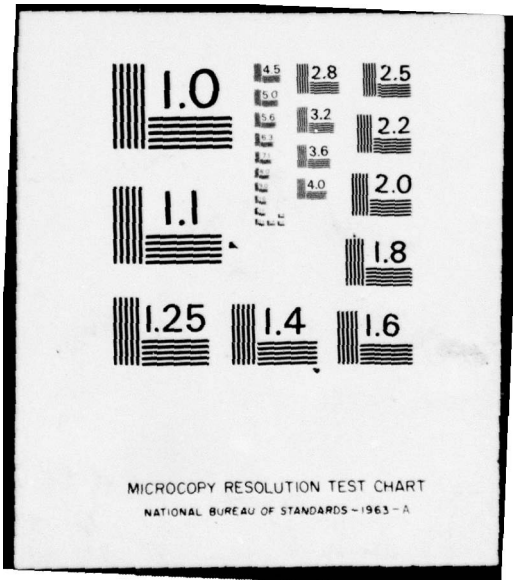
AD-A031 359 HARVARD UNIV CAMBRIDGE MASS F/G 5/3
THE PERMANENT INCOME HYPOTHESIS: A THEORETICAL FORMULATION.(U)
SEP 76 T BEWLEY N00014-67-A-0298-0019
UNCLASSIFIED TR-25 NL

1 OF
AD
A031359

END

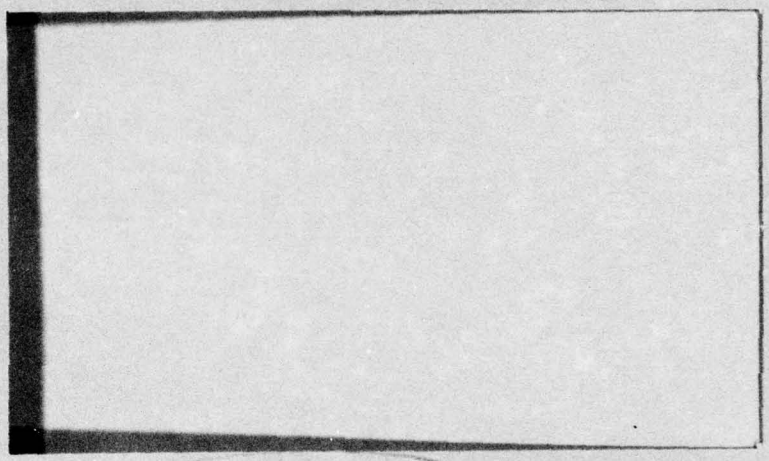
DATE
FILMED
11-76


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS - 1963 - A

PROJECT ON EFFICIENCY OF DECISION MAKING

IN

ECONOMIC SYSTEMS



VE RI TAS

HARVARD UNIVERSITY

THE PERMANENT INCOME HYPOTHESIS

A THEORETICAL FORMULATION*

by Truman Bewley

* The research report here was supported by National Science Foundation Grant No. GS-37899X.

Technical Report No. 25

Prepared under Contact No. N00014-67-A-0298-0019
Project No. NR 047-004
for the Office of Naval Research

Harvard University
1737 Cambridge Street, Room 404
Cambridge, Massachusetts 02138

September, 1976

1. INTRODUCTION

This paper proposes a new formulation of consumer demand functions. The fundamental assumption of the formulation is that a consumer spends his money in such a way that the marginal utility of expenditure equals a constant, which is independent of prices and does not change over time. This assumption will be termed the permanent income hypothesis because of its similarity to the hypothesis of Friedman [7].

The permanent income hypothesis proposed here is somewhat different from that of Friedman. The hypothesis studied here is intended to be a theoretical representation of a consumer's response to short-term fluctuations in prices, his needs, and his income. Because of the theoretical orientation, the hypothesis is stated in terms of an unobservable, fictitious quantity, the marginal utility of expenditure. Friedman is concerned with the relation between consumption spending and current and past income and other variables. His hypothesis is, roughly, that consumption spending consists of a random term plus a multiple of some weighted average of past and expected future incomes (Friedman [7, Chap. III]). Because of Friedman's empirical orientation, his hypothesis is a statement about quantities which are directly or indirectly observable.

Both Friedman's hypothesis and that of this paper are based on the same idea, that consumers try to compensate for economic fluctuations by saving and dissaving. The same idea lies behind the life cycle

model of Modigliani and Brumberg [10], which is very similar to Friedman's hypothesis.

Some economic implications of the permanent income hypothesis of this paper are explored in two other papers [3, 4]. These papers attempt to show that this hypothesis clarifies the economic significance of general equilibrium theory. This paper attempts to clarify and defend the hypothesis itself by stating it precisely and demonstrating when it is applicable.

The permanent income hypothesis may be stated mathematically as follows. Let $U_n(c_n)$ be the utility of the consumption bundle $c_n = (c_{n1}, \ldots, c_{nL})$ purchased in period n. Suppose that U_n is differentiable and defined for all L-dimensional vectors with non-negative components. Let $p_n = (p_{n1}, \ldots, p_{nL})$ be the vectors of prices in period n. The hypothesis asserts that there exists a number $\bar{\lambda}$ such that the following is true for all n and i.

$$\frac{\partial U_n(c_n)}{\partial c_{ni}} \leq \bar{\lambda} p_{ni}, \quad \text{with equality if } c_{ni} > 0.$$

$\bar{\lambda}$ is independent of n, p_n, and of expenditure, $p_n \cdot c_n$.

The permanent income hypothesis may be stated in terms consistent with the usual model of the consumer. If money is included in the list of commodities and if the price of money is set equal to one, then the consumer's transactions must satisfy the accounting identity, $M + p_n \cdot c = M_n$, where M_n is the quantity of money he has at the beginning of period n, c is the vector of commodities purchased, and M is

the quantity of money retained after the purchases. That is, in period n the consumer is restricted to the budget set

$$B_n = \{(M,c) \mid M + p_n \cdot c = M_n,\ M \geq 0,\ c \geq 0\}.$$

The permanent income hypothesis may be interpreted as the following two assertions. (1) The consumer chooses a point in B_n which maximizes a utility function of the form $\bar{\lambda}M + U_n(c)$. (2) M_n is so large that if $(\bar{M},\bar{c})$ is the point chosen in B_n, then $\bar{M} > 0$.

The permanent income hypothesis is a mathematical representation of a consumer who has in mind a definite idea of the value of a dollar and spends money on a commodity up to the point at which the value of the quantity bought with an additional dollar equals the value of the dollar spent. The value of a dollar may also be thought of as the consumer's willingness to spend.

Of course, one cannot believe that in reality a consumer's willingness to spend would be completely fixed. But it is not hard to imagine that under certain conditions it would be nearly fixed, so that it would be reasonable to assume, for simplicity, that it is fixed.

The conditions which could give rise to a nearly constant willingness to spend include the following: (1) fluctuations in the economic environment or the consumer's needs which are frequent, not very large, and tend to offset each other over a period of a few years, (2) nearly correct anticipation by the consumer of the (statistical) nature of the fluctuations, and (3) possession by the consumer of an adequate supply of liquid assets.

Economic common sense indicates that under these conditions,

a consumer would respond to fluctuations largely by saving and dissaving. If he allowed his willingness to spend to fluctuate widely, he could increase his total enjoyment by spending a dollar less when money was worth relatively little to him and a dollar more when money was worth a lot.

However, even if the conditions listed above were satisfied, one would expect the consumer's willingness to spend to vary somewhat. Oscillations in his holdings of liquid assets would certainly affect his attitude toward money. Or, runs of good or bad luck could affect his expectations about the future. But these influences would probably cause only small and gradual changes in the consumer's behavior.

Such changes might be gradual even if the listed conditions were not all satisfied. For instance, though a consumer probably would not be willing or able to compensate for long-term economic oscillations by saving and dissaving, such oscillations would be gradual. A consumer's expectations might be incorrect, but the presence of rapid fluctuations might make it impossible for him to discern his mistake quickly.

The permanent income hypothesis would be completely inappropriate if applied to a consumer living a hand-to-mouth existence or to a consumer subject to abrupt and radical change. A sudden change in expectations or a sudden large scale loss of financial assets certainly could be expected to affect a consumer's willingness to spend. Also, the permanent income hypothesis should not be applied to long-term changes. The consumer's willingness to spend might well change markedly after a long period of time if some long-term change occurred

in his circumstances.

One might accept the permanent income hypothesis on the basis of intuition and common sense alone. But since the hypothesis is stated in terms of marginal utility and since it is traditional to assume that consumers are precise utility maximizers, it is natural to give a demonstration of the hypothesis in terms of the consumer's maximization problem. Most of this paper is devoted to such a demonstration.

There are many ways to formulate the consumer's maximization problem. For instance, the nature of his problem depends on the form of his utility function and on whether he is allowed to borrow.

Since borrowing gives the consumer additional flexibility, it seems clear that the permanent income hypothesis would be easier to demonstrate if borrowing were permitted. For this reason, it is assumed that the consumer cannot borrow.

Since the consumer cannot borrow, his money balances cannot become negative. If he begins period n with M_n units of money and buys a bundle, c_n, during the period at a price vector, p_n, then it must be that $p_n \cdot c_n \leq M_n$. It will be assumed that the consumer receives income at the end of the period. The income received at the end of period n will be denoted by y_{n+1}. (The subscript n+1 appears because the consumer does not necessarily know y_{n+1} at the beginning of period n, but he does know it at the beginning of period n+1.) The consumer begins period n+1 with $M_{n+1} = M_n - p_n \cdot c_n + y_{n+1}$ units of money.

For simplicity, it will be assumed that y_n is a random variable which is not affected by any of the consumer's actions. A more general model would allow the consumer to vary the quantity sold of his endowment

(of labor, for instance) and would allow this quantity to affect his income and utility.

It is convenient, though probably not necessary, to assume that the consumer's utility function is additively separable. That is, the utility of a consumption stream $c_0, c_1, \ldots, c_{N-1}$, is $\sum_{n=0}^{N-1} \delta^n U_n(c_n)$, where c_n is the bundle consumed in period n and δ is the discount rate on utility and $0 < \delta < 1$. It simplifies matters to replace the functions U_n with the indirect utility functions $u_n(x) = \max_{\substack{p_n \cdot c \leq x \\ c \geq 0}} U_n(c)$, where x is the expenditure during period n and p_n is the vector of prices during this period.

u_n will be assumed to be random. Note that this randomness can derive from randomness in both U_n and p_n.

It could be assumed that $\{y_n\}$ and $\{u_n\}$ are sequences of independently and identically distributed random variables. (If this assumption were made, our results would generalize only slightly work of Schechtman [14]. His work is discussed further below.) However, it does not seem appropriate to make such an assumption in the context assumed in this paper. The time periods are to be thought of as short, as days, for instance. Hence, if the random variables were independently and identically distributed, fluctuations would persist for only one day. This seems unreasonable. People learn from the recent past. In a world in which the events of one day are independent of past events, no learning is possible (once the probability distribution of the random variables is known). Also, people have a great deal of information about economic life and use it to anticipate future events. One way to model

this aspect of life is to assume that the consumer's economic environment is described by a stochastic process. The consumer observes this process and knows its probability distribution. Past observations of this process form the consumer's information. Known correlations between past and future events make it possible for him to use this information.

It is assumed here that there is in the background a stationary stochastic process $\{\omega_n\}$, and that u_n and y_n are functions of ω_n. Because y_n and u_n are functions of ω_n, $\{y_n\}$ and $\{u_n\}$ form stationary processes as well. It is to be emphasized that the consumer knows ω_n and that ω_n may contain much information besides knowledge of y_n and u_n. For instance, it may contain the latest weather forecast or knowledge that the consumer has the flu and will be sick for a week. ω_n should be thought of as a vector with many components which express all that the consumer might learn during period n-1. At the beginning of period n, he knows $\{\omega_n: k \leq n\}$.

To assume that a stochastic process $\{\omega_n\}$ is stationary, is to assume, essentially, that the probability distribution of future values of the random variables ω_n is inferrable from knowledge of the (infinite) past. Hence, it is at least somewhat reasonable to assume, as is done here, that the consumer knows the probability distribution of the process, conditional on knowledge of the past.

Stationary processes allow for a wide range of stochastic phenomena. In the next section, an example is given of a process for which the time paths are periodic.

It is natural to formulate the consumer's maximization problem as below. In the expression below, $E(\quad | \mathcal{I}_0)$ denotes the expectation

conditional on the information available to the consumer at the beginning of period zero.

1.1 Problem

$$\max_{x_0,\ldots,x_{N-1}} E\Big(\sum_{n=0}^{N-1} \delta^n u_n(x_n) \,\Big|\, \mathcal{S}_0\Big)$$

subject to:

(1) For all n, x_n is a random variable which depends only on information available at the beginning of period n.

(2) $0 \leq x_n \leq M_n$, for all n.

(3) M_0 is given and for $n > 0$, $M_n = M_{n-1} - x_{n-1} + y_n$.

The discount rate, δ, above, is assumed to be such that $0 < \delta \leq 1$.

If the fluctuations in y_n and u_n are rapid and one is concerned with the behavior of the consumer over short periods of time, then one can think of the horizon N as large and the discount rate δ as close to one. Therefore, the permanent income hypothesis has to do with the asymptotic behavior of the solution to Problem 1.1 as N goes to infinity and δ goes to one.

The results proved in this paper may be sketched easily. Enough assumptions are made so that it is not hard to show that Problem 1.1 has a solution. The u_n are assumed to be differentiable, strictly concave, and increasing, as is usually done in such problems. Certain other more technical assumptions are made. The maximum value of the objective function in Problem 1.1, $V_{N\delta}(M)$, depends differentiably on the initial stock of money, M. The derivative of this function with respect to M, $V'_{N\delta}(M)$, defines the marginal utility of money in period zero for the N-period problem with discount rate δ. This marginal

utility of money is a non-decreasing function of N and δ, as is to be expected. For if δ is increased, greater value is given to future consumption, and if N is increased the consumer has more opportunity to use his money. Certain assumptions are made which put an upper bound on the marginal utility of money. Therefore, $V'_{N\delta}(M)$ converges as N goes to infinity and δ increases to one. The limit, $\lambda_0(M)$, will be called *the limit marginal utility of money*. Clearly, $\lambda_0(M)$ is a non-decreasing function of M.

Observe that $V_{N\delta}(M) \equiv \max_{x_0, \ldots, x_{N-1}} E\left(\sum_{n=0}^{N-1} u_n(x_n) \middle| \mathcal{S}_0\right)$ is a random variable. Hence, $V'_{N\delta}(M)$ and $\lambda_0(M)$ are random variables as well.

If one solves the problem

$$\max_{x_n, \ldots, x_{N+n-1}} E\left(\sum_{k=n}^{N+n-1} \delta^k u_k(x_k) \middle| \mathcal{S}_n\right)$$

for $n > 0$ and goes to the limit with N and δ as above, one obtains a limit marginal utility for period n, $\lambda_n(M)$.

The optimal spending policy for the N-period problem with discount rate δ prescribes expenditures in period n, $x_{N\delta n}(M_n)$, as a function of the consumer's stock of money, M_n, in that period. The functions $x_{N\delta n}$ are non-increasing as functions of N and δ, as is to be expected since money becomes more valuable as N and δ increase. Since $x_{N\delta n}(M) \geq 0$, $\lim_{\substack{N\to\infty \\ \delta\uparrow 1}} x_{N\delta n}(M)$ exists. The limit will be denoted by $x_{\infty n}(M)$.

The functions, $x_{\infty n}$, describe a limit policy. This limit policy corresponds to the limit marginal utility of money, $\lambda_n(M)$, in the sense that $u'_n(x_{\infty n}(M)) \leq \lambda_n(M)$ with equality if $x_{\infty n}(M) > 0$. The limit policy

gives rise to a sequence of money stocks defined as follows: M_0 is given and for $n > 0$, $M_n = M_{n-1} - x_{\infty,n-1}(M_{n-1}) + y_n$. The sequence $\lambda_n(M_n)$ is the time path of the marginal utility of money if the limit program is followed.

To each level, λ, of the marginal utility of money, there corresponds a unique level of expenditure, x, in period n. x is such that $u_n'(x) \leq \lambda$, with equality if $x > 0$. x is, of course, a random variable. It will be shown that there is a unique constant, $\bar{\lambda}$, such that if the marginal utility of money were $\bar{\lambda}$ in every period, then asymptotically the consumer's average spending per period would equal his average income per period. $\bar{\lambda}$ would be the marginal utility of money if the consumer sought to maximize his asymptotic average flow of utility per period and were constrained only to match his asymptotic average spending to his asymptotic average income.

The main theorem is that $\lim_{M\to\infty} \lambda_n(M) = \bar{\lambda}$, for all n. This theorem says that if the horizon were sufficiently distant and the discount rate sufficiently close to one and if the consumer had a sufficient quantity of money, then the optimal marginal utility of expenditure would be nearly constant. Also, the consumer would behave nearly as if he were constrained only to match his average spending and income.

A second theorem (Theorem 3.2) is that $\lim_{n\to\infty} \lambda_n(M_n) = \bar{\lambda}$, for all values of the initial money stock, M_0. This theorem says that if the horizon were sufficiently distant and the discount rate sufficiently close to one and if the consumer followed his optimal program, then he would eventually accumulate enough money to bring the marginal utility of money close to $\bar{\lambda}$.

The interpretation of these theorems would be suspect if the limit policy, $x_{\infty n}$, were somehow wrong. To check that the passage to the limit from $x_{N\delta n}$ to $x_{\infty n}$ is legitimate, it is proved that the limit policy is optimal in the limit average sense (Theorems 3.4 and 3.5).

Most of the results of this paper are proved by Schechtman [14] in a special case. He assumes that the utility functions u_n are not stochastic and do not depend on n. All random variation comes from the y_n, which he assumes to be independently and identically distributed. (Also, he assumes that $\delta = 1$.) He proves that $\lim_{M\to\infty} x_{\infty n}(M) = Ey_0$ and that if y_n is not deterministic, then $\lim_{n\to\infty} M_n = \infty$. The first of these results corresponds to the main theorem here, and the second corresponds to Theorem 3.2 (that $\lim_{n\to\infty} \lambda_n(M_n) = \bar{\lambda}$).

Schechtman's theorem that $\lim_{n\to\infty} M_n = \infty$ is true under the assumptions of this paper, provided that there is sufficient randomness in the incomes or utility functions. This fact is not proved here, but it is worth noticing, for it draws attention to the fact that actual consumer behavior should be thought of as only approximating that implied by the permanent income hypothesis. No consumer would go on accumulating money indefinitely. However, average holdings of liquid assets by American consumers, apparently, are fairly high. Holdings of liquid assets (checking and savings accounts) and investment assets (stocks and bonds) together amount to about 6 to 15 months' income. Holdings of liquid assets alone are equal to one to two months' income [11, pp. 100-103]. However, these figures are very rough, for there is wide variation in the data cited, and the median ratio of assets to income increases sharply with income.

Returning to the work of Schechtman, it is clear that the results reported here simply generalize those of Schechtman. There is, however, an important distinction between his methods and those of this paper. He uses the techniques of dynamic programming. The key arguments of this paper depend on the strong law of large numbers for stationary stochastic processes.

Schechtman's method, essentially, is to study properties of a solution to a limit form of the following dynamic programming equation for $V_{N\delta}(M)$, $V_{N+1,\delta}(M) = \max_{0 \le x \le M} [u(x) + \delta E V_N(M-x+y)]$. In his setting, $V_{N\delta}(M)$ is deterministic, for the random variables, y_n, are mutually independent.

In this paper, $V_{N\delta}(M)$ is a random variable. Because randomness is generated by an arbitrary stochastic process, current information may improve knowledge of the future and so affect the value of $V_{N\delta}(M)$. Since $V_{N\delta}(M)$ is random, the dynamic programming equation for $V_{N\delta}(M)$ loses much of its value.

The proof of the main theorem, given in section 7 of this paper, is the most original part of this paper from a mathematical point of view. It makes no use of dynamic programming, but is based simply on economic common sense and the strong law of large numbers.

The assertion that $\lim_{n\to\infty} \lambda_n(M_n) = \bar{\lambda}$ (Theorem 3.2) is a direct consequence of the main theorem and the convergence theorem for supermartingales. This theorem is proved in section 8. The idea of using the convergence theorem for supermartingales is borrowed from Schectman.

The other two theorems of this paper (Theorems 3.4 and 3.5)

follow directly from the main theorem and Theorem 3.2. They are proved in section 10.

The fact that $V_{N\delta}(M)$ is random introduces certain technical difficulties into the proofs that there exist an optimal N-period program and a limit policy. The difficulties arise from the fact that a conditional expectation appears in the dynamic programming equation for $V_{N\delta}(M)$. This equation is $V_{N\delta}(M) = \max_{0 \leq x \leq M} [u(x) + \delta E(V_{N-1,\delta}(M - x + y_1) | \mathcal{S}_0)]$. Conditional expectations are defined only up to sets of probability zero, and there must be a different such expectation for each of the continuum of possible values of x. If the conditional expectations were not chosen properly, the maximizing value of x might be undefined. (The maximizing value is, of course, a random variable.)

The technical facts needed to deal with this and related problems are gathered in an appendix. With the help of these facts, one can adapt to the context of this paper Schechtman's arguments which prove the existence of optimal and limit policies. These adaptations are made in sections 4 and 5.

Yaari [15] proves another asymptotic result bearing on the permanent income hypothesis. He assumes that the consumer can borrow freely and formulates the consumer's maximization problem as follows.

$$\max_{x_0, \ldots, x_{N-1}} E\left[\sum_{n=0}^{N-2} u(x_n) + u(M_{N-1})\right],$$

where M_0 is given and for $n > 0$

$$M_n = M_{n-1} - x_{n-1} + y_n .$$

Like Schechtman, Yaari assumes that the utility function is deterministic and that the random variables y_n are independently and identically distributed. Yaari's model differs from Schechtman's in that $u(x)$ is defined for all real numbers x and the real variables x_n are not constrained to be non-negative. The only constraint is that consumption spending in the last period be M_{N-1}. Yaari proves that $\lim_{N\to\infty} x_{N0}(M) = E\, y_0$, when $x_{N0}(M)$ is the optimal consumption in the first period if the initial stock of money is M. His methods are similar to Schechtman's in that both use the techniques of dynamic programming.

Yaari's theorem may be generalized to the context of this paper. That is, it may be assumed that utility is random and that all random variation is generated by a stationary process. The proofs are similar to those given here. As is the case here, the key arguments involve only economic intuition and the strong law of large numbers.

The use of stationary processes has appeared elsewhere in economic literature dealing with models related to the one studied here, namely, in Radner [12] and Mirman and Zilcha [9].

2. DEFINITIONS, NOTATION, AND THE MODEL

The Underlying Stochastic Process

The stochastic process known to the consumer is $\{\omega_n : n \text{ is an integer}\}$. (Doob [6] is the reference used for stochastic processes.) The random variables, ω_n, are assumed to take their values in Q-dimensional Euclidean space, R^Q. The underlying sample

space of the process is assumed to be $\Omega = \{(\ldots,\omega_{-1},\omega_0,\omega_1,\ldots): \omega_n \in R^Q \text{ for all } n\}$. ω denotes an element of Ω, and $\omega_n \in R^Q$ denotes the n^{th} subvector of ω. The measurable subsets of Ω are denoted by $\mathcal{S}$, and P denotes the probability on $\mathcal{S}$. $\mathcal{S}$ is the smallest complete σ-field such that all the random variables ω_n are measurable with respect to $\mathcal{S}$. Sets $\mathcal{S}$ such that $P(S) = 0$ are called sets of probability zero. An event occurs almost surely or for almost every ω if it occurs for every ω except for elements of a set of probability zero.

E denotes the expectation operator corresponding to P. That is, if X is a random variable defined on Ω, $EX = \int X(\omega)\,P(d\omega)$.

If $\mathcal{F}$ is a sub-σ-field of $\mathcal{S}$ and X is a random variable on Ω, then $E(X|\mathcal{F})$ denotes the expectation of X conditional on $\mathcal{F}$. That is, $E(X|\mathcal{F})$ is a random variable on Ω which is measurable with respect to $\mathcal{F}$ and satisfies

$$\int_S X(\omega)\,P(d\omega) = \int_S E(X|\mathcal{F})(\omega)P(d\omega)$$

for all $S \in \mathcal{F}$. $E(X|\mathcal{F})$ is well-defined if $E|X| < \infty$. $E(X|\mathcal{F})$ is not uniquely defined, but any two versions of $E(X|\mathcal{F})$ are equal almost surely. (See Doob [6, pp. 17-24] for a discussion of conditional probability.)

$\mathcal{S}_n$ denotes the smallest complete sub-σ-field of $\mathcal{S}$ with respect to which the random variables ω_k, $k \leq n$, are measurable. $\mathcal{S}_n$ represents the information available in period n, and $E(\ \ |\mathcal{S}_n)$ is the expected value operator conditional on this information. For emphasis, it is recalled that $\mathcal{S}_n$ may represent more information than knowledge of past values of y_k and u_k.

The following are the basic assumptions made about the stochastic process $\{\omega_n\}$.

2.1 Assumptions about $\{\omega_n\}$

The stochastic process $\{\omega_n\}$ is stationary and metrically transitive.

Let $\tau : \Omega \to \Omega$ be the shift operator, defined by the formula $(\tau\omega)_n = \omega_{n+1}$. $\{\omega_n\}$ is stationary if and only if τ is probability preserving, that is, $P(\tau S) = P(S) = P(\tau^{-1}S)$, for every measurable set S. A measurable set S is said to be invariant if $P(S) = P(S \cap \tau S) = P(\tau S)$. A process is said to be metrically transitive if every invariant set is of probability zero or one.

Metric transitivity is not a serious restriction, since every process is metrically transitive when restricted to an invariant set.

Stationary processes are quite general. If the ω_n are independently and identically distributed, then the ω_n would form a stationary process. If $\omega_0, \omega_1, \omega_2, \ldots$ are generated by a Markov process with stationary transition probabilities, and if ω_0 is distributed according to a stationary distribution for this Markov process, then $\omega_0, \omega_1, \ldots$ form a stationary process. Every stationary process can be represented as a Markov process with random variables in an infinite dimensional space, the space being $\{(\ldots, \omega_{-1}, \omega_0)\}$. But, many stationary processes cannot be represented as Markov chains with random variables belonging to a finite dimensional vector space. Consider, for instance, the process $\omega_n = \sum_{k=-\infty}^{0} 2^k z_{n-k}$, where the z_k are independently and identically distributed and uniformly bounded.

The realizations of a stationary process may be periodic functions of n. For instance, let u and v be mutually independent, normally distributed random variables with mean zero and variance one. Then, the random variables $\omega_n = u \cos \alpha n + v \sin \alpha n$ form a stationary process where $0 < \alpha < 2\pi$. Observe that one can predict exactly the values of ω_n, for $n > 1$, from knowledge of ω_0 and ω_1.

Processes with independent increments are not stationary. A martingale is not a stationary process.

Frequent use will be made of the following facts.

2.2 Let z be a random variable on Ω and let $z_n(\omega) = z(\tau^n \omega)$. Then, the process $\{ z_n : n \text{ an integer} \}$ is stationary and metrically transitive (Doob [6, p. 458]).

The Economic Model

2.3 Assumptions About Incomes

The income received at the end of period n-1 is a random variable on Ω, $y_n : \Omega \to (-\infty, \infty)$, which is measurable with respect to $\mathcal{F}_n$. The y_n satisfy the following:

i) $y_n(\omega) = y_0(\tau^n \omega)$,

ii) $y_0(\omega) \geq 0$, for all ω, and

iii) $0 < E y_0 < \infty$.

Observe that by 2.2,

2.4 $\{y_n\}$ forms a stationary, metrically transitive process.

$\mathcal{B}$ will always denote the Borel σ-field on $[0,\infty)$ and $\mathcal{B} \otimes \mathcal{S}$ or $\mathcal{B} \otimes \mathcal{S}_n$ will denote the product σ-fields of $\mathcal{B}$ and $\mathcal{S}$ or $\mathcal{B}$ and $\mathcal{S}_n$, respectively. A function $f = [0,\infty) \times \Omega \to [0,\infty)$ will be said to be measurable if it is measurable with respect to $\mathcal{B} \otimes \mathcal{S}$.

2.5 Assumptions About Utility

The consumer's utility function for period n, $u_n: [0,\infty) \times \Omega \to [0,\infty)$, is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_n$ and satisfies the following.

i) $u_n(x,\omega) = u_0(x,\tau^n\omega)$ for all x and ω.

ii) $u_0(0,\omega) = 0$ for all ω.

iii) For every ω, $u_0(\cdot,\omega)$ is strictly increasing, strictly concave, and continuously differentiable. $u_0'(x,\omega)$ will denote $\frac{d}{dx} u_0(x,\omega)$.

iv) $u_0'(0,\omega) \leq 1$ for all ω.

v) $\lim_{x\to\infty} u_0'(x,\omega) = 0$ uniformly in ω.

Assumption ii) involves no loss of generality.

The assumption that u_n is measurable implies that u_n' is measurable, since

$$u_n'(x,\omega) = \lim_{\varepsilon\to 0} \frac{u_n(x+\varepsilon,\omega) - u_n(x,\omega)}{\varepsilon}.$$

Hereafter, proofs that the various functions appearing are measurable will not be provided, since the proofs are routine.

The Optimization Problem

The statement of the consumer's optimization problem given in the Introduction, (1.1), is slightly ambiguous, for the conditional expectation, $E(\quad|\mathcal{S}_0)$, appearing in the objective function, may be defined only up to a set of probability zero. Hence, the objective function may not be defined for every state of the world, ω. For this reason, the conditional expectation is replaced by an ordinary expectation.

2.6 Problem

$$\max_{x_0,\ldots,x_{N-1}} \quad E \sum_{n=0}^{N-1} \delta^n u_n(x_n)$$

subject to: x_n is a random variable measurable with respect to $\mathcal{S}_n$ and $0 \leq x_n \leq M_n$ for all n, where $M_n = M + \sum_{k=0}^{n-1} (y_{k+1} - x_k)$ for all n and M is given.

It is always assumed that $0 < \delta \leq 1$.

In discussing solutions of this problem, it is necessary to require that the random variables, x_n, depend in a measurable way on the initial stock of money M.

Definition. A program, X, is a finite or infinite sequence of functions X_n: $[0,\infty) \times \Omega \longrightarrow [0,\infty)$, $n = 0,1,\ldots$, where X_n is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_n$.

$X_n(M,\omega)$ is the expenditure in period n if the state of the world is ω and the stock of money in period zero is M. Observe that $X_n(M,\omega)$ is always

non-negative. A program $(X_0, \ldots, X_{N-1})$ will be called an N-period program.

A program may be described in a way which makes current expenditures depend on past expenditures as well as on ω and the initial stock of money. But such a program may be reduced easily to one which depends on ω and the initial stock of money only and hence could be described as above.

$M_n(X, M, \omega)$, $n = 0, 1, \ldots$ will denote the sequence of money stocks generated by the program X when the initial stock is M. That is,

$$M_n(X, M, \omega) = M + \sum_{k=0}^{n-1} (y_{k+1}(\omega) - X_k(M, \omega)).$$

Definition. A program is said to be feasible if for every n, $X_n(M, \omega) \leq M_n(X, M, \omega)$ for all M and ω.

If X is an N-period program, $W_{N\delta}(X, M, \omega)$ will denote $\sum_{n=0}^{N-1} \delta^n u_n(X_n(M, \omega), \omega)$.

Definition. An N-period program, $\bar{X} = (\bar{X}_0, \ldots, \bar{X}_{N-1})$, will be called optimal if it is feasible and if for every $M \geq 0$, $E\, W_{N\delta}(\bar{X}, M) = \sup_X E\, W_{N\delta}(X, M)$, where the supremum is taken over feasible N-period programs.

Problem 2.6 may now be stated more compactly as follows.

Problem

$$\max_{X \text{ feasible}} E \sum_{n=0}^{N-1} \delta^n u_n(X_n(M)).$$

It is often more convenient to deal with spending plans which make expenditure depend on the current stock of money rather than the initial stock of money. Such plans will be called policies.

Definition. A policy x is a finite or infinite sequence of functions $x_n: [0,\infty) \times \Omega \longrightarrow [0,\infty)$, $n = 0, 1, \ldots$ such that for each n, x_n is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_n$.

A policy is said to be feasible if $x_n(M,\omega) \leq M$ for all n, M, and ω.

If x is a policy, then $x_n(M,\omega)$ is the expenditure in period n if the current stock of money is M. Policies will be denoted by lower case letters, and programs will be denoted by upper case letters.

A policy x gives rise to a program X in an obvious manner. $X_0(M,\omega) = x_0(M,\omega)$. Suppose that $X_0, \ldots, X_{n-1}$ have been defined. Then, $M_n(X,M,\omega) = M + \sum_{k=0}^{n-1} (y_{k+1}(\omega) - X_k(M,\omega))$ is well-defined and one may let $X_n(M,\omega) = x_n(M_n(X,M,\omega),\omega)$.

A feasible N-period policy is said to be optimal if the program to which it gives rise is optimal.

Optimal Policies and the Limit Policy

It will be proved (Lemma 4.1) that for every N and δ there exists an optimal N-period policy, $x_{N\delta} = (x_{N\delta 0}, \ldots, x_{N\delta, N-1})$. $X_{N\delta} = (X_{N\delta 0}, \ldots, X_{N\delta, N-1})$ will denote the corresponding N-period program. $V_{N\delta}(M, \cdot)$ will denote a version of $E(W_{N\delta}(X_N, M) \mid \mathcal{S}_0)$. $V_{N\delta}(M,\omega)$ is (almost surely) the maximum expected N-period payoff given the information available in period zero and an initial stock of money M.

It will be proved (Lemma 4.1) that $V_{N\delta}$ may be chosen so that for almost every ω, $V_{N\delta}(M,\omega)$ is continuously differentiable as a function of M with the derivative bounded above by one. $V'_{N\delta}(M,\omega)$ will denote a function which is equal to $\frac{d}{dM} V_{N\delta}(M,\omega)$ whenever this derivative is defined.

In Lemmas 5.1, 5.6, and 5.10 it is proved that for almost every ω, $V'_{N\delta}(M,\omega)$ is a non-decreasing function of N and δ, and $x_{N\delta 0}(M,\omega)$ is a non-increasing function of N and δ, for all M. Part iv of Assumption 2.5 implies that $V'_{N\delta}(M,\omega)$ is bounded above by one, so that $\lim_{\substack{\delta\to 1\\ N\to\infty}} V'_{N\delta}(M,\omega)$ exists almost surely. Let λ_0 be any function such that $\lambda_0(M,\omega) = \lim_{\substack{\delta\to 1\\ N\to\infty}} V'_{N\delta}(M,\omega)$ whenever this limit exists. (Then, λ_0 is measurable almost surely.) For each n, let $\lambda_n(M,\omega) = \lambda_0(M,\tau^n\omega)$.

Similarly, $x_{\infty 0}(M,\omega)$ denotes any function such that $0 \le x_{\infty 0}(M,\omega) \le M$ and $x_{\infty 0}(M,\omega) = \lim_{\substack{\delta\to 1\\ N\to\infty}} x_{N\delta 0}(M,\omega)$ whenever this limit exists. For $n>1$, let $x_{\infty n}(M,\omega) = x_{\infty 0}(M,\tau^n\omega)$. Clearly, $0 \le x_{\infty n}(M,\omega) \le M$ everywhere and $x_{\infty n}$ is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_n$. Hence, $x_\infty = (x_{\infty 0}, x_{\infty 1}, \dots)$ is a policy. x_∞ will be called the limit policy and the corresponding program X_∞ will be called the limit program.

If $\delta = 1$, then the subscript for δ will be dropped. Thus, $V_N(M,\omega) \equiv V_{N1}(M,\omega)$, $V'_N(M,\omega) \equiv V'_{N1}(M,\omega)$, $x_{Nn}(M,\omega) \equiv x_{N1n}(M,\omega)$, and $X_{Nn}(M,\omega) \equiv X_{N1n}(M,\omega)$. Similarly, if $0 < \delta < 1$, then $V_\delta(M,\omega) = \lim_{N\to\infty} V_{N\delta}(M,\omega)$, and $x_{\delta n}(M,\omega) = \lim_{N\to\infty} x_{N\delta n}(M,\omega)$. X_δ will denote the program corresponding to the policy x_δ. It will be shown

(Lemma 9.1) that

$$V_\delta(M,\omega) = E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_{\delta n}(M))\,\Big|\,\mathcal{S}_0\Big)(\omega)$$

$$\geq E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_n(M))\,\Big|\,\mathcal{S}_0\Big)(\omega)$$

almost surely for all feasible programs X. That is, x_δ is optimal for the infinite horizon problem with discounted utilities, and $V_\delta(M, \cdot)$ gives the optimal expected total discounted utility, conditional on the information available in period zero.

Policies With a Constant Marginal Utility of Expenditure

Define $\eta_0: (0,1] \times \Omega \longrightarrow [0,\infty)$ as follows. $\eta_0(t,\omega)$ is the unique value of x such that $t \geq u_0'(x,\omega)$ with equality if $x > 0$. $\eta_0(t,\omega)$ is the quantity of money the consumer would spend in period zero if his marginal utility of money were t and he were not constrained by a lack of money. $\eta_0(t,\omega)$ is well-defined, for $u_0'(x,\omega)$ is strictly decreasing as a function of x and $\lim_{x\to\infty} u_0'(x,\omega) = 0$. Clearly, $\eta_0(1,\omega) = 0$, $\lim_{t\to 0} \eta_0(t,\omega) = \infty$ and $\eta_0(t,\omega)$ is continuous as a function of t. Also, if $t_1 > t_2$ and $\eta_0(t_2,\omega) > 0$, then $\eta_0(t_1,\omega) < \eta_0(t_2,\omega)$. It is very easy to see that $\eta_0(t, \cdot)$ is measurable with respect to $\mathcal{S}_0$ for every t. Since $\lim_{x\to\infty} u_0'(x,\omega) = 0$ uniformly in ω, $\eta_0(t,\omega)$ is bounded and so

(2.7) $\qquad E\,\eta_0(t) < \infty \quad$ for all t.

$E\,\eta_0(1) = 0$ and by the monotone convergence theorem (Loève [8, p. 124]), $\lim_{t\to\infty} E\,\eta_0(t) = \infty$.

(2.8) If $t_1 > t_2$ and $E\,\eta_0(t_2) > 0$, then $E\,\eta_0(t_1) < E\,\eta_0(t_2)$.

If $t_n \in (0,1]$ for $n = 0,1,2,\ldots$ and $\lim_{n\to\infty} t_n = t_0$, then $\eta_0(t_n,\omega)$ converges to $\eta_0(t_0,\omega)$ for all ω and $0 \leq \eta_0(t_n,\omega) \leq \eta_0(\min_k t_k,\omega) < \infty$ for all n and ω. It follows from the Lebesque dominated convergence theorem (Loève [8, p. 125]) that $\lim_{n\to\infty} E\,\eta_0(t_n) = E\,\eta_0(t_0)$. Hence,

(2.9) $E\,\eta_0(t)$ is a continuous function of t.

In conclusion, one obtains the following.

(2.10) There exists a unique value of t, $\bar{\lambda}$, such that $E\,\eta_0(\bar{\lambda}) = E\,y_0$. $0 < \bar{\lambda} < 1$.

It will be proved that $\bar{\lambda}$ is the asymptotic value of the marginal utility of money.

$\eta_n(t,\omega)$ will denote $\eta_0(t,\tau^n\omega)$. $(\eta_0(t),\eta_1(t),\ldots)$ forms a program (and also a policy) which will be denoted by $\eta(t)$. This program keeps the marginal utility of expenditure equal to t. $\eta(t)$ is not necessarily feasible.

3. THEOREMS

3.1 Main Theorem

For almost every ω, $\lambda_0(M,\omega) \geq \bar{\lambda}$ for all $M \geq 0$ and

$\lim_{M\to\infty} \lambda_0(M,\omega) = \bar{\lambda}$.

3.2 Theorem

For almost every ω, the following is true for every initial stock of money.

$\lim_{n\to\infty} \lambda_n(M_n(X_\infty,M,\omega),\omega) = \bar{\lambda}$.

It may be said that X_∞ is optimal in the limit average sense if

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_{\infty n}(M)) = \sup_X \lim_{N\to\infty} \inf \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_n(M)), \tag{3.3}$$

where the supremum is taken over feasible policies, X.

Observe that

$$\sup_X \lim_{N\to\infty} \inf \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_n(M)) \le \lim_{N\to\infty} \inf \sup_X \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_n(M))$$

$$= \lim_{N\to\infty} \inf \frac{1}{N} E V_N(M).$$

Therefore, to say that

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_{\infty n}(M)) = \lim_{N\to\infty} \frac{1}{N} E V_N(M)$$

is a stronger assertion about the optimality of X_∞ than is (3.3). Hence, the following theorem asserts that X_∞ is optimal in a limit average sense.

3.4 Theorem

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_{\infty n}(M)) = \lim_{N\to\infty} \frac{1}{N} V_N(M).$$

More generally, the following is true.

3.5 Theorem

For almost every ω, the following equalities hold for all M:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega)$$

$$= \lim_{\delta\to 1} \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} \delta^n u_n(X_{\infty n}(M,\omega),\omega)$$

$$= \lim_{N\to\infty} \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_{\infty n}(M))$$

$$= \lim_{\delta\to 1} \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} \delta^n E\, u_n(X_{\infty n}(M))$$

$$= E\, u_0(\eta_0(\bar{\lambda}))$$

$$= \lim_{N\to\infty} \frac{1}{N} E\, V_N(M)$$

$$= \lim_{\delta\to 1} \frac{1-\delta}{\delta} E\, V_\delta(M)$$

$$= \lim_{N\to\infty} \frac{1}{N} V_N(M,\omega)$$

$$= \lim_{\delta\to 1} \frac{1-\delta}{\delta} V_\delta(M,\omega).$$

4. THE OPTIMAL N-PERIOD PROGRAM

The proofs given in this section and the following one are brief, for they are, for the most part, simply modifications of arguments given by Schechtman [14, section 1].

4.1 Lemma

There exist measurable functions

$V_{N\delta}$: $[0,\infty) \times \Omega \longrightarrow [0,\infty)$,

$q_{N\delta}$: $[0,\infty) \times \Omega \longrightarrow [0,\infty)$, and

$x_{N\delta n}$: $[0,\infty) \times \Omega \longrightarrow [0,\infty)$,

for $n = 0,1,\ldots,N-1$ and $N = 1,2,\ldots$, and $0 < \delta \leq 1$,

which satisfy the following.

i) $x_{N\delta} = (x_{N\delta 0},\ldots,x_{N\delta,N-1})$ is an optimal N-period policy. $x_{N\delta n}(M,\omega) = x_{N-n,\delta 0}(M,\tau^n\omega)$ for all n.

ii) For every $M \geq 0$, $V_{N\delta}(M,\,.\,)$ is a version of $E\Big(\sum_{n=0}^{N-1} \delta^n u_n(X_{N\delta n}(M)) \big| \mathcal{S}_0\Big)$, where $X_{N\delta}$ is the program corresponding to the policy $x_{N\delta}$.

iii) If $M\colon \Omega \longrightarrow [0,\infty)$ is measurable with respect to $\mathcal{S}_0$ and if X is any N-period policy, then $V_{N\delta}(M(\omega),\omega) \geq E\Big(\sum_{n=0}^{N-1} \delta^n u_n(X_n(M)) \big| \mathcal{S}_0\Big)(\omega)$ almost surely.

iv) For almost every ω, $V_{N\delta}(M,\omega)$ and $q_{N\delta}(M,\omega)$ are continuously differentiable as functions of M. The derivatives are positive, less than or equal to one, and strictly decreasing.

$V'_{N\delta}(M,\omega)$ and $q'_{N\delta}(M,\omega)$ denote any functions equal to $\frac{d}{dM} V_{N\delta}(M,\omega)$ and $\frac{d}{dM} q_{N\delta}(M,\omega)$ whenever these derivatives exist.

${}_1V_{N\delta}(M,\omega)$ denotes $V_{N\delta}(M,\tau\omega)$ and

${}_1V'_{N\delta}(M,\omega)$ denotes $V'_{N\delta}(M,\tau\omega)$.

v) $q_{N\delta}(M,\,.\,) = E({}_1V_{N\delta}(M+y_1)\,|\,\mathcal{S}_0)$ and

$q'_{N\delta}(M,\,.\,) = E({}_1V'_{N\delta}(M+y_1\,|\,\mathcal{S}_0)$, for all M.

vi) For almost every ω, $x_{N\delta 0}(M,\omega)$ solves the problem

$\max_{0\le x\le M} [u_0(x,\omega)+\delta q_{N-1,\delta}(M-x,\omega)]$, for all N, δ, and M.

vii) For almost every ω,

$V'_{N+1,\delta}(M,\omega) \ge \delta q'_{N\delta}(M-x_{N+1,\delta 0}(M,\omega),\omega)$ for all M,

with equality if $M - x_{N+1,\delta 0}(M,\omega) > 0$.

viii) For almost every ω, $V'_{N\delta}(M,\omega) \ge u'_0(x_{N\delta 0}(M,\omega),\omega)$

for all M, with equality if $x_{N\delta 0}(M,\omega) > 0$.

ix) For almost every ω, $x_{N\delta 0}(M,\omega)$ and $M - x_{N\delta 0}(M,\omega)$

are continuous, non-decreasing functions of M.

Proof

The proof proceeds by induction on N.

For N=1, let $x_{1\delta 0}(M,\omega) = M$ and $V_{1\delta}(M,\omega) = u_0(M,\omega)$. $x_{1\delta 0}$ and $V_{1\delta}$ clearly satisfy conditions (i)-(iv), (viii), and (ix) of the lemma. Applying Lemma A.3 with $Q(M,\omega) = {}_1V_{1\delta}(M+y_1(\omega),\omega) = u_1(M+y_1(\omega),\omega)$, one obtains $q_{1\delta}$ which satisfies conditions (iv) and (v). This completes the first step of the induction.

Suppose that $N > 1$ and that $V_{k\delta}$, $q_{k\delta}$ and $x_{k\delta n}$ have been defined for $k < N$ and satisfy the conditions of the lemma.

By condition (iv) of the lemma and the assumed properties of u_0, for almost every ω the following problem has a unique solution for each M.

$$\max_{0 \leq x \leq M} [u_0(x,\omega) + \delta q_{N-1,\delta}(M-x,\omega)].$$

Let $x_{N\delta 0}: [0,\infty) \times \Omega \to [0,\infty)$ be any function such that $0 \leq x_{N\delta 0}(M,\omega) \leq M$ and $x_{N\delta 0}(M,\omega)$ solves the above problem whenever the problem has a solution. It is easy to show that $x_{N\delta 0}$ is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_0$. If $1 \leq n \leq N-1$, let $x_{N\delta n}(M,\omega) = x_{N-n,\delta 0}(M, \tau^n\omega)$. Clearly, $x_{N\delta}$ is a feasible policy.

Let $V_{N\delta}(M,\omega) = u_0(x_{N\delta 0}(M,\omega),\omega) + \delta q_{N-1,\delta}(M-x_{N\delta 0}(M,\omega),\omega)$. Since $q_{N-1,\delta}(M-x_{N\delta 0}(M,\,.\,),\,.\,)$ is a version of $E({}_1V_{N-1,\delta}(M-x_{N\delta 0}(M)+y_1)|\mathcal{S}_0)$ by the induction hypothesis, it follows that $V_{N\delta}(M,\,.\,)$ is a version of $E\left(\sum_{n=0}^{N-1} \delta^n u_n(X_{N\delta n}(M))|\mathcal{S}_0\right)$.

It is now proved that $V_{N\delta}$ satisfies condition (iii). Let $M: \Omega \to [0,\infty)$ be measurable with respect to $\mathcal{S}_0$ and let X be any feasible N-period program. By condition (iii) applied to $V_{N-1,\delta}$,

$${}_1V_{N-1,\delta}(M(\omega) - X_0(M(\omega),\omega) + y_1(\omega),\omega) \geq E\left(\sum_{n=1}^{N-1} \delta^{n-1} u_n(X_n(M)|\mathcal{S}_0\right)(\omega)$$

almost surely. By condition (v) and Lemma A.4,

$$q_{N-1,\delta}(M(\omega) - X_0(M(\omega),\omega),\omega) = E\left({}_1V_{N-1,\delta}(M - X_0(M) + y_1)|\mathcal{S}_0\right)(\omega).$$

Hence,

$$q_{N-1,\delta}(M(\omega) - X_0(M(\omega),\omega),\omega) \geq E\left(\sum_{n=1}^{N-1} \delta^{n-1} u_n(X_n(M))|\mathcal{S}_0\right)(\omega)$$

almost surely.

It follows that

$$E\Big(\sum_{n=0}^{N-1}\delta^n u_n(X_n(M))\,\Big|\,\mathcal{S}_0\Big)(\omega)$$

$$\leq u_0(X_0(M(\omega),\omega),\omega) + \delta q_{N-1,\delta}(M(\omega) - X_0(M(\omega),\omega),\omega)$$

$$\leq u_0(X_{N0}(M(\omega),\omega),\omega) + \delta q_{N-1,\delta}(M(\omega) - X_{N\delta 0}(M(\omega),\omega))$$

$$= V_{N\delta}(M(\omega),\omega)$$

almost surely, where $X_{N\delta}$ *is the program associated with the policy* $x_{N\delta}$. The second inequality follows from the definition of $X_{N\delta 0}$. This completes the proof that $V_{N\delta}$ satisfies (iii).

Clearly, (iii) implies that $x_{N\delta}$ is an optimal N-period policy.

It follows from elementary calculus and the definition of $x_{N\delta 0}$ that $x_{N\delta 0}$ and $V_{N\delta}$ satisfy conditions (iv), (vi) - (viii). Again by elementary calculus, for almost every ω, $x_{N\delta 0}(M,\omega)$ and $M - x_{N\delta 0}(M,\omega)$ are non-decreasing functions of M. Since the sum of these functions is M, it follows that $x_{N\delta 0}(M,\omega)$ is Lipschitz with constant 1 and hence continuous. (This argument is borrowed from Brock and Mirman [5, p. 490].) This proves (ix).

By Lemma A.3 applied to the function $V_{N\delta}(M + y_1(\omega), \tau\omega)$, there exists a function $q_{N\delta}$ satisfying conditions (iv) and (v).

Q.E.D.

5. THE LIMIT PROGRAM

In order to avoid problems with sets of measure zero, from now on it will be assumed that δ is restricted to the following countable set $\{\delta \mid 0 < \delta \leq 1,\ \delta \text{ is rational}\}$.

5.1 Lemma

For almost every ω, $V'_{N\delta}(M,\omega)$ is a non-decreasing function of N for all M.

Proof

In what follows, it will be assumed that ω is such that the following are true for all N, δ, and M:

(5.2) $V'_{N\delta}(M,\omega) \geq u'_0(x_{N\delta 0}(M,\omega),\omega)$, *with equality if* $x_{N\delta 0}(M,\omega) > 0$ and

(5.3) $V'_{N+1,\delta}(M,\omega) \geq q'_{N\delta}(M - x_{N+1,\delta 0}(M,\omega),\omega)$, with equality if $M - x_{N+1,\delta 0}(M,\omega) > 0$.

By Lemma 4.1, almost every ω satisfies the above.

The lemma will be proved by proving that almost surely both $V'_{N\delta}(M,\omega)$ and $q'_{N\delta}(M,\omega)$ are non-decreasing functions of N for all M. The proof is by induction on N, for $N \geq 2$.

Suppose N = 2. By (5.2),

$$V'_{2\delta}(M,\omega) = u'_0(x_{2\delta 0}(M,\omega),\omega) \geq u'_0(M,\omega) = V'_{1\delta}(M,\omega), \text{ for all M.}$$

It is next shown that

(5.4) for almost every ω, $q'_{2\delta}(M,\omega) \geq q'_{1\delta}(M,\omega)$ for all M.

By what has just been shown, for each M

$$q'_{2\delta}(M, .) = E({}_1V'_{2\delta}(M+y_1)|\mathcal{S}_0) \geq E({}_1V'_{1\delta}(M+y_1)|\mathcal{S}_0) = q'_{1\delta}(M, .)$$

almost surely. Hence, $q'_{2\delta} - q'_{1\delta}$ satisfies the conditions of Lemma A.6 and so (5.4) is true.

Suppose that $N > 2$ and it has been shown that for almost every ω, $V'_{k\delta}(M,\omega)$ and $q'_{k\delta}(M,\omega)$ are non-decreasing functions of k for all M and for $k < N$. It will now be assumed that ω satisfies the following as well as (5.2) and (5.3).

(5.5) $q'_{k,\delta}(M,\omega)$ is a non-decreasing function of k for all M and for $k < N$.

It is now proved that $V'_{N\delta}(M,\omega) \geq V'_{N-1,\delta}(M,\omega)$ for all M.

Suppose that $x_{N\delta 0}(M,\omega) > 0$. Then, $x_{N\delta 0}(M,\omega) \leq x_{N-1,\delta 0}(M,\omega)$. For if $x_{N\delta 0}(M,\omega) > x_{N-1,\delta 0}(M,\omega)$, then by (5.2), (5.3), and (5.5)

$$\begin{aligned} u'_0(x_{N-1,\delta 0}(M,\omega),\omega) &> u'_0(x_{N\delta 0}(M,\omega),\omega) = V'_{N\delta}(M,\omega) \\ &\geq \delta q'_{N-1,\delta}(M-x_{N\delta 0}(M,\omega),\omega) \geq \delta q'_{N-2,\delta}(M-x_{N\delta 0}(M,\omega),\omega) \\ &> \delta q'_{N-2,\delta}(M-x_{N-1,\delta 0}(M,\omega),\omega). \end{aligned}$$

This implies that $x_{N-1,\delta 0}(M,\omega) = M$, which is impossible.

Since $x_{N-1,\delta 0}(M,\omega) \geq x_{N\delta 0}(M,\omega) > 0$,

$V'_{N\delta}(M,\omega) = u'_0(x_{N\delta 0}(M,\omega),\omega) \geq u'_0(x_{N-1,\delta 0}(M,\omega),\omega) = V'_{N-1,\delta}(M,\omega)$, as is to be proved.

Suppose now that $x_{N\delta 0}(M,\omega) = 0$. If $x_{N-1,\delta 0}(M,\omega) > 0$, then $V'_{N-1,\delta}(M,\omega) = u'_0(x_{N-1,\delta 0}(M,\omega),\omega) < u'_0(0,\omega) \leq V'_{N\delta}(M,\omega)$. If

$x_{N-1,\delta 0}(M,\omega) = 0$ as well and $M > 0$, then $V'_{N-1,\delta}(M,\omega) = q'_{N-2,\delta}(M,\omega)$ $\leq q'_{N-1,\delta}(M,\omega) = V'_{N,\delta}(M,\omega)$. This completes the proof that $V'_{N\delta}(M,\omega)$ $\geq V'_{N-1,\delta}(M,\omega)$ if $M > 0$. Since $V'_{N\delta}(\,\cdot\,,\omega)$ and $V'_{N-1,\delta}(\,\cdot\,,\omega)$ are continuous almost surely, $V'_{N\delta}(0,\omega) \geq V'_{N-1,\delta}(0,\omega)$ almost surely.

This completes the proof that for almost every ω, $V'_{N\delta}(M,\omega)$ $\geq V'_{N-1,\delta}(M,\omega)$ for all M.

The argument that proves (5.4) proves that for almost every ω, $q_N(M,\omega) \geq q_{N-1}(M,\omega)$ for all M.

Q.E.D.

5.6 Lemma

For almost every ω, $V'_{N\delta}(M,\omega)$ is a non-decreasing function of δ for all M.

Proof

By Lemma 4.1, it may be assumed that ω is such that the following are true for all N, δ, and M.

(5.7) $V_{N\delta}(M,\omega)$ and $q_{N\delta}(M,\omega)$ are differentiable and strictly concave.

(5.8) $$V_{N\delta}(M,\omega) = u_0(x_{N\delta}(M,\omega),\omega) + \delta q_{N-1,\delta}(M - x_{N\delta 0}(M,\omega),\omega).$$

(5.9) $x_{N\delta 0}(M,\omega)$ solves the problem

$$\max_{0\leq x\leq M} \left[u_0(x,\omega) + \delta q_{N-1,\delta}(M-x,\omega)\right].$$

It is now proved by induction on N that for almost every ω, $V'_{N\delta}(M,\omega)$ and $q'_{N\delta}(M,\omega)$ are non-decreasing functions of δ for all N and M.

If N = 1, there is nothing to prove since $V'_{1\delta}(M,\omega) = u'_1(M,\omega)$ and $q'_{1\delta}(M,\omega) = E(u'_1(M+y_1)|\mathcal{S}_0)(\omega)$ do not depend on δ.

Suppose by induction that almost surely $q'_{N-1,\delta}(M,\omega)$ is non-decreasing in δ for all M. From (5.7)-(5.9) it follows by elementary calculus that almost surely $V'_{N\delta}(M,\omega)$ is non-decreasing in δ. Since $q'_{N\delta}(M,\omega) = E({}_1V'_{N\delta}(M+y_1)|\mathcal{S}_0)(\omega)$, if follows (using Lemma A.6) that for almost every ω, $q'_{N\delta}(M,\omega)$ is a non-decreasing function of δ for all M.

Q.E.D.

5.10 Lemma

For almost every ω, $x_{N\delta 0}(M,\omega)$ is a non-increasing function of N and δ for all M.

Proof

This lemma follows immediately from the previous two and the fact that almost surely, $V'_{N\delta}(M,\omega) \geq u'_0(x_{N\delta 0}(M,\omega),\omega)$ for all M with equality if $x_{N\delta 0}(M,\omega) > 0$.

Q.E.D.

Recall that λ_0 and $x_{\infty 0}$ are defined to be functions such that for almost every ω, $\lambda_0(M,\omega) = \lim_{\substack{N\to\infty \\ \delta\to 1}} V'_{N\delta}(M,\omega)$ and $x_{\infty 0}(M,\omega) = \lim_{\substack{N\to\infty \\ \delta\to 1}} x_{N\delta 0}(M,\omega)$ for all $M \geq 0$. Also, $\lambda_n(M,\omega) = \lambda_0(M,\tau^n\omega)$ and $x_{\infty n}(M,\omega) = x_{\infty 0}(M,\tau^n\omega)$. The following lemma says that λ_n gives the marginal utility of money in period n consistent with use of the limit policy $x_\infty = (x_{\infty 0}, x_{\infty 1}, \ldots)$.

5.11 Lemma

For almost every ω, $\lambda_n(M,\omega) \geq u_n'(x_{\infty n}(M,\omega),\omega)$ with equality if $x_{\infty n}(M,\omega) > 0$ for $n = 0, 1, \ldots$ and for all M.

Proof

Since $\lambda_n(M,\omega) = \lambda_0(M,\tau^n\omega)$ and $u_n'(x_{\infty n}(M,\omega),\omega) = u_0'(x_{\infty 0}(M,\tau^n\omega),\tau^n\omega)$, it is sufficient to prove the lemma for $n = 0$.

By Lemma 4.1, for almost every ω, $V_{N\delta}'(M,\omega) \geq u_0'(x_{N\delta 0}(M,\omega),\omega)$ for all M and N with equality if $x_{N\delta 0}(M,\omega) > 0$. Since $u_0'(M,\omega)$ is a continuous function of M, it follows that

(5.12) for almost every ω,

$$\lambda_0(M,\omega) = \lim_{\substack{N\to\infty \\ \delta\to 1}} V_{N\delta}'(M,\omega) \geq \lim_{\substack{N\to\infty \\ \delta\to 1}} u_0'(x_{N\delta 0}(M,\omega),\omega)$$

$$= u_0'(x_{\infty 0}(M,\omega),\omega), \quad \text{for all } M.$$

If $x_{\infty 0}(M,\omega) > 0$, $x_{N\delta 0}(M,\omega) > 0$ for all N and δ so that one has equality throughout (5.12).

Q.E.D.

6. THE ERGODIC THEOREM

In this section, a precise statement is made of the strong law of large numbers or ergodic theorem for stationary processes and also of all the applications of this theorem made in this paper.

Ergodic Theorem (Doob [6, p. 465])

Let z_n, $n = 0,1,2,\ldots$ be a stationary metrically transitive stochastic process with $E|z_0| < \infty$. Then, $\lim_{n\to\infty} \frac{1}{n}(z_0 + \ldots + z_{n-1}) = E\, z_0$ almost surely.

Recall the definition of the program $\eta(t) = (\eta_0(t), \eta_1(t), \ldots)$ given at the end of section 2. Let $M_N(\eta(t), M, \omega)$ be as in section 2, that is,

$$M_N(\eta(t), M, \omega) = M + \sum_{n=0}^{N-1} (y_{n+1}(\omega) - \eta_n(t,\omega)).$$

By (2.2), $\{\eta_n(t,\omega)\}$ forms a stationary and metrically transitive process. By (2.7), $E|\eta_0(t)| < \infty$. By (2.4), $\{y_n\}$ is a stationary and metrically transitive process. Therefore, by the ergodic theorem

(6.1) for all $M \geq 0$,

$$\lim_{N\to\infty} \frac{1}{N} M_N(\eta(t), M, \omega) = E(y_0 - \eta_0(t)) \text{ almost surely.}$$

By (2.8) - (2.10),

(6.2) $E(y_0 - \eta_0(t))$ is a non-increasing, continuous function of t, for $0 < t < 1$, and $E(y_0 - \eta_0(t)) = 0$ if and only if $t = \bar{\lambda}$.

Consequently,

(6.3) for all $M \geq 0$,

$$\lim_{N\to\infty} M_N(\eta(t), M, \omega) = \infty \text{ almost surely if } t > \bar{\lambda},$$

$$\lim_{N\to\infty} M_N(\eta(t), M, \omega) = -\infty \text{ almost surely if } t < \bar{\lambda}, \text{ and}$$

$$\lim_{N\to\infty} \frac{1}{N} M_N(\eta(\bar{\lambda}), M, \omega) = 0 \text{ almost surely.}$$

It will be necessary to use the fact that

(6.4) if $t > \bar{\lambda}$, then

$$\lim_{N\to\infty} [M_N(\eta(t),0,\omega) - \eta_N(\eta(t),\omega)] = \infty \text{ almost surely.}$$

(6.4) follows from the obvious fact that

$$\frac{1}{N}[M_N(\eta(t),0,\omega) - \eta_N(\eta(t),\omega)] = \frac{1}{N}\left[\sum_{n=1}^{N} y_n(\omega) - \sum_{n=0}^{N} \eta_N(\eta(t),\omega)\right]$$

converges almost surely to the positive number $E(y_0 - \eta_0(t))$.

The ergodic theorem also implies that

(6.5) $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(\eta_n(t,\omega),\omega) = E\, u_0(\eta_0(t)) \text{ almost surely for all } t.$$

7. PROOF OF THE MAIN THEOREM

The proof of the main theorem is broken up into two lemmas. The economic intuition underlying the argument is explained in the course of proving the lemmas.

Recall that $V_N = V_{N1}$, $V'_N = V'_{N1}$, $x_{Nn} = x_{N1n}$ and $X_{Nn} = X_{N1n}$. That is, V_N, etc. correspond to the N-horizon problem with no discounting of utility. This notation will be used frequently in the next two sections.

7.1 Lemma

For almost every ω, $\lambda_0(M,\omega) \geq \bar{\lambda}$ for all M.

Proof

By Lemma 4.1, $V_N(M,\omega)$ is concave as a function of M almost surely, so that $V'_N(M,\omega) \geq V_N(M+1,\omega) - V_N(M,\omega)$. Since $\lambda_0(M,\omega) = \lim_{N\to\infty} V'_N(M,\omega)$, it is sufficient to prove the following.

(7.2) Given $\varepsilon > 0$, for almost every ω

$$\limsup_{N\to\infty} [V_N(M+1,\omega) - V_N(M,\omega)] \geq \bar{\lambda} - \varepsilon \quad \text{for all } M.$$

The following is proved.

(7.3) Given $\varepsilon > 0$, there exists a program $\bar{X}_N$ such that for each M,

$$E(W_N(\bar{X}_N, M+1)\,|\,\mathcal{S}_0)(\omega) - V_N(M,\omega) \geq \bar{\lambda} - \varepsilon - \delta_N(M,\omega)$$

almost surely, where the functions δ_N are measurable and are such that for almost every ω, 1) $\delta_N(M,\omega)$ is continuous from the right as a function of M and 2) $\lim_{N\to\infty} \delta_N(M,\omega) = 0$.

Clearly, $V_N(M+1,\omega) \geq E(W_N(\bar{X}_N, M+1)\,|\,\mathcal{S}_0)(\omega)$ almost surely. Since almost surely $V_N(M+1,\omega)$ and $\delta_N(M,\omega)$ are continuous from the right as functions of M, it follows from Lemma A.6 and (7.3) that for almost every ω, $V_{N+1}(M+1,\omega) - V_N(M,\omega) \geq \bar{\lambda} - \varepsilon - \delta_N(M,\omega)$ for all M. Hence, (7.2) follows from (7.3).

The idea of the proof of (7.3) is as follows. Suppose that the consumer has M+1 units of money initially. $\bar{X}_N$ treats one unit of money as extra money to be spent at an opportune time. $\bar{X}_N$ requires the consumer to follow his optimal N-period program with initial stock

M until the marginal utility of expenditure exceeds $\bar{\lambda} - \varepsilon$. He then spends as much of his extra money as he can without bringing the marginal utility of expenditure below $\bar{\lambda} - \varepsilon$. He continues in this way until he spends all his extra money or reaches the last period. If he spends all his extra money before the last period, he will increase his total utility by at least $\bar{\lambda} - \varepsilon$. If he reaches the last period without having spent all the extra money, then the marginal utility of expenditure must always have been at most $\bar{\lambda} - \varepsilon$. This implies that he will have spent at least as much as he would have if he had followed the program $\eta(\bar{\lambda} - \varepsilon)$ described in the previous section. By (6.3), the stock of money left by this policy, $M_{N-1}(\eta(\bar{\lambda}-\varepsilon), M+1, \omega)$, converges to minus infinity almost surely. Since the consumer can never let the stock of money become negative, it is very unlikely that if N is large, the consumer could reach period $N-1$ without having spent the extra unit of money.

This argument is now made precise.

$\bar{X}_{Nn}$ is defined by induction on n as follows.

(i) If $u_0'(X_{N0}(M,\omega),\omega) \leq \bar{\lambda} - \varepsilon$, then

$$\bar{X}_{N0}(M+1,\omega) = X_{N0}(M,\omega).$$

(ii) If $u_0'(X_{N0}(M,\omega),\omega) > \bar{\lambda} - \varepsilon$, then

$$\bar{X}_{N0}(M+1,\omega) = \min(X_{N0}(M,\omega)+1,\ \eta_0(\bar{\lambda}-\varepsilon,\omega)).$$

Suppose that $\bar{X}_{Nk}$ has been defined for $k < n$ and that $n > 0$.

(iii) If $u_n'(X_{Nn}(M,\omega),\omega) \leq \bar{\lambda} - \varepsilon$, then

$$\bar{X}_{Nn}(M+1,\omega) = X_{Nn}(M,\omega).$$

(iv) If $u_n'(X_{Nn}(M,\omega),\omega) > \bar{\lambda} - \varepsilon$, then

$$\bar{X}_{Nn}(M+1,\omega) = \min\Big(X_{Nn}(M,\omega) + M_n(\bar{X}_N, M+1, \omega) - M_n(X_N, M, \omega), \eta_n(\bar{\lambda}-\varepsilon,\omega)\Big).$$

The following facts are immediate consequences of the definition of $\bar{X}_N$.

$$M_n(\bar{X}_N, M+1, \omega) \geq M_n(X_N, M, \omega) \quad \text{for all } n.$$

(7.4) If ω is such that $M_{N-1}(\bar{X}_N, M+1, \omega) = M_{N-1}(X_N, M, \omega)$, then

$$W_N(\bar{X}_N, M+1, \omega) - W_N(X_N, M, \omega) \geq \bar{\lambda} - \varepsilon.$$

(7.5) If ω is such that $M_{N-1}(\bar{X}_N, M+1, \omega) > M_{N-1}(X_N, M, \omega)$, then

$$u_n'(\bar{X}_{Nn}(M,\omega),\omega) \leq \bar{\lambda} - \varepsilon \quad \text{for all } n < N-1, \text{ so that}$$

$$\bar{X}_{Nn}(M+1,\omega) \geq \eta_n(\bar{\lambda}-\varepsilon,\omega) \quad \text{for all } n < N-1.$$

(7.5) implies that

(7.6) If ω is such that $M_{N-1}(\bar{X}_N, M+1, \omega) > M_{N-1}(X_N, M, \omega)$, then

$$0 \leq M_{N-1}(\bar{X}_N, M+1, \omega) \leq M_{N-1}(\eta(\bar{\lambda}-\varepsilon), M+1, \omega).$$

Let $f_N(M,\omega)$ be the indicator function of the set $S_N(M) = \{\omega \mid M_{N-1}(\eta(\bar{\lambda}-\varepsilon), M+1, \omega) \geq 0\}$. That is,

$$f_N(M,\omega) = \begin{cases} 1, & \text{if } \omega \in S_N(M) \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (7.4) and (7.6) that

$$(7.7) \qquad W_N(\bar{X}_N, M+1, \omega) - W_N(X_N, M, \omega) \geq (\bar{\lambda}-\varepsilon)(1-f_N(M,\omega)).$$

Applying $E(\ \ |\mathcal{S}_0)$ to both sides of (7.7), one obtains

$$(7.8) \qquad E(W_N(\bar{X}_N, M+1)|\mathcal{S}_0)(\omega) - V_N(M,\omega) \geq \bar{\lambda} - \varepsilon - (\bar{\lambda}-\varepsilon)E(f_N(M)|\mathcal{S}_0)(\omega)$$

almost surely.

Observe that for every ω, $f_N(M,\omega)$ is continuous from the right and non-decreasing as a function of M. Also, since $\lim_{N\to\infty} M_N(\eta(\bar{\lambda}-\varepsilon), M+1, \omega) = -\infty$ almost surely, $\lim_{N\to\infty} f_N(M,\omega) = 0$ almost surely for every M. Hence, by Lemma A.5, there exists a sequence of measurable functions $\delta_N(M,\omega)$ which satisfy the following conditions.

(i) $\delta_N(M, .) = (\bar{\lambda}-\varepsilon)E(f_N(M)|\mathcal{S}_0)$ for all M.

(ii) For almost every ω, $\delta_N(M,\omega)$ is continuous from the right as a function of M.

(iii) For almost every ω, $\lim_{N\to\infty} \delta_N(M,\omega) = 0$ for all M.

By (i) and (7.8),

$$E(W_N(\bar{X}_N, M+1)|\mathcal{S}_0)(\omega) - V_N(M,\omega) \geq \bar{\lambda} - \varepsilon - \delta_N(M,\omega) \text{ almost surely.}$$

This completes the proof of (7.3).

Q.E.D.

7.9 Lemma

$$\lim_{M\to\infty} \lambda_0(M,\omega) \leq \bar{\lambda} \quad \text{almost surely.}$$

Proof

By the almost sure concavity of $V_N(M,\omega)$, for almost every ω, $V'_N(M,\omega) \leq V_N(M,\omega) - V_N(M-1,\omega)$ for all $M \geq 1$. Since almost surely $\lambda_0(M,\omega) = \lim_{N\to\infty} V'_N(M,\omega)$ for all M, it follows that for almost every ω, $\lambda_0(M,\omega) \leq \liminf_{N\to\infty} [V_N(M,\omega) - V_N(M-1,\omega)]$ for all $M \geq 1$. Therefore, it is sufficient to prove the following.

(7.10) Given $\varepsilon > 0$, for almost every ω

$$V_N(M,\omega) - V_N(M-1,\omega) \leq \bar{\lambda} + \varepsilon + \delta(M,\omega)$$

for all $M \geq 1$ and for all N, where

$$\lim_{M\to\infty} \delta(M,\omega) = 0 \text{ almost surely.}$$

The following will be proved.

(7.11) Given $\varepsilon > 0$, there exists a feasible policy $\bar{X}_N$ such that for each $M \geq 1$

$$V_N(M,\omega) - E(W_N(\bar{X}_N, M-1)\,|\,\mathcal{S}_0)(\omega) \leq \bar{\lambda} + \varepsilon + \delta(M,\omega)$$

almost surely and for all N, where for almost every ω

$$\lim_{M\to\infty} \delta(M,\omega) = 0 \text{ and } \delta(M,\omega) \text{ is continuous from the left}$$

as a function of M.

Clearly, $V_N(M-1,\,.\,) \geq E(W_N(\bar{X}_N, M-1)\,|\,\mathcal{S}_0)$ almost surely. Hence, (7.11) implies that

(7.12) For each $M \geq 1$, $V_N(M,\omega) - V_N(M-1,\omega) \leq \bar{\lambda} + \varepsilon + \delta(M,\omega)$

almost surely for all N.

Since for almost every ω, the functions $V_N(M,\omega) - V_N(M-1,\omega)$ and $\delta(M,\omega)$ are continuous from the left as functions of M, it follows from (7.12) and Lemma A.6 that for almost every ω, $V_N(M,\omega) - V_N(M-1,\omega) \leq \bar{\lambda} + \varepsilon + \delta(M,\omega)$ for all $M \geq 1$ and for all N. That is, (7.11) implies (7.10).

The idea of the proof of (7.11) is quite similar to that of (7.3). The object of $\bar{X}_N$ is to enable the consumer to give up one unit of money while losing no more than $\bar{\lambda} + \varepsilon + \delta(M)$ units of utility, where M is the initial stock of money and where $\lim_{M\to\infty} \delta(M) = 0$.

$\bar{X}_N$ instructs the consumer to reduce his expenditure, if possible, only when the marginal utility of money is less than $\bar{\lambda}+\varepsilon$. If he spends less only at such times, he gives up no more than $\bar{\lambda}+\varepsilon$ units of utility. The consumer could be prevented from waiting for times of low marginal utility of expenditure only if lack of money or arrival of the final time period forced him to spend less. Let n be a period in which he is forced to spend less. In period n, he spends all his money. Also, his expenditure in this period is at most $\eta_n(\bar{\lambda}+\varepsilon,\omega)$. Otherwise, he would not be losing utility in period n at a rate greater than $\bar{\lambda}+\varepsilon$ units per dollar. In the periods preceding n, the marginal utility of expenditure must be at least $\bar{\lambda}+\varepsilon$, so that expenditure in these periods does not exceed that prescribed by the program $\eta(\bar{\lambda}+\varepsilon)$. Hence, his stock of money in period n is at least $M_n(\eta(\bar{\lambda}+\varepsilon),M-1,\omega)$ and $M_n(\eta(\bar{\lambda}+\varepsilon),M-1,\omega) \leq \eta_n(\bar{\lambda}+\varepsilon,\omega)$.

Observe that no matter what happens, the consumer gives up no more than one unit of utility, since the marginal utility of expenditure is bounded above by one.

In conclusion, either the consumer gives up $\bar{\lambda}+\varepsilon$ units of utility or he gives up at most one unit of utility and an event occurs, the probability of which is at most

$$\delta(M) \equiv \text{Prob}\{\inf_{1\leq n<\infty}[M_n(\eta(\bar{\lambda}+\varepsilon),M-1,\omega)-\eta_n(\bar{\lambda}+\varepsilon)]\leq 0\}$$

That is, the consumer's loss is dominated by $\bar{\lambda}+\varepsilon+\delta(M)$.

Since by (6.4), $\lim_{n\to\infty}[M_n(\eta(\bar{\lambda}+\varepsilon),M-1,\omega)-\eta_n(\bar{\lambda}+\varepsilon,\omega)]=\infty$ almost surely, it follows easily that $\lim_{M\to\infty}\delta(M)=0$.

This argument is now made precise.

$\bar{X}_{Nn}$ is defined below by induction on n. It may be assumed that $M\geq 1$.

(i) If $u_0'(X_{N0}(M,\omega),\omega)\geq\bar{\lambda}+\varepsilon$, let

$$\bar{X}_{N0}(M-1,\omega)=\min(X_{N0}(M,\omega),M-1).$$

(ii) If $u_0'(X_{N0}(M,\omega),\omega)<\bar{\lambda}+\varepsilon$, let

$$\bar{X}_{N0}(M-1,\omega)=\min\{\max[X_{N0}(M,\omega)-1,\eta_0(\bar{\lambda}+\varepsilon,\omega)],M-1\}.$$

Suppose that $\bar{X}_{Nk}$ has been defined for $k<n$ and $n>0$. Then, $M_n(\bar{X}_N,M-1,\omega)$ is defined. It may be assumed that $M_n(\bar{X}_N,M-1,\omega)\leq M_n(X_n,M,\omega)$.

(iii) If $M_n(\bar{X}_N,M-1,\omega)=M_n(X_N,M,\omega)$, let

$$\bar{X}_N(M-1,\omega)=X_N(M,\omega).$$

Suppose that $M_n(\bar{X}_N,M-1,\omega)<M_n(X_N,M,\omega)$.

(iv) If $u_n'(X_{Nn}(M,\omega),\omega)\geq\bar{\lambda}+\varepsilon$, let

$$\bar{X}_{Nn}(M-1,\omega)=\min(X_{Nn}(M,\omega),M_n(\bar{X}_N,M-1,\omega)).$$

(v) If $u_n'(X_{Nn}(M,\omega),\omega) < \bar{\lambda} + \varepsilon$, let

$$\bar{X}_{Nn}(M-1,\omega) = \min\{\max[X_{Nn}(M,\omega) - M_n(X_N,M,\omega) + M_n(\bar{X}_N,M-1,\omega), \eta_n(\bar{\lambda}+\varepsilon,\omega)], M_n(\bar{X}_N,M-1,\omega)\}.$$

The following facts should be clear.

$$M_n(\bar{X}_N,M-1,\omega) \leq M_n(X_N,M,\omega).$$

(7.13) If $M_n(\bar{X}_N,M-1,\omega) = M_n(X_N,M,\omega)$, then

$$M_k(\bar{X}_N,M-1,\omega) = M_k(X_N,M,\omega) \text{ for } k > n.$$

(7.14) If $M_n(\bar{X}_N,M-1,\omega) < M_n(X_N,M,\omega)$ and

$\bar{X}_{Nn}(M-1,\omega) > \eta_n(\bar{\lambda}+\varepsilon,\omega)$, *then*

$$M_k(\bar{X}_N,M-1,\omega) = M_k(X_N,M,\omega) \text{ for } k > n.$$

Define $k(M,\omega)$ as follows. If for some integer $n = 0,1,\ldots,N-1$, $M_n(\bar{X}_N,M-1,\omega) < M_n(X_N,M,\omega)$ and $M_n(\bar{X}_N,M-1,\omega) \leq \eta_n(\bar{\lambda}+\varepsilon,\omega)$, let $k(M,\omega)$ be the smallest such integer. Otherwise, let $k(M,\omega) = \infty$. $k(M,\omega)$ is, roughly speaking, the first period in which the consumer could be forced to spend less at a time of high marginal utility of expenditure.

It follows immediately from the definition of $k(M,\omega)$ and from (7.13) that if $k(M,\omega) < \infty$, then $M_n(\bar{X}_N,M-1,\omega) < M_n(X_N,M,\omega)$, for all $n \leq k(M,\omega)$. Consequently, by (7.14) and the definition of $k(M,\omega)$, $\bar{X}_{Nn}(M-1,\omega) \leq \eta_n(\bar{\lambda}+\varepsilon,\omega)$ for all $n < k(M,\omega)$, and so

$$M_{k(M,\omega)}(\eta(\bar{\lambda}+\varepsilon),M-1,\omega) \leq M_{k(M,\omega)}(\bar{X}_N,M-1,\omega) \leq \eta_{k(M,\omega)}(\bar{\lambda}+\varepsilon,\omega).$$

In summary,

(7.15) if $k(M,\omega) < \infty$, then

$$M_{k(M,\omega)}(\eta(\bar{\lambda}+\varepsilon), M-1, \omega) - \eta_{k(M,\omega)}(\bar{\lambda}+\varepsilon, \omega) \leq 0.$$

Since $u_n'(x,\omega) \leq 1$,

(7.16) $$W_N(X_N, M, \omega) - W_N(\bar{X}_N, M-1, \omega) \leq 1 \quad \text{for all } \omega.$$

Clearly,

(7.17) if $k(M,\omega) = \infty$, then

$$W_N(X_N, M, \omega) - W_N(\bar{X}_N, M-1, \omega) \leq \bar{\lambda} + \varepsilon.$$

Let $f(M,\omega)$ be the indicator function of the set

$$S(M) = \{\omega \mid \inf_n [M_n(\eta(\bar{\lambda}+\varepsilon), M-1, \omega) - \eta_n(\bar{\lambda}+\varepsilon, \omega)] \leq 0\}.$$

(7.15) - (7.17) imply that

(7.18) $$W_N(X_N, M, \omega) - W_N(\bar{X}_N, M-1, \omega) \leq (\bar{\lambda}+\varepsilon) + f(M, \omega).$$

Applying $E(\ \ |\mathcal{S}_0)$ to both sides of (7.18), one obtains

(7.19) $$V_N(M,\omega) - E(W(\bar{X}_N, M-1)|\mathcal{S}_0)(\omega) \leq \bar{\lambda} + \varepsilon + E(f(M)|\mathcal{S}_0)(\omega)$$
almost surely.

Since $S(M) = \{\omega \mid M-1+\inf_n [M_n(\eta(\bar{\lambda}+\varepsilon), 0, \omega) - \eta_n(\bar{\lambda}+\varepsilon, \omega)] \leq 0\}$, it follows that for every ω, $f(M,\omega)$ is continuous from the left and non-increasing as a function of M. Therefore, by Lemma A.1, there exists a measurable function $\delta: [0,\infty) \times \Omega \to [0,1]$ such that

(i) $\delta(M, .) = E(f(M)|\mathcal{S}_0)$ for all M and

(ii) for almost every ω, $\delta(M,\omega)$ is continuous from the left and non-

decreasing as a function of M. By (7.19),

$V_N(M,\omega) - E(W(\bar{X}_N, M-1)\,|\,\mathcal{S}_0)(\omega) \leq \bar{\lambda} + \varepsilon + \delta(M,\omega)$ almost surely.

In order to complete the proof of (7.11), it is necessary to show that $\lim_{M\to\infty} \delta(M,\omega) = 0$ almost surely. By (6.4),

$\lim_{n\to\infty} [M_n(\eta(\bar{\lambda}+\varepsilon), 0, \omega) - \eta_n(\bar{\lambda}+\varepsilon, \omega)] = \infty$ almost surely. It follows that

$\lim_{M\to\infty} f(M,\omega) = 0$ almost surely. Therefore, by (A.2),

$\lim_{\substack{M\to\infty \\ M \text{ an integer}}} \delta(M,\omega) = 0$ almost surely. Since for almost every ω, $\delta(M,\omega)$ is non-increasing as a function of M, it follows that almost surely

$\lim_{M\to\infty} \delta(M,\omega) = 0.$

Q.E.D.

8. PROOF OF THEOREM 3.2

The outline of the proof is as follows. The fact that $V'_{N+1}(M,\omega) \geq q'_N(M - x_{N+1,0}(M,\omega), \omega)$ implies that the random variables $\lambda_n(M_n(X_n, M, \omega), \omega)$ form a supermartingale. By the convergence theorem for supermartingales, these random variables converge almost surely. Since $\lambda_n(M,\omega) \geq \bar{\lambda}$, the limit is at least $\bar{\lambda}$. If it exceeds $\bar{\lambda}$, then for some $\varepsilon > 0$, $X_{\infty,n}(M,\omega)$ will be less than $\eta_n(\bar{\lambda} - \varepsilon, \omega)$ for all n. Hence, $M_n(X_\infty, M, \omega)$ converges to infinity. Consequently, by the main theorem $\lambda_n(M_n(X_\infty, M, \omega), \omega)$ will converge to $\bar{\lambda}$. This contradiction implies that $\lambda_n(M_n(X_\infty, M, \omega), \omega)$ converges to $\bar{\lambda}$.

8.1 Lemma

The random variables $\lambda_n(M_n(X_\infty, M, \omega), \omega)$ form a supermartingale. That is, for all n, $\lambda_n(M_n(X_n, M, \omega), \omega) \geq E[\lambda_{n+1}(M_{n+1}(X_\infty, M)) | \mathcal{S}_0](\omega)$ almost surely.

Proof

For brevity, $M_n(X_\infty, M, \omega)$ will be denoted by $M_n(\omega)$. Let ${}_nV'_N(M, \omega) = V'_N(M, \tau^n\omega)$ and ${}_nq'_N(M, \omega) = q'_N(M, \tau^n\omega)$.

$$(8.2) \qquad {}_nV'_N(M_n(\omega), \omega) \geq {}_nq'_{N-1}(M_n(\omega) - x_{Nn}(M_n(\omega), \omega), \omega)$$

$$\geq {}_nq'_{N-1}(M_n(\omega) - x_{\infty n}(M_n(\omega), \omega), \omega)$$

$$= E({}_{n+1}V'_{N-1}(M_{n+1}) | \mathcal{S}_n)(\omega).$$

The first inequality above follows from Lemma 4.1, part vii and from Lemma A.4. The second follows from the inequality $M - x_{\infty n}(M, \omega) \geq M - x_{Nn}(M, \omega)$ and the fact that ${}_nq'_{N-1}(M, \omega)$ is decreasing in M almost surely. The equality follows from Lemma 4.1, part v and from Lemma A.4. Since $\lim_{N\to\infty} {}_{n+1}V'_{N-1}(M_{n+1}(\omega), \omega) = \lambda_{n+1}(M_{n+1}(\omega), \omega)$ almost surely it follows by (A.2) that $\lim_{N\to\infty} E({}_{n+1}V'_{N-1}(M_{n+1}) | \mathcal{S}_n) = E\lambda_{n+1}(M_{n+1}) | \mathcal{S}_n)$ almost surely. Also, $\lim_{N\to\infty} {}_nV'_N(M_n(\omega), \omega) = \lambda_n(M_n(\omega), \omega)$ almost surely, so that by passage to the limit in (8.2), $\lambda_n(M_n(\omega), \omega) \geq E(\lambda_{n+1}(M_{n+1}) | \mathcal{S}_n)(\omega)$ almost surely.

Q.E.D.

By the convergence theorem for supermartingales (Doob [6, p. 324, Theorem 4.10]), $\lambda_n(M_n(X_\infty, M, \omega), \omega)$ converges almost surely. Let

$\lambda_\infty(M,\omega)$ be the limit. Clearly, $\lambda_\infty(M,\tau\omega) = \lambda_\infty(M,\omega)$ almost surely. That is, $\lambda_\infty(M, \cdot)$ is an invariant function. It is easy to see that because the stochastic process $\{\omega_n\}$ is metrically transitive, every invariant function is equal to a constant almost surely. Let λ_∞ be the constant corresponding to $\lambda_\infty(M, \cdot)$.

By the main theorem, $\lambda_n(M_n(X_\infty,M,\omega),\omega) \geq \bar{\lambda}$ almost surely, so that $\lambda_\infty \geq \bar{\lambda}$. It must be shown that $\lambda_\infty = \bar{\lambda}$.

In what follows, use will be made of the concept of convergence in probability. A sequence of random variables, z_n, is said to converge to z in probability if for every $\varepsilon > 0$, $\lim_{n\to\infty} \text{Prob}\{|z_n - z| > \varepsilon\} = 0$. z_n converges to infinity in probability if for every $K > 0$, $\lim_{N\to\infty} \text{Prob}\{z_n < K\} = 0$. If z_n converges almost surely, z_n converges in probability.

Suppose that $\lambda_\infty > \bar{\lambda}$. Observe that $X_{\infty n}(M,\omega) = \eta_n(\lambda_n(M_n(X_\infty,M,\omega),\omega),\omega)$ by Lemma 5.11. Since $\lim_{n\to\infty} \lambda_n(M_n(X_\infty,M,\omega),\omega) = \lambda_\infty$ almost surely, it follows that

(8.3) for almost every ω, $X_{\infty n}(M,\omega) \leq \eta_n\left(\frac{\bar{\lambda}+\lambda_\infty}{2}, \omega\right)$ for n sufficiently large.

Since $\lim_{n\to\infty} M_n\left(\eta\left(\frac{\bar{\lambda}+\lambda_\infty}{2}\right),\omega\right) = \infty$ almost surely (by (6.3)), it follows from (8.3) that $M_n(X_\infty,M,\omega)$ converges to infinity almost surely and hence in probability. Since τ is probability preserving, it follows that $M_n(X_\infty,M,\tau^{-n}\omega)$ converges to infinity in probability.

By the main theorem, $\lim_{M\to\infty} \lambda_0(M,\omega) = \bar{\lambda}$ almost surely and hence in probability. It follows easily that the sequence of random variables $\lambda_0(M_n(X_\infty, M, \tau^{-n}\omega), \omega)$ converges to $\bar{\lambda}$ in probability. Again using the fact that τ is probability preserving, it follows that $\lambda_0(M_n(X_\infty, M, \omega), \tau^n\omega) = \lambda_n(M_n(X_\infty, M, \omega), \omega)$ converges to $\bar{\lambda}$ in probability. But this is impossible, since $\lambda_\infty > \bar{\lambda}$ and $\lambda_n(X_\infty, M, \omega), \omega)$ converges to λ_∞ almost surely and hence in probability. This proves that $\lambda_\infty = \bar{\lambda}$.

It has been proved that for each M, $\lim_{n\to\infty} \lambda_n(M_n(X_\infty, M, \omega), \omega) = \bar{\lambda}$ almost surely. Clearly, $\lim_{n\to\infty} \lambda_n(M_n(X_\infty, M, \omega), \omega)$ is non-increasing as a function of M, for almost every ω. Therefore, for almost every ω, $\lim_{n\to\infty} \lambda_n(M_n(X_\infty, M, \omega), \omega) = \bar{\lambda}$ for all M.

Q.E.D.

9. THE INFINITE HORIZON, DISCOUNTED PROGRAM

The lemma below is needed in the proof of Theorem 3.5. Recall that if $\delta < 1$, $V_\delta(M,\omega) = \lim_{N\to\infty} V_{N\delta}(M,\omega)$ almost surely and $x_{\delta n}(M,\omega) = \lim_{N\to\infty} x_{N\delta n}(M,\omega)$ almost surely. Since $V_{N\delta}(M,\omega)$ is a non-decreasing function of N almost surely, $\lim_{N\to\infty} V_{N\delta}(M,\omega)$ exists almost surely. X_δ is the program corresponding to the plan x_δ.

9.1 Lemma

If $\delta < 1$, then

$$V_\delta(M,\omega) = E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_{\delta n}(M)) \,\Big|\, \mathcal{S}_0\Big)(\omega)$$

almost surely, and if X is any feasible program,

$$V_\delta(M,\omega) \geq E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_n(M)) \,\Big|\, \mathcal{S}_0\Big)(\omega)$$

almost surely.

This lemma says, of course, that X_δ is an optimal program for the infinite horizon problem with discounted utilities.

Proof

If X is any feasible program, then

$$\begin{aligned}\sum_{n=0}^{\infty} \delta^n u_n(X_n(M,\omega),\omega) &\leq \sum_{n=0}^{\infty} \delta^n X_n(M,\omega) \\ &\leq \sum_{n=0}^{\infty} \delta^n M_n(X,M,\omega) \\ &\leq \sum_{n=0}^{\infty} \delta^n \Big(M + \sum_{k=1}^{n} y_k(\omega)\Big).\end{aligned}$$

The terms of this last series are non-negative, so that it converges, though perhaps to infinity. However, by the monotone convergence theorem (Loève [8, p. 124]),

$$E \sum_{n=0}^{\infty} \delta^n \Big(M + \sum_{k=1}^{n} y_k \Big) = \sum_{n=0}^{\infty} \delta^n (M + (n-1)Ey_0) < \infty ,$$

so that the series $\sum_{n=0}^{\infty} \delta^n \Big(M + \sum_{k=1}^{n-1} y_k(\omega) \Big)$ converges to a finite limit almost surely. It has been proved that

(9.2) For almost every ω, there is a convergent series which dominates $\sum_{n=0}^{\infty} \delta^n u_n(X_n(M,\omega),\omega)$ for every feasible policy X.

It follows that $\sum_{n=0}^{\infty} \delta^n u_n(X_n(M,\omega),\omega)$ converges almost surely.

Next observe that since $\lim_{N\to\infty} x_{N\delta n}(M,\omega) = x_{\delta n}(M,\omega)$ almost surely, $\lim_{N\to\infty} M_n(X_{N\delta}, M, \omega) = M_n(X_\delta, M, \omega)$ almost surely. Hence,

$$\lim_{N\to\infty} u_n(X_{N\delta n}(M,\omega),\omega) = u_n(X_{\delta n}(M,\omega),\omega) \tag{9.3}$$

almost surely for all n.

It follows from (9.2) and (9.3) that

$$\lim_{N\to\infty} \sum_{n=0}^{N-1} \delta^n u_n(X_{N\delta n}(M,\omega),\omega) = \sum_{n=0}^{\infty} \delta^n u_n(X_{\delta n}(M,\omega),\omega) \tag{9.4}$$

almost surely.

(9.4) and (A.2) imply that

(9.5) $$\lim_{N\to\infty} V_{N\delta}(M,\omega) = E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_{\delta n}(M)) \,\Big|\, \mathcal{S}_0\Big)(\omega) \text{ almost surely.}$$

Let X be any feasible policy. Then, by Lemma 4.1,

$$V_{N\delta}(M,\omega) \geq E\Big(\sum_{n=0}^{N-1} \delta^n u_n(X_n(M)) \,\Big|\, \mathcal{S}_0\Big)(\omega) \quad \text{almost surely.}$$

Hence, (using (A.2) and (9.2))

$$V_\delta(M,\omega) \equiv \lim_{N\to\infty} V_{N\delta}(M,\omega) \geq E\Big(\sum_{n=0}^{\infty} \delta^n u_n(X_n(M)) \,\Big|\, \mathcal{S}_0\Big)(\omega) \text{ almost surely.}$$

Q.E.D.

10. PROOF OF THEOREM 3.5

In this section use will be made of the following well-known theorem.

Abel's Limit Theorem (Ahlfors [1, p. 42])

If a_n is a sequence of numbers such that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n$ exists,

then $\lim_{\delta\uparrow 1} \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} \delta^n a_n = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n$.

10.1 Lemma

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega)$$

$$= \lim_{\delta\uparrow 1} \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} \delta^n u_n(X_{\infty n}(M,\omega),\omega)$$

$$= E\, u_0(\eta_0(\bar{\lambda})) \quad \text{almost surely.}$$

Proof

By Abel's theorem, it is sufficient to prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega) = E\, u_0(\eta_0(\bar{\lambda})) \quad \text{almost surely.}$$

By the main theorem, $\lambda_n(M,\omega) \geq \bar{\lambda}$ for all M almost surely. Therefore, $X_{\infty n}(M,\omega) = \eta_n(\lambda_n(M_n(X_\infty, M,\omega),\omega),\omega) \overset{\leq}{=} \eta_n(\bar{\lambda},\omega)$ almost surely for all n. Hence,

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega)$$

$$\leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(\eta_n(\bar{\lambda},\omega),\omega) = E\, u_0(\eta_0(\bar{\lambda}))$$

almost surely. The equality follows from (6.5).

Since $E\, u_0(\eta_0(t))$ is a continuous function of t, it is sufficient to prove that

(10.2) for every $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega) \geq E\, u_0(\eta_0(\bar{\lambda}+\varepsilon))$$

almost surely.

Since $\lim_{n\to\infty} \lambda_n(M_n(X_\infty, M,\omega),\omega) = \bar{\lambda}$ almost surely, it follows that for almost every ω, $X_{\infty n}(M,\omega) \geq \eta_n(\bar{\lambda}+\varepsilon,\omega)$ if n is sufficiently large. Therefore, for almost every ω,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(X_{\infty n}(M,\omega),\omega)$$

$$\geq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n(\eta_n(\bar{\lambda}+\varepsilon,\omega),\omega) = E\, u_0(\eta_0(\bar{\lambda}+\varepsilon)).$$

This proves (10.2). Q.E.D.

Since $\eta_0(\bar{\lambda},\omega)$ is uniformly bounded (by Assumption 2.5, part v) and since $X_{\infty n}(M,\omega) \leq \eta_n(\bar{\lambda},\omega) = \eta_0(\bar{\lambda},\tau^n\omega)$ almost surely, it follows that the functions

$\frac{1}{N}\sum_{n=0}^{N-1} u_n(X_\infty(M,\omega),\omega)$ are uniformly bounded. Therefore, by the Lebesque dominated convergence theorem and Lemma 10.1,

$$\lim_{N\to\infty} \frac{1}{N} E \sum_{n=0}^{N-1} u_n(X_{\infty n}(M)) = E\, u_0(\eta_0(\bar{\lambda})).$$

By Abel's theorem, it follows that

$$\lim_{\delta\uparrow 1} \frac{1-\delta}{\delta} \sum_{n=0}^{\infty} \delta^n E\, u_n(X_{\infty n}(M)) = E\, u_0(\eta_0(\bar{\lambda})).$$

This proves the first four equalities of Theorem 3.5 for fixed M.

It is next shown that

(10.3) for each M, $\liminf_{N\to\infty} \frac{1}{N} V_N(M,\omega) \geq E\, u_0(\eta_0(\bar{\lambda}))$ almost surely.

By part iii of Lemma 4.1,

$$V_N(M,\omega) \geq E\Big(\sum_{n=0}^{N-1} u_n(X_{\infty n}(M))\Big|\mathcal{S}_0\Big)(\omega)$$

almost surely. Therefore,

$$\liminf_{N\to\infty} \frac{1}{N} V_N(M,\omega) \geq \liminf_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n(X_{\infty n}(M))\Big|\mathcal{S}_0\Big)(\omega)$$

almost surely. By (A.2) and Lemma 10.1,

$$\liminf_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n(X_{\infty n}(M)) \Big| \mathcal{S}_0 \Big) = E\, u_0(\eta_0(\bar{\lambda}))$$

almost surely. This proves (10.3).

Similar arguments prove that

$$\liminf_{N\to\infty} \frac{1}{N} E\, V_N(M,\omega) \geq E\, u_0(\eta_0(\bar{\lambda})), \tag{10.4}$$

$$\liminf_{\delta\uparrow 1} \frac{1-\delta}{\delta} V_\delta(M,\omega) \geq E\, u_0(\eta_0(\bar{\lambda})) \quad \text{almost surely, and} \tag{10.5}$$

$$\liminf_{\delta\uparrow 1} \frac{1-\delta}{\delta} E\, V_\delta(M,\omega) \geq E\, u_0(\eta_0(\bar{\lambda})). \tag{10.6}$$

The proofs of (10.5) and (10.6) require the use of Lemma 9.1.

Next, it is proved that

$$\limsup_{N\to\infty} \frac{1}{N} V_N(M,\omega) \leq E\, u_0(\eta_0(\bar{\lambda})). \tag{10.7}$$

Since

$$E\, u_0(\eta_0(\bar{\lambda})) = \lim_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n(\eta_n(\bar{\lambda})) \Big| \mathcal{S}_0 \Big)$$

almost surely, in order to prove (10.7) it is sufficient to prove that

$$\begin{aligned} 0 &\geq \limsup_{N\to\infty} \frac{1}{N} V_N(M,\omega) - \lim_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n(\eta_n(\bar{\lambda})) \Big| \mathcal{S}_0 \Big) \\ &= \limsup_{N\to\infty} \frac{1}{N} \left[V_N(M,\omega) - E\Big(\sum_{n=0}^{N-1} u_n(\eta_n(\bar{\lambda})) \Big| \mathcal{S}_0 \Big) \right] \\ &= \limsup_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} (u_n(X_{Nn}(M)) - u_n(\eta_n(\bar{\lambda}))) \Big| \mathcal{S}_0 \Big) \quad \text{almost surely.} \end{aligned}$$

For each N, n, M and ω, there exists a unique number $Z_{Nn}(M,\omega)$ such that

$$u_n(X_{Nn}(M,\omega),\omega) - u_n(\eta_n(\bar{\lambda},\omega),\omega) = u_n'(Z_{Nn}(M,\omega),\omega)(X_{Nn}(M,\omega) - \eta_n(\bar{\lambda}),\omega)).$$

Therefore,

$$\limsup_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} (u_0(X_{Nn}(M)) - u_n(\eta_n(\bar{\lambda})))\big|\mathcal{S}_0\Big)$$

$$= \limsup_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n'(Z_{Nn}(M))(X_{Nn}(M) - \eta_n(\bar{\lambda}))\big|\mathcal{S}_0\Big).$$

Since

$$\lim_{N\to\infty} \frac{1}{N} \Big(\sum_{n=0}^{N-1} (X_{Nn}(M) - \eta_n(\bar{\lambda}))\big|\mathcal{S}_0\Big) = 0 \text{ almost surely,}$$

$$\limsup_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} u_n'(Z_{Nn}(M))(X_{Nn}(M) - \eta_n(\bar{\lambda}))\big|\mathcal{S}_0\Big)$$

$$= \limsup_{N\to\infty} \frac{1}{N} E\Big(\sum_{n=0}^{N-1} (u_n'(Z_{Nn}(M) - \bar{\lambda})(X_{Nn}(M) - \eta_n(\bar{\lambda}))\big|\mathcal{S}_0\Big)$$

$$\leq 0 \text{ almost surely.}$$

The last inequality follows from the fact that

$$(u_n'(Z_{Nn}(M,\omega),\omega) - \bar{\lambda})(X_{Nn}(M,\omega) - \eta_n(\bar{\lambda},\omega)) \leq 0.$$

This completes the proof of (10.7).

Similar arguments prove that

$$\text{(10.8)} \qquad \limsup_{N\to\infty} \frac{1}{N} E\, V_N(M) \leq E\, u_0(\eta_0(\bar{\lambda})),$$

$$\text{(10.9)} \qquad \limsup_{\delta\uparrow 1} \frac{1-\delta}{\delta} V_\delta(M,\omega) \leq E\, u_0(\eta_0(\bar{\lambda})), \text{ and}$$

$$\text{(10.10)} \qquad \limsup_{\delta\uparrow 1} \frac{1-\delta}{\delta} E\, V_\delta(M) \leq E\, u_0(\eta_0(\bar{\lambda})).$$

(10.3) - (10.10) imply that for each value of M, the last four equalities of Theorem 3.5 hold almost surely. Since all of the limits appearing in the theorem are almost surely non-decreasing functions of M, it follows that for almost every ω, all the equalities of the theorem hold for all values of M.

11. REMOVING THE UPPER BOUND ON $u'(x,\omega)$

It has been assumed that $u'(0,\omega) \leq 1$ for all ω. This assumption was made in order to put an upper bound on the marginal utility of money. This bound may also be obtained by assuming that income is bounded away from zero. (This point was made by Schechtman in his dissertation [13].) Specifically, if it is assumed that for some $a > 0$, $y_0(\omega) \geq a$ almost surely, then one may allow $\lim_{x \to 0} u_0'(x)$ to be infinity and one may even allow $\lim_{x \to 0} u_0(x)$ to be minus infinity. It is, however, necessary to require that $\sup_{\omega} u_0'(a,\omega) < \infty$. If these assumptions are made, all the theorems of section 3 remain true. Their proofs require only a few modifications.

The following example shows that if the restriction on y_0 is not imposed, then the main theorem is false. The problem is that if the consumer is not guaranteed a positive income and if running out of income is disastrous, then the limit policy could require the consumer to save all his income, no matter how rich he might be.

Example

For $0 < x < \infty$, let $u(x) = \int_1^x e^{t^{-2}}\, dt$.

Let the incomes, y_n, be independently and identically distributed and suppose that $\text{Prob}(y_n = 0) = \text{Prob}(y_n = 1) = \frac{1}{2}$.

Let $V_N(M) = \max E \sum_{n=0}^{N-1} u(x_n)$, subject to $0 \leq x_n \leq M_n$, where $M_0 = M$ and $M_{n+1} = M_n - x_n + y_{n+1}$ for $n > 0$. It will be shown that $\lim_{N\to\infty} V'_N(M) = \infty$.

Clearly, $V'_N(M) \geq \text{Prob}(y_0 = \ldots = y_{N-1} = 0) E(u'(M_{N-1}) \mid y_0 = \ldots = y_{N-1} = 0)$. Also, if $y_0 = \ldots = y_{N-1} = 0$, then $M_{N-1} \leq \frac{M}{N}$. Therefore,

$$V'_N(M) \geq 2^{-N} u'\left(\frac{M}{N}\right) = 2^{-N} e^{\left(\frac{N}{M}\right)^2} = \left(\frac{e^{N/M^2}}{2}\right)^N.$$ Clearly,

$$\lim_{N\to\infty} \left(\frac{e^{N/M^2}}{2}\right)^N = \infty, \text{ for all } M.$$

12. POSSIBLE GENERALIZATIONS

It has been assumed throughout this paper that the utility function is differentiable, strictly concave, and additively separable. It should be possible to eliminate the differentiability and concavity assumptions altogether and to weaken the separability assumption considerably. Elimination of the differentiability and concavity assumptions introduces only technical difficulties. If the separability assumption is eliminated, it must be replaced by some other set of restrictions. It is not clear what these should be.

The intuitive arguments underlying the proof of the main theorem do not depend on the differentiability and concavity assumptions at all. It might seem that the proof would collapse because the policies $\eta(t)$ would be undefined. This is not so. It is true that if $u_n(\,\cdot\,,\omega)$ were not strictly concave, $\eta_n(t,\omega)$ could be multiple valued. Nevertheless, $E\,\eta_0(t)$ could be defined in the usual way as $\{Ef \mid f(\omega) \in \eta_0(t,\omega)$ almost surely and f is measurable$\}$. (See Aumann [2] for a discussion of integration of multiple-valued functions.) By choosing points from $\eta_n(t,\omega)$ randomly, it is possible to define a policy X for which $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} X_n(M,\omega)$ would exist almost surely and would equal any preassigned point in $E\,\eta_0(t)$. Also, $E\,\eta_0(t)$ would be a decreasing, upper-semi-continuous multiple-valued function. For these reasons, it seems that it would be possible to carry out the proof of the main theorem and the other theorems as well. However, the author has not checked the proof in detail.

If the separability assumption were eliminated, some restriction would have to be placed on interactions across time within the utility function. The utility function should be time invariant. The effect on the consumer's decisions in one period of changes in consumption in another period should become negligible as the time separating the periods goes to infinity. Similarly, the effect on the consumer's near term plans of a change in the time horizon should become negligible as the horizon goes to infinity. It is not clear how these conditions should be formulated.

APPENDIX

CONDITIONAL EXPECTATIONS OF RANDOM VARIABLES WITH A PARAMETER

Let C be some set of properties of a real valued function of a real variable.

Definition

C is said to be conservable if the following is true.

Let $Q: [0,\infty) \times \Omega \to [0,1]$ be any measurable function such that for almost every ω, the function $Q(\,.\,,\omega) = [0,\infty) \to [0,1]$ has properties C. Then, there exists a measurable function $q: [0,\infty) \times \Omega \to [0,1]$ which satisfies the following:

(i) $q(M,\,.\,) = E(Q(M)|\mathcal{S}_0)$ for all M.

(ii) For almost every ω, $q(\,.\,,\omega): [0,\infty) \to [0,1]$ has properties C.

A.1 Lemma

Any of the following pairs of properties is conservable: continuous and strictly decreasing; continuous from the right and non-decreasing; continuous from the left and non-increasing.

Frequent use will be made of the following fact.

A.2 Let $\mathcal{F}$ be a sub-σ-field of $\mathcal{S}$ and let z_k be a sequence of random variables on Ω such that $\lim_{k\to\infty} z_k = z$ almost surely and $|z_k(\omega)| \leq Z(\omega)$, almost surely for all k, where $EZ < \infty$. Then, $\lim_{k\to\infty} E(z_k|\mathcal{F}) = E(z|\mathcal{F})$ almost surely. (Doob [6, p. 23].)

Proof of Lemma A.1

Continuous and Strictly Decreasing

For every non-negative rational number M let $q(M, .)$ be any version of $E(Q(M)|\mathcal{S}_0)$.

For each pair of positive integers, k and N, let

$$f_N(k,\omega) = \sup\{|Q(M_2,\omega) - Q(M_1,\omega)| \mid 0 \leq M_1 \leq N,\ M_1 \leq M_2 \leq M_1 + k^{-1}\}.$$

It is easy to show that for every k, $f_N(k,\omega)$ is a measurable function of ω. Also, since for almost every ω, $Q(M,\omega)$ is continuous in M, it follows that $\lim_{k\to\infty} f_N(k,\omega) = 0$ almost surely. Therefore, by (A.2), $\lim_{k\to\infty} E(f_N(k)|\mathcal{S}_0) = 0$ almost surely. Also, if M_1 and M_2 are rational and $0 \leq M_1 \leq N$ and $M_1 \leq M_2 \leq M_1 + k^{-1}$, then $|q(M_1,\omega) - q(M_2,\omega)| \leq E(f_N(k)|\mathcal{S}_0)(\omega)$ almost surely. *It follows that for almost every* ω,

$$\lim_{\substack{M_1\to M \\ M_1 \text{ rational}}} \ \lim_{\substack{M_2\to M \\ M_2 \text{ rational}}} |q(M_1,\omega) - q(M_2,\omega)| = 0$$

for every M such that $0 \leq M < N$. Since this statement is true for each of the countably many values of N, it follows that it is true with $N = \infty$.

If M is irrational, let

$$q(M,\omega) = \lim_{\substack{M'\to M \\ M' \text{ rational}}} q(M',\omega),$$

provided that this limit exists. Otherwise, let $q(M,\omega)$ be arbitrary. It follows from what has just been proved that for almost every ω, $q(M,\omega)$ is everywhere well-defined and continuous as a function of M. It is easy to see that q is measurable.

It is next proved that $q(M, .) = E(Q(M)|\mathcal{S}_0)$ for all M. Let M_n be

a sequence of non-negative rational numbers converging to M. By definition, $q(M_n, .) = E(Q(M_n)|\mathcal{S}_0)$ for all n. Since almost surely, $q(t,\omega)$ and $Q(t,\omega)$ are continuous as functions of t, $q(M_n,\omega)$ and $Q(M_n,\omega)$ converge to $q(M,\omega)$ and $Q(M,\omega)$, respectively, almost surely. Therefore, by (A.2) $q(M, .) = E(Q(M)|\mathcal{S}_0)$.

It remains to be proved that for almost every ω, $q(.,\omega)$ is strictly decreasing as a function of M. If M_1 and M_2 are rational and $M_1 < M_2$, then $q(M_1,\omega) > q(M_2,\omega)$ almost surely. Since there are only countably many pairs of rational numbers, it follows that for almost every ω, $q(M_1,\omega) > q(M_2,\omega)$ whenever $M_1 < M_2$ and M_1 and M_2 are rational. Since $q(.,\omega)$ is continuous for almost every ω, it follows that $q(.,\omega)$ is strictly decreasing for almost every ω.

Continuous from the Right and Non-Decreasing

For every rational M, let $q(M, .)$ be a version of $E(Q(M)|\mathcal{S}_0)$. Since $Q(M,\omega)$ is almost surely non-decreasing in M, it follows by the argument just made that for almost every ω, $q(M,\omega)$ is non-decreasing as M varies over rationals.

If M is irrational, let $q(M,\omega) = \lim_{\substack{M' \downarrow M \\ M' \text{ rational}}} q(M,\omega)$. For almost every ω, this limit exists for all M, since $q(M,\omega)$ is almost surely non-decreasing on rationals.

By definition, $q(M,\omega)$ is almost surely continuous from the right at irrational values of M. It is now proved that the same is true at rational values. For each rational M and each positive integer k, let $f(M,k,\omega) = \sup\{Q(M',\omega) - Q(M,\omega) | M \le M' \le M + k^{-1}\}$. Since $Q(M,\omega)$ is

continuous from the right almost surely, $\lim_{k\to\infty} E(f(M,k)|\mathcal{S}_0) = 0$ almost surely. Therefore, for almost every ω, $\lim_{\substack{M'\downarrow M \\ M' \text{ rational}}} (q(M',\omega)-q(M,\omega)) = 0$. Since there are only countably many rational numbers, M, it follows that for almost every ω, $\lim_{\substack{M'\downarrow M \\ M' \text{ rational}}} (q(M',\omega) - q(M,\omega)) = 0$ for all rational M.

In order to prove that $q(M, .) = E(Q(M)|\mathcal{S}_0)$ for all M, let M_n be a sequence of rational numbers converging to M from the right and use (A.2).

Arguments similar to those just given prove the lemma for the case of functions which are continuous from the left and non-increasing.

Q.E.D.

A.3 Lemma

Let $Q: [0,\infty) \times \Omega \to [0,\infty)$ be measurable and suppose that for almost every ω, $Q(M,\omega)$ is continuously differentiable as a function of M and such that $\frac{d}{dM} Q(M,\omega)$ is positive, less than or equal to 1, and strictly decreasing. Let $Q'(M,\omega)$ be any function which equals $\frac{d}{dM} Q(M,\omega)$ whenever this derivative exists.

There exists a measurable function $q: [0,\infty) \times \Omega \to [0,\infty)$ satisfying the following.

(i) For almost every ω, $q(M,\omega)$ is continuously differentiable as a function of M. The derivative is positive, less than or equal to 1, and strictly decreasing.

Let $q'(M,\omega)$ denote any function equal to $\frac{d}{dM} q(M,\omega)$ whenever this derivative exists.

(ii) $q(M, .) = E(Q(M)|\mathcal{S}_0)$ and

$q'(M, .) = E(Q'(M)|\mathcal{S}_0)$, for all M.

Proof

Since Q is measurable, Q′ is so as well. Therefore, one may apply Lemma A.1 to Q′ and obtain q′ as in the lemma to be proved. Let $q(M,\omega) = \int_0^M q'(t,\omega)\,dt$. It follows from the Fubini theorem (Loève [8, p. 125]) that q satisfies property (ii) in the statement of the lemma.

Q.E.D.

A.4 Lemma

Let $Q: [0,\infty) \times \Omega \to [0,\infty)$ and $q: [0,\infty) \times \Omega \to [0,\infty)$ be measurable and such that for almost every ω, $Q(M,\omega)$ and $q(M,\omega)$ are continuous functions of M. Suppose that for every M, $q(M, .) = E(Q(M)|\mathcal{S}_0)$. If $z: \Omega \to [0,\infty)$ is measurable with respect to $\mathcal{S}_0$ and such that $E|Q(z)| < \infty$, then $q(z(.), .) = E(Q(z)|\mathcal{S}_0)$.

Proof

Let $z_n: \Omega \to [0,\infty)$ be a sequence of simple functions, which are measurable with respect to $\mathcal{S}_0$ and converge to z almost surely. Since z_n is simple, $q(z_n(.), .) = E(Q(z_n)|\mathcal{S}_0)$. By (A.2) and the continuity properties of q and Q, $q(z(.), .) = E(Q(z)|\mathcal{S}_0)$.

Q.E.D.

A.5 Lemma

Let $f_n: [0,\infty) \times \Omega \to [0,1]$ be a sequence of measurable functions. Suppose that

(a) for almost every ω, $f_n(M,\omega)$ is continuous from the right and non-decreasing as a function of M, and

(b) for almost every ω, $\lim_{n\to\infty} f_n(M,\omega) = 0$.

Then, there exists a sequence of measurable functions $\delta_n: [0,\infty) \times \Omega \to [0,1]$ with the following properties.

(i) $\delta_n(M) = E(f_n(M) | \mathcal{S}_0)$ for all M.

(ii) For almost every ω, $\delta_n(M,\omega)$ is continuous from the right as a function of M.

(iii) For almost every ω, $\lim_{n\to\infty} \delta_n(M,\omega) = 0$ for all M.

Proof

By applying Lemma A.1 to f_n, one obtains δ_n satisfying (i) and (ii) above. Clearly, for almost every ω, $\lim_{n\to\infty} \delta_n(M,\omega) = 0$ for every positive integer M. Since $\delta_n(M,\omega)$ is non-decreasing in M for almost every ω, it follows that $\lim_{n\to\infty} \delta_n(M,\omega) = 0$ for all M.

Q.E.D.

A.6 Lemma

Let $f: [0,\infty) \times \Omega \to (-\infty,\infty)$ be measurable and such that

(i) for each M, $f(M,\omega) \geq 0$ almost surely and

(ii) for almost every ω, $f(M,\omega)$ is continuous from the right as a function of M, or

(iii) for almost every ω, $f(M,\omega)$ is continuous from the left as a function of M.

Then, for almost every ω, $f(M,\omega) \geq 0$ for all M.

Proof

The lemma follows at once from the fact that for almost every ω, $f(M,\omega) \geq 0$ for all rational values of M.

Q.E.D.

BIBLIOGRAPHY

1. Ahlfors, Lars, Complex Analysis, third edition (McGraw-Hill, New York: 1966).

2. Aumann, Robert, "Integrals of Set-Valued Functions," Journal of Mathematical Analysis and Applications, 12 (1965) pp. 1-12.

3. Bewley, Truman, "Price Stability and the Permanent Income Hypothesis."

4. ———————, "Welfare Economics and the Permanent Income Hypothesis."

5. Brock, William and Leonard Mirman, "Optimal Growth and Uncertainty: The Discounted Case," Journal of Economic Theory, 4 (1972) pp. 479-513.

6. Doob, J.L., Stochastic Processes (Wiley and Sons, New York: 1963).

7. Friedman, Milton, A Theory of the Consumption Function (Princeton University Press, Princeton, N.J.: 1957).

8. Loève, Michel, Probability Theory, third edition (D. van Nostrand, New York: 1955).

9. Mirman, Leonard and Itzhak Zilcha, "Characterizing Optimal Policies in a One Sector Model of Economic Growth Under Uncertainty," mimeographed (University of Illinois, Urbana-Champaign, Jan. 1976).

10. Modigliani, F. and R. Brumberg, "Utility Analysis and the Consumption Function: An Interpretation of Cross-Section Data," in K. Kurihara, ed. Post Keynesian Economics (New Brunswick: 1954).

11. Projector, Dorothy and Gertrude Weiss, "Survey of Financial Characteristics of Consumers " (Board of Governors of the Federal Reserve System: Aug. 1966).

12. Radner, Roy, "Optimal Stationary Consumption with Stochastic Production and Resources," Journal of Economic Theory 6 (Feb. 1973) pp. 68-90.

13. Schechtman, Jack, "Some Applications of Competitive Prices to Dynamic Programming Problems Under Uncertainty," mimeographed, Operations Research Center, University of California, Berkeley, 1973.

14. ————————, "An Income Fluctuation Problem," Journal of Economic Theory 12 (1976) pp. 218-241.

15. Yaari, Menahem, "A Law of Large Numbers in the Theory of Consumer's Choice Under Uncertainty," Journal of Economic Theory 12 (1966) pp. 202-217.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Project on Efficiency of Decision Making in Economic Systems, 1737 Cambridge St., #404, Harvard University, Cambridge, Mass. 02138

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

(6) THE PERMANENT INCOME HYPOTHESIS: A THEORETICAL FORMULATION.

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

(9) Technical Report, No. 25

5. AUTHOR(S) (First name, middle initial, last name)

(10) Truman/Bewley

(12) 71p. (14) TR-25

6. REPORT DATE

(11) September, 1976

7a. TOTAL NO. OF PAGES

68

7b. NO. OF REFS

15

8a. CONTRACT OR GRANT NO.

(15) N00014-67-A-0298-0019 / NSF-GS-37899

b. PROJECT NO.

NR-47-004

c. tp

d. (16) NR-047-004

9a. ORIGINATOR'S REPORT NUMBER(S)

Technical Report No. 25

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its publication is unlimited. Reproduction in whole or in part is permitted for any purpose of the United States Government.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Logistics and Mathematics Statistics Branch, Department of the Navy, Office of Naval Research, Washington, D. C.

13. ABSTRACT

This paper defends the view that in dealing with a consumer's reponse to short-term changes, it is reasonable to assume that the marginal utility of money is constant. A theoretical defense of this view is made in terms of the consumer's intertemporal maximization problem. It is assumed that the consumer may hold money, but may not borrow. The consumer's utility function is assumed to be additively separable with respect to time. Prices, the consumer's income, and his utility function for each period are assumed to fluctuate according to a stationary stochastic process. It is proved that if the time horizon of the consumer's problem is sufficiently distant, if his discount rate for future utility is sufficiently small, and if he has a sufficient quantity of money, then the marginal utility of money is nearly independent of current prices and income and is nearly constant over time. The proof of these facts is based on economic common sense and the strong law of large numbers for stationary processes.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Unclassified 163700

Security Classification