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ON PERIODIC AND MULTIPLE AUTOREGRESSIONS*

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Abstract

A methodology is presented for analyzing periodic autoregressions which is also applicable when inferring the second order properties of periodically correlated processes. In addition, capitalizing on the duality between periodic and multiple autoregressions, a method is set forth for analyzing the latter, which overcomes the usual requirements of a large number of both parameters and computer storage locations.

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1. Introduction

Given a time series $\{Y(t) ; t = 0, \pm 1, \ldots\}$ whose second order moments exist, define its mean function, $m(t) = EY(t)$, and its covariance kernel, $R(s,t) = E\left[Y(s) - m(s)\right]\left[Y(t) - m(t)\right]$. A class of nonstationary processes which readily lends itself to analysis, and is of practical importance (see [9], [5], [18]) is that of periodically correlated processes (see [2]).

Definition. The process $Y(\cdot)$ is said to be periodically correlated of period d, if for some positive integer d and for all integers s,t,

$$m(t) = m(t + d) , \quad R(s,t) = R(s + d, t + d) \quad .$$

Since we propose dealing with the second order properties of the process, without loss of generality take $m(t) = 0$.

Periodically correlated processes are not only of interest in their own right, but, because of their duality with multivariate covariance stationary time series, they also provide insight into and modeling facility for these series. This claim is based, in the main, upon the following construction: define the jth component of the d-dimensional vector $\underset{\sim}{X}(t)$ by (see [2]),

$$(1.1) \qquad X_j(t) = Y\left[j + d(t-1)\right]$$

and the covariance kernel of $\underset{\sim}{X}(\cdot)$ by

$$R_{jk}(s,t) = EX_j(s)\, X_k(t) , \quad 1 \le j,k \le d , \quad s,t = 0, \pm 1, \ldots \quad .$$

By noting that

(1.2) $\quad R_{jk}(s,t) = R(j + ds, k + dt)$,

we nave,

<u>Theorem</u> (Gladyshev [2]). The process $Y(\cdot)$ is periodically correlated of period d if, and only if, the process $\underset{\sim}{X}(\cdot)$ is covariance stationary.

This theorem shows that, although not covariance stationary, the process $Y(\cdot)$, nonetheless, does not deviate too much from stationarity; and, in fact, its second order properties may be deduced from those of $\underset{\sim}{X}(\cdot)$. We can further capitalize on this duality but in the other direction; given a covariance stationary process $\underset{\sim}{X}(\cdot)$ define the associated periodically correlated process, $Y(\cdot)$, by (1.1). This is especially useful when $\underset{\sim}{X}(\cdot)$ is a multiple autoregression. For then by investigating $Y(\cdot)$, we actually effect an easily calculated orthogonal decomposition of the process $\underset{\sim}{X}(\cdot)$, thus achieving the usual gain in simplicity and power associated with orthogonalization. This point is amplified in the remainder of the paper.

2. Autoregressions

Given a sample $\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(T)$ from a d-dimensional zero mean covariance stationary time series with spectral density matrix $f(\omega)$, $-\pi \le \omega \le \pi$, the problem is to estimate $f(\cdot)$. Under some very mild restrictions on $f(\cdot)$, [7], it can be written as the spectral density of an infinite order autoregression. Parzen [13] thus proposes treating the sample as one from a pth order autoregression,

$$(2.1) \qquad \underset{\sim}{X}(t) + \sum_{j=1}^{p} A(j)\, \underset{\sim}{X}(t-j) = \underset{\sim}{\varepsilon}(t)$$

where $\operatorname{cov}(\underset{\sim}{\varepsilon}(t)) = \Sigma$, and by choosing p large enough, $f(\cdot)$ can be arbitrarily closely approximated. Parzen [15] gives a method for choosing p which accomodates both the numerical approximation, which argues for a large p, and the requisite statistical estimation, which argues for a small p. The autoregressive approximants have been used in practice with success, (see [4], [14] and references therein, also in geophysics [16] where a closely analogous method, the method of maximal entropy, is used). For these reasons it seems important to study autoregressive processes.

The periodically correlated analogue of an autoregression is given by (see [5]),

<u>Definition</u>. A process $Y(\cdot)$ is said to be a <u>periodic autoregression</u> of period d and order $(p_1,\ldots,p_d)$ if for all integer t,

$$(2.2) \qquad Y(t) + \sum_{j=1}^{p_t} \alpha_t(j)\, Y(t-j) = \varepsilon(t)$$

where the $\epsilon(\cdot)$ are uncorrelated with mean zero and $E\epsilon^2(t) = \sigma_t^2$, and, $p_t = p_{t+d}$, $\sigma_t^2 = \sigma_{t+d}^2$, and, $\alpha_t(j) = \alpha_{t+d}(j)$, $j = 1,\ldots,p_t$.

<u>Theorem 1</u>. If $\underset{\sim}{X}(\cdot)$ and $Y(\cdot)$ are associated by (1.1), then, $\underset{\sim}{X}(\cdot)$ is an autoregression of order p with positive definite Σ if, and only if, $Y(\cdot)$ is a periodic autoregression of period d and order $(p_1,\ldots,p_d)$ with positive $\sigma_1^2,\ldots,\sigma_d^2$, and,

$$p = [\max_j (p_j - j)/d] + 1 \quad ,$$

where, for integral j, $[x] = j$ for $j \le x < j+1$.

<u>Proof</u>. If $Y(\cdot)$ is a periodic autoregression of order $(p_1,\ldots,p_d)$ then it may be written as (see Lemma 2)

$$L\underset{\sim}{X}(t) + \sum_{j=1}^{p} A'(j)\,\underset{\sim}{X}(t-j) = \underset{\sim}{\epsilon}'(t) \tag{2.3}$$

where L is a unit lower triangular matrix, p satisfies the condition of the theorem, and the $\underset{\sim}{\epsilon}'(\cdot)$ are uncorrelated with a diagonal covariance matrix, $E\underset{\sim}{\epsilon}'(t)\,\underset{\sim}{\epsilon}'^T(t) = D = \operatorname{diag}(\sigma_1^2,\ldots,\sigma_d^2)$. Therefore,

$$\underset{\sim}{X}(t) + \sum_{j=1}^{p} A(j)\,\underset{\sim}{X}(t-j) = \underset{\sim}{\epsilon}(t)$$

where

$$A(j) = L^{-1}A'(j) \qquad j = 1,\ldots,p$$
$$\underset{\sim}{\epsilon}(t) = L^{-1}\underset{\sim}{\epsilon}'(t) \qquad t = 0,\pm 1,\ldots,$$

and the $\underset{\sim}{\varepsilon}(\cdot)$ are thus uncorrelated and have covariance matrix $\Sigma = L^{-1} D L^{-T}$, which is positive definite since D is.

Conversely, suppose

$$\underset{\sim}{X}(t) + \sum_{j=1}^{p} A(j)\, \underset{\sim}{X}(t-j) = \underset{\sim}{\varepsilon}(t)$$

where the $\underset{\sim}{\varepsilon}(\cdot)$ are uncorrelated with positive definite covariance matrix Σ. Define the unique modified Cholesky decomposition of Σ by $\Sigma = L D L^T$, where L is unit lower triangular and D is positive definite diagonal, $D = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_d^2)$. Then L^{-1} is unit lower triangular and $\underset{\sim}{X}(\cdot)$ satisfies

$$L^{-1}\underset{\sim}{X}(t) + \sum_{j=1}^{p} A'(j)\, \underset{\sim}{X}(t-j) = \underset{\sim}{\varepsilon}'(t)$$

where

$$A'(j) = L^{-1} A(j) \quad , \qquad j = 1, \ldots, p \quad ,$$

and the $\underset{\sim}{\varepsilon}'(t) = L^{-1}\underset{\sim}{\varepsilon}(t)$, and are thus uncorrelated with diagonal positive definite covariance matrix, and the theorem is proved.

Note that without too much difficulty we could accomodate a process for which Σ is not of full rank.

<u>Definition</u>. A process $Y(\cdot)$ is said to be a <u>covariance stationary periodic autoregression</u> if it is a periodic autoregression and, furthermore, the associated vector process $\underset{\sim}{X}(\cdot)$, (1.1), is covariance stationary.

By a stationary multiple autoregression, we mean a process $\underset{\sim}{X}(\cdot)$ which obeys (2.1) and admits to a solution, in the mean square sense, in terms of the "past" $\underset{\sim}{\varepsilon}(\cdot)$, i.e.

$$\underset{\sim}{X}(t) = \underset{\sim}{\varepsilon}(t) + \sum_{k=1}^{\infty} B(k)\, \underset{\sim}{\varepsilon}(t-k) \quad .$$

For this to be true, the zeroes of the determinental polynomial, $\det\left(I_d + \sum_{j=1}^{p} A(j)\, z^j\right)$ must lie outside the unit circle. This does not translate into the intuitively expected conditions for the periodic autoregression. For example, suppose $d = 3$ and $p_1 = p_2 = p_3 = 1$, then for stationarity we require $|\alpha_1(1)\,\alpha_2(1)\,\alpha_3(1)| < 1$.

One of the values of an autoregressive process is, of course, the facility with which one may perform linear prediction. It can easily be shown, see [18], that the least squares predictor of $Y(t+h)$, for positive integer h , on the basis of $Y(t), Y(t-1), \ldots,$ is

$$Y(t+h/t) = -\sum_{j=1}^{h-1} \alpha_{t+h}(j)\, Y(t+h-j/t) - \sum_{j=h}^{p_{t+h}} \alpha_{t+h}(j)\, Y(t+h-j)$$

if $Y(\cdot)$ is a covariance stationary periodic autoregression. The one-step-ahead $(h = 1)$ prediction error is simply,

$E\{Y(t+1/t) - Y(t+1)\}^2 = \sigma^2_{t+1}$. For $h > 1$ one must formally solve (see (3.8)),

$$Y(t+h) = -\sum_{j=1}^{p_{t+h}} \alpha_{t+h}(j)\, Y(t+h-j) + \epsilon(t+h)$$

$$= \sum_{k=0}^{\infty} \beta_{t+h}(k)\, \epsilon(t+h-k)$$

with $\beta_{t+h}(0) = 1$, in which case

$$E\{Y(t+h/t) - Y(t+h)\}^2 = \sum_{k=0}^{h-1} \beta_{t+h}^2(k)\, \sigma_{t+h-k}^2 \ .$$

A side benefit is that the above method yields a solution to the problem of predicting only a subset of $\underset{\sim}{X}(t+1), \underset{\sim}{X}(t+2), \ldots,$ in terms of $\underset{\sim}{X}(t), \underset{\sim}{X}(t-1), \ldots$, for an autoregressive $\underset{\sim}{X}(\cdot)$.

3. Parameter Estimation

Given a sample $Y(1),\ldots,Y(T)$ from a zero mean Gaussian covariance stationary periodic autoregression of order $(p_1,\ldots,p_d)$ and period d, we wish to make inference about the parameters $\alpha_k(j)$ and σ_j^2, $j=1,\ldots,p_k$, $k=1,\ldots,d$. We show that the results of Mann and Wald [6] extend to this case.

To simplify the notation we take $T = Nd$, where N is a natural number. Then the problem is equivalent to having a sample $\underset{\sim}{X}(1),\ldots,\underset{\sim}{X}(N)$ from a stationary autoregression, where $\underset{\sim}{X}(\cdot)$ and $Y(\cdot)$ are related by (1.1).

Define

$$(3.1) \qquad R_N(k,v) = N^{-1} \sum_{j=0}^{m} Y(k+d\,j)\; Y(k+v+d\,j)$$

where $m = N - [(k+v)/d]$, for $k = 1,\ldots,d$, $v = 0,1,\ldots,T-k-1$.

Lemma 1. If $Y(\cdot)$ is a periodically correlated Gaussian process, then, with $R_N(k,v)$ defined by (3.1), as $T \to \infty$, the $R_N(k,v)$ converge almost surely and in mean square to $R(k,v)$, and

$$(3.2) \qquad N\,\mathrm{Cov}\left[R_N(k_1,v_1),\, R_N(k_2,v_2)\right] \to \sum_{u=-\infty}^{\infty} \{R(k_1,k_2+du)\; R(v_1,v_2+du)$$

$$+\, R(k_1,v_2+du)\; R(v_1,k_2+du)\} \;.$$

Proof. By using (1.1) and (1.2) and Gladyshev's theorem, we see that this is merely the extension of Slutsky's result; see [3], pp. 209, 210.

In a periodic autoregression, the α and σ are related to the covariance kernel R in a modified Yule-Walker form;

Theorem 2. If $Y(\cdot)$ is a covariance stationary periodic autoregression of order $(p_1,\ldots,p_d)$ with covariance kernel $R(\cdot,\cdot)$, then for $k = 1,\ldots,d$,

$$R(k,k-v) + \sum_{j=1}^{p_k} \alpha_k(j)\, R(k-j,k-v) = \delta_{v0}\,\sigma_k^2 \quad , \qquad v \geq 0 \quad , \tag{3.3}$$

where we use the Kronecker delta.

Proof. We must first show that

$$Y(t) = -\sum_{j=1}^{p_t} \alpha_t(j)\, Y(t-j) + \epsilon(t)$$

$$= \epsilon(t) + \sum_{k=1}^{\infty} \beta_t(k)\, \epsilon(t-k) \quad .$$

This result is immediately available from Theorem 1. Therefore, for positive v, $Y(t-v)$ is uncorrelated with $\epsilon(t)$. Multiplying both sides of (2.2) by $Y(t-v)$ and taking expected values yields the theorem.

We are thus immediately led to Fisher—consistent estimators of the α and σ^2 ; in (3.3) replace the R by the appropriate R_N and solve the resulting linear equations. Indeed, one can show after some laborious algebra that these estimators are the same as those one would obtain by solving the appropriately constrained multivariate Yule-Walker equations. Thus, the above suggests a possible method for finding the properties of the estimators, but, since the constraints are unpleasant, we prefer to proceed directly.

If $Y(\cdot)$ is a periodically correlated process, denote by $\mathcal{J}$ the Fisher information matrix, and by $\mathcal{J}(\alpha,\beta)$ the element in this matrix corresponding to the parameters α and β .

Lemma 2. If $Y(\cdot)$ is a Gaussian covariance stationary periodic auto-regression of order $(p_1,\ldots,p_d)$, then the information matrix is block diagonal,

$$\mathcal{J}(\alpha_k(j),\alpha_m(\ell)) = \delta_{km}R(k-j,k-\ell)/\sigma_k^2 \quad ,$$

$$\mathcal{J}(\alpha_k(j),\sigma_m^2) = 0$$

$$\mathcal{J}(\sigma_k^2,\sigma_m^2) = \delta_{km}/2\sigma_k^4 \quad ,$$

for $j = 1,\ldots,p_k$, $\ell = 1,\ldots,p_m$, $k,m = 1,\ldots,d$.

Proof. Using (1.1) then,

$$L\,\underset{\sim}{X}(t) + \sum_{j=1}^{p} A(j)\,\underset{\sim}{X}(t-j) = \underset{\sim}{\varepsilon}(t)$$

where

$$L_{kj} = \alpha_k(j) \quad , \qquad j < k$$

$$A_{kj}(v) = \alpha_k(dv - k + j) \quad , \qquad v = 1,2,\ldots,p$$

and p as in Theorem 1. Then, defining $D = \mathrm{diag}\,(\sigma_1^2,\ldots,\sigma_d^2)$, and,

$$G(z) = L + \sum_{j=1}^{p} A(j)\,z^j \quad , \qquad z = e^{i\omega} \quad ,$$

we have that the spectral density matrix of the $\underset{\sim}{X}(\cdot)$ process is

$$f(\omega) = G^{-1}(z)\,D\,G^{-*}(z)/2\pi \quad ,$$

where the asterisk denotes the complex conjugate transpose. We have, see [17], that

$$(3.4) \qquad \mathcal{J}(\alpha,\beta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{tr}\left[\frac{\partial f^{-1}}{\partial\alpha}\,f\,\frac{\partial f^{-1}}{\partial\beta}\,f\right](\omega)\,d\omega \quad .$$

Let E_{jk} denote the d-dimensional square matrix with a one in the (j,k) component and zeroes elsewhere. Now

$$(3.5) \qquad \frac{\partial f^{-1}}{\partial A_{kj}(v)}\,f = E_{jk}\,G^{-*}\,z^{-v} + G^{*}\,D\,E_{kj}\,f\,z^{v}$$

(3.6) $$\frac{\partial f^{-1}}{\partial L_{kj}} f = E_{jk} G^{-*} + G^{*} D E_{kj} f \quad , \qquad j < k$$

(3.7) $$\frac{\partial f^{-1}}{\partial \sigma_k^2} f = - G^{*} E_{kk} G^{-*} / \sigma_k^2 \quad .$$

Using the formula,

(3.8) $$G^{-1}(z) = L^{-1} + \sum_{k=1}^{\infty} B(k)\, z^k \quad ,$$

where the $B(\cdot)$ decay exponentially to zero because of the assumption on the zeroes of $\det\left[G(z)\right]$, we see that the terms of the form

$$\mathrm{tr} \int_{-\pi}^{\pi} E_{kj} G^{-1} E_{mn} G^{-1} e^{\pm i(v+u)\omega} d\omega = 0$$

for all k,j,m,n if $v + u \neq 0$, and for $k > j$ or $m > n$ if $v + u = 0$ (see [10]). Upon substitution of the appropriate (3.5), (3.6) and, or, (3.7) into (3.4), and some straightforward algebra, the lemma follows.

If, with the R_N defined in (3.1), we define the $\hat{\alpha}$ and $\hat{\sigma}$ as solutions to the normal equations,

(3.9) $$R_N(k,k-v) + \sum_{j=1}^{p_k} \hat{\alpha}_k(j)\, R_N(k-j,k-v) = \delta_{v0}\, \hat{\sigma}_k^2$$

for $v = 0,\ldots,p_k$ and $k = 1,\ldots,d$, and the vectors $\hat{\underset{\sim}{\alpha}}_k = \left[\hat{\alpha}_k(1),\ldots,\hat{\alpha}_k(p_k)\right]^T$, then, Theorem 3. If $Y(1),\ldots,Y(T)$ is a sample from a covariance stationary Gaussian periodic autoregression of order $(p_1,\ldots,p_d)$, then the $\hat{\alpha}$ and $\hat{\sigma}$ defined by (3.9) are almost surely consistent estimators $(T \to \infty)$. And $\sqrt{N}\,(\hat{\underset{\sim}{\alpha}}_k - \underset{\sim}{\alpha}_k)$ $k = 1,\ldots,d$ have an asymptotic distribution which is Gaussian with mean zero and covariance matrix $\mathcal{J}^{-1}$, where $\mathcal{J}$ is the appropriate block of the information matrix given in Lemma 2. Thus the estimators are, in this sense, asymptotically efficient.

Proof. The consistency follows from Lemma 1 and Theorem 2. To show the asymptotic distribution, as in [12], we have from (3.3) and (3.9) for $k = 1,\ldots,d$ and $v = 1,\ldots,p_k$,

$$(3.10) \qquad \sum_{j=0}^{p_k} R_N\{k-j,k-v\}\{\hat{\alpha}_k(j) - \alpha_k(j)\} = \sum_{j=0}^{p_k} \alpha_k(j)\{R_N(k-j,k-v) - R(k-j,k-v)\}$$

defining $\alpha_k(0) = \hat{\alpha}_k(0) = 1$. From Lemma 1 the random variables on the right of (3.10) are asymptotically Gaussian with mean zero. To find their asymptotic covariance matrix, consider for v_1 and v_2 positive,

$$N \sum_{j_1=0}^{p_{k_1}} \sum_{j_2=0}^{p_{k_2}} \alpha_{k_1}(j_1)\, \alpha_{k_2}(j_2)\, \mathrm{Cov}\left[R_N(k_1 - j_1, k_1 - v_1), R_N(k_2 - j_2, k_2 - v_2)\right]$$

$$= \sum_{j_1=0}^{p_{k_1}} \sum_{j_2=0}^{p_{k_2}} \alpha_{k_1}(j_1)\, \alpha_{k_2}(j_2) \Big\{ R(k_1 - j_1, k_2 - j_2)\, R(k_1 - v_1, k_2 - v_2)$$

$$+ R(k_1 - v_1, k_2 - j_2)\, R(k_2 - v_2, k_1 - j_1)$$

$$+ \sum_{u=1}^{\infty} \Big[R(k_1 - j_1, k_2 - j_2 + du)\, R(k_1 - v_1, k_2 - v_2 + du)$$

$$+ R(k_2 - j_2, k_1 - j_1 + du)\, R(k_2 - v_2, k_1 - v_1 + du)$$

$$+ R(k_1 - v_1, k_2 - j_2 + du)\, R(k_2 - v_2, k_1 - j_1 - du)$$

$$+ R(k_1 - v_1, k_2 - j_2 - du)\, R(k_2 - v_2, k_1 - j_1 + du) \Big] \Big\}$$

$$= \sigma_{k_1}^2\, \delta_{k_1,k_2}\, R(k_1 - v_1, k_1 - v_2) \quad ,$$

from Lemma 1 and a repeated application of (3.3). Now, from Lemma 1 and Cramér's theorem, ([1] p. 254) the random variables on the left of (3.10) have the same joint asymptotic distribution as

$$\sum_{j=0}^{p_k} R\{k-j,k-v\}\{\hat{\alpha}_k(j) - \alpha_k(j)\} \quad , \quad \begin{array}{l} v = 1,\ldots,p_k \\ k = 1,\ldots,d \end{array}$$

and the theorem is proved.

A noteworthy result contained in the theorem is that, asymptotically, the $\hat{\underset{\sim}{\alpha}}_k$, $k = 1,\ldots,d$, are independent. Thus, when analyzing a multivariate autoregression, the parametrization in terms of the $\underset{\sim}{\alpha}_k$, as opposed to the $A(j)$ in (2.1), allows us to analyze each channel separately. In addition, since the different orders of the channels are not necessarily such that $dp = p_k - k + 1$, $k = 1,\ldots,d$, we can model a multivariate autoregression of order p (with $A(p) \neq 0$) with fewer than $d^2p + \{d(d+1)\}/2$ parameters, and we thus have a general methodology for systematically reducing the number of parameters required. The number of parameters in terms of the $A(j)$ can be sizable; for example, if $d = 12$ (monthly data), to fit a zero order multiple autoregression requires 78 parameters; a first order, 222 parameters. We have thus overcome the difficulty expressed by Whittle [17], "... of analyzing a d-tuple series [which] may be said to increase roughly as d^2 (the number of auto- and cross-correlograms which must be calculated, and the order of the number of parameters to be estimated), while the number of observations increases only as d ." It could be said that we have defined multiple autoregressions of non-integer orders.

4. Deciding the Order

When using autoregressions to model data, an important question has been what order to use. To answer this for periodic autoregressions, we expand the notation in (2.1) and (2.2) to make explicit the orders, by using Σ_p , $\alpha_k(j,p_k)$ and $\sigma_k^2(p_k)$. Parzen [15] has introduced the CAT criterion to decide the order of a fitted multiple autoregression,

$$\mathrm{CAT}(m) = \mathrm{tr}\{dN^{-1} \sum_{p=1}^{m} \Sigma_p^{-1} - \Sigma_m^{-1}\} , \qquad m = 0,1,\ldots,$$

which measures the mean square error (plus a constant) of approximating the infinite order autoregressive transfer function by a fitted $\underline{m}$th order autoregression. The chosen order is that which minimizes an estimate of CAT(m) . Rewriting CAT we see that it equals

$$dN^{-1} \sum_{p=1}^{m} \sum_{k=1}^{d} \sum_{j=0}^{k} \frac{\alpha_k^2(j,p_k)}{\sigma_k^2(p_k)} - \sum_{k=1}^{d} \sum_{j=0}^{k} \frac{\alpha_k^2(j,m_k)}{\sigma_k^2(m_k)}$$

where $dp = p_k - k + 1$, $dm = m_k - k + 1$, $k = 1,\ldots,d$.

To remove these constraints on the individual orders, and thus allow for noninteger order multiple autoregressions, we can define

$$(4.1) \qquad \mathrm{P\,CAT}(\underset{\sim}{m}) = \sum_{k=1}^{d} \mathrm{P\,CAT}(m_k)$$

where $\underset{\sim}{m} = (m_1,\ldots,m_d)$, and,

$$(4.2) \qquad \mathrm{P\,CAT}(m_k) = \sum_{j=0}^{k} \left\{ N^{-1} \sum_{p=1}^{m_k} \frac{\alpha_k^2(j,p)}{\sigma_k^2(p)} - \frac{\alpha_k^2(j,m_k)}{\sigma_k^2(m_k)} \right\}$$

$$m_k = 0,1,\ldots.$$

We see that PCAT$(\underset{\sim}{m})$ is minimized when each PCAT(m_k) is minimized. The PCAT(m_k) may be estimated by replacing the α by the $\hat{\alpha}$ in (3.8) and the $\sigma_k^2(p)$ by $\left[N/(N-p)\right] \hat{\sigma}_k^2(p)$ obtained in (3.8). In this form, we can use PCAT to decide the order of a periodic autoregression.

An alternative is to use the Akaike criterion [4], which presupposes that the sample is indeed from a multiple autoregression of order m,

$$\text{AIC}(m) = \log \det \hat{\Sigma}_m + (2d^2 m/N) \quad ,$$

and minimize it with respect to m. This function can be rewritten as

$$\text{AIC}(m) = \sum_{k=1}^{d} \log \hat{\sigma}_k^2(m_k) + (2d^2 m/N)$$

with $dm = m_k - k + 1$, $k = 1,\ldots,d$. This leads to the generalization,

$$\text{(4.3)} \qquad \text{PAIC}(\underset{\sim}{m}) = \sum_{k=1}^{d} \text{PAIC}(m_k)$$

where $\underset{\sim}{m} = (m_1,\ldots,m_d)$, and,

$$\text{(4.4)} \qquad \text{PAIC}(m_k) = \log \hat{\sigma}_k^2(m_k) + 2m_k/N \quad , \qquad m_k = 0,1,\ldots ,$$

which is a straight generalization of the univariate AIC . This is indeed approximately proportional to,

$$-2 \ln (\text{maximum likelihood function}) + (2 \text{ number of parameters}) / N \quad ,$$

agreeing with the definition given in [4].

We note that, as for scalar autoregressions, the periodically correlated autoregression allows us to put a prediction theoretic interpretation on the multivariate AIC . If we let

$$\tilde{\sigma}_k^2(m_k) = N\hat{\sigma}_k^2(m_k)/(N - m_k) \quad , \qquad k = 1,\dots,d \quad ,$$

i.e., introduce the "unbiased" estimator of the residual variance, then

$$\text{AIC}(m) = \sum_{k=1}^{d} \log \tilde{\sigma}_k^2(m_k)\left[1 + m_k N^{-1} + O(m_k^2 N^{-2})\right]$$

with $dm = m_k - k + 1$, $k = 1,\dots,d$. Thus, asymptotically, AIC is the sum of the logs of the estimated one-step-ahead prediction variances of $X_1(t+1)$ given $\underset{\sim}{X}(t),\underset{\sim}{X}(t-1),\dots,$ plus that of predicting $X_2(t+1)$ given $X_1(t+1),\underset{\sim}{X}(t),\underset{\sim}{X}(t-1),\dots,$ plus that of predicting $X_3(t+1)$ given $X_2(t+1),X_1(t+1),\underset{\sim}{X}(t),\underset{\sim}{X}(t-1),\dots,$ etc. ... .

5. Discussion

Viewing a multiple autoregression as a periodic autoregression clearly displays the effects of prewhitening each channel before doing a joint analysis. If the prewhitening is done with an autoregressive filter on each channel, then this can be viewed as, first, fitting a periodic autoregression with $\alpha_k(d), \alpha_k(2d), \ldots$, $k = 1,\ldots,d$, as the only nonzero coefficients, and, second, performing a periodic autoregression on the residuals. An alternative approach would be to include a subset regression option when fitting the periodic autoregression (such as in [8], for example) and this would obviate the need for prewhitening.

In analyzing atmospheric data, Jones and Brelsford [5] achieved a reduction in the number of parameters by expanding the α in Fourier series,

$$(5.1) \qquad \alpha_k(j) = \sum_{n=0}^{m} \{c_{jn} \cos(2\pi n k/d) + s_{jn} \sin(2\pi n k/d)\} ,$$

taking m small (relative to $d/2$), arguing that the α were slowly varying (with respect to k) and periodic of period d. Using the asymptotic distribution in Theorem 3, one can obtain efficient estimators of the c's and s's by performing a weighted regression of the $\hat{\alpha}$, in (3.9), on the c's and s's (as in [11]). Indeed, these estimators are not too different from those in [5], but this approach would provide a method for systematically testing the hypothesis exhibited in (5.1).

The Gaussian assumptions made in Section 3 can clearly be relaxed. Another generalization can be achieved by considering periodically correlated q-dimensional vector processes $\underset{\sim}{Y}(\cdot)$, in which case $\underset{\sim}{X}(\cdot)$ would be a dq-dimensional covariance stationary time series [2].

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A methodology is presented for analyzing periodic autoregressions which is also applicable when inferring the second order properties of periodically correlated processes. In addition, capitalizing on the duality between periodic and multiple autoregressions, a method is set forth for analyzing the latter, which overcomes the usual requirements of a large number of both parameters and computer storage locations.

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