional time delay beamformer steered to an angle θ with respect to broadside is

$$b (\xi , t) = \int_{-\infty}^{\infty} dx \, h(x, t + \xi x) \, p(x), \tag{1}$$

where $\xi \equiv \dfrac{\sin \theta}{c}$ and c is the speed of sound in the medium.

Furthermore, consider that this beam output is spectrum analyzed over a sliding temporal window of duration T. The spectrum analyzed beam output is then described by

$$B = 1/T \int_{t' - T/2}^{t' + T/2} dt\ b\ (\xi,\ t)\ e^{-i2\pi\ ft}. \tag{2}$$

Substituting (1) into (2) and assuming* that $T \gg L/c$ yields

$$B = 1/T \iint_{-\infty}^{\infty} dx\ dt\ h(x,\ t)\ p(x)\ \Pi\left(\frac{t-t'}{T}\right) e^{-i2\pi ft}\ e^{i2\pi f\xi x}, \tag{3}$$

where $\Pi\ \frac{(t-t')}{T}$ is the familiar rectangle function centered about t' and of unit height and duration T.

For comparison purposes, the direct two-dimensional Fourier transform of the acoustic field $h(x,t)$, as seen through a spatial aperture described by $p(x)$ and a temporal aperture described by $1/T\ \Pi\ \frac{(t-t')}{T}$, is

$$\mathcal{B} = 1/T \iint_{-\infty}^{\infty} dx\ dt\ h(x,\ t)\ p(x)\ \Pi\ \frac{(t-t')}{T}\ e^{-i2\pi ft}\ e^{-i2\pi f\xi x}, \tag{4}$$

which represents the acoustic field in k-ω space, that is, at a temporal frequency $\omega = 2\pi f$ and a spatial frequency $k = \xi\omega = \frac{\omega \sin \theta}{c}$. Comparison of (3) and (4) shows that $B(k, \omega) = \mathcal{B}(-k, \omega)$. Thus, spectrum analyzing the various beams will generate the acoustic field in k-ω space, which merely reaffirms the fact that time delay beamforming is equivalent to performing a spatial Fourier transform.

In practice, (3) must be modified by an appropriate sampling function since beams are formed only at discrete values of ξ and are spectrally analyzed only at discrete temporal frequencies. For the initial analysis, however, the spectrum analyzed beam output $B\ (\xi,\ f)$ is considered to be a continuous function of ξ and f. Appropriate modifications are made later.

In general, the acoustic field $h(x,\ t)$ will consist of a signal and noise component. If the signal is a plane wave arriving at an angle θ_0 with respect to the normal of the x-axis, as shown in figure 1, $h(x,t)$ may be expressed as

$$h\ (x,\ t) = s\ (t-\xi_0 x) + n\ (x,\ t),$$

where $\xi_0 = \frac{\sin \theta_0}{c}$. Thus, provided the noise exhibits no spatial structure, the

---

*This assumption implies that the observation time T is much greater than the travel time of the signal wavefront across the array.

acoustic energy distribution $y(\xi, f) = |B(\xi, f)|^2$ will exhibit a distinct maximum when $\xi = \xi_0$. In accordance with the postulate mentioned earlier, it is desired to obtain the mean and variance of the estimate $\hat{\xi}_0$ at which y is a maximum.
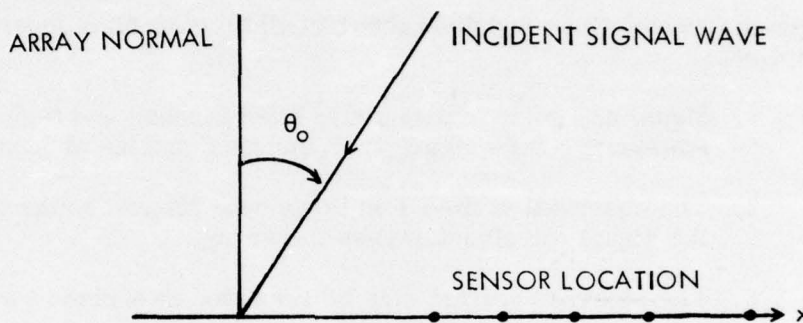
ARRAY NORMAL   INCIDENT SIGNAL WAVE

θ_0

SENSOR LOCATION

x

Figure 1. Array Geometry

MEAN AND VARIANCE OF ESTIMATING THE LOCATION OF THE GLOBAL PEAK OF ENERGY DISTRIBUTION IN k-ω SPACE

Consider the spectrum analyzed beam output $B(\xi, f)$, as given by (3), to be a continuous function of ξ and f. As shown in figure 2, the proposed processor searches for the global peak of the square of the magnitude of B.

$B(\xi, f)$ → $|\ |^2$ → $y(\xi, f)$ → SEARCH FOR PEAK → $\hat{\xi}_0$
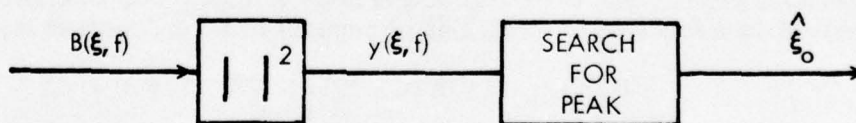
Figure 1. Proposed Processing Scheme

Accordingly, at a given temporal frequency f, the value of $\xi = \hat{\xi}_0$ at which y is a peak is considered to be an estimate of the target location. The mean and variance of this estimate are given by

$$\text{var}(\hat{\xi}_0) = \frac{\text{var}(y')}{\left[\dfrac{\partial E(y')}{\partial \xi}\right]^2}\Bigg|_{\xi = \xi_0} \qquad E(\hat{\xi}_0) = \xi_0 - \frac{E(y')}{\dfrac{\partial E(y')}{\partial \xi}}\Bigg|_{\xi = \xi_0},$$

where $y' = \frac{\partial}{\partial \xi} y(\xi, f)$. Since taking the expectation is a linear operation, it may readily be shown that $\frac{\partial E(y')}{\partial \xi} = E(y'')$, so that the above relations become

3

$$\text{var}(\hat{\xi}_0) = \frac{E(y'^2) - E^2(y')}{E^2(y'')}\Bigg|_{\xi=\xi_0} \qquad E(\hat{\xi}_0) = \xi_0 - \frac{E(y')}{E(y'')}\Bigg|_{\xi=\xi_0} \cdot (6)$$

The various expectations required above shall be computed under the following assumptions:

1. Signal and noise are spatially homogeneous and temporally stationary, zero mean, independent Gaussian processes.

2. The observation time T is large with respect to the travel time of the signal wavefront across the array.

3. The signal wavefront may be regarded as a plane wave.

Later assumptions shall include:

4. The noise is spatially incoherent and uniform.

5. The effective sensor density is discrete and uniform (no shading).

6. The observation time is large with respect to the signal and noise correlation times.

Throughout this analysis, one will require the crosscorrelation between two separate points in the space-time aperture. The assumption of spatial homogeneity and temporal stationarity allows this correlation to be expressed as a function of spatial and temporal differences; that is,

$$E\left[h(x_1, t_1)\, h(x_2, t_2)\right] = R_h(x, \tau), \qquad (7)$$

where $x = x_1 - x_2$ and $\tau = t_1 - t_2$.

Assumption 3 allows the further simplification:

$$R_h(x, \tau) = R_s(\tau - \xi_0 x) + R_n(x, \tau), \qquad (8)$$

where R_s and R_n are the signal and noise correlation functions, respectively. Notice that R_s depends only on the difference $(\tau - \xi_0 x)$. Finally, assumption 4 (uncorrelated noise) allows $R_h(x, \tau)$ to be expressed as

$$R_h(x, \tau) = R_s(\tau - \xi_0 x) + \delta(x) R_n(\tau), \qquad (9)$$

where $\delta(x)$ is the impulse function and $R_n(\tau)$ refers to the noise autocorrelation function at each sensor.

The derivation of the expectations required in equation (6), which is relatively straightforward but tedious, is presented in appendix A. The results are quite general—being subject only to the first two assumptions listed above. In particular, it is shown that, provided $f > 1/T$, the expectation and variance of $\hat{\xi}_0$ is

$$E\,(\hat{\xi}_0) = \xi_0 - \left.\frac{\iint dx d\tau \phi(p * p)\,(i\omega x)}{\iint dx d\tau \phi(p * p)\,(i\omega x)^2}\right|_{\xi=\xi_0} \tag{10}$$

$$\mathrm{var}\,(\hat{\xi}_0) = \frac{1}{\omega^2 \left[\iint dx d\tau \phi(p * p)\,x^2\right]^2}$$

$$\cdot \left\{ 2\,\mathrm{Re}\left[\left|\iint dx d\tau\,\phi\,(p * xp)\right|^2 - 2\left(\iint dx d\tau \phi(p * xp)\right)^2\right.\right. \tag{11}$$

$$\left.\left. + \iint dx d\tau \phi(p * p) \iint dx d\tau \phi(xp * xp)\right] - \left(\iint dx d\tau \phi x\,(p * p)\right)^2\right\}\Bigg|_{\xi=\xi_0},$$

where

$$\phi = \phi(x, \tau) \equiv R_h(x, \tau)\,\Lambda(\tau/T)\,e^{-i\omega\tau}e^{i\omega\xi x} \tag{12}$$

$$\Lambda(\tau/T) = \begin{cases} 1 - |\tau|/T & |\tau| < T \\ 0 & \text{otherwise} \end{cases} \quad \text{(Triangle Function)}$$

Re( ) denotes the real part of ( )

* denotes correlation; e.g., $g(x) * u(x) = \int g(t)\,u(x + t)\,dt$

T = observation time

$R_h(x, \tau) = E\left[h(x_1, t_1)\,h(x_1 - x, t_1 - \tau)\right]$ = crosscorrelation between the acoustic field at two separate points in the space-time aperture.

The integrals are to be evaluated at the value of ξ corresponding to the true target bearing θ_0 and, unless otherwise noted, all integrations are from $-\infty$ to $+\infty$.

These expressions show the effect of an arbitrary spatial weighting function $p(x)$ on estimating the location of the peak of the energy distribution at a given temporal frequency $f > 1/T$. As indicated in appendix A, the

expressions may also be modified to include any arbitrary temporal weighting function $w(\tau)$ by simply replacing $\Lambda(\tau/T)$ in (12) by $1/T\left[w(\tau) * w(\tau)\right]$ and restricting temporal frequencies to those greater than the width of the corresponding spectral window.

To gain further insight into the performance of this estimator, it shall now be assumed that the signal is a plane wave and that the noise is spatially uncorrelated and uniform. As shown in appendix B, these assumptions allow each double integral in (10) and (11) to be expressed as a sum of the product of two single integrals. Furthermore, the estimator becomes unbiased since, under these conditions,

$$\iint dx d\tau \phi (p * p) \quad x \Big|_{\xi = \xi_0} = 0.$$

The result, as obtained from appendix B, is that $E(\hat{\xi}_0) = \xi_0$ and

$$\text{var}(\hat{\xi}_0) = \frac{1}{\omega^2 \left[\int x^2 (p * p) \, dx\right]^2}$$

$$\bullet \left\{ N/S \left[2 \int p \, dx \int x^2 (p * p^2) \, dx - \int p^2 dx \int x^2 (p * p) \, dx \right] \right.$$

$$\left. + \left(N/S\right)^2 \int x^2 (p^2 * p^2) \, dx \right\}, \tag{13}$$

where

$$S = S(f) = \int d\alpha G_s(\alpha) \, \text{sinc}^2 T (f-\alpha) = G_s(f) * \text{sinc}^2 Tf$$

$$N = N(f) = \int d\alpha G_n(\alpha) \, \text{sinc}^2 T (f-\alpha) = G_n(f) * \text{sinc}^2 Tf$$

$G_s(f)$ is the signal spectral density

$G_n(f)$ is the noise spectral density.

Note that one may regard $S(f)$ and $N(f)$ as being the signal and noise power, respectively, within the analysis band which is centered on frequency f.

Thus far, no particular restriction has been placed on the sensor density $p(x)$. However, the uncorrelated noise condition (which has been applied) is rather unrealistic unless $p(x)$ is discrete. If it is further assumed that $p(x)$ is uniform (no shading), then p^2 in (13) may be replaced by p. The variance then reduces to

$$\text{var}(\xi_0) = \frac{1}{\omega^2 \left[\int x^2 (p * p) \, dx\right]} \left[(N/S) \int p \, dx + (N/S)^2\right]. \tag{14}$$

Mathematically, a uniform, discrete array consisting of M sensors may be described by a density

$$p(x) = \sum_{j=1}^{M} \delta(x - x_j), \tag{15}$$

where $\delta(x)$ is the impulse function and x_j is the coordinate of the jth sensor.

Substituting (15) into (14) yields

$$\text{var}(\hat{\xi}_0) = \frac{1}{\omega^2 \sum\limits_{j=1}^{M} \sum\limits_{r=1}^{M} (x_j - x_r)^2} \left[M(N/S) + (N/S)^2 \right]. \tag{16}$$

This expression represents the uncertainty, under the first five assumptions listed previously, of locating the global peak of the energy distribution in k–ω space at a single temporal frequency ω. Since $\xi_0 = \dfrac{\sin \theta_0}{c}$, it may readily be shown that

$$\text{var}(\hat{\xi}_0) \cong \left(\frac{\cos \theta_0}{c} \right)^2 \text{var}(\hat{\theta}_0), \tag{17}$$

where $\hat{\theta}_0$ is the estimate of the signal bearing angle with respect to broadside. Thus, the variance of $\hat{\theta}_0$ is

$$\text{var}(\hat{\theta}_0) = \left(\frac{c}{\omega \cos \theta_0} \right)^2 \frac{\left[M(N/S) + (N/S)^2 \right]}{\sum\limits_{j=1}^{M} \sum\limits_{r=1}^{M} (x_j - x_r)^2}. \tag{18}$$

In these expressions, notice that the effect of the signal and noise spectrum is included through the relationships

$$S(f) = G_s(f) * \text{sinc}^2 Tf \tag{19}$$

$$N(f) = G_n(f) * \text{sinc}^2 Tf.$$

In particular, if the observation time is large with respect to the signal and noise correlation times (assumption 6), that is, if G_s and G_n are broad with respect to $\text{sinc}^2 Tf$, then $S(f) \approx 1/T\, G_s(f)$ and $N(f) \approx 1/T\, G_n(f)$. Substituting

7

these spectral densities into (18) and comparing the result with expression (16) of MacDonald and Schultheiss[1] shows that the two expressions are equivalent at a single temporal frequency. Therefore, at a single temporal frequency, the procedure of forming an infinite number of beams and searching for the peak of the energy distribution is optimum in the sense that the uncertainty of this estimate is equal to the Cramer-Rao lower bound (CRLB).

The question of combining the bearing estimates obtained at other temporal frequencies shall now be discussed. For this purpose, assume that each of the beams is spectrum analyzed according to (2) at evenly spaced, discrete frequencies $f_n = n/T$ with $n = 1, 2, 3, \ldots, N_o$. The reason for this choice is that if the temporal periodicity of the acoustic field is matched to the observation time T, then the minimum sampling interval required to properly describe the acoustic field is $1/T$.

Consider now that the peak of the energy distribution is estimated at each temporal frequency f_n. According to the previous analysis, each estimate $\hat{\xi}_n$ may be considered a random variable whose mean is $\xi_o = \dfrac{\sin \theta_o}{c}$ and whose variance σ_n^2 is given by (16) with f replaced by f_n. Furthermore, if T is large with respect to the signal and noise correlation times, it is reasonable to assume that each estimate $(\hat{\xi}_n)$ is statistically independent.

A natural scheme for combining these estimates is to form a new random variable $\xi_{eq.}$, which consists of a linear combination of $\hat{\xi}_n$; that is, let

$$\xi_{eq.} = \sum_{n=1}^{N_o} a_n \hat{\xi}_n , \qquad (20)$$

where N_o represents the number of independent estimates available and a_n is a weighting factor. Under the constraint that the mean of $\xi_{eq.}$ also be equal to ξ_o and that its variance be a minimum, it may be shown that $\xi_{eq.}$ should be formed according to

$$\xi_{eq.} = \frac{1}{\displaystyle\sum_{n=1}^{No} 1/\sigma_n^2} \cdot \sum_{n=1}^{N_o} 1/\sigma_n^2 \, \hat{\xi}_n, \qquad (21)$$

and that the variance of $\xi_{eq.}$ is

$$\text{var}\,(\xi_{eq.}) = \frac{1}{\sum\limits_{n=1}^{N_o} 1/\sigma_n^2} \;. \tag{22}$$

Relation (21) shows that each estimate $\hat{\xi}_n$ should be weighted by the inverse of its variance σ_n^2 . Referring to (16), this implies that estimates having a large ratio,

$$\frac{\omega_n^2}{M\,(N/S) + (N/S)^2} \approx \frac{\omega_n^2}{M}\,(S/N) \quad \text{for S/N} \gg 1/M \;, \tag{23}$$

should be weighted more heavily.

Substituting (16) into (22) and using (17) and the fact that $S\,(f_n) \approx 1/T\,G_s\,(f_n)$ and $N\,(f_n) \approx 1/T\,G_n\,(f_n)$, one obtains

$$\tag{24}$$

$$\text{var}\,(\theta_{eq.}) = \left\{ \frac{\cos^2\theta_o}{c^2} \sum_{j=1}^{M} \sum_{r=1}^{M} (x_j - x_r)^2 \sum_{n=1}^{N_o} \omega_n^2 \frac{G_s^2\,(\omega_n)/G_n^2\,(\omega_n)}{1 + M\,G_s[(\omega_n)/G_n\,(\omega_n)]} \right\}^{-1},$$

where $\theta_{eq.}$ is the best estimate of the target bearing

$$\omega_n = 2\pi f_n = \frac{2\pi n}{T} \quad n = 1,\,2,\,3,\,\ldots,\,N_o$$

ω_{N_o} = the highest frequency processed.

The above expression is identical to the CRLB derived by MacDonald and Schultheiss[1] (their equation 16) and agrees with Carter[3] when M=2. It follows that, under the six conditions listed earlier, the CRLB can be achieved by a processor that estimates the angle θ_n of the peak of the energy distribution in k-ω space at each processed frequency $f_n = n/T$, n = 1, 2, ..., N_o and obtains a weighted sum of the N_o bearing estimates. The particular weighting to be used in the averaging is proportional to the inverse of the uncertainty associated with each estimate.

Thus far it has been assumed that a continuous beamformer is available for estimating ξ_o. Obviously, in practice, only a finite discrete number

of beams can be formed. This dilemma leads directly into the classical inter-polation problem. One practical solution is to form a "large number" of beams and to use a simple interpolation scheme, such as 3-point interpolation, in order to estimate the location of the global peak. Since the beam pattern is known, another approach is to crosscorrelate the finite number of spectrum analyzed beam outputs with the amplitude pattern of the sensor density. This crosscorrelation must be performed at each processed temporal frequency and must precede the squaring operation shown in figure 2. Obviously, the amount of processing required is quite extensive, but it is possible to com-pletely recover the continuous beam output if the original, finite beams are formed at the Nyquist rate.

The interpolation problem serves to highlight the fact that the CRLB can only be <u>approached</u> by any practical estimator.

APPLICATION OF BEARING ESTIMATION SCHEME TO RECENT SONAR SYSTEMS

The "k-ω space" concept has led to an alternate processing scheme which, under the condition of a broad signal spectrum, is capable of achieving the CRLB. In view of the large amount of required processing and the fact that existing split-beam correlators are nearly optimum, this processing scheme is not recommended for the sole purpose of estimating the bearing of a broadband signal. However, the technique is quite attractive for recent sonar systems, particularly since the input quantities required for this estimator, e.g., spectrum analyzed beam outputs, are already available.

When the signal spectrum becomes more impulsive (that is, when one can no longer assume that the observation time T is large with respect to the signal autocorrelation time), (16) and (19) show (with f replaced by f_n) that the variance of ξ_n becomes unduly large, because of the reduced signal-to-noise ratio, unless the processing frequency f_n is aligned on the particular signal tonal frequency f_o. Hence, additional frequency tracking is required. This leads directly to the concept that bearing estimation is actually a two-dimensional search (in k and ω) for the primary peaks of the energy distribution. If the temporal frequency rate is sufficiently high, this frequency tracking may take the form of simple interpolation between fre-quency bins. Notice that the variance of the resultant bearing estimate $\hat{\theta}_n$ at the nominal frequency f_n should correspond to that obtained by DIFAR*-like automatic target followers (ATF) operating on a single signal tonal frequency. As before, if the frequency f_n associated with each estimate is sufficiently separated so that the $\hat{\xi}_n$ are independent, the estimates may be combined "across frequency," in the manner indicated by (21), to obtain an

*DIrection Finding And Ranging.

overall "best" estimate whose variance is given by (22), where now N_o represents the number of independent estimates.

From the previous comments, it is clear that the estimator is also attractive from the viewpoint of replacing the conventional DIFAR-like ATF because

1. The estimator operates in an open loop and is not afflicted by the acquisition and gate widths associated with closed loop trackers.

2. The estimation scheme combines bearing estimates obtained from multiple tonal frequencies. The estimator is therefore more immune to fading tonals than is the conventional ATF.

3. The tracking threshold equals the detection threshold since the estimation scheme merely locates the peak (in frequency and bearing) of detected tonals.

The effects of multiple targets and frequency and bearing dynamics have, thus far, been purposely avoided. It is clear, however, that although the basic bearing estimation procedure remains the same, the effects give rise to a sorting problem. Before the various estimates $\hat{\xi}_n$ (or $\hat{\theta}_n$) can be combined "across frequency," one must ensure that they are, in fact, from the same target.

CONCLUSIONS

It has been shown that the processing scheme suitable for estimating the bearing of a plane wave in spatially incoherent noise is one that locates and linearly combines the estimates of each global peak of the energy distribution in k-ω space. For sonar systems that spectrum analyze each of a finite number of preformed beams, this amounts to estimating, at each processed frequency, the value of $\sin \theta$ at which the magnitude squared of the beam outputs is a maximum. These estimates may then be combined by forming a weighted average where the weighting is proportional to the inverse of the uncertainty associated with each estimate. It has been shown that, for broad signal spectra, such an estimator is capable of achieving the Cramer-Rao lower bound and is, therefore, optimum.

For impulse signal spectra, although a similar procedure is indicated, an analytical expression for the variance of the overall estimate is difficult to obtain — partly because of the unknown statistics of adjacent spectral estimates.

Even for general use with multiple targets and frequency dynamics, the bearing estimation scheme is considerably complicated by the necessary sorting problem.

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Appendix A

DERIVATION OF THE EXPECTATIONS AND VARIANCES

In this appendix we wish to derive the expectation and variance of the estimated coordinate of the global peak in $y = |B|^2$, where B is the spectrum analyzed beam output as expressed by (3) in the text, which is

$$B = B (\xi, f) = 1/T \int\limits_{-\infty}^{+\infty}\!\!\int dx dt\, h(x,t)\, p(x)\, \Pi\, \frac{(t-t')}{T}\, e^{-i2\pi f(t-\xi x)}\,. \qquad \text{(A-1)}$$

For this purpose, only the following two assumptions are made:

1. Signal and noise are spatially homogeneous and temporally stationary, zero mean, Gaussian processes.

2. The observation time T is large with respect to the travel time of the signal wavefront across the array. (This assumption has already been made in expression (A-1).)

In accordance with (6) of the text, we need to evaluate the expectation of derivatives of y as well as the expectation of $(y')^2$. The expectation of derivatives of y may be determined in the following direct manner. From (A-1),

$$y = BB^* = 1/T^2 \iiiint dx_1 dx_2\, dt_1 dt_2\, h(x_1,t_1)\, h(x_2,t_2)\, p(x_1)\, p(x_2)$$

$$\bullet\ \Pi\, \frac{(t_1-t')}{T}\, \Pi\, \frac{(t_2-t')}{T}\, e^{-i\omega(t_1-t_2)}\, e^{i\omega\xi\,(x_1-x_2)}\,, \qquad \text{(A-2)}$$

where $\omega = 2\pi f$ and all integrations are from $-\infty$ to $+\infty$. Now the nth derivative of y with respect to ξ is

$$y^{(n)} = 1/T^2 \iiiint dx_1 dx_2 dt_1 dt_2\, h(x_1,t_1)\, h(x_2,t_2)\, p(x_1)\, p(x_2)\, \Pi\, \frac{(t_1-t')}{T}$$

$$\bullet\ \Pi\, \frac{(t_2-t')}{T}\, \left[i\omega(x_1-x_2)\right]^n\, e^{-i\omega(t_1-t_2)}\, e^{-i\omega\xi(x_1-x_2)}\,, \qquad \text{(A-3)}$$

and the expectation of the nth derivative of y is

$$E\left[y^{(n)}\right] = 1/T^2 \iiint dx_1\, dx_2\, dt_1\, dt_2 \; E\left[h(x_1,t_1)\, h(x_2,t_2)\right] p(x_1)\, p(x_2)$$

$$\bullet \; \Pi \frac{(t_1-t')}{T} \Pi \frac{(t_2-t')}{T}\left[i\omega(x_1-x_2)\right]^h e^{-i\omega(t_1-t_2)}\, e^{i\omega\xi(x_1-x_2)} \quad . \tag{A-4}$$

Because of the spatial homogeneity and temporal stationarity of the acoustic field $h(x,t)$, we may write

$$E\left[h(x_1,t_1)\, h(x_2,t_2)\right] = R_h(x_1-x_2, t_1-t_2) , \tag{A-5}$$

and introduce the following integration variables in (A-4):

$$x = x_1-x_2 \text{ in the integration over } x_2$$

$$\tau = t_1-t_2 \text{ in the integration over } t_2.$$

Then, (A-4) becomes

$$E\left[y^{(n)}\right] = 1/T^2 \iiint dx_1\, dx\, dt_1\, d\tau\, R_h(x,\tau)\, p(x_1)\, p(x_1-x)$$

$$\bullet \; \Pi \frac{(t_1-t')}{T} \, \Pi \frac{(t_1-\tau-t')}{T}\, (i\omega x)^n\, e^{-i\omega\tau}\, e^{i\omega\xi x} \quad . \tag{A-6}$$

If we perform the integration over x_1 and t_1 and note that

$$\int dx_1\, p(x_1)\, p(x_1-x) = p(x) * p(x) \tag{A-7}$$

and

$$\int dt_1\, \Pi \frac{(t_1-t')}{T} \, \Pi \frac{(t_1-t'-\tau)}{T} = \Pi(\tau/T) * \Pi(\tau/T) = T \; \Lambda(\tau/T) , \tag{A-8}$$

where $\Lambda(\tau/T) = \begin{cases} 1-|\tau|/T & |\tau|\le T \\ \\ 0 & \text{otherwise} \end{cases}$

and that $*$ denotes correlation, (A-6) reduces to

$$E\left[y^{(n)}\right] = 1/T \iint dx d\tau \, \phi(x, \tau) \, (i\omega x)^n \left[p(x) * p(x)\right] , \qquad (A-9)$$

where

$$\phi(x, \tau) \equiv R_h(x, \tau) \; \Lambda(\tau/T) \, e^{-i\omega\tau} \, e^{i\omega\xi x} . \qquad (A-10)$$

The evaluation of $E(y'^2)$ is more readily accomplished in an indirect manner. Thus,

$$y = |B|^2 = BB^*$$

and

$$(y')^2 = \left(\frac{\partial y}{\partial \xi}\right)^2 = (BB^{*\prime} + B^*B')^2 = 2\,\text{Re}(B^*B'B^*B' + BB^*B'B'^*) ,$$

where $\text{Re} (\;\;)$ denotes the real part of $(\;\;)$. The expectation of $(y')^2$ then becomes

$$E(y'^2) = 2\,\text{Re}\left[E(B^*B'B^*B') + E(BB^*\,B'B'^*)\right] . \qquad (A-11)$$

Since the spectrum analyzed beam output B is also a zero mean Gaussian process, we may use the familiar "chain rule" for the expectation of four products to obtain

$$E(y'^2) = 2\,\text{Re}\Big\{ 2\left[E(B^*B')\right]^2 + \left[E(BB)\right]^* E(B'B')$$
$$+ E(BB^*)\,E(B'B'^*) + \left|E(BB')\right|^2 + \left|E(B^*B')^2\right| \Big\} . \qquad (A-12)$$

In a manner completely analogous to the derivation of $E\left[y^{(n)}\right]$, it may be shown that

$$E\left[B^{(k)}\overset{*}{B}^{(n)}\right] = 1/T \iint dx d\tau \, \phi(x, \tau) \left[(-i\omega x)^n \, p(x) * (i\omega x)^k \, p(x)\right] \qquad (A-13)$$

and

$$E\left[B^{(k)}B^{(n)}\right] = \frac{e^{-i2\omega t'}}{T} \iint dx d\tau \, \psi(x, \tau) \left[(i\omega x)^n \, p(x)\, e^{i\omega\xi x} * (i\omega x)^k \, p(x)\, e^{i\omega\xi x}\right], \qquad (A-14)$$

where

$$\psi(x, \tau) \equiv R_h(x, \tau) \; \Lambda(\tau/T) \, \text{sinc } 2f\,(T - |\tau|) \qquad (A-15)$$

15

$$\text{sinc } Q \equiv \frac{\sin \pi Q}{\pi Q}$$

$B^{(k)} =$ kth derivative of B with respect to ξ .

Expressions (A-13) and (A-14) allow the evaluation of each of the expectations in (A-12). For example, $E(B'B'^*)$ is obtained from (A-13) with n=k=1 and $E(BB')$ is obtained from (A-14) with k=0, n=1. The complete expression for $E(y'^2)$, in shorthand notation, is

$$E(y'^2) = \frac{2\omega^2}{T^2} \text{ Re } \left\{ \left| \iint \text{dxd}\tau \phi(p * xp) \right|^2 \right.$$

$$- \left| \iint \text{dxd}\tau \ \psi (pe^{i\omega\xi x} * x p e^{i\omega\xi x}) \right|^2$$

$$+ \iint \text{dxd}\tau \phi(p * p) \iint \text{dxd}\tau \phi(xp * xp) \qquad \text{(A-16)}$$

$$- \iint \text{dxd}\tau \ \psi (pe^{-i\omega\xi x} * pe^{-i\omega\xi x}) \iint \text{dxd}\tau \ \psi(xpe^{i\omega\xi x} * xpe^{i\omega\xi x})$$

$$\left. - 2\left[\iint \text{dxd}\tau \phi(p * xp) \right]^2 \right\} .$$

This relationship may be simplified considerably via the following argument. The second term of (A-16) may be expressed as

$$\left| \iint \text{dxd}\tau \ \psi e^{i\omega\xi x} (pe^{i2\omega\xi x} * xp) \right|^2 .$$

Then using (A-15) and (A-10), this may be written as

$$\left| \iint \text{dxd}\tau \ \phi(p * xp) \left[e^{i\omega\tau} \text{ sinc } 2f \ (T - |\tau|) \ \frac{(pe^{i2\omega\xi x} * xp)}{(p * xp)} \right] \right|^2 .$$

Now the magnitude of the term in the brackets [ ] is always much less than one provided $f > 1/T$. Thus, provided we use temporal frequencies in excess of 1/T,

$$\left| \iint \text{dxd}\tau \ \phi(p * xp) \right|^2 \gg \left| \iint \text{dxd}\tau \ \psi(pe^{i\omega\xi x} * xpe^{i\omega\xi x}) \right|^2 .$$

Similarly, the third term of (A-16) will always be much greater than the fourth term if $f > 1/T$. Hence, under this minor restriction of temporal frequencies, (A-16) reduces to

$$E(y'^2) = \frac{2\omega^2}{T^2} \, \mathrm{Re} \left\{ \left| \iint dx d\tau \phi (p * xp) \right|^2 - 2 \left[\iint dx d\tau \phi \, p * xp) \right]^2 \right.$$

$$\left. + \iint dx d\tau \phi (p * p) \bullet \iint dx d\tau \phi (xp * xp) \right\}, \tag{A-17}$$

and from (A-9),

$$E(y') = \frac{i\omega}{T} \iint dx d\tau \phi \, x(p * p) \tag{A-18}$$

and

$$E(y'') = \frac{-\omega^2}{T} \iint dx d\tau \phi \, x^2 (p * p). \tag{A-19}$$

Finally, the substitution of (A-17) through (A-19) into (6) of the text leads to expressions (10) and (11) of the text.

It should be mentioned that the final results may readily be extended to include any arbitrary temporal weighting function $w(\tau)$. This is most easily accomplished by replacing the uniform weighting function $\Pi \frac{(t-t')}{T}$ in (A-1) by $w \frac{(t-t')}{T} = \eta(t-t') \, \Pi \frac{(t-t')}{T}$. This substitution carries through to (A-10), where $\Lambda (\tau/T)$ must be replaced by $1/T \left[\eta(\tau) \, \Pi \, (\tau/T) * \eta(\tau) \, \Pi \, (\tau/T) \right] = 1/T \left[w(\tau) * w(\tau) \right]$. Similarly, $\Lambda(\tau/T)$ sinc $2f \, (T - |\tau|)$ in (A-15) must be replaced by $1/T \left[w(\tau) \, e^{-i\omega\tau} * w(\tau) \, e^{-i\omega\tau} \right]$.

Furthermore, the assumption in (A-16) that the contribution from integrals involving $\psi(x, \tau)$ is negligible in comparison to the contribution from corresponding integrals involving $\phi(x, \tau)$ remains valid provided that f is greater than the width of the spectral window. Thus, in most practical situations, the only modification required to expressions (A-17) through (A-19) is that $\Lambda(\tau/T)$ in (A-10) be replaced by $1/T \left[w(\tau) * w(\tau) \right]$.

Appendix B

SIMPLIFICATION OF THE BIAS AND
RANDOM BEARING ERROR EXPRESSIONS

In this appendix we wish to simplify expressions (A-17) through (A-19) or (10) and (11) in the text by using the following two additional assumptions of the text:

3. The signal wavefront is a plane wave and is incident on the array at an angle θ_0 with respect to the normal of the x-axis.

4. The noise is spatially uncorrelated and uniform.

These two assumptions allow us to express the crosscorrelation between any two points in the space-time aperture as

$$R_h(x, \tau) = R_s (\tau - \xi_0 x) + \delta(x) R_n(\tau), \tag{B-1}$$

where R_s and R_n are the signal and noise autocorrelation functions, respectively; $\delta(x)$ is the familiar impulse function; and ξ_0 is defined as $\frac{\sin\theta_0}{c}$. This effectively uncouples each of the double integrals in (A-17) through (A-19) and allows each to be expressed as the sum of the product of two single integrals-one integration being over space and the other time. To illustrate this, consider an integral of the form

$$I = \iint dx d\tau \; \phi(x, \tau) \; g(x), \tag{B-2}$$

where $g(x)$ depends on the sensor density $p(x)$. Substituting (A-10) and (B-1) into (B-2) yields

$$I = \iint dx d\tau R_s (\tau - \xi_0 x) \; \Lambda (\tau/T) \; e^{-i\omega\tau} \; e^{i\omega\xi x} \; g(x)$$

$$+ \iint dx d\tau \; \delta(x) R_n (\tau) \; \Lambda(\tau/T) \; e^{-i\omega\tau} e^{i\omega\xi x} \; g(x) \; . \tag{B-3}$$

Letting $\tau_1 = \tau - \xi_0 x$ in the first double integral, we obtain

$$I = \iint dx d\tau_1 R_s (\tau_1) \; \Lambda \frac{(\tau_1 + \xi_0 x)}{T} \; e^{-i\omega(\tau_1 + \xi_0 x)} \; e^{i\omega\xi x} \; g(x) \tag{B-4}$$

$$+ \iint dx d\tau \; R_n (\tau) \; \Lambda(\tau/T) \; e^{-i\omega\tau} \delta(x) \; g(x) \; e^{i\omega\xi x} \; .$$

Now, the maximum value that x can attain is $\pm L/2$, where L is the total length of the array. Thus, provided $T \gg L/c$,

$$\Lambda \frac{(\tau_1 + \xi_0 x)}{T} \approx \Lambda(\tau_1/T) \, .$$

Using

$$\int dx \delta(x) \, e^{i\omega\xi x} \, g(x) = g(0)$$

reduces (B-4) to

$$I = \int d\tau R_s(\tau) \quad \Lambda(\tau/T) \, e^{-i\omega\tau} \bullet \int dx \, g(x) \, e^{i\omega(\xi-\xi_0)x}$$

$$+ \, g(0) \quad \bullet \int d\tau R_n(\tau) \quad \Lambda(\tau/T) \, e^{-i\omega\tau} \, . \tag{B-5}$$

An equivalent expression for the above integral in terms of the signal and noise autospectral densities, $G_s(f)$ and $G_n(f)$, can be obtained by noting that

$$1/T \int d\tau R_s(\tau) \quad \Lambda(\tau/T) \, e^{-i\omega\tau} = \int d\alpha G_s(\alpha) \, \text{sinc}^2 T(d-f)$$

$$\tag{B-6}$$

$$= G_s(f) * \text{sinc}^2 Tf \equiv S(f)$$

and

$$1/T \int d\tau R_n(\tau) \quad \Lambda(\tau/T) \, e^{-i\omega\tau} = \int d\alpha G_n(f) \, \text{sinc}^2 T(\alpha-f)$$

$$\tag{B-7}$$

$$= G_n(f) * \text{sinc}^2 Tf \equiv N(f) \, ,$$

where S(f) and N(f) are the respective signal and noise power within the processing band centered at frequency f. Using these definitions and evaluating the integral at $\xi = \xi_0$, we obtain

$$\iint dx d\tau \phi(x, \tau) \, g(x) \Big|_{\xi=\xi_0} = \, T \, S(f) \int g(x) dx \, + \, T \, N(f) \, g(0) \, . \tag{B-8}$$

Expression (B-8) may now be used to evaluate each of the double integrals in (A-17) through (A-19). Accordingly, one obtains (in shorthand notation)

$$E(y'^2)\Big|_{\xi=\xi_o} = 2\omega^2 \left\{ \left[S\int dx\, (p*p) + N\, (p*p)_o \right]\left[S\int dx\, (xp*xp) + N\, (xp*xp)_o \right] \right.$$
$$\left. - \left[S\int dx\, (p*xp) + N\, (p*xp)_o \right]^2 \right\} \tag{B-9}$$

$$E(y')\Big|_{\xi=\xi_o} = 0 \tag{B-10}$$

$$E(y'')\Big|_{\xi=\xi_o} = -\omega^2 \, S\int dx \quad x^2\, (p*p), \tag{B-11}$$

where $(\ )_o$ denotes that the resulting function is to be evaluated at $x = 0$.

The fact that $E(y')\big|_{\xi=\xi_o} = 0$ implies that the estimator is unbiased under the assumed conditions.

In a lengthy but straightforward operation, expression (B-9) may be expanded and rewritten as

$$E(y'^2)\Big|_{\xi=\xi_o} = \omega^2 \left\{ SN \left[2\int p dx \int x^2\, (p*p^2)\ dx - \int p^2 dx \int x^2\, (p*p)\ dx \right] \right.$$
$$\left. + N^2 \int x^2\, (p^2 * p^2)\ dx \right\}. \tag{B-12}$$

Notice that the coefficients of S^2 have canceled.

Finally, substitution of (B-10) through (B-12) into (6) in the text yields

$$E\,(\hat{\xi}_o) = \xi_o \quad \text{(estimator is unbiased)} \tag{B-13}$$

$$\text{var}\,(\hat{\xi}_o) = \frac{1}{\omega^2 \left[\int x^2\, (p*p)\ dx \right]^2} \bullet \left\{ (N/S) \left[2\int p dx \int x^2\, (p*p^2)\ dx \right. \right.$$
$$\left. \left. - \int p^2 dx \int x^2 (p*p)\ dx \right] + (N/S)^2 \int x^2\, (p^2 * p^2)\ dx \right\}. \tag{B-14}$$

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