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LIMIT MODELING
OF THERMAL TURBULENCE
IN UNSTABLY STRATIFIED FLOWS

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Technical Report

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by

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Abstract

Studied and developed is a new, relatively simple, rational, and viable theory of phenomenological thermal turbulence occurring in incident unstably stratified fluid flows. The subject of stratified flow has consisted largely of the study of stably stratified incident flows and of unstable thermal convection of a fluid heated from below with no incident flow. The new turbulence theory is intermediate in complexity between the venerable but generally inapplicable turbulent eddy-transport theory and the very complex generic theory known as "turbulence modeling" enjoying much current attention. We call our method "a limit theory of phenomenological thermal turbulence modeling"; based on a system of plausibly large and small parameters, it is more suitable in many applications where the said parameters behave as required; moreover, the theory lends itself to systematic improvement. The work proposed is preceded by experimental probing in a low-speed wind tunnel and use is always made of the general equations as a continuing guide to the specializations made in our specific formulation that sidesteps the attendant major computation in the general unstable, turbulence formulation. The research consists of a study of turbulent cellular flow due to an unstable temperature stratification underneath a flow with a stable temperature stratification. These two layers are separated by an interface whose detailed structure is a pausal boundary layer. The turbulent thermal convection below buffets the upper layer by means of "penetrative convection" and heats the upper layer at the interface because the turbulent heat flux mechanism must switch off. The penetrative convection and heat switch must amount to a "heat engine" whereby the rate of work of thermal origin is done on the bottom of the upper layer. This power source can then produce internal, surface, and Kelvin-Helmholtz waves.

1. Introduction

Studied is a relatively simple, rational, and viable theory of thermally turbulent incident stratified flow, dominantly controlled by the Rayleigh number R_a (although other parameters are important), in contrast to the classical inertially turbulent flow, controlled by the Reynolds number R_e. The application given of the theory is kept reasonably specific in order to avoid the pretentiousness of studying "the turbulence problem." The work is accompanied by prior experimental probing in a low-speed wind tunnel (see Appendix) in order to provide satisfaction that the theoretical simplifications presented here are commonly grounded in laboratory fact. Use is always made of the general equations, within the Boussinesq approximation, as a continuing guide to the specializations made in the specific formulation that sidesteps the attendant major computation identified with turbulence modeling as generally practiced.

Our prior theoretical and experimental probing suggests notable promise for a new phenomenological turbulence theory intermediate in complexity between the venerable but generally inapplicable turbulent eddy-transport theory ("K-theory") and the generic theory known as "turbulence modeling" enjoying much current attention. We call our method "a limit theory of phenomenological thermal turbulence modeling"; based on a system of plausibly large and small parameters, it is more suitable in many applications where the said parameters behave as required; moreover, the theory is subject to systematic (i.e. rational) improvement. The turbulence modeling schemes are very complicated but are still flawed by the weak link of any of the quite crude and arbitrary closure relations.

Thus we develop a new theory of phenomenological thermal turbulence occurring in incident unstably stratified fluid flows; it contains both new ideas and new methods. If the unstable flow lies under a stable flow, a heat-switch mechanism can produce a heat engine near the interface; we show this result explicitly.

Previously the subject of stratified flow has consisted largely of the study of stably stratified incident flows, with consequent wave phenomena, and of unstable thermal convection with no oncoming stream. The study of unstably stratified incident flow has not received commensurate attention (although its importance is adequately known) presumably because one must cope simultaneously (as we do here) with the cells, turbulence, and the oncoming stream. Thus the objective of the proposed work is to do just that. We provide these new ideas and new methods to deal with a little explored area to develop significant new results.

There are many areas of relevance in geophysical fluid dynamics. For example, the unstable turbulent "mixed layer" on the ocean surface largely drives the layers of abyssal sea water beneath. Further, in addition to the tides, the water above the continental shelf is well stirred by the surface wind stress and driven by the esturine currents. Moreover, these results have application to improved technology for off-shore oil drilling on the continental shelf: near the surface, near the bottom, and in between. Submersible objects are anticipated at all three levels — staged or unstaged. Thus the results can provide aid with respect to fossil fuel recovery. In the atmosphere, the surface wind layer of the planetary boundary layer is often unstable and turbulent and this fact is relevant to the energy and pollution problems plaguing modern society. Further, in the atmosphere there are the flow complexities, in the vertical, of frontal systems, and also the convective flow field associated with clouds outside the visible cloud structure: these two problems are relevant to weather forecasting.

Conerning the theory, it is appropriate to begin with the original work (Goldstein 1938 and Schichting 1968) in turbulence wherein the density ρ is constant and the temperature T is absent. For the case of incident flow, one decomposes the pressure and velocity variables into a time mean plus an instantaneous fluctuation thus:

$$p = \bar{p} + p' \quad \text{and} \quad u_i = \bar{u}_i + u_i', \quad \text{where} \quad (i = 1,2,3)$$

in cartesian tensor analysis. One then forms the time mean of the continuity and momentum equations, the inertia terms of which contribute the extra six turbulent Reynolds' stresses

$$\Sigma_{ij} \equiv -\rho \, \overline{u_i' u_j'} \, ; \quad (i,j = 1,2,3)$$

These stresses can be evaluated in terms of the local mean flow by eddy transport considerations, viz., the Prandtl mixing length theory, von Karman's hypothesis, or the vorticity transfer theories of Taylor and Goldstein. However, to provide instead for the fact that the turbulence generally has dynamics of its own, one multiplies the momentum equations by u_k' and then forms the time mean to obtain the six fluctuation-moment equations; these equations now govern the Reynolds' stresses but contain other complex mean correlations in addition; the problem of relating these extra correlations to the existing mean variables above, in order to close the system mathematically, is the original and perplexing closure problem by "turbulence modeling," say, for which no fully satisfying arguments exist. These incident flow considerations of turbulence are entirely inertial, whereas the present research emphasizes thermal turbulent correlations.

If we pass to a flow initially at rest (no incident flow) but heated from below at sufficient R_a, then gravitational instability overcomes molecular transport and cellular motion ensues. In its linear version we have the famous Bénard problem (Chandrasekar 1961); as R_a is continuously increased, the flow becomes increasingly nonlinear (and very sensitive to Prandtl number P_r unless P_r is very small) and finally turbulent (Turner 1973). The proposed problem has antecedents in this problem of thermal convection.

Current approaches to the problem of turbulence in an incident flow representing research in progress by turbulence modeling are prominently represented by the works of Lumley and Khajeh-Nouri (1973), Launder and Spalding (1972),

Donaldson (1972a), and Daly and Harlow (1970); Deardorff (1973) employs direct numerical simulation of turbulence but with turbulence modeling on the sub-grid scale, where the grid scale is chosen no smaller than practicable. Other current research in progress concerning incident turbulence flow includes direct numerical solution of the Navier-Stokes equations by Orszag (1972).

Current research in progress in stably or unstably stratified incident turbulent flow conspicuously includes the studies of Lumley (1973, 1975), Donaldson (1972b, 1973), Tennekes (1973a,1973b),and Wyngaard (1973). These works make use of turbulence modeling and/or similarity laws. Background reference books are those by O. M. Phillips (1969), P. Morel (ed.) (1973), and especially D. A. Haugen (ed.) (1973).

The turbulence modeling in the fluctuation-moment equations of the extra, complex, mean turbulent correlations is performed in order to close mathematically the system of relations in use. All of the known turbulence modelings are rather arbitrary and questionable, and certainly very crude when compared to the closure relations of kinetic theory or even radiative transfer by differential approximation. Whereas there is merit in the argument (Lumley and Khajeh-Nouri 1973) that the crude third-order closure schemes are attenuated in their influence on the first-order equations of mean motion, the closure schemes still serve to call in question the common-sense worth of the extremely extensive calculations with 20 to 30 simultaneous partial differential equations that follow; moreover, the computer and fluid-dynamicist team can raise a serious question concerning which one is slave and which one master?

2.   The General Problem

In Sects. 2.1 and 2.2 we present the general problem as it might be formulated by workers with a hyperthetically large preceding literature and very extensive computing and experimental capabilities. In Sects. 2.3 and 2.4 we present what we shall call the "heat switch mechanism" and the "heat engine,"

which we show occur when the gravitationally unstable fluid is bounded by stable fluid above. We later obtain the heat engine operation explicitly.

2.1 Physical Statement of General Problem

We consider a moderately unstable stratified shear flow furnished with a lid and floor. In the atmosphere or ocean (salinity aside), we expect that this flow will not only be cellular but also turbulent for thermal buoyancy reasons dominantly controlled by R_a.

From the geophysical point of view, a relevant unstably stratified shear flow is the one shown in Fig. 1 where the interacting lid is a stably stratified shear flow above and the floor is a solid boundary endowed with temperature T_ℓ or heat transfer rate q_ℓ that drive or do not drive the initial flow; the non-monotonic temperature profile $T(x_3)$ and velocity profile $U(x_3)$, together with a turbulence field are introduced initially at $x_1 = 0$. This problem is clearly difficult but the unstable layer can represent the following: (i) in the atmosphere, an unstable earth surface layer interacting with a stable upper planetary boundary layer and possible producing internal waves, Kelvin-Helmholtz waves or just a general lift of the interface, and (ii) the upper ocean (inverted), where the mixed layer (with incompletely modeled surface wind effects) interacts with the stably stratified fluid below and can produce internal waves and Kelvin-Helmholtz waves again. This problem introduces the notable physical phenomena discussed in Sects. 2.3 and 2.4 below.

2.2 Formulation of the General Problem

Let u_i be the velocity, ρ be the density, p be the pressure, T be the temperature, β be the thermal coefficient of expansion, μ be the viscosity, and $(\ )_o$ denote the constant common reference state. The strain rate is given by $e_{ij} \equiv u_{i,j} + u_{j,i}$. In accord with a tenet of the Boussinesq approximation, we split out the dominant part of the body force in the momentum equation $\underline{M}$ by introducing the neutral reference hydrostatic equilibrium

$$- \rho_o g \delta_{3i} \equiv P,_i \; ; \quad (i = 1,2,3), \tag{2.2.1}$$

where the dominant $P \sim x_3$ is a reference hydrostatic pressure.

We introduce the departure pressure $\tilde{p}$ by

$$p \equiv P + \tilde{p} \tag{2.2.2}$$

and the departure temperature $\tilde{T}$ by

$$T \equiv T_o + \tilde{T}, \tag{2.2.3}$$

where the tilde ($\tilde{}$) is omitted in turbulence equations. The thermal equation of state in the Boussinesq approximation is

$$\rho = \rho_o (1 - \beta_o \tilde{T}). \tag{2.2.4}$$

The general departure equations of continuity $\underset{\sim}{C}$, momentum $\underset{\sim}{\tilde{M}}$, and energy $\underset{\sim}{\tilde{E}}$ are then, for constant properties,

$$\underset{\sim}{C}: \qquad\qquad\qquad u_j,_j = 0, \tag{2.2.5}$$

$$\underset{\sim}{\tilde{M}}: (\rho_o u_i),_t + (\rho_o u_i u_j),_j = \rho_o \beta_o g \tilde{T} \delta_{3i} - \tilde{p},_i + \mu_o u_i,_{jj} \; ; \quad (i,j = 1,2,3),$$

and
$$\tag{2.2.6}$$

$$\underset{\sim}{\tilde{E}}: \qquad (\rho_o c_{po} \tilde{T}),_t + (\rho_o c_{po} \tilde{T} u_j),_j = - F_j,_j + k_o \tilde{T},_{jj} + \frac{1}{2} \mu_o e_{ij} e_{ij}, \tag{2.2.7}$$

where c_p is the specific heat, k is the thermal conductivity, and F_j is the radiative flux vector. These equations are identified with unsteady thermal convection.

By writing $u_i = \bar{u}_i + u_i'$, $T = \bar{T} + T'$, etc. and taking the time mean of the above equations, we obtain the general mean turbulent departure equations. These are

$$\underset{\sim}{\bar{C}}: \qquad\qquad\qquad \bar{u}_j,_j = 0, \tag{2.2.8}$$

$$\underset{\sim}{\bar{M}}: \quad (\rho_o \bar{u}_i),_t + (\rho_o \bar{u}_i \bar{u}_j),_j = \rho_o \beta_o g \bar{T} \delta_{3i} - \bar{p},_i + \mu_o \bar{u}_i,_{jj} - (\rho_o \overline{u_i' u_j'}),_j$$

$$; \quad (i,j = 1,2,3), \tag{2.2.9}$$

and

$$\bar{\bar{E}}: \quad (\rho_o c_{po} \bar{T})_{,t} + (\rho_o c_{po} \bar{T}\bar{u}_j)_{,j} = -\bar{F}_{j,j} + k_o \bar{T}_{,jj} - (\rho_o c_{po} \overline{T'u'_j})_{,j}$$

$$+ \frac{1}{2} \mu_o \bar{e}_{ij}\bar{e}_{ij} + \frac{1}{2} \mu_o \overline{e'_{ij}e'_{ij}} , \qquad (2.2.10)$$

where the "extra" terms are the turbulent correlations

$$\Sigma_{ij} \equiv -\rho_o \overline{u'_i u'_j} = \text{turbulent stress tensor for constant}$$
$$\text{density flow,}$$

$$Q_j \equiv \rho_o c_{po} \overline{T'u'_j} = \text{turbulent heat-flux vector for constant}$$
$$\text{density flow, and}$$

$$\frac{1}{2} \mu_o \overline{e'_{ij}e'_{ij}} = \text{viscous turbulent dissipation per unit}$$
$$\text{volume for constant density flow.}$$

It is observed that ten temporal correlations stand unevaluated and these are not taken herein to depend on or track the local mean flow field; the problem is then very formidable and not herewith closed mathematically.

We can form the general mean fluctuation departure mechanical energy relation by multiplying (2.2.6) by u'_k; by writing the variables as $u_i = \bar{u}_i + u'_i$, $T = \bar{T} + T'$, etc.; and by taking the time mean of the resulting equation. Then integrate over the finite volume τ bounded by the fixed surface A to obtain finally

$$\oint_A (-\overline{p'u'_j} + \mu_o \overline{e'_{ij}u'_i})n_j dA - \frac{1}{2} \oint_A \rho_o \overline{u'_i u'_i u'_j}n_j dA =$$

$$\qquad\qquad (2.2.11)$$

$$\int_\tau \rho_o \frac{D}{Dt}\left(\frac{\overline{u'_i u'_i}}{2}\right)d\tau - \int_\tau \rho_o \beta_o g\overline{T'u'_3}d\tau + \frac{1}{2}\mu_o\int_\tau \overline{e'_{ij}e'_{ij}}d\tau - \frac{1}{2}\int_\tau \Sigma_{ij}\bar{e}_{ij}d\tau.$$

If T_i is the departure traction on A, then the first integral on the left can be shown to be $\oint_A \overline{T'_i u'_i}dA$, where $T'_i \equiv \sigma'_{ij}n_j$. In stratified flow a key point is that the departure body force per unit volume ($\equiv$ buoyancy force) in (2.2.6) is nonconservative; key points in the heat-switch mechanism presented next are that

$\overline{T'u_3'} > 0$ in the unstable flow and that the second integral on the right makes a large contribution in the unstable flow.

2.3 Heat-Switch Mechanism

Consider, at time $t = 0$, the initial vertical mean turbulent profiles of temperature $\overline{T}(x_3)$ and horizontal stream $\overline{U}(x_3)$ shown in Fig. 2 (derived from Fig. 1). A pause layer is shown and the temperature increases from this value in both the upper layer, denoted by subscript u, and the lower layer, denoted by subscript ℓ. Then the upper layer of fluid is gravitationally stable because $\partial \overline{T}_u/\partial x_3 > 0$ and the lower layer is gravitationally unstable because $\partial \overline{T}_\ell/\partial x_3 < 0$; these two segments of the initial temperature profile have no special shape, e.g. they are not straight lines. The initial horizontal stream profile has a shape that is not of special importance to the present consideration excepting that the profile in the lower layer is nearly vertical. One deals with the departures in pressure, temperature, and density from the reference state given in Sect. 2.2.

In the general mean turbulent departure energy equation given, the presently relevant term is the turbulent heat flux vector

$$Q_j \equiv \rho_o c_{po} \overline{T'u_j'} \ . \tag{2.3.1}$$

In the general departure mechanical energy relation for the mean fluctuation turbulence, the presently relevant term is the thermal buoyancy power expended in unit volume

$$\rho_o g \beta_o \overline{T'u_3'} \ . \tag{2.3.2}$$

The lower layer is gravitationally unstable, so that, in the atmosphere and ocean, the Rayleigh number would be sufficiently high to give immediate transition to turbulence. The buoyant parcels of fluid shown in Fig. 2 are turbulent eddies; whether the eddies move up or down, the associated turbulent heat correlation

$$\overline{T'u_3'} > 0 \qquad\qquad (2.3.3)$$

for unstably stratified fluid, whereas

$$\overline{T'u_3'} = 0 \qquad\qquad (2.3.4)$$

to excellent approximation for the stably stratified fluid flow. In Fig. 2 it might be supposed that a loose turbulent analogue to the Fourier heat law prevails so that $Q_j = - k_T \partial \overline{T} \partial x_j$, where k_T is a Boussinesq eddy conductivity. But this loose analogue ignores the connection between buoyancy force and heat transfer. Moreover, the turbulent heat flux correlation $\overline{T'u_j'}$ does not depend on the local $\overline{T}$ and its derivatives (tracking) but has an identity of its own in the fluid.

Let us examine the two implications of the preceding paragraph in Fig. 2. In stratum P, the pause layer, we see that it is heated from below but no heat leaves it from above. The turbulent heat correlation has switched off. We then ask, what is the fate of the heated pause layer? In the extensive lower unstable layer, the turbulent heat flux $\overline{T'u_3'}$ is clearly driven by $\overline{T}(x_3)$ in a bulk fashion but there is no one-to-one correspondence, in general, between the two. The stratum L shown in Fig. 2 is heated from below by the turbulent heat correlation and cooled from above thereby, but not to the same extent; we cannot expect that $\partial(\overline{T'u_j'})/\partial x_j = 0$ in general. In both the strata P and L we see a turbulent heat correlation imbalance that must be resolved.

To this end, we turn to the general mean turbulent energy equation. If we neglect therein the viscous dissipation, the viscous turbulent dissipation, the molecular conductivity, and radiative transfer, we arrive at the buoyancy equation (that governs the scalar part of the departure buoyancy force per unit volume in the momentum equation), as follows:

$$\frac{D}{Dt}(\rho_o \beta_o g\overline{T}) \equiv \frac{\partial}{\partial t}(\rho_o \beta_o g\overline{T}) + \frac{\partial}{\partial x_j}(\rho_o \beta_o g\overline{T}\overline{u}_j) = -\frac{\partial}{\partial x_j}(\rho_o \beta_o g\overline{T'u_j'}), \quad (j = 1,2,3).$$

$$(2.3.5)$$

For the initial-value problem under discussion in Fig. 2, we have first to look at the term corresponding to turbulent heat flux in the $j = 3$ direction, viz.

$$- \frac{\partial}{\partial x_3} (\rho_o \beta_o g \overline{T'u_3'}).$$
(2.3.6)

Evidently, this term is positively infinite in the pause layer (if the pause layer is thin), nonzero in the lower layer, and zero in the upper layer. On the other hand, the convective term in (2.3.5) is satisfied everywhere by the initial conditions given in Fig. 2. The nonvanishing of term (2.3.6) in the pause layer and the lower layer is crucial; we see in (2.3.5) that it leads initially to local heating (or cooling) because we must get

$$\partial \overline{T}/\partial t \neq 0.$$

In the momentum equation, if the departure buoyancy terms $\sim \overline{T}$ becomes fully time dependent, then hydrostatic equilibrium in the x_3 direction becomes impossible so that mean motion ensues in the lower layer.

To see what sort of mean motion results in the steady-state operation, suppose that the flow is approximately steady in a moving frame attached to $\overline{U}_\ell$. Then (2.3.5) gives

$$\frac{\partial}{\partial x_j} (\rho_o \beta_o g \overline{T} \overline{u}_j + \rho_o \beta_o g \overline{T'u_j'}) \doteq 0.$$
(2.3.7)

This conservation equation means that the nonvanishing of (2.3.6) must be counterbalanced by considering other components of the correlation to be nonzero and especially by mean convective motion in the interior of the lower layer.

Suppose $\overline{u}_1 = \overline{T'u_1'} = \frac{\partial}{\partial x_1} = 0$ at first. Then $j = 3$ in (2.3.7) and (2.3.7) is integrable, to give

$$\overline{T}(x_3)\overline{u}_3(x_3) + \overline{T'u_3'} (x_3) = \overline{T'u_3'} (x_3 = 0),$$

if $\overline{u}_3(x_3) = 0$. The result signifies that mean convective motion extends from the pause layer vertically downward to infinity. Clearly, this result is crude

and we should consider next the case where $\bar{u}_1$, $\overline{T'u_1'}$, and $\partial/\partial x_1$ are nonzero. If we examine (2.3.5), using the substantial derivative on the left and the positive value of term (2.3.6) on the right, then at the top of the lower layer near the pause layer we see that $\bar{T}$ for a moving fluid element must increase.

The two considerations of the last paragraph lead us from vertical convection to the type of cellular convection sketched in Fig. 2, where the cells corotate, not counterrotate. This cellular motion in the extensive lower layer is the direct consequence of the nonvanishing of term (2.3.6) in the pause layer and lower layer. Noting that we deal always with departures from the reference state, we observe that the cellular motion implies nonzero values of $\overline{T'u_1'}$ and $\overline{T'u_2'}$ as well as $\overline{T'u_3'} > 0$.

We may now generalize Eqs. (2.3.3) and (2.3.4) to read that, for the heat switch,

$$\overline{T'u_3'} > 0, \quad \overline{T'u_\alpha'} \neq 0, \quad (\alpha = 1,2) \tag{2.3.8}$$

for unstably stratified flow, whereas

$$\overline{T'u_j'} = 0, \quad (j = 1,2,3) \tag{2.3.9}$$

to excellent approximation for the stably stratified flow.

2.4 Heat Engine

At the structured interface (called the pause layer) in Fig. 3 two events happen. First, the thermal cells and turbulent eddies buffet the stable upper layer by "convective penetration" (the phenomenon was first noted in a tank of water by Townsend 1964), delivering power to the upper layer by the process. Moreover, it is to be borne in mind that the turbulent heat flux must switch off rather abruptly upon reaching strong, stable stratification. The second event at the interface is therefore the formation of a thermal layer that can lower the local Richardson number R_i in a thin layer from a stable value to a value less than the critical value 0.25 for which the layer becomes dynamically unstable.

The two events produce respectively internal waves and Kelvin-Helmholtz waves. The breaking of these waves ultimately absorbs the thermal buoyancy power delivered at the interface from below. These two phenomena are depicted in Fig. 3. Work is therefore being done on the upper stable layer by the thermal phenomena in the lower layer so we have what might be called a "heat engine."

3. The Specific Problem

Our prior experimental probing in a low-speed wind tunnel (cf. Appendix) with our concomitant theoretical considerations suggest the common existence of a system of large and small parameters. These plus other considerations permit the formulation of our relatively simple, rational theory of thermally turbulent incident stratified flow that is turbulent due to gravitational instability. Specifically, outside any interfacial pause layer, turbulent stresslike terms, turbulent kinetic energy terms, and viscous dissipation terms can be removed mainly because of certain large correlation ratio parameters J_F, I, and H to be defined later; moreover, it is common that the relevant values of the reciprocal internal Froude number are small and it is convenient to our purposes to develop the square of the reciprocal internal Froude number in ascending series in this section as an added simplification and elucidation. We present in Sect. 3 the theory and its application to the problem that will now be described.

3.1 Physical Statement of Specific Problem

We return now to Fig. 1 and the discussion in Sect. 2.1. We undertake the considerations of turbulent flow due to large R_a (but not large R_e) due to the gravitationally unstable region depicted therein; the equations also hold, often trivially, in the nonturbulent stable region where the fluctuations are identically zero so that the entire fluid domain is treated except the interface in Fig. 1 when the thin pause layer structure of the interface must be considered (see Fig. 2).

For $x_3 = -d$ and for $x_3 = +D$, we provide boundary conditions of zero heat transfer on the solid boundaries over the interval $0 \leq x_1 < \infty$. At $x_1 = 0$, the depicted temperature profiles $T(x_3)$ and velocity profiles $U(x_3)$ are the upstream boundary conditions introduced, superposed on which, and not shown, are suitable turbulence fields. For $x_1 = 0$, the boundary condition is that turbulence vanishes in the stable layer. In the unstable layer, the boundary condition at $x_1 = 0$ is taken to be that of turbulent thermal convection produced by an extensive uniformly heated surface for small Prandtl number P_r, large R_a, large flux Richardson number J_F, large I, and large H. Of course, no requirement is set on internal Froude number. The requisite turbulent correlations can be taken from experiment and/or theory. These data, in truth, fix the problem. The boundary data are then set in semi-infinite parabolic form. The problem has its antecedents in the highly nonlinear extension of the Bénard problem but is aperiodic in x_1, driven only by the upstream boundary conditions, and approaches some equilibrium solution as $x_1 \to \infty$.

3.2 Formulation and Procedures of Specific Problem

3.2.1 *Ansatz*

We begin with the general unsteady laminar departure Eqs. (2.2.5), (2.2.6), and (2.2.7) with $\mu_o = 0$, $k_o = 0$, $F_j = 0$, and $(\mu_o/2)e_{ij}e_{ij} = 0$. The energy equation $\underline{E}$ reduces to the "buoyancy equation" $\underline{B}$ that amounts to the substantial derivative of the buoyancy force magnitude in $\underline{M}$.

The departure variables are u_i, $\tilde{p}$, and $\tilde{T}$ and the Boussinesq thermal state equation remains as

$$\rho = \rho_o(1 - \beta_o\tilde{T}) \qquad (3.2.1)$$

(For a perfect gas, $\beta_o = 1/T_o$.)

We decompose the departure variables into the two parts:

$$\tilde{T} \equiv T(x_3) \quad + \quad \hat{T}(x_1, x_2, x_3, t)$$

$$\tilde{p} \equiv \Pi(x_3) \quad + \quad \hat{p}(x_1, x_2, x_3, t) \qquad (3.2.2)$$

$$u_j \equiv U(x_3)\delta_{1j} \quad + \quad \hat{u}_j(x_1, x_2, x_3, t)$$

Sheared	"Excursion Flow"
hydrosta-	due to cells,
tic given	turbulence, and
boundary	waves
conditions	

This decomposition serves to introduce the boundary conditions into the differential equations as well as the Brunt-Väisälä parameter

$$N^2(x_3) \equiv g\beta_o \frac{\partial T}{\partial x_3} \lessgtr 0.$$

We form dimensionless equations from the definitions

$$\hat{u}_i^* \equiv \frac{\hat{u}_i}{U_c}, \quad \hat{p}^* \equiv \frac{\hat{p}}{\rho_o U_c^2}, \quad \hat{T}^* \equiv \frac{\hat{T}}{\hat{T}_c}, \quad U^* \equiv \frac{U}{U_c}, \quad t^* \equiv \frac{U_c t}{d}, \quad x_i^* \equiv \frac{x_i}{d}. \qquad (3.2.3)$$

where U_c is a constant characteristic velocity difference in the x_3 direction,

d is a constant characteristic associated distance in the x_3 direction,

and $\hat{T}_c$ is a constant characteristic value of $\hat{T}$.

We obtain the following excursion-flow equations in the dimensionless excursion-flow variables $\hat{u}_i^*$, $\hat{p}^*$, and $\hat{T}^*$:

$$\hat{\underline{C}}^*: \qquad\qquad \frac{\partial}{\partial x_j^*}(\hat{u}_j^*) = 0, \qquad (3.2.4)$$

$$\hat{\underline{M}}^*: \quad \frac{\partial}{\partial t^*}(\hat{u}_i^*) + U^*(x_3^*)\frac{\partial}{\partial x_1^*}(\hat{u}_i^*) + \hat{u}_3^*\delta_{1i}\frac{\partial}{\partial x_3^*}U^*(x_3^*) + \frac{\partial}{\partial x_j^*}(\hat{u}_i^*\hat{u}_j^*)$$
$$= \varepsilon\hat{T}^*\delta_{3i} - \frac{\partial\hat{p}^*}{\partial x_i^*}; \quad (i,j = 1,2,3), \qquad (3.2.5)$$

and

$$\hat{\underline{B}}^*: \qquad \frac{\partial\hat{T}^*}{\partial t^*} + U^*(x_3^*)\frac{\partial\hat{T}^*}{\partial x_1^*} + \frac{\partial}{\partial x_j^*}(\hat{T}^*\hat{u}_j^*) = -\left[J(x_3^*)/\varepsilon\right]\hat{u}_3^*. \qquad (3.2.6)$$

For similitude parameters, we find above

$$F_I^{-2} \equiv \left[\frac{\hat{T}_c \beta_o g d}{U_c^2}\right] \equiv \epsilon = (\text{internal Froude no.})^{-2} \gtrless 0, \qquad (3.2.7)$$

and

$$J(x_3) \equiv \left[\frac{N^2(x_3)}{U_c^2/d^2}\right] = \text{a Richardson no.} \gtrless 0. \qquad (3.2.8)$$

Now $\beta_o \hat{T}_c$ is small in the Boussinesq approximation, tending to make ϵ small. Moreover $|J/\epsilon| \sim 1$ for $\hat{\underline{\underline{B}}}^*$ to balance for small ϵ. Hereafter we shall exploit by series expansion the small parameters

$$\epsilon \approx 10^{-1} \quad \text{and} \quad |J/\epsilon| \sim 1, \qquad (3.2.9)$$

noting that the smallness of these parameters affects the magnitude of the mean flow variables but not the relevant turbulent correlations $= O(\epsilon^o)$ introduced by the upstream boundary conditions at full strength for all ϵ. The relative importance of the turbulent correlations is determined by additional considerations for certain large correlation ratio parameters J_F, I, and H.

Henceforth, let the notation simplify to

$$\hat{u}_i^* \rightarrow u_i, \quad \hat{p}^* \rightarrow p, \quad \hat{T}^* \rightarrow T, \quad U^*(x_3) \rightarrow U(x_3), \quad t^* \rightarrow t, \quad x_i^* \rightarrow x_i.$$
$$\qquad (3.2.10)$$

3.2.2 Mean Turbulent Equations

We decompose the excursion flow into cells or waves $\overline{(\ )}$ plus temporal turbulent fluctuations $(\ )'$ according to

$$u_j = \bar{u}_j + u_j', \quad p = \bar{p} + p', \quad T = \bar{T} + T', \qquad (3.2.11)$$

and time mean the preceding $\underline{\underline{C}}$, $\underline{\underline{M}}$, and $\underline{\underline{B}}$ to obtain

$$\bar{\underline{C}}: \qquad\qquad \bar{u}_j,_j = 0, \qquad\qquad (3.2.12)$$

$$\underline{\bar{M}}: \bar{u}_i,_t + U(x_3)\bar{u}_i,_1 + \bar{u}_3\delta_{1i}U,_3(x_3) + (\bar{u}_i\bar{u}_j),_j + \overline{(u_iu_j)},_j = \epsilon\bar{T}\delta_{3i} - \bar{p},_i;$$

$$(i,j = 1,2,3), \qquad (3.2.13)$$

$$\underline{\bar{B}}: \qquad \bar{T},_t + U(x_3)\bar{T},_1 + (\bar{T}\bar{u}_j),_j = - (\overline{T'u'_j}),_j - \left[J(x_3)/\epsilon \right] \bar{u}_3. \qquad (3.2.14)$$

The variables are $\bar{u}_i$, $\bar{p}$, $\bar{T}$, and $(\overline{T'u'_j})$ and the equations do not close because the three correlations $(\overline{T'u'_j})$ are as yet unspecified. In the mean equations of motion, a necessary condition for neglecting the Reynolds' stress term in $\underline{M}$ is that the flux Richardson number J_F is large.

3.2.3 *Series Procedure in* ϵ *for Solving Mean Turbulent Equations*

(a) Zeroth-order solution:

If $\epsilon = 0$, then $\bar{\underline{C}}$ and $\underline{\bar{M}}$ uncouple from $\underline{\bar{B}}$. Considering the boundary conditions, the solution is

$$\bar{u}_i^{(o)} \equiv 0 \quad \text{and} \quad \bar{p}^{(o)} \equiv 0. \qquad (3.2.15)$$

As long as turbulence is present, $\underline{B}$ is nontrivial and $\underline{\bar{B}}$ becomes the following brief equation in $\bar{T}^{(o)}$ and $(\overline{T'u'_j})^{(o)}$:

$$\underline{\bar{B}}^{(o)}: \qquad \bar{T},_t^{(o)} + U(x_3)\bar{T},_1^{(o)} = - (\overline{T'u'_j}),_j^{(o)} = O(1) \qquad (3.2.16)$$

on measure ϵ. This relation shows no mean mechanical activity but a mere heating up; notice that $J \sim N^2$ is absent and the interface can merely rise and erode the upper stably stratified layer.

(b) First-order solution:

If $O(\epsilon) \ll 1$, then $\bar{\underline{C}}$ and $\underline{\bar{M}}$ are weakly coupled to $\underline{\bar{B}}$ and $\underline{\bar{B}}$ is modified by the factor $\left[1 + O(\epsilon) \right]$. It will be observed that $J \sim N^2$ now appears.

Now we arrive at the peculiar series

$$\bar{u}_i = 0 + \bar{u}_i^{(1)} \cdot \epsilon + \ldots$$

$$\bar{p} = 0 + \bar{p}^{(1)} \cdot \epsilon + \ldots$$

$$\bar{T} = \bar{T}^{(o)} + \bar{T}^{(1)} \cdot \epsilon + \ldots \tag{3.2.17}$$

$$(\overline{T'u_j'}) = (\overline{T'u_j'})^{(o)} + (\overline{T'u_j'})^{(1)} \cdot \epsilon + \ldots$$

Then the first-order equations are

$$\underline{\underline{C}}^{(1)}: \qquad\qquad \bar{u}_{j,j}^{(1)} = 0, \tag{3.2.18}$$

$$\underline{\underline{M}}^{(1)}: \quad \bar{u}_{i,t}^{(1)} + U(x_3)\bar{u}_{i,1}^{(1)} + \bar{u}_3^{(1)}\delta_{1i}U_{,3}(x_3) = \bar{T}^{(o)}\delta_{3i} - \bar{p}_{,i}^{(1)}, \tag{3.2.19}$$

and

$$\underline{\underline{B}}^{(1)} - \underline{\underline{B}}^{(o)}: \qquad \bar{T}_{,t}^{(1)} + U(x_3)\bar{T}_{,1}^{(1)} + \bar{u}_j^{(1)}\bar{T}_{,j}^{(o)}$$

$$= - (\overline{T'u_j'})_{,j}^{(1)} - \left[J(x_3)/\epsilon\right]\bar{u}_3^{(1)}. \tag{3.2.20}$$

The first two equations of this trio uncouple from the third; they are linear and we solve for $\bar{u}_i^{(1)}$ and $\bar{p}^{(1)}$. Subsequently, we are able to solve the third equation; it is linear and we ultimately solve for $\bar{T}^{(1)}$ and $(\overline{T'u_j'})^{(1)}$.

3.2.4 *Mean Turbulent Fluctuation-Moment Equations*

Begin with $\underline{C}$, $\underline{M}$, and $\underline{B}$, Eqs. (3.2.4) to (3.2.6), write $u_i = \bar{u}_i + u_i'$, $T = \bar{T} + T'$, etc., and next multiply $\underline{M}$ and $\underline{B}$ each by u_k' and then T' to form the following mean turbulent fluctuation-moment equations indicated in symbolic form by

$$\overline{\underline{\underline{M}} \cdot u_k'}, \quad \overline{\underline{\underline{M}} \cdot T'}, \quad \overline{\underline{\underline{B}} \cdot T'}, \quad \text{and} \quad \overline{\underline{\underline{B}} \cdot u_i'}. \tag{3.2.21}$$

In closing the mean turbulent equations for solution, we work <u>not</u> with the traditional $\overline{\underline{\underline{M}} \cdot u_k'}$ but with the three equations

$$[\overline{\underline{\underline{M}} \cdot T'} + \overline{\underline{\underline{B}} \cdot u_i'}] \tag{3.2.22}$$

and the one equation

$$[\overline{\underline{\underline{B}} \cdot T'}] \tag{3.2.23}$$

governing as variables the two appearing correlations $(\overline{T'u_i'})$ and $(\overline{T'T'})$; we neglect $(\overline{u_i'u_j'})$ compared to $(\overline{T'u_i'})$ as they occur in (3.2.22) and justify this strongest assumption made by use of the argument mentioned in the last paragraph of Sect. 1 and attributed to Lumley and Khajeh-Nouri (1973).

For the "extra" correlations in these fluctuation-moment equations, we arbitrarily adopt from the literature the turbulence modeling of Donaldson (1973) for $(\overline{p'T'})$, $(\overline{p'T_{,i}'})$, $(\overline{T'u_i'u_j'})$, and $(\overline{T'T'u_j'})$; this modeling introduces the length-scale parameters $\Lambda_\alpha (\alpha = 1,2,3)$ and the velocity parameter V. We emphasize that our contribution in this paper is not dependent on the individual choice made or preferred for the turbulence modeling. Using the nondimensionalization method of Sect. 3.2.1, the above cited turbulence modeling gives

$$(\overline{p'T'}) = -\Lambda_3 V(\overline{T'u_j'})_{,j}$$

$$(\overline{p'T_{,i}'}) = -(V/\Lambda_1)(\overline{T'u_i'})$$

$$(\overline{T'u_i'u_j'}) = -V\Lambda_2\left[(\overline{T'u_i'})_{,j} + (\overline{T'u_j'})_{,i}\right] \tag{3.2.24}$$

$$(\overline{T'T'u_j'}) = -V\Lambda_2(\overline{T'T'})_{,j}$$

The four fluctuation-moment equations discussed above are now written explicitly. From $[\underline{\underline{M}}\cdot\overline{T'} + \underline{\underline{B}}\cdot\overline{u_i'}]$ we obtain the three explicit equations

$$(\overline{T'u_i'})_{,t} + \overline{u_j}(\overline{T'u_i'})_{,j} + U(x_3)(\overline{T'u_i'})_{,1} + \delta_{1i}U_{,3}(x_3)(\overline{T'u_3'}) + \overline{u_i}_{,j}(\overline{T'u_j'})$$

$$- \{\Lambda_2 V[(\overline{T'u_i'})_{,j} + (\overline{T'u_j'})_{,i}]\}_{,j} = \varepsilon\delta_{3i}(\overline{T'T'}) + [\Lambda_3 V(\overline{T'u_j'})_{,j}]_{,i} - \frac{V}{\Lambda_1}(\overline{T'u_i'});$$

$$(i,j = 1,2,3), \tag{3.2.25}$$

and from $[\underline{\underline{B}}\cdot T']$ we get one explicit equation

$$\frac{1}{2}(\overline{T'T'})_{,t} + \frac{1}{2}\overline{u_j}(\overline{T'T'})_{,j} + \frac{1}{2}U(x_3)(\overline{T'T'})_{,1} + \overline{T}_{,j}(\overline{T'u_j'}) - \frac{1}{2}[\Lambda_2 V(\overline{T'T'})_{,j}]_{,j}$$

$$= -[J(x_3)/\varepsilon](\overline{T'u_3'}). \tag{3.2.26}$$

The four fluctuation-moment equations above plus $\underline{\bar{C}}$, $\underline{\bar{M}}$, $\underline{\bar{B}}$ close the thermal turbulence system under study. The unknowns are $\bar{u}_i$, $\bar{p}$, $\bar{T}$, $(\overline{T'u_i'})$, and $(\overline{T'T'})$. (3.2.27)

3.2.5 *Series Procedure in* ε *for Solving Mean Turbulent Fluctuation-Moment Equations*

(a) Zeroth-order solution:

If $\varepsilon = 0$, we have already found that

$$\bar{u}_i^{(o)} \equiv 0 \quad \text{and} \quad \bar{p}^{(o)} \equiv 0. \qquad (3.2.15 \text{ bis})$$

Then the fluctuation-moment equations above give for $[\overline{\underline{M} \cdot T'} + \overline{\underline{B} \cdot u_i'}]^{(o)}$ the result

$$(\overline{T'u_i'})_{,t}^{(o)} + U(x_3)(\overline{T'u_i'})_{,1}^{(o)} + \delta_{1i} U_{,3}(x_3)(\overline{T'u_3'})^{(o)} - \{\Lambda_2 V[(\overline{T'u_i'})_{,j} + (\overline{T'u_j'})_{,i}]\}_{,j}^{(o)}$$
$$= + [\Lambda_3 V(\overline{T'u_j'})_{,j}]_{,i}^{(o)} - [\frac{V}{\Lambda_1}(\overline{T'u_i'})]^{(o)}, \qquad (3.2.28)$$

and for $[\overline{\underline{B} \cdot T'}]^{(o)}$ the result

$$\frac{1}{2}(\overline{T'T'})_{,t}^{(o)} + \frac{1}{2} U(x_3)(\overline{T'T'})_{,1}^{(o)} + \bar{T}_{,j}^{(o)}(\overline{T'u_j'})^{(o)} - \frac{1}{2}[\Lambda_2 V(\overline{T'T'})_{,j}]_{,j}^{(o)}$$
$$= [J(x_3)/\varepsilon](\overline{T'u_3'})^{(o)}. \qquad (3.2.29)$$

The first of this pair is uncoupled from the second. The first is to be solved simultaneously with Eq. (3.2.16) for $\underline{\bar{B}}^{(o)}$ for the variables $\bar{T}^{(o)}$ and $(\overline{T'u_i'})^{(o)}$. After this solution, the second of this pair can be solved for $(\overline{T'T'})^{(o)}$ if desired.

(b) First-order solution:

The complete series is

$$\bar{u}_i = \quad 0 \quad + \quad \bar{u}_i^{(1)} \cdot \epsilon \quad + \ldots$$

$$\bar{p} = \quad 0 \quad + \quad \bar{p}^{(1)} \cdot \epsilon \quad + \ldots$$

$$\bar{T} = \quad \bar{T}^{(o)} \quad + \quad \bar{T}^{(1)} \cdot \epsilon \quad + \ldots \tag{3.2.30}$$

$$(\overline{T'u_i'}) = (\overline{T'u_i'})^{(o)} + (\overline{T'u_i'})^{(1)} \cdot \epsilon + \ldots$$

$$(\overline{T'T'}) = (\overline{T'T'})^{(o)} + (\overline{T'T'})^{(1)} \cdot \epsilon + \ldots$$

The moment equations give for $[\overline{\underline{\underline{M}} \cdot T'} + \overline{\underline{\underline{B}} \cdot u_i'}]^{(1)} - [\overline{\underline{\underline{M}} \cdot T'} + \overline{\underline{\underline{B}} \cdot u_i'}]^{(o)}$ the three following linear equations in the variables $(\overline{T'u_i'})^{(1)}$ and $(\bar{u}_i)^{(1)}$:

$$(\overline{T'u_i'})_{,t}^{(1)} + \bar{u}_j^{(1)}(\overline{T'u_i'})_{,j}^{(o)} + U(x_3)(\overline{T'u_i'})_{,1}^{(1)} + \delta_{1i}U_{,3}(x_3)(\overline{T'u_3'})^{(1)} + \bar{u}_{i,j}^{(1)}(\overline{T'u_j'})^{(o)}$$
$$- \{\Lambda_2 V[(\overline{T'u_i'})_{,j} + (\overline{T'u_j'})_{,i}]\}_{,j}^{(1)} = \delta_{3i}(\overline{T'T'})^{(o)} + [\Lambda_3 V(\overline{T'u_j'})_{,j}]_{,i}^{(1)} - [\frac{V}{\Lambda_1}(\overline{T'u_i'})]^{(1)}$$
$$\tag{3.2.31}$$

In the above, from $[\overline{\underline{\underline{B}} \cdot T'}]^{(o)}$ under (a), $(\overline{T'T'})^{(o)}$ is given by the prior result

$$\frac{1}{2}(\overline{T'T'})_{,t}^{(o)} + \frac{1}{2}U(x_3)(\overline{T'T'})_{,1}^{(o)} - \frac{1}{2}[\Lambda_2 V(\overline{T'T'})_{,j}]_{,j}^{(o)}$$
$$\approx - \bar{T}_{,j}^{(o)}(\overline{T'u_j'})^{(o)} - [J(x_3)/\epsilon](\overline{T'u_3'})^{(o)}.$$
$$\tag{3.2.29 bis}$$

4. Evaluation of Power Delivered to Upper Layer by Heat Engine

Before considering from Sect. 3.2, the solution equations we can give some explicit results for the heat switch of Sect. 2.3 and the heat engine of Sect. 2.4 as they pertain to the second paragraph of Sect. 2.1 and to Fig. 3.

Let us write out the first system of fluctuation-moment equations given in Eqs. (3.2.21). These were indicated in Sect. 3 as part of the dimensionless excursion-flow discussion but not displayed or used. For $\overline{\underline{\underline{M}} \cdot u_k'}$ then we have the relevant Reynolds' stress equations

$$\overline{u_k'u_i'}_{,t} + U(x_3)\overline{u_k'u_i'}_{,1} + \bar{u}_j\overline{u_i'_{,j}u_k'} + \delta_{1i}U_{,3}(x_3)\overline{u_3'u_k'} + \bar{u}_{i,j}\overline{u_j'u_k'} + \overline{u_j'u_i'_{,j}u_k'}$$
$$= \epsilon\delta_{3i}\overline{T'u_k'} - \overline{p'_{,i}u_k'} ; \quad (i,j,k = 1,2,3). \tag{4.1}$$

After solution, we may return to this equation, contract k to i, and obtain the mechanical-energy relation for the mean fluctuations according to our dimensionless excursion-flow equations. Suppose we neglect

$$\frac{D}{Dt}\left[\frac{1}{2}\left(\overline{u_i' u_i'}\right)\right], \quad \left(\overline{u_i' u_j'}\right)\bar{e}_{ij}, \quad \text{and} \quad \frac{1}{2}\left(\overline{u_i' u_i' u_j'}\right)_{,j} \tag{4.2}$$

on the grounds that they are presently and commonly small as argued. We then obtain <u>without any modeling</u> of the correlation $\overline{p_{,j}' u_j'}$ the result

$$\left(\overline{p' u_j'}\right)_{,j} = \varepsilon\left(\overline{T' u_3'}\right). \tag{4.3}$$

Now integrate over the finite volume τ and use the identity

$$\int_\tau b_{,j}\,d\tau \equiv \oint_A b n_j\,dA$$

to obtain the result

$$-\oint_A \left(\overline{p' u_j'}\right)n_j\,dA = -\varepsilon\int_\tau \left(\overline{T' u_3'}\right)d\tau \tag{4.4}$$

for a finite fluid aggregate.

Therefore, the mechanical power delivered to the upper stable layer at the interface is equal to the thermal power of the thermal turbulence in the lower layer and vanishes if $\varepsilon = 0$ (a result self-consistant with our other results).

This result may be compared to the <u>dimensional</u> general mean fluctuation departure energy relation (2.2.11) above. By comparison, we can infer that the assumptions of our excursion-flow theory are

$$J_F \equiv \left|\frac{\rho_o \beta_o g\left(\overline{T' u_3'}\right)}{(1/2)\Sigma_{ij}\bar{e}_{ij}}\right| \gg 1, \tag{4.5}$$

$$I \equiv \left|\frac{-\beta_o g\left(\overline{T' u_3'}\right)}{\dfrac{D}{Dt}\left(\dfrac{\overline{u_i' u_i'}}{2}\right)}\right| \gg 1, \tag{4.6}$$

$$H \equiv \left| \frac{\overline{(p'u_j')}}{(\rho_o/2)(\overline{u_i'u_i'u_j'})} \right| \gg 1, \tag{4.7}$$

plus the assumptions of negligible viscous dissipation

$$\left| \frac{-\rho_o\beta_o g(\overline{T'u_3'})}{(\mu_o/2)(\overline{e_{ij}'e_{ij}'})} \right| \gg 1, \tag{4.8}$$

and

$$\left| \frac{-(\overline{p'u_j'})}{\mu_o(\overline{e_{ij}'u_i'})} \right| \gg 1. \tag{4.9}$$

5.   The Solution Equations with Arbitrary Shear $U = U(x_3)$

The requisite equations were developed in Sect. 3 but are not yet in a convenient form for manipulation. It is noted, however, that all equations are linear — in some cases with variable coefficients — even with arbitrary shear.

5.1 The Zeroth-Order Equations in ε

We gather up the results of Sects. 3.2.3(a) and 3.2.5(a). We have noted that, to zeroth order, there exists no $J \sim N^2$. Let

$$\vec{Q} \doteq Q_i = (\overline{T'u_i'})^{(o)}, \qquad \Theta = \bar{T}^{(o)}, \tag{5.1.1}$$

and

$$\Lambda_1, \ \Lambda_2, \ \Lambda_3, V = \text{assumed consts.} \tag{5.1.2}$$

The last four are the empirical functions of Donaldson (1973).

Then from $\underline{\underline{B}}^{(o)}$,

$$\Theta,_t + U(x_3)\Theta,_1 = - Q_j,_j \tag{5.1.3}$$

or, in vector notation,

$$\frac{\partial\Theta}{\partial t} + U(z)\frac{\partial\Theta}{\partial x} = - \nabla\cdot\vec{Q} \tag{5.1.4}$$

From $[\overline{\underline{M}\cdot T'} + \overline{\underline{\underline{B}}\cdot u_i'}]^{(o)}$,

$$Q_{i,t} + U(x_3)Q_{i,1} + \delta_{1i}U_{,3}(x_3)Q_3 = \Lambda_2 VQ_{i,jj} + (\Lambda_2 + \Lambda_3)VQ_{j,ji} - (V/\Lambda_1)Q_i$$

(5.1.5)

or, in vector notation,

$$\frac{\partial \vec{Q}}{\partial t} + U(z)\frac{\partial \vec{Q}}{\partial x} + \vec{1}\frac{\partial}{\partial z}U(z)\vec{k}\cdot\vec{Q} = \Lambda_2 V\nabla^2 \vec{Q} + (\Lambda_2 + \Lambda_3)V\nabla(\nabla\cdot\vec{Q}) - (V/\Lambda_1)\vec{Q}$$

(5.1.6)

Given $U(z)$, Eq. (5.1.6) can be solved as it stands for $\vec{Q}$. We then solve Eq. (5.1.4) for the other unknown, viz Θ.

5.2 The First-Order Equations in ε

We gather up the results of Sects. 3.2.3(b) and 3.2.5(b). We repeat that, to first order, there appears a $J \sim N^2$. Let

$$\vec{q} \doteq q_i = (\overline{T'u_i'})^{(1)}, \quad \tau = (\overline{T'T'})^{(o)}, \quad \theta = \overline{T}^{(1)}, \quad \vec{v} \doteq v_i = \overline{u}_i^{(1)}, \quad \pi = \overline{p}^{(1)},$$

(5.2.1)

and

$$\Lambda_1, \Lambda_2, \Lambda_3, V = \text{assumed consts.}$$

(5.2.2)

Then the problem is given by solving simultaneously the equations $\overline{\underline{\underline{C}}}^{(1)}$, $\underline{\underline{M}}^{(1)}$, and $[\overline{\underline{M}\cdot T'} + \overline{\underline{\underline{B}}\cdot u_i'}]^{(1)} - [\overline{\underline{M}\cdot T'} + \overline{\underline{\underline{B}}\cdot u_i'}]^{(o)}$ since the equations $[\underline{\underline{B}}^{(1)} - \underline{\underline{B}}^{(o)}]$ and $[\overline{\underline{\underline{B}}\cdot T'}]^{(o)}$ are to be solved aside. We therefore write

$$\overline{\underline{\underline{C}}}^{(1)}: \qquad v_{j,j} = 0 \quad \text{or} \quad \nabla\cdot\vec{v} = 0,$$

(5.2.3)

$$\underline{\underline{M}}^{(1)}: \qquad v_{i,t} + U(x_3)v_{i,1} + v_3\delta_{1i}U_{,3}(x_3) = \Theta\delta_{3i} - \pi_{,i}$$

(5.2.4)

or

$$\frac{\partial \vec{v}}{\partial t} + U(z)\frac{\partial \vec{v}}{\partial x} + \vec{1}(\vec{k}\cdot\vec{v})\frac{\partial}{\partial z}U(z) = \Theta\vec{k} - \nabla\pi,$$

(5.2.5)

and

$$[\overline{\underline{M}\cdot T'} + \overline{\underline{\underline{B}}\cdot u_i'}]^{(1)} - [\overline{\underline{M}\cdot T'} + \overline{\underline{\underline{B}}\cdot u_i'}]^{(o)}:$$

$$q_{i,t} + v_j Q_{i,j} + U(x_3)q_{i,1} + \delta_{1i}U_{,3}(x_3)q_3 + Q_j v_{i,j} - \Lambda_2 Vq_{i,jj}$$

$$= \delta_{3i}\tau + (\Lambda_2 + \Lambda_3)Vq_{j,ji} - (V/\Lambda_1)q_i \quad (5.2.6)$$

or

$$\frac{\partial \vec{q}}{\partial t} + (\vec{v}\cdot\nabla)\vec{Q} + U(z)\frac{\partial \vec{q}}{\partial x} + \vec{1}\frac{\partial}{\partial z}U(z)(\vec{k}\cdot\vec{q}) + (\vec{Q}\cdot\nabla)\vec{v} - \Lambda_2 V\nabla^2\vec{q}$$

$$= \vec{k}\tau + (\Lambda_2 + \Lambda_3)V\nabla(\nabla\cdot\vec{q}) - (V/\Lambda_1)\vec{q}. \qquad (5.2.7)$$

The equation for τ can be solved alone once the right member has been determined from the zeroth-order solution. Then, from $[\overline{\underline{B}\cdot T'}]^{(o)}$,

$$\frac{1}{2}\tau_{,t} + \frac{1}{2}U(x_3)\tau_{,1} - \frac{1}{2}\Lambda_2 V\tau_{,jj} = -Q_j\theta_{,j} - [J(x_3)/\epsilon]Q_3 \qquad (5.2.8)$$

or

$$\frac{1}{2}\frac{\partial \tau}{\partial t} + \frac{1}{2}U(z)\frac{\partial \tau}{\partial x} - \frac{1}{2}\Lambda_2 V\nabla^2\tau = -\vec{Q}\cdot\nabla\Theta - [J(z)/\epsilon]\vec{k}\cdot\vec{Q}. \qquad (5.2.9)$$

The variable θ does not appear in the above equations but the buoyancy equation depends on $\vec{v}$ and $\vec{q}$ obtained above. To be solved last then is

$$\underline{\underline{B}}^{(1)} - \underline{\underline{B}}^{(o)}: \quad \theta_{,t} + U(x_3)\theta_{,1} + \theta_{,j}v_j + [J(x_3)/\epsilon]v_3 = -q_j{}_{,j} \qquad (5.2.10)$$

or

$$\frac{\partial \theta}{\partial t} + U(z)\frac{\partial \theta}{\partial x} + (\nabla\Theta)\cdot\vec{v} + [J(z)/\epsilon]\vec{k}\cdot\vec{v} = -\nabla\cdot\vec{q}. \qquad (5.2.11)$$

6. Discussion

What is now required is to solve the equations of Sect. 5 subject to the boundary conditions given plus the boundary conditions at the upper lid or interface near $x_3 = 0$. The difficulty is that the upper layer and lower layer are closely coupled together; although the lower layer is the driver as per Sect. 4, there will be feedback from the upper layer. Thus we must deal above with internal, surface, and Kelvin-Helmholtz waves simultaneously. Then a wide variety of forms of cellular structure below can be obtained by suitable adjustment of the shear rate $U(x_3)$. A survey of the limited literature on this coupled problem is given by Turner (1973); see Ch. 7 and especially paragraph 7.3, p. 226. There is an extensive literature on linear waves in stably stratified flows and we have rendered our unstably stratified lower turbulent flow linear

and therefore compatible. (The problem is still well formulated if the expansion in ϵ is not made.) The detailed solution is left for further research.

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Appendix: Prior Experiments

We have mentioned that the work proposed is backed up by prior experimental probing in a low-speed wind tunnel at Brown. The experiments were exploratory and thus produced qualitative smoke photographs resembling Fig. 3 as well as semi-quantitative tunnel data such as shown in Table 1 to be discussed below. The research was reported in Clarke et al. (1973) and Clarke (1975). The semi-quantitative data (only) corresponded physically to the case $\epsilon \equiv F_I^{-2} \doteq 0$ in the sense that the general resemblance was remarkable; as is indicated in Eq. (4.4), the heat engine did not operate detectably; the phenomena observed was thermal in nature with little mechanical activity of wave or cell type (in the excursion theory, N^2 is absent if $\epsilon = 0$). On the other hand, the qualitative data displayed a great deal of wave activity of all types and suggested strongly the operation of the heat engine as in Fig. 3.

Both the qualitative and semi-quantitative data indicated strongly that the flow was essentially free of stress-like turbulent correlations and dissipation outside of the thin pause layer (and wall boundary layers) over a test length of 25 feet within a tunnel cross-section of 32 inches in width and 22 inches in height. If $x_1 = x$ in the flow direction, $x_2 = y$ in the horizontal direction measured from one vertical wall, $x_3 = z$ measured from the tunnel floor, and since $\beta_o^{-1} = T_o$ for a perfect gas, then we can display

some of the semi-quantitative data in Table 1. The lower unstable layer

corresponds to $0 < z \lesssim 12.19$ inches. All of the symbols have already been

defined. It is seen that $\epsilon \equiv F_I^{-2} < 10^{-2}$, $|J/F_I^{-2}| \sim 1$, and the four measured

values of $|J_F|$ are comfortably greater than unity (the values not shown are

larger). All of the hot-wire data taken displayed a decay of activity at

points more and more above the pause-layer zone of activity.

It was not possible to raise the quantitative values of $\epsilon \equiv F_I^{-2}$ to the

more interesting range of values, clearly perceived theoretically, from the

tunnel results given above in the Brown University wind tunnel without sacrifice

of flow uniformity. On the other hand, the experimental and theoretical probing

seemed to open up entire new theoretical avenues for unstable turbulent flow --

not only for the specific problem just discussed, but for similar ones. None of

this probing suggested that the large and small parameters observed were limited

in occurrence to the wind tunnel experiences but should be quite common in

nature. Accordingly, we undertook to develop the new simplified theory of

phenomenological thermal turbulence suitable whenever the parameters behave as

required and subject to systematic (i.e. rational) improvement in this respect.

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z (inches)	$N^2(z) = \dfrac{g}{T_o}\dfrac{\Delta T}{\Delta z}\left[\dfrac{1}{sec^2}\right]$	$\hat{T}_c$ (°C)	$F_I^{-2}(z) = \dfrac{(\hat{T}_c/T_o)gd}{U_c^2}$	$J(z)/F_I^{-2} = \dfrac{N^2(z)d}{(\hat{T}_c/T_o)g}$	$J_F = \dfrac{\rho_o g}{T_o}\dfrac{\overline{T'u_3'}}{(-\rho_o\overline{u_1'u_3'})\bar{u}_{1,3}}$
7.70	- .412	.90	.0030	- 3.730	-
9.24	- .721	1.60	.0052	- 3.671	-
9.98	-2.046	1.88	.0062	- 8.867	-
10.84	-1.798	1.97	.0065	- 7.436	-
11.26	-1.368	1.93	.0063	- 5.775	- 3.38
11.51	-1.044	1.85	.0061	- 4.598	-15.80
11.76	-0.716	1.82	.0060	- 3.205	-10.65
12.01	0	1.75	.0058	0	- 5.29
12.51	+0.792	1.60	.0052	+ 4.033	- 0.04
13.01	+1.332	1.50	.0049	+ 7.235	+ 0.04
13.51	+1.728	1.30	.0043	+10.830	- 0.09
13.76	+1.800	-	-		- 0.79
14.76	+2.209	-	-		-
16.70	+1.703	-	-		-
18.27	+1.093	-	-		-

NOTES: T_o = 291.5°C, x = 209 in., y = 16 in., U_c = 5.5 ft/sec., d = 10.8 in.,

The pausal layer is at z = 12.19 inches.

Table 1: Experimentally Determined Values of Parameters Relevant to Theory.
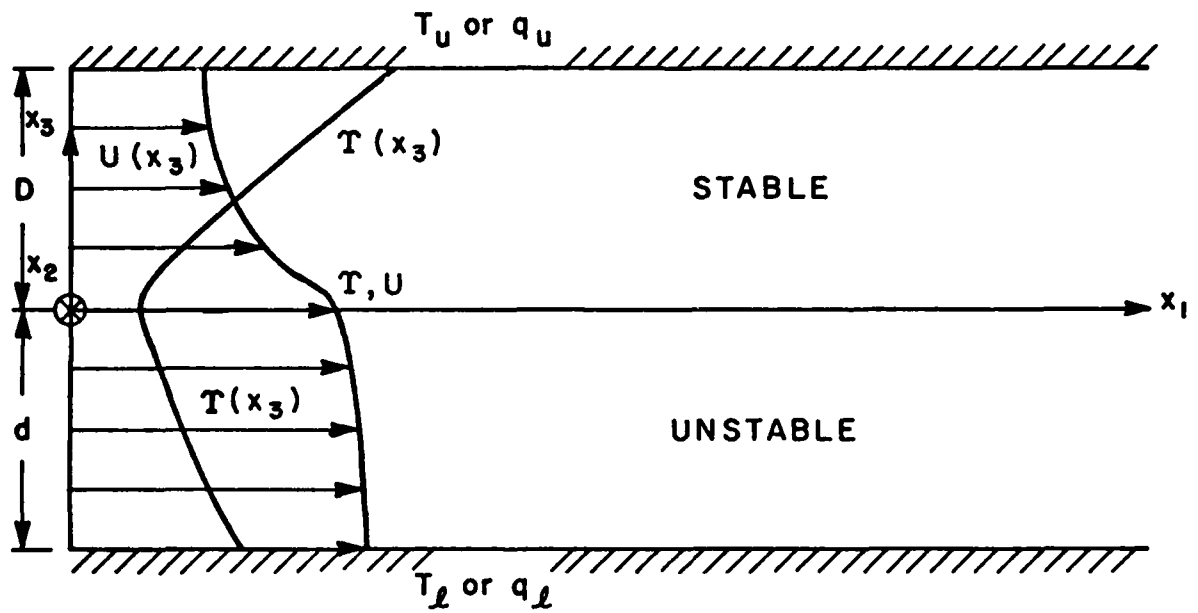
FIG. 1: UNSTABLY STRATIFIED FLOW BOUNDED BELOW BY
SOLID BOUNDARY & ABOVE AT $x_3 \approx 0$ BY STABLY
STRATIFIED FLOW
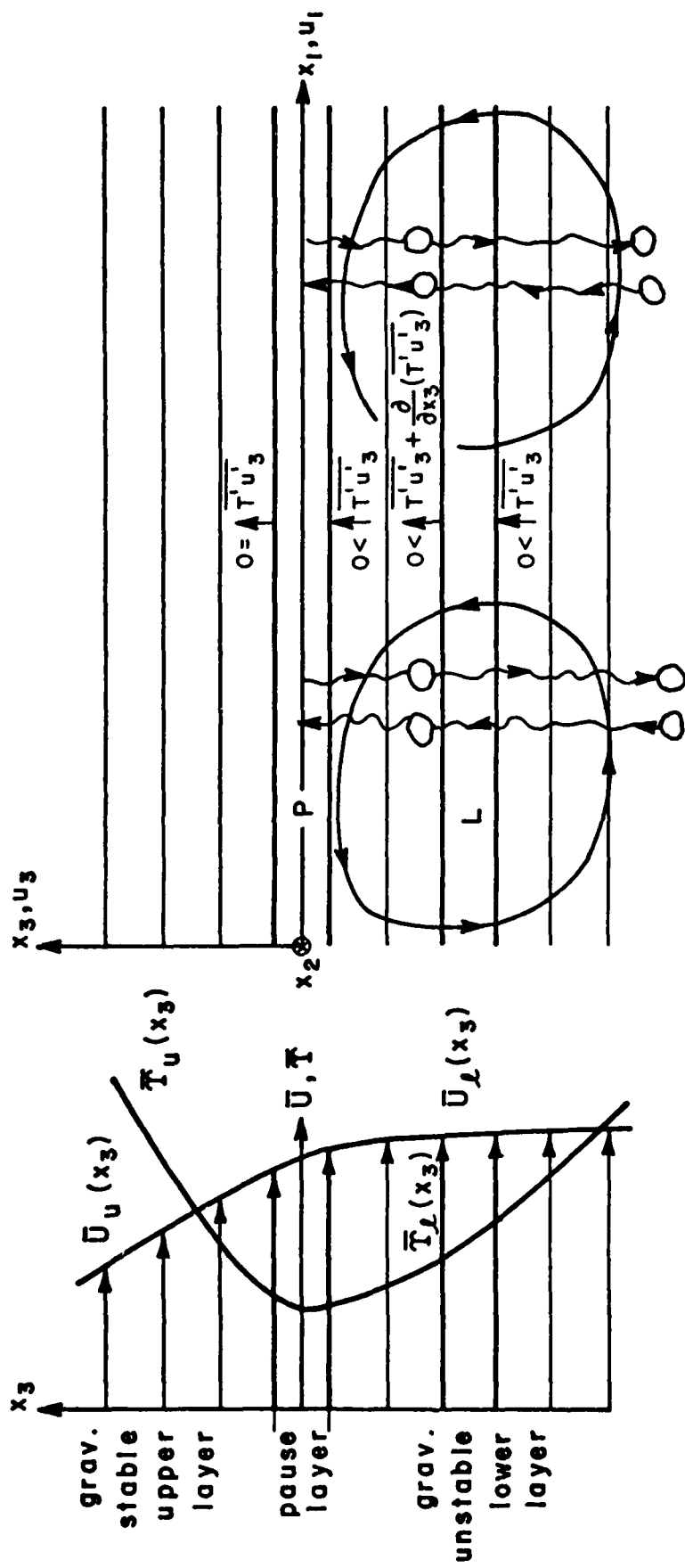
FIG. 2: HEAT-SWITCH MECHANISM WITH NO POWER EXTRACTED
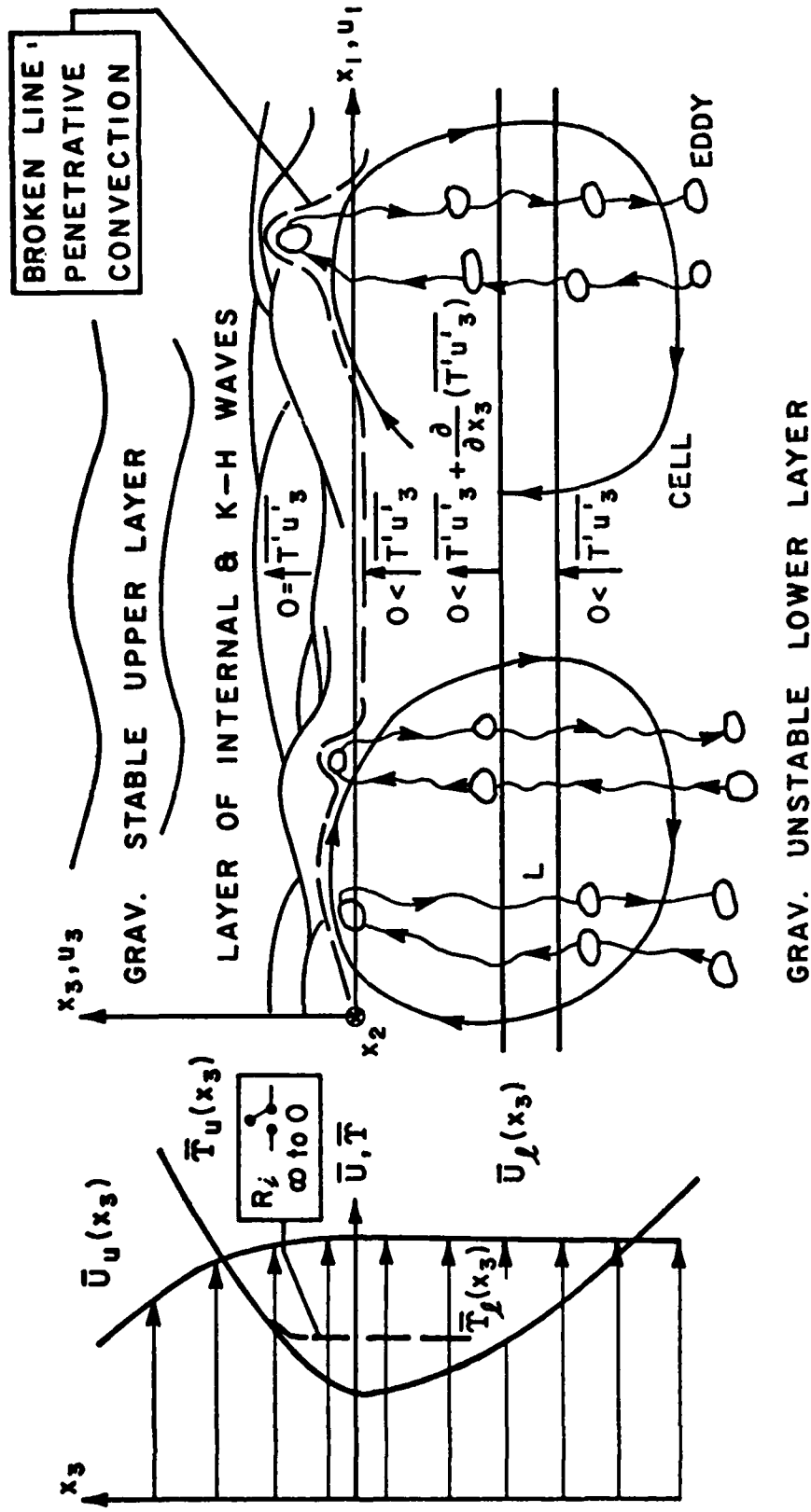
FIG. 3: TURBULENT HEAT SWITCHES OFF PRODUCING LAYER OF INTERNAL AND K-H WAVES BY PENETRATIVE CONVECTION AND BY HEATING

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Penetrative Convection	Oceanic Turbulence and waves
Internal Wave Sources	

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Studied and developed is a new, relatively simple, rational, and viable theory of phenomenological thermal turbulence occurring in incident unstably stratified fluid flows. The subject of stratified flow has consisted largely of the study of stably stratified incident flows and of unstable thermal convection of a fluid heated from below with no incident flow. The new turbulence theory is intermediate in complexity between the venerable but generally inapplicable turbulent eddy-transport theory and the very complex generic theory known (cont.)

(cont.) 20.

as "turbulence modeling", enjoying much current attention. We call our method
"a limit theory of phenomenological thermal turbulence modeling"; based on a
system of plausibly large and small parameters, it is more suitable in many
applications where the said parameters behave as required; moreover, the theory
lends itself to systematic improvement. The work proposed is preceded by
experimental probing in a low-speed wind tunnel and use is always made of the
general equations as a continuing guide to the specializations made in our
specific formulation that sidesteps the attendant major computation in the
general unstable, turbulence formulation. The research consists of a study of
turbulent cellular flow due to an unstable temperature stratification underneath
a flow with a stable temperature stratification. These two layers are separated
by an interface whose detailed structure is a pausal boundary layer. The
turbulent thermal convection below buffets the upper layer by means of
"penetrative convection" and heats the upper layer at the interface because
the turbulent heat flux mechanism must switch off. The penetrative convection
and heat switch must amount to a "heat engine" whereby the rate of work of
thermal origin is done on the bottom of the upper layer. This power source
can then produce internal, surface, and Kelvin-Helmholtz waves.