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A LIKELIHOOD RATIO STATISTIC FOR TESTING GOODNESS OF FIT WITH RANDOMLY CENSORED DATA.

by

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SUMMARY

A likelihood ratio statistic is proposed for testing goodness of fit with grouped data which are subject to random right censoring. It is shown that, under appropriate conditions, this statistic has an asymptotic chi-square distribution which is non-central under contiguous alternatives. Some examples are given including one on marijuana usage which needs an extension of the test to the doubly censored case.

Some key words: Likelihood ratio; Goodness of fit test; Grouping; Random censoring; Multinomial distribution; Kaplan-Meier product limit estimator; Self-consistency; The EM method; Double censoring.

1. INTRODUCTION

In this paper we consider the problem of testing "goodness of fit" when some of the data may be subject to random censoring. Single right censoring occurs commonly in response time data. Here each lifetime X may be observed exactly or, alternatively, may be known only to exceed a certain value. These situations occur, for instance in industrial life-testing, medical follow-up and recidivism studies. Some examples are given in Kaplan and Meier (1958). We shall concentrate on this case of single censoring but there are obvious extensions of the methods we shall propose to more complicated censoring patterns such as interval or double censoring (see respectively Peto (1973), and Turnbull (1974, 1976)).

The question of whether the observations can be explained by a particular mathematical model is an important problem. For instance, if an exponential model can be accepted, then further analysis - e.g. estimation and testing - is simplified considerably. In life-testing it becomes meaningful to employ Standards like MIL STD 690B and MIL STD 781 B, which assume a constant failure rate. If a goodness-of-fit test can lead the investigator to accept a certain parametric mathematical or physical model, then this can enable him to glean some information about the tail of the response time distribution, which is often important in reliability studies. Non-parametric methods usually reveal little about the tail behaviour.

We will consider the following random censorship model. There are N pairs of random variables $(X_1,Y_1), (X_2,Y_2)\ldots(X_N,Y_N)$. Usually these represent response times. The observed data, however, consist only of $\min(X_i,Y_i)$ and $I_{\{X_i \leq Y_i\}}$ for $1 \leq i \leq N$. (Here I_A denotes the indicator of the set A.) If $I_{\{X_i \leq Y_i\}} = 0$, we say that X_i has been censored by Y_i, otherwise X_i has been observed exactly. Assume that $X_1,X_2,\ldots,X_N$ are i.i.d. with survivor function $F(x) = P(X > x)$, and similarly $Y_1,Y_2,\ldots,Y_N$ are iid with survivor function $G(y) = P(Y > y)$. Assume also that X_i and Y_i are independent $(1 \leq i \leq N)$; without this assumption there are identifiability problems (Tsiatis, 1975). Both F and G are unknown. Here G is a "nuisance parameter" and the goal is to make inferences about F, specifically to test some null hypothesis H_0: $F = F_0$. H_0 may be simple or composite, i.e. F_0 may be completely specified or may depend on some parameters which are left unspecified.

Goodness of fit analysis is substantially complicated by the presence of censoring, and most researchers have only considered the singly censored case. Among graphical methods, Nelson (1972) describes methods for plotting the cumulative hazard function, while Barlow and Campo (1975) have proposed "total time on test" plots. In the analysis of heart transplant survival data Turnbull, Brown and Hu (1974) also used graphical methods to compare the best fitting exponential and Pareto model curves with the product-limit estimate curve as defined by Kaplan and Meier (1958). Lamborn (1969) described a Pearson type statistic and, extending the methods of Roy (1956) and Watson (1958), she established that this is asymptotically distributed as a linear combination of independent χ_1^2 random variables. The test is difficult to use in practice because this asymptotic distribution is different for each problem. Greenberg, Bayard and Byar (1974) also used a Pearson χ^2 statistic but obviated the difficulties, because for each censored obervation, it was known into which class interval it fell. Barr and Davidson (1973) described a Kolmogorov-Smirnov (KS) test for data which, if censored, are censored at the <u>same</u> fixed point (i.e. $Y_1 = Y_2 = \ldots = Y_N$). This situation is common in life-testing but not in medical follow-up or recidivism data. The asymptotic distribution of the Barr-Davidson statistic has been obtained and tabulated by Koziol and Byar (1975). Cramér-von Mises (CM) type statistics have been investigated by Pettit and Stephens (1976) for the case with all Y_i equal (they also consider double censoring); and by Koziol and Green (1975) for the particular model $G = F_0^\beta$ under H_0 for some $\beta > 0$. In practice this assumption must be verified first and β estimated. A limitation of these KS and CM type tests is that H_0 must be simple, i.e. F_0 completely specified. Finally, Barlow and Proschan (1969) have described a test for the exponential model which is unbiased against IFRA alternatives (see also Harris (1976)). Stollmack and Harris (1974) have applied this last test to the analysis of recidivism data.

In this paper we present a likelihood ratio statistic for testing goodness of fit of hypothesis H_0, which may be simple or composite, and is applicable when the ranges of the random variables X,Y are discrete or when the observations are grouped into discrete intervals. Using the general results of Weiss (1975) concerning the likelihood ratio in non-standard cases, the statistic is shown to have an asymptotic χ^2 distribution.

Specifically we assume that X is discrete with finite range $t_1 < t_2 < \dots < t_m$, which occurs, for instance in response time data if there were a natural discrete time scale, e.g. see Klotz (1976). Alternatively we can assume that the data are grouped and the lifetimes recorded only as belonging to one of the m intervals $(t_0,t_1]$, $(t_1,t_2]$, ..., $(t_{m-1},t_m]$, where usually $t_0 = 0$ and often $t_m = +\infty$. Further we assume that the range of Y is $\{t_1+,t_2+,\dots,t_m+\}$, i.e. any right censored observation (or "<u>losses</u>") at t_i occur immediately after any observed deaths. This setup is often appropriate in studies with periodic inspection (or "snapshots") - see the discussion in Section 1.4 of Kaplan and Meier (1958). Essentially the two problems (X discrete, or X continuous with grouped observations) are the same with slight modifications, but it is easier to think of X taking on discrete values and we shall do this in our further discussion. In Section 3 we will give examples of both situations.

2. THE LIKELIHOOD RATIO TEST

For $1 \leq i \leq m$, define $F_i = F(t_i) = P(X > t_i)$, $s_i = F_{i-1} - F_i$, $G_i = G(t_i+) = P(Y > t_i+)$, $u_i = G_{i-1} - G_i$, with $\sum_{i=1}^{m} s_i = \sum_{i=1}^{m} u_i = 1$, $F_m = G_m = 0$. Following the notation of Kaplan and Meier (1958), we define frequencies δ_i, λ_i $(1 \leq i \leq m)$ where there are δ_i pairs with $X = t_i$ and $Y > t_i$ (exact observations at t_i) and λ_i observations with $X > t_i$ and $Y = t_i+$ (losses at t_i). Note that since $F_m = 0$, we must have $\lambda_m = 0$. (The assumption $F_m = 0$ is no loss of generality; if $F_m > 0$, $\lambda_m > 0$ we can add an extra "bin" or time point, t_{m+1} say.) Note also that $\sum_{i=1}^{m} (\delta_i + \lambda_i) = N$.

Define $\underset{\sim}{s} = (s_1, \ldots, s_{m-1})$, $\underset{\sim}{u} = (u_1, \ldots, u_{m-1})$. Then the likelihood L is given by

$$L(\underset{\sim}{s}, \underset{\sim}{u}) = [u_m s_m]^{\delta_m} \prod_{i=1}^{m-1} [s_i(1 - u_i - \ldots - u_{i-1})]^{\delta_i} [u_i(1 - s_1 - \ldots - s_i)]^{\lambda_i} \tag{1}$$

where $s_m = 1 - s_1 - \ldots - s_{m-1}$ and $u_m = 1 - u_1 - \ldots u_{m-1}$. Define $\Omega = \{\underset{\sim}{s}: s_i \geq 0, \sum_{i=1}^{m-1} s_i \leq 1\} \subseteq R^{m-1}$. Kaplan and Meier (1958) show that L is maximised in Ω by $\underset{\sim}{s} = \hat{\underset{\sim}{s}} = (\hat{s}_1, \ldots, \hat{s}_{m-1})$, where $\hat{s}_i = \hat{F}_{i-1} - \hat{F}_i$ and

$$\hat{F}_i = \hat{F}_{i-1}[1 - \{\delta_i / \sum_{j=i}^{m} (\delta_i + \lambda_i)\}] \quad (1 \leq i \leq m) \text{ with } \hat{F}_0 = 1. \tag{2}$$

Suppose we wish to test the hypothesis

$$H_0: \; s_i = Q_i(\underset{\sim}{\phi}) \quad \text{for } 1 \leq i \leq m-1$$

for some unspecified $\underset{\sim}{\phi} = (\phi_1, \ldots, \phi_a) \in R^a$ where $1 \leq a \leq m-2$ and the $\{Q_i\}$ are given functions from R^a into R^{m-1}.

We assume that $s_i = Q_i(\underset{\sim}{\phi})$ for $1 \leq i \leq a$ defines a 1-1 relation between $\underset{\sim}{\phi}$ and $(s_1, s_2, \ldots, s_a)$. We can cater for the case of H_0 simple i.e. when the s_i are completely specified by formally allowing the case $a = 0$. For example if we wish to test that X is geometrically distributed we have $Q_i(\phi) = \phi(1-\phi)^{i-1}$ for $1 \leq i \leq m-1$. Here $a = 1$ if ϕ is unspecified, while $a = 0$ if ϕ is specified by H_0.

The hypothesis H_0 implies that $\underset{\sim}{s}$ belongs to some subspace, Ω_0 say, of the parameter space Ω. Let $\tilde{\underset{\sim}{\phi}}$ denote the MLE of $\underset{\sim}{\phi}$ under H_0 and thus $\tilde{\underset{\sim}{s}} = \underset{\sim}{Q}(\tilde{\underset{\sim}{\phi}})$ will maximise the likelihood (1) under the restriction $\underset{\sim}{s} \in \Omega_0$. The problem of calculating $\tilde{\underset{\sim}{\phi}}$ and $\tilde{\underset{\sim}{s}}$ can be quite difficult computationally (except of course when H_0 is simple, $a = 0$) and there are many papers in the literature concerned with estimating parameters with grouped and censored data in special cases e.g. Weibull or Poisson; perhaps the best general references are to Blight (1970) and to Dempster, Laird and Rubin (1976). Once we have obtained $\tilde{\underset{\sim}{s}}$, we can form the generalised likelihood ratio:

$$\frac{\max\limits_{\underset{\sim}{\mu} \in \Omega,\, \underset{\sim}{s} \in \Omega_0} L(\underset{\sim}{s},\underset{\sim}{\mu})}{\max\limits_{\underset{\sim}{\mu},\, \underset{\sim}{s} \in \Omega} L(\underset{\sim}{s},\underset{\sim}{\mu})} = \frac{L(\tilde{\underset{\sim}{s}},\hat{\underset{\sim}{\mu}})}{L(\hat{\underset{\sim}{s}},\hat{\underset{\sim}{\mu}})}$$

$$= \prod_{i=1}^{m} \left(\frac{\tilde{s}_i}{\hat{s}_i}\right)^{\delta_i} \prod_{i=1}^{m-1} \left(\frac{1-\tilde{s}_1-\ldots-\tilde{s}_i}{1-\hat{s}_1-\ldots-\hat{s}_i}\right)^{\lambda_i}$$

$$= \Lambda_N(\tilde{\underset{\sim}{s}},\hat{\underset{\sim}{s}}), \quad \text{say.} \qquad (3)$$

In (3), we have defined $\hat{s}_m = 1 - \hat{s}_1 - \ldots - \hat{s}_{m-1}$ and similarly $\tilde{s}_m$. Note that Λ_N does not depend on $\hat{\underset{\sim}{\mu}}$.

Further, define $\phi_{a+1},\phi_{a+2},\ldots,\phi_{m-1}$ *by the relations*

$$s_i = \phi_i + Q_i(\phi_1,\phi_2,\ldots,\phi_a) \qquad (4)$$

$$\text{for } i = a+1,\ldots,m-1.$$

We have now reparametrized: our new parameters are $u_1, u_2, \ldots, u_{m-1}$, $\phi_1, \phi_2, \ldots, \phi_{m-1}$. Of these $u_1, u_2, \ldots, u_{m-1}, \phi_1, \ldots, \phi_a$ are nuisance parameters and the hypothesis H_0 which we wish to test becomes

$$H_0: \ \phi_{a+1} = \phi_{a+2} = \cdots = \phi_{m-1} = 0.$$

Let $\underset{\sim}{\theta}^0$ denote a particular configuration $(u_1^0, \ldots, u_{m-1}^0, \phi_1^0, \ldots, \phi_a^0, 0, \ldots 0)$. We assume that throughout a neighborhood of $(\phi_1^0, \ldots, \phi_a^0)$, all third derivatives of $Q_i(\phi_1, \ldots, \phi_a)$ are bounded in absolute value, and that throughout a neighborhood of $\underset{\sim}{\theta}^0$, the 2m quantities $u_1, \ldots, u_m, s_1, \ldots s_m$ are all bounded away from zero. Here and in Appendix, when we speak of a point $\underset{\sim}{\theta} = (u_1, \ldots, u_{m-1}, \phi_1, \ldots, \phi_{m-1})$, it is understood that s_i is given by $Q_i(\phi_1 \ldots, \phi_a)$ for $i = 1, \ldots, a$; by $\phi_i + Q_i\ (\phi_1, \ldots, \phi_a)$ for $i = a+1, \ldots, m-1$; and $s_m = 1 - (s + \ldots + s_{m-1})$. Then when $\underset{\sim}{\theta}^0$ is the true parameter vector, $-\frac{1}{N} \left.\frac{\partial^2 \log L}{\partial u_\alpha \partial u_\beta}\right|_{\underset{\sim}{\theta}^0}$ tends stochastically to a limit, $V_{\alpha\beta}(\underset{\sim}{\theta}^0)$ say, and $-\frac{1}{N} \left.\frac{\partial^2 \log L}{\partial \phi_\alpha \partial \phi_\beta}\right|_{\underset{\sim}{\theta}^0}$ tends stochastically to a limit, $W_{\alpha\beta}(\underset{\sim}{\theta}^0)$ say, for $1 \le \alpha \le m-1$, $1 \le \beta \le m-1$. These define $(m-1)\times(m-1)$ matrices $\underset{\sim}{V}(\underset{\sim}{\theta}^0)$ and $\underset{\sim}{W}(\underset{\sim}{\theta}^0)$. The details of this and expressions for the $\{V_{\alpha\beta}\}$ and $\{W_{\alpha\beta}\}$ are given in the Appendix. We are now ready to state the following:

THEOREM: Suppose $\underset{\sim}{V}(\underset{\sim}{\theta}^0)$ and $\underset{\sim}{W}(\underset{\sim}{\theta}^0)$ are both non-singular, and consider the contiguous alternative where the true parameter values are $(u_1^0, \ldots, u_{m-1}^0, \phi_1^0, \ldots, \phi_a^0, c_{a+1}/\sqrt{N}, \ldots, c_{m-1}/\sqrt{N})$. Then the asymptotic distribution of $-2\log \Lambda_N$ is $\chi^2_{m-a-1}(\delta^2)$. Here the non-centrality parameter δ^2 is

$$(c_{a+1}, \ldots, c_{m-1})\ \underset{\sim}{Z}^{-1}(\underset{\sim}{\theta}^0)(c_{a+1}, \ldots, c_{m-1})'$$

where $\underset{\sim}{Z}(\underset{\sim}{\theta}^0)$ denotes the $(m-a-1)\times(m-a-1)$ matrix derived by deleting the

The proof is given in the Appendix.

REMARK 1. The asymptotic distribution of $-2\log \Lambda_N$ under H_0 is χ^2_{m-a-1}, which is as might be expected. Thus knowledge of V,W, or Z is not required to carry out a test of significance. An asymptotically level α test of H_0 will reject when $-2\log \Lambda_N$ exceeds the $(1 - \alpha)$ quantile of the χ^2_{m-a-1} distribution.

REMARK 2. Generally, the matrices $\underset{\sim}{V}(\underset{\sim}{\theta}^0)$, $\underset{\sim}{W}(\underset{\sim}{\theta}^0)$ are non-singular as long as it is known that none of the true $\{s_i\}$ or $\{u_i\}$ $(1 \leq i \leq m)$ are zero. However if it is known that some of these parameters must be zero, then the analysis can go through with the appropriate reduction in dimensionality. For instance, as in the second example of the next section, the experiment may be designed so that all but a certain number, b say, of the u_i $(1 \leq i \leq m)$ are constrained to be zero. These zero u_i are just omitted from $\underset{\sim}{\theta}^0$ and L; $\underset{\sim}{V}$ is a $(b-1)\times(b-1)$ matrix, and otherwise the statement of the theorem is unchanged.

REMARK 3. The contiguous or "challenging" alternatives in the theorem are commonly used in large sample theory in order to keep Type I and Type II error probabilities bounded away from nought and one. For finite sample sizes, we should modify our test as in Weiss (1975) to guard against non-contiguous alternatives. We do this by also rejecting H_0 if, for any i $(a+1 \leq i \leq m-1)$

$$|\hat{s}_i - Q_i(\underset{\sim}{t}^*)| > k \cdot N^{-\Delta}$$

where $k > 0$, $2/3 < \Delta < 1$ and (when $a \neq 0$) $\underset{\sim}{t}^* \in R^a$ satisfies $\hat{s}_i = Q_i(\underset{\sim}{t}^*)$ $(1 \leq i \leq a)$. This modification will not affect the asymptotic properties of the test under H_0 or contiguous alternatives.

3. EXAMPLES

Example 1. Suppose we consider the geometric example mentioned earlier. Here

$$H_0: \; s_i = \phi(1 - \phi)^{i-1} \quad (1 \le i \le m-1) \quad s_m = (1 - \phi)^{m-1},$$

for some $0 \le \phi \le 1$ unspecified. Such an H_0 would be applicable to testing a constant hazard rate with $t_1, t_2, \ldots, t_{m-1}$, all equally spaced. ($\delta_m$ represents those items still alive at t_{m-1}.)

Under H_0, the likelihood is proportional to

$$L^*(\phi) = \prod_{i=1}^{m} [\phi(1 - \phi)^{i-1}]^{\delta_i} \prod_{i=1}^{m-1} (1 - \phi)^{i\lambda_i} .$$

Setting the derivative of $L^*(\phi)$ equal to zero, we see that the maximising value of ϕ is $\tilde{\phi} = \sum_{i=1}^{m} \delta_i / [\sum_{j=1}^{m} j(\lambda_j + \delta_j)]$. Then we can obtain the likelihood ratio Λ_N by substituting in (3) for $\hat{\underset{\sim}{s}}$ by (2) and for $\tilde{s}_i$ by $\tilde{\phi}(1 - \tilde{\phi})^{i-1}$ $(1 \le i \le m-1)$. Here $a = 1$, and so the cutoff point will be based on the percentage points of the χ^2_{k-2} distribution. (Of course, if H_0 also specified the value of ϕ we would take $\tilde{\phi}$ as that value and then $a = 0$).

Example 2. Consider the following grouped data taken from Table 465 of Kaplan and Meier (1958).

TABLE 1.

i	1	2	3	4	5	6	7	8
t_i	1	1.7	2	3	3.6	4	5	T
δ_i	3	5	4	10	9	6	15	16
λ_i	0	20	0	0	12	0	0	0

Take $t_0 = 0$, $T > 5$. Thus $m = 8$ and $N = 100$. Here we could say that losses could only occur at times 1.7 and 3.6 because of the design of the experiment. (Actually, the δ_8 deaths at T were really reported as losses at 5+. Clearly it makes no difference how we apportion the 16 losses at time 5+ between δ_8 and λ_7.) By Remark 3 however, the fact that only u_2, u_5 and u_8 can be non-zero does not affect the test. In Table 1, δ_i is interpreted as the frequency of observed response times occurring in $(t_{i-1}, t_i]$.

Consider the hypothesis H_0 that X is exponentially distributed. Thus

$$H_0: \quad s_i = e^{-\theta t_{i-1}} - e^{-\theta t_i} \qquad (1 \leq i \leq m-1).$$

Under H_0, the likelihood is proportional to

$$L^*(\phi) = \prod_{i=1}^{m} [(e^{-\phi t_{i-1}} - e^{-\phi t_i})^{\delta_i} e^{-\phi t_i \lambda_i}]$$

where we have used the convention $t_m = +\infty$ and $t_m \lambda_m = 0$. Setting $\frac{\partial \log L^*}{\partial \phi} = 0$, we have that $\tilde{\phi}$ satisfies

$$\sum_{i=1}^{m-1} \lambda_i t_i = \sum_{i=1}^{m} \delta_i \left[\frac{-t_i \exp(-\phi t_{i-1}) + t_i \exp(-\phi t_i)}{\exp(-\phi t_{i-1}) - \exp(-\phi t_i)} \right]$$

Solving numerically we obtain $\tilde{\phi} = .1638$ and using $\tilde{s}_i = \exp(-\tilde{\phi} t_{i-1}) - \exp(-\tilde{\phi} t_i)$

we have $\tilde{s} = (.151, .092, .036, .109, .057, .035, .078, .441)$. From Kaplan and Meier (p. 465) we have

$$\hat{s} = (0.3, .05, .05, .13, .11, .11, .25) \text{ and } \hat{s}_m = .27.$$

Evaluating $-2\log \Lambda_N(\tilde{s},\hat{s}) = 46.1$, and comparing this with the tables of χ^2_6, we see that H_0 is rejected at any reasonable significance level.

With a little more numerical work, one could, in a similar fashion, investigate the hypothesis that X was Weibull or Pareto distributed, for example. In the Weibull case $s_i = \exp[-(\eta t_{i-1})^\beta] - \exp[-(\eta t_i)^\beta]$ where H_0, while under the Pareto model $s_i = [\eta/(\eta + t_{i-1})]^\beta - [(\eta/(\eta + t_i)]^\beta$ where $\phi = (\eta,\beta)$ is unspecified. In both cases $a = 2$, and the percentage points of χ^2_5 are applicable.

Example 3. (Doubly censored data). The techniques described in this paper can be extended analogously to handle doubly censored data. Here in addition the frequencies $\{\delta_i\}$ and $\{\lambda_i\}$, there are frequencies $\{\mu_i\}$, where μ_i $(1 \leq i \leq m)$ represents the number of observations left censored at t_i. (For a more detailed discussion of double censoring see Turnbull (1974).) As an example consider Table 2 below which summarizes the answers of [illegible] California high school students to the question "When did you first use marijuana?" (This was part of a large study on the Stanford-Palo Alto Peer Counseling Program, which has been reported by Hamburg, Kraemer and Jahnke (1975).) Any direct answer such as "12,14,15,..." gives rise to an exact observation. If the student answered: "I have never used it" then this gives rise to an observation which is censored on the right at his/her present age. The final possibility was someone who answered "I have used it but cannot recall just when the first time was." This gives rise to a left censored observation where age of first use is known only to be prior to the student's current age.

TABLE 2

i	1	2	3	4	5	6	7	8	9	10
t_i	10	11	12	13	14	15	16	17	18	> 18
δ_i	4	12	19	24	20	13	3	1	0	4
λ_i	0	0	2	15	24	18	14	6	0	0
μ_i	0	0	0	1	2	3	2	3	1	0

Now many of the frequencies in the table are zero or very small and so the asymptotic χ^2 distribution is unlikely to be a meaningful approximation to the distribution of the likelihood ratio. Nevertheless we shall proceed in order to show how the calculations are carried out in the doubly censored case.

Let us consider the problem of testing goodness of fit of a negative binomial distribution: (This hypothesis was suggested to us by one of the investigators and is based on a model in which opportunities to start taking drugs occur at random times and that different children have varying susceptibilities. This theory leads to mixture of Poissons of which the negative binomial is a particular example.) Thus

$$H_0: \quad s_i = \left(\frac{\beta}{\beta+1}\right)^{i-1} \left(\frac{1}{\beta+1}\right)^{\eta} \prod_{j=0}^{i-2} (\eta + j) \qquad (1 \leq i \leq 9)$$

$$s_{10} = 1 - \sum_{1}^{9} s_i$$

(The empty product is defined to be unity.)

Using the method of self-consistency (Turnbull, 1974) or equivalently the EM algorithm (Dempster et al., 1976) one obtains $\tilde{\eta} = 6.75$, $\tilde{\beta} = 0.979$

$$\tilde{s} = (.010, .033, .064, .092, .111, .118, .114, .103, .088, .267)$$

and

$$\hat{s} = (.023, .070, .112, .143, .136, .124, .047, .037, .001, .307)$$

Noting that there is now an extra factor

$$\prod_{i=1}^{m} \left(\frac{\tilde{s}_1 + \ldots + \tilde{s}_i}{\hat{s}_1 + \ldots + \hat{s}_i} \right)^{\mu_i}$$

in the expression (3) for Λ_N due to the left censored observations we calculate the value of $-2\log\Lambda_N$ to be 32.5.

Comparing this with the percentage points of χ_7^2, we see that the negative binomial does not fit the observed data, (that is assuming that the asymptotic approximation is valid, which assumption we make only for pedagogic reasons).

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APPENDIX

This appendix is devoted to a proof of the Theorem. The proof consists of a verification that the assumptions of Weiss (1975) hold in the present case.

Fix a value Δ in the open interval $(0, \frac{1}{6})$, and define $J_N(\theta^0)$ as

$$\left\{ (u_1,\ldots,u_{m-1},\phi_1,\ldots,\phi_{m-1}) \;\middle|\; \begin{array}{ll} |u_i - u_i^0| \le \dfrac{N^{1/6-\Delta}}{\sqrt{N}}, & i = 1,\ldots, m-1 \\ |\phi_i - \phi_i^0| \le \dfrac{N^{1/6-\Delta}}{\sqrt{N}}, & i = 1\ldots,a \\ |\phi_j| \le \dfrac{N^{1/6-\Delta}}{\sqrt{N}}, & j = a+1,\ldots,m-1 \end{array} \right\}.$$

It is easily verified that

$$-\frac{1}{N}\frac{\partial^2}{\partial u_\alpha \partial\phi_\beta} \log L(\theta) = 0 \qquad \text{for } \alpha,\beta = 1,\ldots,m-1;$$

$$-\frac{1}{N}\frac{\partial^2}{\partial u_\alpha \partial u_\beta} \log L(\theta) = \sum_{i=1}^{m}\left(\frac{\delta_i}{N}\right)\Gamma_{\alpha,\beta,i}(\theta) + \sum_{i=1}^{m-1}\left(\frac{\lambda_i}{N}\right) D_{\alpha,\beta,i}(\theta) \qquad \text{for } \alpha,\beta = 1,\ldots,m-1;$$

$$-\frac{1}{N}\frac{\partial^2}{\partial \phi_\alpha \partial\phi_\beta} \log L(\theta) = \sum_{i=1}^{m}\left(\frac{\delta_i}{N}\right)\Gamma^*_{\alpha,\beta,i}(\theta) + \sum_{i=1}^{m-1}\left(\frac{\lambda_i}{N}\right) D^*_{\alpha,\beta,i}(\theta) \qquad \text{for } \alpha,\beta = 1,\ldots,m-1;$$

where $\Gamma_{\alpha,\beta,i}(\theta)$, $D_{\alpha,\beta,i}(\theta)$, $\Gamma^*_{\alpha,\beta,i}(\theta)$, $D^*_{\alpha,\beta,i}(\theta)$ do not depend on N, and are rational functions of u,ϕ, and first and second derivatives of $\{Q_i(\phi_1\ldots,\phi_a)\}$, all the denominators being first and second powers of sums of one or more of $(u_1,\ldots,u_m,s_1,\ldots,s_m)$. Thus these denominators are bounded away from zero if the derivatives are taken for $\underset{\sim}{\theta}$'s in $J_N(\underset{\sim}{\theta}^0)$.

Define $v_{\alpha\beta}(\underset{\sim}{\theta}^0)$ as

$$\sum_{i=1}^{m-1} [s_i^0(1 - u_1^0 \ldots - u_{i-1}^0)]\, \Gamma_{\alpha,\beta,i}(\underset{\sim}{\theta}^0) + \sum_{i=1}^{m-1} [u_i^0(1 - s_1^0 \ldots - s_i^0)]\, D_{\alpha,\beta,i}(\underset{\sim}{\theta}^0)$$

$$+ (u_m^0\, s_m^0)\Gamma_{\alpha,\beta,m}(\underset{\sim}{\theta}^0)$$

and $w_{\alpha\beta}(\underset{\sim}{\theta}^0)$ as

$$\sum_{i=1}^{m-1} [(s_i^0(1 - u_1^0 \ldots - u_{i-1}^0)]\, \Gamma^*_{\alpha,\beta,i}(\underset{\sim}{\theta}^0) + \sum_{i=1}^{m-1} [(u_i^0\ (1 - s_i^0 \ldots - s_i^0)]\, D^*_{\alpha,\beta,i}(\underset{\sim}{\theta}^0)$$

$$+ (u_m^0\, s_m^0\ )\ \Gamma^*_{\alpha,\beta,m}(\underset{\sim}{\theta}^0)$$

for $\alpha,\ \beta = 1,\ldots,m-1$.

Suppose the true parameter point depends on N, and is $\underset{\sim}{\theta}^*(N)$ in $J_N(\underset{\sim}{\theta}^0)$. Define $\sigma_i(N)$ as $\left|\frac{\delta_i}{N} - s_i(\underset{\sim}{\theta}^*(N))\, [(1 - u_1(\underset{\sim}{\theta}^*(N)) - \ldots - u_{i-1}(\underset{\sim}{\theta}^*(N))]\right|$, for $i = 1,\ldots,m-1$, $\sigma_m(N)$ as $\left|\frac{\delta_m}{N} - u_m(\underset{\sim}{\theta}^*(N))s_m(\underset{\sim}{\theta}^*(N))\right|$, $\sigma_i^*(N)$ as $\left|\frac{\lambda_i}{N} - u_i(\underset{\sim}{\theta}^*(N))\, [(1 - s_1(\underset{\sim}{\theta}^*(N)) - \ldots - s_i(\underset{\sim}{\theta}^*(N))]\right|$ for $i = 1,\ldots,m-1$. By the properties of the multinomial distribution, $\sqrt{N}\ \sigma_i(N)$, $\sqrt{N}\ \sigma_i^*(N)$ are all finite with probability one. It follows that we have

$$\frac{\delta_i}{N} = s_i^0(1 - u_1^0 \ldots - u_{i-1}^0) + \bar{\sigma}_i(N)\left(\frac{N^{1/6-\Delta}}{\sqrt{N}}\right) \qquad (i = 1,\ldots,m-1)$$

$$\frac{\delta_m}{N} = u_m^0\, s_m^0 + \bar{\sigma}_m(N)\left(\frac{N^{1/6-\Delta}}{\sqrt{N}}\right),$$

$$\frac{\lambda_i}{N} = u_i^0(1 - s_1^0 - \ldots - s_i^0) + \bar{\sigma}_i^*(N)\left(\frac{N^{1/6-\Delta}}{\sqrt{N}}\right) \qquad (i = 1,\ldots,m-1)$$

where $P_{\theta^*(N)}\{|\bar{\sigma}_i(N)| < \bar{\bar{\sigma}}(\theta^0) \cap |\bar{\sigma}_i^*(N)| < \bar{\bar{\sigma}}(\theta^0);\ \text{all}\ i\} > 1 - q_N(\theta^0)$, where $\bar{\bar{\sigma}}(\theta^0)$ is finite, and $\lim_{N\to\infty} q_N(\theta^0) = 0$. Also, by our assumptions about $Q_i(\phi_1,\ldots,\phi_a)$, we have, for any $\bar{\theta}(N)$ in $J_N(\theta^0)$, that

$$|\Gamma_{\alpha,\beta,i}(\bar{\theta}(N)) - \Gamma_{\alpha,\beta,i}(\theta^0)| < \bar{q}(\theta^0)\,\frac{N^{1/6-\Delta}}{\sqrt{N}}$$

$$|D_{\alpha,\beta,i}(\bar{\theta}(N)) - D_{\alpha,\beta,i}(\theta^0)| < \bar{q}(\theta^0)\,\frac{N^{1/6-\Delta}}{\sqrt{N}}$$

$$|\Gamma^*_{\alpha,\beta,i}(\bar{\theta}(N)) - \Gamma^*_{\alpha,\beta,i}(\theta^0)| < \bar{q}(\theta^0)\,\frac{N^{1/6-\Delta}}{\sqrt{N}}$$

$$|D^*_{\alpha,\beta,i}(\bar{\theta}(N)) - D^*_{\alpha,\beta,i}(\theta^0)| < \bar{q}(\theta^0)\,\frac{N^{1/6-\Delta}}{\sqrt{N}}$$

for all α,β,i, where $\bar{q}(\theta^0)$ is finite.

The verification that the assumptions of Weiss (1975) holds is now immediate: the matrix $B(\theta^0)$ in Weiss (1975) is the $2(m-1)$ by $2(m-1)$ matrix written in partitioned form as

$$\begin{pmatrix} V(\theta^0) & 0 \\ 0 & W(\theta^0) \end{pmatrix}.$$

The continuity of $B(\theta^0)$ follows from our assumptions about $\{Q_i(\phi_1,\ldots,\phi_a)\}$, and the positive definiteness from the nonsingularity of $V(\theta^0)$ and $W(\theta^0)$. $K_i(N)$ and $M_i(N)$ of Weiss (1975) are in our present case $\sqrt{N}$, $N^{1/6-\Delta}$ respectively.

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Likelihood ratio; Goodness of fit test; Grouping; Random censoring; Multinomial distribution; Kaplan-Meier product limit estimator; Self-consistency; The EM method; Double censoring.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A likelihood ratio statistic is proposed for testing goodness of fit with grouped data which are subject to random right censoring. It is shown that, under appropriate conditions, this statistic has an asymptotic chi-square distribution which is non-central under contiguous alternatives. Some examples are given including one on marijuana usage which needs an extension of the test to the doubly censored case.

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