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CONDITIONED LIMIT THEOREMS IN QUEUEING THEORY

BY

PEICHUEN KAO

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CHAPTER 1

INTRODUCTION

1. Introduction and Summary

In this dissertation we study stochastic processes which arise out of a single GI/G/1 queueing system. We are concerned here with the weak convergence of each process conditioned on an event whose probability is converging to zero as time becomes large.

The basic processes we are going to discuss are $\{W_n: n \geq 0\}$, $\{Q(t): t \geq 0\}$ and $\{W(t): t \geq 0\}$, where W_n represents the time which customer number n must wait before being serviced, $Q(t)$ is the number of customers present in the system at time t, and $W(t)$ denotes the workload of the server at time t.

The setup for our problems is the classical one. In a single GI/G/1 queueing system, customer number 0 arrives at time $t_0 = 0$, finds a free server, and experiences a service time v_0. The n^{th} customer arrives at time t_n and experiences a service time v_n. Let the interarrival times $t_n - t_{n-1} = u_n$, $n \geq 1$. We assume the sequence of random vectors $\{(v_{n-1}, u_n): n \geq 1\}$ is independent and identically distributed (i.i.d.). Let $E\{u_n\} = \lambda^{-1}$ and $E\{v_n\} = \mu^{-1}$, where $0 < \lambda, \mu < \infty$. In addition, we shall always assume that $E\{u_n^2\} + E\{v_n^2\} < \infty$ and that the deterministic system in which both u_n and v_n are degenerate is excluded. The natural measure of congestion for this system is the traffic intensity $\rho = \lambda/\mu$. In our study we shall consider systems in which ρ is greater than, equal to or less than unity.

If we define $\xi_n = v_{n-1} - u_n$, then W_n can be defined recursively by the relations

$$W_0 = 0 ,$$

$$W_{n+1} = [W_n + \xi_{n+1}]^+, \qquad n \geq 0.$$

Now $\{\xi_n : n \geq 1\}$ is a sequence of i.i.d. random variables with $E\{\xi_1\} \equiv \mu_1$ and $E\{\xi_1 - \mu_1\}^2 = \sigma^2$, $0 < \sigma^2 < \infty$. Form the random walk $\{S_n : n \geq 0\}$ by setting $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$. It is easy to show by induction that

$$W_n = \max\{S_n - S_k : k = 0, 1, \ldots, n\}, \quad n \geq 0$$

Next, let N_x be the hitting time of the set $(-\infty, x]$ by the random walk:

$$N_x = \inf\{n > 0: S_n \leq x\} ,$$

where the infimum of the empty set is taken to be $+\infty$. Set $N = N_0$. In the queueing context N is the number of customers served in the first busy period.

What sort of limiting results would we hope to provide for $\{W_n : n \geq 0\}$ and the other processes? One would like to have results of convergence in distribution. For example, does there exist a random variable $S^{\#}$, whose distribution is proper, such that

$$\lim_{n\to\infty} P\{S_n \leq x \mid n < N < \infty\} = P\{S^{\#} \leq x\}$$

for each $x \geq 0$ when $\rho > 1$ (i.e., $\mu_1 > 0$)? We observe that $W_n = S_n$ on $\{N > n\}$. Thus $S^{\#}$ can be thought of as a limit law for the waiting time process in the first busy period of a general single server queue with $\rho > 1$.

Let $[x]$ be the greatest integer in x. Define the random function X_n by

$$X_n(t) = \frac{S_{[nt]}}{\alpha n^{\frac{1}{2}}}, \qquad 0 \le t \le 1,$$

where α is a constant to be specified later. Then, we would like to know whether one can obtain a functional central limit theorem as

$$(X_n \mid n < N < \infty) \Rightarrow X,$$

where X is certain random function. The symbol $\Rightarrow$ means weak convergence in the Skorohod topology in $D[0,1] \equiv D$; see BILLINGSLEY (1968), p. 137, for further discussion of this result.

The limiting random function X that we shall encounter in our study can be identified in terms of Brownian motion B. The limit is either <u>Brownian meander</u> or <u>Brownian excursion</u> depending upon whether $\rho = 1$ or $\rho \neq 1$.

Brownian meander, W^+, is first recognized in BELKIN (1972), Theorem 3.1. Let $\{B(t) : t \ge 0\}$ be the standardized Brownian motion. Let $\tau_1 = \sup\{t \in [0,1]: B(t) = 0\}$. Set $\Delta_1 = 1 - \tau_1$. Then

$$W^+(t) = |B(\tau_1 + t\Delta_1)| \,/\, \Delta_1^{\frac{1}{2}}, \qquad 0 \le t \le 1.$$

It is a continuous, non-homogeneous Markov process and has transition density given by

$$(1.1) \qquad P\{W^+(t) \in dy\} = p^+(0,0,t,y)dy$$
$$= t^{-\frac{3}{2}} y \exp\left(-\frac{y^2}{2t}\right) |N|\left(\frac{y}{(1-t)^{\frac{1}{2}}}\right) dy$$

for $0 < t \le 1$ and $y > 0$; for $0 < s < t \le 1$, and $x, y > 0$

$$(1.2) \qquad P\{W^+(t) \in dy \mid W^+(s) = x\}$$
$$= p^+(s,x,t,y)dy = g(t-s,x,y) \frac{|N|\left(\frac{y}{(1-t)^{\frac{1}{2}}}\right)}{|N|\left(\frac{x}{(1-s)^{\frac{1}{2}}}\right)} dy ,$$

where $g(t,x,y) = (2\pi t)^{-\frac{1}{2}} [\exp(-\frac{(y-x)^2}{2t}) - \exp(-\frac{(y+x)^2}{2t})]$ and

$$|N|(x) = (2/\pi)^{\frac{1}{2}} \int_0^x \exp(-\frac{u^2}{2})du, \qquad x \ge 0 .$$

For the derivation of this transition density, see BELKIN (1972), p. 61. Note that for $x \ge 0$,

$$P\{W^+(1) \le x\} = \int_0^x R(y)dy ,$$

where $R(x) \equiv x \exp(-x^2/2)$, $x \ge 0$, is the density of the Raleigh distribution.

Another process which we will be interested in is the <u>Browman excursion</u>, W_0^+. Let $\tau_2 = \inf\{t \ge 1: W(t) = 0\}$. Set $\Delta_2 = \tau_2 - \tau_1$ and

$$W_0^+(t) = |B(\tau_1 + t\Delta_2)| / \Delta_2^{\frac{1}{2}} , \qquad 0 \le t \le 1.$$

Brownian excursion is also a continuous, non-homogeneous Markov process with transition density given by

$$(1.4) \qquad P\{W_0^+(t) \in dy\} = p_0^+(0,0,t,y)dy = \frac{2y^2 \exp\left(-\frac{y^2}{2t(1-t)}\right)dy}{\sqrt{2\pi t^3(1-t)^3}}$$

for $0 < t < 1$ and $y > 0$; for $0 < s < t < 1$ and $x, y > 0$

$$(1.5) \qquad P\{W_0^+(t) \in dy \mid W_0^+(s) = x\}$$

$$= p_0^+(s,x,t,y)dy = g(t-s, x,y)\left(\frac{1-s}{1-t}\right)^{\frac{3}{2}} \frac{y \exp\left(-\frac{y^2}{2(1-t)}\right)}{x \exp\left(-\frac{x^2}{(2(1-s))}\right)} dy ;$$

see ITÔ-McKEAN (1965), p. 76, for this result. Clearly, the distribution of $W_0^+(0)$ and $W_0^+(1)$ are degenerate at 0.

In Chapter 2 we shall show that, if $\rho \neq 1$, the finite-dimensional distributions (f.d.d.'s) of $(X_n \mid n < N < \infty)$ converge weakly to those of W_0^+.

Because of the degeneracy at point $t = 1$, the limiting behavior $(W_n \mid n < N < \infty)$ is not revealed. When $\rho > 1$, the existence of

$$\lim_{n\to\infty} (W_n \mid n < N < \infty)$$

will be established and its distribution function will be exhibited. This is carried out in Chapter 3.

In Chapter 4 we restrict our attention to the case that $\rho = 1$. We establish results for the workload process, $W(t)$, and the queue-length process, $Q(t)$. Results for certain multiple channel queues are also given.

To close this section, we make a comment regarding the notation in this thesis. The basic notation adhered throughout the paper is developed in Chapter 1 and the first sections of Chapters 2 and 4. Bracket , [ ], refers to the bibliography.

2. Related Work

The history of conditioned limit results for various stochastic processes does not extend over the past twenty years. The existence of contioned limit distributions in both discrete and continuous time Markov chains is extensively discussed by MANDEL (1959, 1960), DARROCH and SENETA (1965, 1967), SENETA and VERE-JONES (1967).

Results on conditioned random walks have been obtained by DWASS and KARLIN (1963), LIGGETT (1968), DALEY (1969), BELKIN (1970, 1972), KAIGH (1974), and IGLEHART (1974a,b).

For the $M/G/1$ queueing system KYPRIANOU (1971) first established the existence of

$$\lim_{t\to\infty} P\{W(t) \le x \mid t < T(a) < \infty\}, \quad 0 < x < \infty,$$

where $T(a) = \inf\{t : W(t) = 0, W(0) = a\}$ for $a \neq 0$. It is further demonstrated by KENNEDY (1974) that as time becomes large $W(t)$, suitably scaled and normed, converges to Brownian excursion or Brownian meander depending upon whether $\rho \neq 1$ or $\rho = 1$.

The first conditioned limit result for the $GI/G/1$ queue is contained in IGLEHART (1974a). In [17] he established the following result.

(1.6) THEOREM. [IGLEHART (1974a), Theorem 3.4]. If $\mu_1 = 0$, $0 < \sigma^2 < \infty$, $E\{|\xi_1|^3\} < \infty$, and ξ_1 is nonlattice or integer-valued with span 1, then $(X_n \mid N > n) \Rightarrow W^+$.

The third moment condition was later removed, see DURRETT (1976).

When $\mu_1 < 0$, which corresponds to a stable queue, the random walk $\{S_n: n \geq 0\}$ is strongly attracted to the origin and the limit results in this case do not depend only on μ_1 and σ^2 but on the entire distribution of ξ_1. Assuming that the distribution of ξ_1 satisfies the following conditions:

(1.7) $$-\infty \leq \mu_1 < 0;$$

(1.8) $\theta(s) = E\exp(s\cdot\xi_1)$ converges for real $s \in [0,a)$, for some $a > 0$;

(1.9) $\theta(s)$ attains its infimum at a point τ, $0 < \tau < a$, where $\theta(\tau) \equiv \gamma < 1$ and $\theta'(\tau) = 0$;

and

(1.10) if ξ_1 is lattice, then $P\{\xi_1 = 0\} > 0$,

IGLEHART (1974b) established the following result:

(1.11) THEOREM. [IGLEHART (1974b), Theorem 2.3]. If conditions (1.7) - (1.10) hold and $u \geq 0$, then

(1.12) $$\lim_{n\to\infty} E\{\exp(-uW_n)|N > n\} = \frac{\tau}{\tau+u}\exp\left\{\sum_{n=1}^{\infty}\frac{\gamma^{-n}}{n}[E\{\exp(-uS_n^+)\}-1]\right\}$$
$$\equiv f(u) .$$

The distribution function of the limit random variable S^* is later given by VERAVERBEKE and TEUGELS (1975). Let $U(s, x) = \sum_{n=1}^{\infty} P\{S_n \leq x, N > n\} s^n$ and

$$(1.13) \qquad V(x) = 1 - e^{-\tau x} + \tau \int_{0+}^{x} U(\tfrac{1}{\gamma}, x-y) e^{-\tau y}\, dy.$$

Then under the same conditions, (1.7) - (1.10), it is shown in [30], p. 283, that

$$(1.14) \qquad \lim_{n\to\infty} P\{W_n \leq x | N > n\} = \exp\left[- \sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n > 0\} \right] \cdot V(x).$$

Acknowledgment: We close this introductory chapter by acknowledging a heavy debt to the two above-mentioned papers of IGLEHART (1974a, b) on conditioned limit theorems for random walks.

CHAPTER 2

LIMITING DIFFUSION FOR RANDOM WALKS WITH DRIFT CONDITIONED TO STAY POSITIVE

1. Introduction and Probability Space

Let $\{\xi_k: k \geq 1\}$ be a sequence of independent, identically distributed (i.i.d.) random variables with $E\{\xi_1\} = \mu_1$ and $E\{\xi_1 - \mu_1\}^2 = \sigma^2$, $0 < \sigma^2 < \infty$. Form the random walk $\{S_n: n \geq 0\}$ by setting $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$. Let N_x be the hitting time of the set $(-\infty, x]$ by the random walk:

$$N_x = \inf\{n > 0: S_n \leq x\} ,$$

where the infimum of the empty set is taken to be $+\infty$. Set $N = N_0$.

Throughout this chapter, we shall assume that the distribution of ξ_1 satisfies the following conditions

(2.1) $\mu_1 \neq 0$;

(2.2) $\theta(s) = E\exp(s\xi_1)$ converges either for real $s \in [0,a)$, if $\mu_1 < 0$, or for real $s \in (-a,0]$, if $\mu_1 > 0$, for some $a > 0$;

(2.3) $\theta(s)$ attains its infimum either at a point τ, if $\mu_1 < 0$, or at a point $-\tau$, if $\mu_1 > 0$, where $0 < \tau < a$, and $\theta(\pm\tau) \equiv \gamma < 1$ and $\theta'(\pm\tau) = 0$;

and

(2.4) ξ_1 is nonlattice .

Define the random function X_n by

$$X_n(t) = \frac{S_{[nt]}}{\alpha n^{\frac{1}{2}}}, \qquad 0 \le t \le 1,$$

where $\alpha^2 = \theta''(\tau)/\gamma$, $0 < \alpha < \infty$, and $[x]$ is the greatest integer in x.

Our goal in this chapter is to prove that the finite-dimensional distributions (f.d.d.'s) of the random function X_n, conditioned on $n < N < \infty$, converge weakly to those of Brownian excursion, W_0^+.

To be more specific, we assume that $\{\xi_k: k \ge 1\}$ are the coordinate functions defined on the product space

$$(\Omega, \mathfrak{F}, P) = \bigtimes_{n=1}^{\infty} (R, \mathfrak{R}, \pi),$$

where $R = (-\infty, \infty)$, $\mathfrak{R}$ is the σ-field of Borel sets of R, and π is the common probability measure of the ξ_k's. If $\Lambda_n = \{n < N < \infty\}$, then we let $(\Lambda_n, \Lambda_n \cap \mathfrak{F}, P_n)$ be the trace of $(\Omega, \mathfrak{F}, P)$ on Λ_n, where $\Lambda_n \cap \mathfrak{F}$ $= \{\Lambda_n \cap F, F \in \mathfrak{F}\}$ and $P_n(A) = P(A)\ P(\Lambda_n)$ for $A \in \Lambda_n \cap \mathfrak{F}$.

Our result has an application to queueing theory, as well as the obvious interpretation as the fortune of a gambler or insurance company prior to ruin. If W_n is the waiting time of n^{th} customer in a general single server queue, then the ξ's in this application are differences of service and interarrival times. In this context N is the number of customers served in the first busy period. Observe that $W_n = S_n$ on $\{N > n\}$. Thus conditioning on $\{n < N < \infty\}$ will yield limit theorems for the waiting time process, given that the first busy period has not ended but will end eventually.

We close this section by noting a well known fact that $P\{N < \infty\} = 1$ when $\mu_1 < 0$ [see CHUNG (1968) p. 244]. In this case we can write $\{N > n\}$ instead of $\{n < N < \infty\}$.

2. Some Asymptotic Results

Conditions (2.1) through (2.4) imply that as $n \to \infty$,

(2.5) $$P\{S_n \geq 0\} \sim (2\pi n)^{-\frac{1}{2}} \gamma^n (\alpha\tau)^{-1}, \quad \text{if } \mu_1 < 0,$$

and

(2.6) $$P\{S_n \leq 0\} \sim (2\pi n)^{-\frac{1}{2}} \gamma^n (\alpha\tau)^{-1}, \quad \text{if } \mu_1 > 0.$$

These results are contained in BAHADUR and RAO (1960); also see IGLEHART (1974b).

Our analysis relies heavily on two lemmas, which appear in IGLEHART (1974b).

(2.7) LEMMA [IGLEHART (1974b), Lemma 2.1]. Let

$$\sum_{n=0}^{\infty} d_n \Delta^n = \exp\left\{\sum_{n=1}^{\infty} b_n \Delta^n\right\}$$

for $|\Delta| \leq 1$. If $b_n = O(n^{-\frac{3}{2}})$, then

$$d_n = O(n^{-\frac{3}{2}}) \quad \text{as } n \to \infty.$$

(2.8) LEMMA [IGLEHART (1974b), Lemma 2.2]. Let $c_n, d_n \geq 0$, $c_n \sim cn^{-\frac{1}{2}}$, $\sum_{n=0}^{\infty} d_n = d < \infty$, and $d_n = O(n^{-1})$. If $a_n = \sum_{j=0}^{n-1} c_{n-j} d_j$, then

$$a_n \sim cdn^{-\frac{1}{2}} \quad \text{as } n \to \infty.$$

Let $f_n = P\{N = n\}$ and $r_n = P\{n < N < \infty\}$. Our first result yields the asymptotic behavior of the sequence of $\{f_n: n \geq 1\}$ and $\{r_n: n \geq 1\}$.

(2.9) THEOREM. *If conditions (2.1) - (2.4) hold, then as* $n \to \infty$

$$f_n \sim \frac{\gamma^n}{n^{\frac{3}{2}}} \frac{A}{(2\pi)^{\frac{1}{2}}\alpha\tau} , \tag{2.10}$$

$$r_n \sim \frac{\gamma^n}{n^{\frac{3}{2}}} \frac{A}{(2\pi)^{\frac{1}{2}}\alpha\tau(\gamma^{-1} - 1)} , \tag{2.11}$$

where

$$A = \begin{cases} \exp\left[- \sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\}\right], & \text{if } \mu_1 > 0, \\ (\gamma^{-1} - 1) \exp\left[\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n > 0\}\right], & \text{if } \mu_1 < 0. \end{cases}$$

Proof. When $\mu_1 < 0$, the asymptotic relation (2.11) is proved in IGLEHART (1974b), Theorem 2.1. Because $f_n = r_{n-1} - r_n$, it follows immediately that given $\epsilon > 0$, there exists n_0 such that for all $n > n_0$

$$\frac{f}{(\gamma^{-1}-1)}\frac{\gamma^n}{n^{\frac{3}{2}}}\left[\gamma^{-1}\left(\frac{n}{n-1}\right)^{\frac{3}{2}}-1\right] + o(\epsilon) \le f_n \le \frac{f}{\gamma^{-1}-1}\frac{\gamma^n}{n^{\frac{3}{2}}}\left[\gamma^{-1}\left(\frac{n}{n-1}\right)^{\frac{3}{2}}-1\right] + o(\epsilon),$$

where $f = (2\pi)^{-\frac{1}{2}}(\alpha\tau)^{-1}A$. Since ϵ is arbitrary, we obtain (2.10) for $\mu_1 < 0$.

To obtain (2.10) and (2.11) for $\mu_1 > 0$, we use an identity from random walk theory [see CHUNG (1968), p. 256-258],

$$1 - \sum_{n=1}^{\infty} s^n \int_{\{N=n\}} \exp(itS_n)dP = \exp\left[- \sum_{n=1}^{\infty} \frac{s^n}{n} \int_{\{S_n \le 0\}} \exp\{itS_n\}dP\right]$$

for $|s| < 1$. Setting $t = 0$ yields

$$(2.12) \qquad 1 - \sum_{n=1}^{\infty} f_n s^n = \exp\left[-\sum_{n=1}^{\infty} \frac{s^n}{n} P\{S_n \le 0\}\right].$$

Differentiating both sides of (2.12) with respect to s yields

$$(2.13) \qquad \sum_{n=1}^{\infty} nf_n \gamma^{-n} s^n = \exp\left[-\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\} s^n\right] \sum_{n=1}^{\infty} \gamma^{-n} P\{S_n \le 0\} s^n,$$

where $|s| < 1$. Set $a_n = nf_n \gamma^{-n}$, $c_n = \gamma^{-n} P\{S_n \le 0\}$, and

$$\sum_{n=0}^{\infty} d_n s^n = \exp\left[-\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\} s^n\right].$$

Then from (2.13), we see that $a_n = \sum_{j=0}^{n-1} c_{n-j} d_j$. Now $c_n \sim (2\pi n)^{-\frac{1}{2}} (\alpha\tau)^{-1}$, because of (2.6). So $b_n \equiv n^{-1} c_n = O(n^{-\frac{3}{2}})$. Also,

$$\sum_{n=0}^{\infty} d_n = \exp\left[-\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\}\right] < \infty.$$

Hence we can apply Lemmas 2.7 and 2.8 to obtain (2.10).

Now for any $\epsilon > 0$, there exists n_0 such that for all $n \ge n_0$,

$$(2.14) \qquad (1 - \epsilon) f \frac{\gamma^n}{n^{\frac{3}{2}}} \le f_n \le (1 + \epsilon) f \frac{\gamma^n}{n^{\frac{3}{2}}} .$$

Hence

$$n^{\frac{3}{2}} \gamma^{-n} r_n \le (1 + \epsilon) f \sum_{k=n+1}^{\infty} \gamma^{-n+k} (\tfrac{n}{k})^{\frac{3}{2}}$$

$$\le (1 + \epsilon) f \sum_{k=1}^{\infty} \gamma^{k} .$$

Letting $n \to \infty$, we obtain

$$\text{(2.15)} \qquad \limsup_{n\to\infty} n^{\frac{3}{2}} \gamma^{-n} f_n \le (1 + \epsilon) f (\gamma^{-1} - 1)^{-1} .$$

Similarly, using the first inequality of (2.14), we have

$$\text{(2.16)} \qquad \liminf_{n\to\infty} n^{\frac{3}{2}} \gamma^{-n} f_n \ge (1 - \epsilon) f (\gamma^{-1} - 1)^{-1} .$$

Combining (2.15) and (2.16), we obtain (2.11), since ϵ is arbitrary.

Our problem is closely related to the asymptotic analysis of the distribution function of M_n, where $M_n = \max\{S_k: 0 \le k \le n\}$. This subject has been treated extensively by BOROVKOV (1965). The following two theorems are immediate results of BOROVKOV (1965), Theorem 7. Denote $M = \sup\{S_k: k \ge 0\}$.

(2.17) THEOREM. If conditions (2.1) - (2.4) hold, then for $x > 0$ as $n \to \infty$

$$\text{(2.18)} \qquad P\left\{\frac{M_n}{\alpha\sqrt{n}} \le x\right\} \sim \frac{\gamma^n}{n} \exp(x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1} \quad \text{for } \mu_1 > 0,$$

and

$$P\left\{\frac{M_n}{\alpha\sqrt{n}} \le x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} \le x\right\} \sim \frac{\gamma^n}{n} \exp(-x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1}, \tag{2.19}$$

for $\mu_1 < 0$.

<u>The limits in</u> (2.18) <u>and</u> (2.19) <u>are uniform for</u> x <u>in compact sets.</u>

(2.20) THEOREM. <u>If conditions</u> (2.1) - (2.4) <u>hold, then for</u> $x > 0$ <u>as</u> $n \to \infty$

$$P\left\{n < N_{-x\alpha\sqrt{n}} < \infty\right\} \sim \frac{\gamma^n}{n} \exp(-x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} \frac{A}{(\gamma^{-1} - 1)}, \tag{2.21}$$

for $\mu_1 > 0$,

and

$$P\left\{N_{-x\alpha\sqrt{n}} > n\right\} \sim \frac{\gamma^n}{n} \exp(x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} \frac{A}{(\gamma^{-1} - 1)}, \tag{2.22}$$

for $\mu_1 < 0$.

<u>The limits in</u> (2.21) <u>and</u> (2.22) <u>are uniform for</u> x <u>in compact sets.</u>

Recall that $R(x) = x \exp(-x^2/2)$, $x \ge 0$, is the density of the Raleigh distribution. Theorem 2.17 enables us to understand the asymptotic behavior of S_n, conditioned on $\{N > n\}$, as $n \to \infty$.

(2.23) THEOREM. <u>If conditions</u> (2.1) - (2.4) <u>hold, then for</u> $x > 0$ <u>as</u> $n \to \infty$

$$P\left\{\frac{S_n}{\alpha\sqrt{n}} \le x,\ N > n\right\} \sim \frac{\gamma^n}{n} \exp(x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau}, \quad \text{if } \mu_1 > 0, \tag{2.24}$$

and

$$(2.25) \qquad P\left\{\frac{S_n}{\alpha\sqrt{n}} > x,\ N > n\right\} \sim \frac{\gamma^n}{n} \exp(-x\alpha\tau\sqrt{n}) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau}, \qquad \text{if } \mu_1 < 0.$$

The limits in (2.24) and (2.25) are uniform for x in compact sets.

Proof. We shall prove (2.24) first. Recall that $S_n = W_n$ on $\{N > n\}$. So if we can show (2.24) with S_n replaced by W_n, the result will be established. A simple path decomposition yields

$$(2.26) \qquad P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x,\ N > n\right\} = P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x\right\} - \sum_{k=1}^{n} P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x,\ N = k\right\}.$$

Using the fact that N is an optional random variable and that $W_N = 0$, we can rewrite (2.26) as

$$(2.27) \qquad P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x,\ N > n\right\} = P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x\right\} - \sum_{k=1}^{n} f_k P\left\{\frac{W_{n-k}}{\alpha\sqrt{n}} \le x\right\}.$$

For $n \ge 1$ and $x \ge 0$, let

$$h(n,x) = n\gamma^{-n} \exp(-x\alpha\tau\sqrt{n})\, P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x\right\}.$$

For $\epsilon > 0$ and a fixed $0 < n_0 < [n(1-\epsilon)]$ we can write, because of (2.27),

$$(2.28) \qquad n\gamma^{-n} \exp(-x\alpha\tau\sqrt{n})\ P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x,\ N > n\right\}$$

$$= h(n,x) - \sum_{k=1}^{n-1}\left[\gamma^{-k} f_k \cdot h\left(n-k,\ x\sqrt{\frac{n}{n-k}}\right) \cdot \left(\frac{n}{n-k}\right)\right] - n\gamma^{-n} e^{-x\alpha\tau\sqrt{n}} f_n$$

$$= h(n,x) - \sum_{k=1}^{n_o} \quad \sum_{k=n_o+1}^{[n(1-\epsilon)]} \quad - \sum_{k=[n(1-\epsilon)]+1}^{n-1} - n\gamma^{-n} e^{-x\alpha\tau\sqrt{n}} f_n$$

$$= h(n,x) - I_n - J_n - K_n - n\gamma^{-n} e^{-x\alpha\tau\sqrt{n}} f_n .$$

Now

$$(2.29) \qquad \lim_{n\to\infty} h(n,x) = \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1}$$

and

$$(2.30) \qquad \lim_{n\to\infty} I_n = \left(\sum_{k=1}^{n_o} \gamma^{-k} f_k\right) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1}$$

because of (2.18). Since $R(x) = xe^{-x^2/2}$ is bounded, relation (2.18) also implies that $h(n,x) \le G$ $(0 < G < \infty)$ for all $n \ge 1$ and $x \ge 0$. Using this fact, we can conclude that

$$(2.31) \qquad J_n \le G \sum_{k=n_o+1}^{[n(1-\epsilon)]} \gamma^{-k} f_k \left(\frac{n}{n-k}\right) \le G\, \epsilon^{-1} \sum_{k=n_o+1}^{\infty} \gamma^{-k} f_k .$$

Because of (2.10), $\gamma^{-n} f_n \le Hn^{-\frac{3}{2}}$ $(0 < H < \infty)$ for $n \ge 1$. Thus we have

$$(2.32) \qquad K_n \le GH \sum_{k=[n(1-\epsilon)]+1}^{n-1} k^{-\frac{3}{2}} \left(\frac{n}{n-k}\right)$$

$$\le GH \frac{n}{[n(1-\epsilon)]^{\frac{3}{2}}} \sum_{k=[n(1-\epsilon)]+1}^{n-1} (n-k)^{-1}$$

$$= GH(1-\epsilon)^{-\frac{3}{2}} n^{-\frac{1}{2}} \sum_{k=1}^{[n\epsilon]} \frac{1}{k}$$

$$\le GH(1-\epsilon)^{-\frac{3}{2}} n^{-\frac{1}{2}}(1 + \ln \epsilon + \ln n) \ .$$

Finally,

$$(2.33) \qquad \lim_{n\to\infty} n\gamma^{-n} e^{-x\alpha\tau\sqrt{n}} f_n = 0$$

because of (2.10) and τ being positive. Using (2.28) - (2.33), and selecting n_0 sufficiently large we can make

$$\left| n\gamma^{-n} e^{-x\alpha\tau\sqrt{n}} P\left\{\frac{W_n}{\alpha\sqrt{n}} \le x,\ N > n\right\} - \left(1 - \sum_{k=1}^{\infty} \gamma^{-k} f_k\right) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1} \right|$$

arbitrarily small for large n. Since

$$A \equiv \exp\left[-\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\}\right] = 1 - \sum_{k=1}^{\infty} \gamma^{-k} f_k$$

by virtue of (2.12), this completes the proof of (2.24). Because (2.24) is a direct consequence of (2.18), that the limit in (2.24) is uniform for x in compact sets follows immediately from Theorem 2.17.

To establish (2.25) for the case $\mu_1 < 0$, we can write from (2.26)

$$(2.34)\qquad P\left\{\frac{W_n}{\alpha\sqrt{n}} > x,\ N>n\right\} = P\left\{\frac{W_n}{\alpha\sqrt{n}} > x\right\} - \sum_{k=1}^{n} f_k P\left\{\frac{W_{n-k}}{\alpha\sqrt{n}} > x\right\}$$

$$= \left[P\left\{\frac{M_n}{\alpha\sqrt{n}} > x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\}\right]$$

$$- \sum_{k=1}^{n} f_k \cdot \left[P\left\{\frac{M_{n-k}}{\alpha\sqrt{n}} > x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\}\right]$$

$$+ P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\} P\{N > n\}$$

$$= - \left[P\left\{\frac{M_n}{\alpha\sqrt{n}} \le x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} \le x\right\}\right]$$

$$+ \sum_{k=1}^{n} f_k \left[P\left\{\frac{M_{n-k}}{\alpha\sqrt{n}} \le x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} \le x\right\}\right] + P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\} P\{N>n\}.$$

For $x \ge 0$ and $n \ge 1$, let

$$g(n,x) = n\gamma^{-n} \exp(x\alpha\tau\sqrt{n}) \left[P\left\{\frac{M_n}{\alpha\sqrt{n}} \le x\right\} - P\left\{\frac{M}{\alpha\sqrt{n}} \le x\right\}\right].$$

Then from (2.19)

$$(2.35)\qquad \lim_{n\to\infty} g(n,x) = \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1}.$$

Multiplying both sides of (2.34) by $n\gamma^{-n} \exp(x\alpha\tau\sqrt{n})$, we have

$$(2.36)\qquad n\gamma^{-n}\exp(x\alpha\tau\sqrt{n})\, P\left\{\frac{W_n}{\alpha\sqrt{n}} > x,\ N > n\right\}$$

$$= -g(n,x) + \sum_{k=1}^{n-1} \gamma^{-k} f_k \cdot g(n-k,\ x\sqrt{n/n-k}) \cdot \left(\frac{n}{n-k}\right)$$

$$- n\gamma^{-n}\exp(x\alpha\tau\sqrt{n})\, P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\} P\{N > n\}.$$

To estimate the last term appearing in (2.36), let κ be the nontrivial solution of $\theta(\delta) = 1$. Then for $x > 0$ as $n \to \infty$

$$(2.37)\qquad P\{M > x\alpha\sqrt{n}\} \sim D\exp(-x\alpha\kappa\sqrt{n})$$

where D is a positive constant whose precise value need not concern us; see IGLEHART (1972), Lemma 1 for this result. Hence,

$$(2.38)\qquad \lim_{n\to\infty} n\gamma^{-n}\exp(x\alpha\tau\sqrt{n})\, P\left\{\frac{M}{\alpha\sqrt{n}} > x\right\} P\{N > n\} = 0$$

because of (2.11), (2.37), and $\kappa > \tau > 0$. The summation term in (2.36) can be broken into three parts and be taken care of in the same manner as we have done in (2.28) - (2.32). And we can conclude this time that

$$(2.39)\qquad \lim_{n\to\infty} n\gamma^{-n}\exp(x\alpha\tau\sqrt{n})\, P\left\{\frac{W_n}{\alpha\sqrt{n}} > x,\ N > n\right\}$$

$$= \left(-1 + \sum_{k=1}^{\infty} \gamma^{-k} f_k\right) \frac{R(x)}{\pi^{\frac{1}{2}}\alpha\tau} A^{-1}.$$

The power series $\sum_{n=1}^{\infty} \frac{\delta^n}{n} P\{S_n > 0\}$ has $\gamma^{-1} > 1$ as radius of convergence and $\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n > 0\}$ is finite; see VERAVERBEKE and TEUGELS (1975), p. 281, for a discussion of this result. Therefore, from (2.12) we have

$$(2.40) \qquad -1 + \sum_{n=1}^{\infty} \gamma^{-n} f_n = (\gamma^{-1} - 1) \exp\left[\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n > 0\}\right] .$$

Combining (2.39) and (2.40) we obtain (2.25). Uniform convergence is a direct consequence of Theorem 2.17. This completes our proof.

For $x \geq 0$ and $n \geq 1$, we define

$$(2.41) \qquad F_n(x) = (n\pi)^{\frac{1}{2}} \gamma^{-n} \int_{(o,x]} P\left\{\frac{S_n}{\alpha\sqrt{n}} \in dy,\ N > n\right\} \exp(\mp y\alpha\tau\sqrt{n}) ,$$

where the "-" sign is taken for $\mu_1 > 0$, and the "+" sign is taken for $\mu_1 < 0$.

(2.42) COROLLARY. <u>For all</u> $x \geq 0$,

$$\lim_{n\to\infty} F_n(x) = 1 - \exp\left(-\frac{x^2}{2}\right),$$

<u>the Raleigh distribution</u>.

<u>Proof</u>. Integration by parts yields

$$F_n(x) = (n\pi)^{\frac{1}{2}} \gamma^{-n} P\left\{\frac{S_n}{\alpha\sqrt{n}} \leq x,\ N > n\right\} \exp(-x\alpha\tau\sqrt{n})$$

$$+ \pi^{\frac{1}{2}} \alpha\tau\, n\gamma^{-n} \int_{(o,x]} P\left\{\frac{S_n}{\alpha\sqrt{n}} \leq y,\ N > n\right\} \exp(-y\alpha\tau\sqrt{n})\, dy ,$$

if $\mu_1 > 0$. Hence, by Theorem 2.23, we have

$$F_n(x) = o(1) + \int_{(o,x]} y \exp\left(-\frac{y^2}{2}\right) dy .$$

If $\mu_1 < 0$, integration by parts will yield

$$F_n(x) = (n\pi)^{\frac{1}{2}}\gamma^{-n}\left[P\{N > n\} - P\left\{\frac{W_n}{\alpha\sqrt{n}} > x\ N > n\right\} \exp(x\alpha\tau\sqrt{n})\right]$$

$$+ \pi^{\frac{1}{2}}\alpha\tau n\gamma^{-n} \int_{(o,x]} P\left\{\frac{W_n}{\alpha\sqrt{n}} > y,\ N > n\right\} \exp(y\alpha\tau\sqrt{n})dy$$

$$= o(1) + \int_{(o,x]} y \exp\left(-\frac{y^2}{2}\right) dy$$

because of (2.11) and Theorem 2.23 .

The next result is needed in proving the convergence of high dimensional distributions.

(2.43) THEOREM. Assume conditions (2.1) – (2.4) are satisfied. For $x, y > 0$ as $n \to \infty$

(2.44) $$P\left\{\frac{S_n}{\alpha\sqrt{n}} \le x - y,\ N_{-y\alpha\sqrt{n}} > n\right\}$$

$$\sim \frac{\gamma^n}{\alpha\tau n^{\frac{3}{2}}} \exp\{(x - y)\alpha\tau\sqrt{n}\}\ g(1, y, x),$$

where
$$g(t,y,x) = (2\pi t)^{-\frac{1}{2}}\left\{\exp\left[-\frac{(y-x)^2}{2}\right] - \exp\left[-\frac{(y+x)^2}{2}\right]\right\}.$$

The convergence in (2.44) is uniform for x in compact sets.

Proof. Let $M_n^- = \max\{-S_k : 0 \le k \le n\}$ and $M^- = \sup\{-S_k : k \ge 0\}$. If we set

$$a_n(y, x) = P\left\{\frac{S_n}{\alpha\sqrt{n}} \le x - y,\ N_{-y\alpha\sqrt{n}} > n\right\}.$$

then

$$a_n(y, x) = P\left\{\frac{M_n^-}{\alpha\sqrt{n}} < y,\ \frac{(-S_n)}{\alpha\sqrt{n}} \ge y - x\right\}.$$

Using the simple identity $P(A \cap \bar{B}) = P(A) + P(\bar{A} \cap B) - P(B)$, we can write

$$(2.45)\qquad a_n(y,x) = P\left\{\frac{M_n^-}{\alpha\sqrt{n}} < y\right\} + P\left\{\frac{M_n^-}{\alpha\sqrt{n}} \ge y,\ \frac{(-S_n)}{\alpha\sqrt{n}} < y-x\right\} - P\left\{\frac{(-S_n)}{\alpha\sqrt{n}} < y-x\right\}$$

$$= I_n + J_n - K_n .$$

But we have

$$(2.46)\qquad I_n = P\left\{\frac{M^-}{\alpha\sqrt{n}} \ge y\right\} + o(1)\,\frac{\gamma^n}{n^{\frac{1}{2}}}\exp(-y\alpha\tau\sqrt{n}) ,$$

$$(2.47)\qquad J_n = \left\{P\left(\frac{M^-}{\alpha\sqrt{n}} < y\right) - \frac{\gamma^n}{(2\pi n)^{\frac{1}{2}}\alpha\tau}\exp\left[-(y-x)\alpha\tau\sqrt{n} - \frac{(y+x)^2}{2}\right]\right\}[1 + o(1)] ,$$

$$(2.48)\qquad K_n = \left\{i - \frac{\gamma^n}{(2\pi n)^{\frac{1}{2}}\alpha\tau}\exp\left[-(y-x)\alpha\tau\sqrt{n} - \frac{(y-x)^2}{2}\right]\right\}[1+ o(1)] ;$$

see BOROVKOV (1965), Theorem 7, for the above results. Combining (2.45) to (2.48), we obtain (2.44). Uniform convergence also follows directly from BOROVKOV (1965), Theorem 7.

3. The Main Limit Theorems

Having obtained various asymptotic results in the last section, we are now able to establish the main result of this chapter. We shall show the convergence of one-dimensional distribution of $(X_n \mid n < N < \infty)$, then extend the result to higher dimensions. First, we state a standard result in the theory of weak convergence used by IGLEHART (1974a), Lemma (2.18). Alternatively, see BELKIN (1972), p. 54, or BILLINGSLEY (1968), Theorem 5.5 .

(2.49) LEMMA. *Let* $\{\mu_n : n \geq 1\}$ *be a sequence of finite measures on* $\mathfrak{R}$, *the Borel sets of* $R = (-\infty, +\infty)$. *Suppose* $\mu_n \Rightarrow \mu$, *finite. If* $\{f_n : n \geq 1\}$ *is a sequence of uniformly bounded, Borel measurable functions converging uniformly on compact sets to an everywhere bounded continuous limit* f, *then*

$$\lim_{n\to\infty} \int_B f_n(x)\mu_n(dx) = \int_B f(x)\mu(dx)$$

for $B \in \mathfrak{R}$ *provided* $\mu(\partial B) = 0$.

The following two results are the main objective of this chapter.

(2.50) THEOREM. *If conditions* (2.1) - (2.4) *are satisfied, then for all* $x \geq 0$ *and* $0 \leq t \leq 1$ *as* $n \to \infty$

$$(X_n(t) \mid n < N < \infty) \Rightarrow W_0^+(t) \ . \tag{2.51}$$

Proof. The claim (2.51) is trivial for $t = 0$. Next, consider $t = 1$, and take $0 < a < b$. We want to estimate

$$P\left\{a < \frac{S_n}{\alpha\sqrt{n}} \leq b \mid n < N < \infty\right\}, \tag{2.52}$$

which is equal to

(2.53) $$r_n^{-1} P\left\{a < \frac{S_n}{\alpha\sqrt{n}} \le b, \quad n < N < \infty\right\}$$

$$= r_n^{-1} \int_{(a,b]} P\left\{\frac{S_n}{\alpha\sqrt{n}} \in dx, \ N > n\right\} P^{x\alpha\sqrt{n}} \{N < \infty\} .$$

If $\mu_1 > 0$, then (2.52) is furthermore equal to

(2.54) $$\frac{\gamma^n}{(\pi n^3)^{\frac{1}{2}} r_n} \int_{(a,b]} (n\pi)^{\frac{1}{2}}\gamma^{-n} \cdot \exp(-x\alpha\tau\sqrt{n}) \cdot P\left\{\frac{S_n}{\alpha\sqrt{n}} \in dx, \ N > n\right\}$$

$$\cdot \ n \exp(x\alpha\tau\sqrt{n}) \ P\{N_{-x\alpha\sqrt{n}} < \infty\}$$

$$= \frac{\gamma^n}{(\pi n^3)^{\frac{1}{2}} r_n} \int_{(a,b]} F_n(dx) \cdot n \exp(x\alpha\tau\sqrt{n}) \ P\{N_{-x\alpha\sqrt{n}} < \infty\} .$$

Now

(2.55) $$n \exp(x\alpha\tau\sqrt{n}) \ P\{N_{-x\alpha\sqrt{n}} < \infty\} = n \exp(x\alpha\tau\sqrt{n}) \ P\{M^- > x\alpha\sqrt{n}\}$$

$$\sim D \cdot n \exp[-x\alpha(\kappa-\tau)\sqrt{n}\ ].$$

Combining (2.11), (2.52) - (2.55), and using Corollary 2.42 and Lemma 2.49, we conclude that for $\mu_1 > 0$

(2.56) $$\lim_{n\to\infty} P\left\{a < \frac{S_n}{\alpha\sqrt{n}} \le b \ \middle| \ n < N < \infty\right\} = 0.$$

If $\mu_1 < 0$, then (2.52) is less than or equal to

$$r_n^{-1} P\left\{\frac{S_n}{\alpha\sqrt{n}} > a, \quad N > n\right\}$$

which is asymptotically as $n \to \infty$ equal to

$$\left(\frac{2}{n}\right)^{\frac{1}{2}} \exp(-x\alpha\tau\sqrt{n}) \ R(x) \cdot (r^{-1} - 1)A^{-1}$$

because of (2.11) and (2.25). This shows that (2.56) also holds when $\mu_1 < 0$.

Next, consider $0 < t < 1$ and take $0 < a < b$. For all choices of a, b, and t it will suffice to show that

$$(2.57) \qquad \lim_{n\to\infty} P\left\{a < \frac{S_{[nt]}}{\alpha\sqrt{n}} \le b \mid n < N < \infty\right\}$$

$$= P\{a < W_0^+(t) \le b\} .$$

If $\mu_1 > 0$,

$$(2.58) \qquad P\left\{a < \frac{S_{[nt]}}{\alpha\sqrt{n}} \le b \mid n < N < \infty\right\}$$

is equal to $\quad r_n^{-1} \int_{(a,b]} P\left\{\frac{S_{[nt]}}{\alpha\sqrt{n}} \in dx,\ N > [nt]\right\} P^{x\alpha\sqrt{n}} \{n-[nt] < N < \infty\}$

$$= \frac{\gamma^n}{[\pi t(1-t)n^3]^{\frac{1}{2}} r_n} \int_{(a,b]} (nt\pi)^{\frac{1}{2}} \gamma^{-nt} \exp(-x\alpha\tau\sqrt{n})$$

$$\cdot\ P\left\{\frac{S_{[nt]}}{\alpha\sqrt{nt}} \in \frac{dx}{\sqrt{t}},\quad N > [nt]\right\}$$

$$\cdot\ n(1-t)\gamma^{-n(1-t)} \exp(x\alpha\tau\sqrt{n}) P\{n-[nt] < N_{-x\alpha\sqrt{n}} < \infty\} .$$

Appealing to Corollary 2.42, we see that (2.57) equals further to

$$(2.59) \quad \frac{\gamma^n}{n^{\frac{3}{2}} r_n} \cdot \frac{1}{(\pi t)^{\frac{1}{2}}(1-t)} \int_{(a,b]} F_{[nt]}\left(\frac{dx}{\sqrt{t}}\right) \cdot \frac{n(1-t)}{\gamma^{n(1-t)}} \exp(x\alpha\tau\sqrt{n}) P\{n-[nt] < N_{-x\alpha\sqrt{n}} < \infty\} .$$

Because of (2.21),

$$(2.60) \quad P\{n-[nt] < N_{-x\alpha\sqrt{n}} < \infty\} \sim \frac{\gamma^{n(1-t)}}{n(1-t)} \exp(-x\alpha\tau\sqrt{n}) \frac{R(x)\left(\frac{x}{\sqrt{1-t}}\right)}{\pi^{\frac{1}{2}}\alpha\tau} \cdot \frac{A}{(\gamma^{-1}-1)} .$$

Combining (2.11), (2.41), (2.59), (2.60) and using Lemma 2.49, we obtain

$$\lim_{n\to\infty} P\left\{a < \frac{S_{[nt]}}{\alpha\sqrt{n}} \le b \mid n < N < \infty\right\}$$

$$= \frac{(2\pi)^{\frac{1}{2}}\alpha\tau(\gamma^{-1}-1)}{A} \cdot \frac{1}{(\pi t)^{\frac{1}{2}}(1-t)} \int_{(a,b]} \frac{x}{\sqrt{t}} e^{-\frac{x^2}{2t}} \cdot \frac{\frac{x}{\sqrt{1-t}}\exp\left(-\frac{x^2}{2(1-t)}\right)}{\pi^{\frac{1}{2}}\alpha\tau(\gamma^{-1}-1)} \cdot \frac{A}{(\gamma^{-1}-1)} \frac{dx}{\sqrt{t}}$$

$$= \int_{(a,b]} \frac{2x^2\exp\left(-\frac{x^2}{2t(1-t)}\right)dx}{[2\pi t^3(1-t)^3]^{\frac{1}{2}}} = \int_{(a,b]} P_0^+(0,0,t,x)\, dx ,$$

which proves (2.57) for $\mu_1 > 0$.

Similarly, if $\mu_1 < 0$, the term (2.58) becomes

$$P\left\{a < \frac{S_{[nt]}}{\alpha\sqrt{n}} \le b \mid N > n\right\} ,$$

which is equal to

$$(2.61) \quad \frac{\gamma^n}{n^{\frac{3}{2}} r_n} \frac{1}{(\pi t)^{\frac{1}{2}}(1-t)} \int_{(a,b]} F_{[nt]}\left(\frac{dx}{\sqrt{t}}\right) \frac{n(1-t)}{\gamma^{n(1-t)}} \exp(-x\alpha\tau\sqrt{n}) P\left\{N_{-x\alpha\sqrt{n}} > n-[nt]\right\}.$$

Using the same argument, we see (2.61) converge as $n \to \infty$ to

$$\int_{(a,b]} P_0^+(0,\ 0,\ t,\ x)dx .$$

This completes the proof.

Finally, we can plunge into the proof of convergence of the f.d.d.'s.

(2.62) THEOREM. _If conditions_ (2.1) - (2.4) _hold, then for_ $k > 1$ _and_ $0 \le t_1 < t_2 < \dots < t_k \le 1$ _as_ $n \to \infty$.

$$(2.63) \quad (X_n(t_1),\ \dots,\ X_n(t_k) \mid n < N < \infty) \Rightarrow (W_0^+(t_1),\ \dots,\ W_0^+(t_k)) .$$

Proof. Theorem (2.50) takes care of the case $k = 1$. Now suppose (2.63) is true for $k = m-1$, we show next that it can be extended to $k = m$. We begin by writing

(2.64)
$$P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ldots, \frac{S_{[nt_m]}}{\alpha\sqrt{n}} \le x_m \mid n < N < \infty\right\}$$

$$= r_n^{-1} \int_{0+}^{x_{m-1}} \int_{0+}^{x_m} P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ldots, \frac{S_{[nt_{m-2}]}}{\alpha\sqrt{n}} \le x_{m-2},\right.$$

$$\left.\frac{S_{[nt_{m-1}]}}{\alpha\sqrt{n}} \in dy_{m-1}, \frac{S_{[nt_m]}}{\alpha\sqrt{n}} \in dy_m; \quad n < N < \infty\right\}$$

$$= r_n^{-1} \int_{0+}^{x_{m-1}} \int_{0+}^{x_m} P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ldots, \frac{S_{[nt_{m-1}]}}{\alpha\sqrt{n}} \in dy_{m-1}, N > [nt_{m-1}]\right\}$$

$$\cdot P^{y_{m-1}\alpha\sqrt{n}}\left\{\frac{S_{[nt_m]-[nt_{m-1}]}}{\alpha\sqrt{n}} \in dy_m, \min_{0 \le k \le [nt_m]-[nt_{m-1}]} \left(\frac{S_k}{\alpha\sqrt{n}}\right) > 0\right\}$$

$$\cdot P^{y_m\alpha\sqrt{n}} \{n - [nt_m] < N < \infty\}$$

$$= \int_{0+}^{x_{m-1}} \int_{0+}^{x_m} P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ldots, \frac{S_{[nt_{m-1}]}}{\alpha\sqrt{n}} \in dy_{m-1} \mid n < N < \infty\right\}$$

$$\cdot P\left\{\frac{S_{[nt_m]-[nt_{m-1}]}}{\alpha\sqrt{n}} \in dy_m, N_{-y_{m-1}\alpha\sqrt{n}} > [nt_m] - [nt_{m-1}]\right\}$$

$$\cdot \frac{P\left\{n-[nt_m] < N_{-y_m\alpha\sqrt{n}} < \infty\right\}}{P\left\{n-[nt_{m-1}] < N_{-y_{m-1}\alpha\sqrt{n}} < \infty\right\}} .$$

From (2.60) we see that the quotient term in the last product term is asymptotically equal to

$$(2.65)\qquad \left(\frac{1-t_{m-1}}{1-t_m}\right)^{\frac{3}{2}} \frac{y_m \exp\left[-\frac{y_m^2}{2(1-t_m)}\right]}{y_{m-1}\exp\left[-\frac{y_{m-1}^2}{2(1-t_{m-1})}\right]} \gamma^{-n(t_m-t_{m-1})} \exp\left\{\mp(y_m-y_{m-1})\alpha\tau\sqrt{n}\right\} .$$

Let

$$G_n(y_m) = \int_{0+}^{y_m} P\left\{\frac{S_{[nt_m]-[nt_{m-1}]}}{\alpha\sqrt{n}} \in dx,\ N_{-y_{m-1}\alpha\sqrt{n}} > [nt_m]-[nt_{m-1}]\right\}$$

$$\gamma^{[nt_m]-[nt_{m-1}]} \exp\left\{\pm (x-y_{m-1})\alpha\tau\sqrt{n}\right\} .$$

Integration by parts yields

$$(2.66)\qquad \lim_{n\to\infty} G_n(y_m) = \int_0^{y_m} g(t_m - t_{m-1}, y_{m-1}, x)\,dx,$$

because of Theorem (2.43). Finally, by our induction assumption

$$(2.67)\qquad \lim P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ldots, \frac{S_{[nt_{m-1}]}}{\alpha\sqrt{n}} \le x_{m-1} \,\middle|\, n < N < \infty\right\}$$

$$= \int_0^{x_1}\cdots\int_0^{x_{m-1}} p_0^+(0,0,t_1,y_1)p_0^+(t_1,y_1,t_2,y_2)\cdots p_0^+(t_{m-2},y_{m-2},t_{m-1},y_{m-1})$$

$$dy_{m-1}\cdots dy_1 .$$

Combining (2.64) - (2.67) and using Lemma (2.49) twice, we obtain

$$\lim P\left\{\frac{S_{[nt_1]}}{\alpha\sqrt{n}} \le x_1, \ \ldots, \ \frac{S_{[nt_m]}}{\alpha\sqrt{n}} \le x_m \ \middle|\ n < N < \infty\right\}$$

$$= \int_0^{x_{m-1}} \int_0^{x_m} \int_0^{x_1} \cdots \int_0^{x_{m-2}} p_0^+(0,0,t_1,y_1)p_0^+(t_1,y_1,t_2,y_2)\cdots p_0^+(t_{m-2},y_{m-2},t_{m-1},y_{m-1})$$

$$\cdot \left(\frac{1-t_{m-1}}{1-t_m}\right)^{\frac{3}{2}} \cdot \frac{y_m \exp\left(-\frac{y_m^2}{2(1-t_m)}\right)}{y_{m-1}\exp\left(-\frac{y_{m-1}^2}{2(1-t_{m-1})}\right)} g(t_m - t_{m-1}, y_{m-1}, y_m)$$

$$dy_{m-1}dy_{m-2} \cdots dy_1 \cdot dy_m .$$

Relation (2.63) follows immediately, which completes the proof.

CHAPTER 3

CONDITIONED LIMIT THEOREMS FOR RANDOM WALKS WITH POSITIVE DRIFT

1. Introduction

Let $\{X_k: k \geq 1\}$ be a sequence of independent random variables, identically distributed with common distribution $F(x)$ on $(-\infty, +\infty)$. Let $E\{X_1\} = \mu_1$ and $E\{X_1 - \mu_1\}^2 = \sigma^2$, $0 < \sigma^2 < \infty$. Form the random walk $\{S_n: n \geq 0\}$ by setting $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Next let N_x be the hitting time of the set $(-\infty, x]$ by the random walk

$$N_x = \inf\{n > 0: S_n \leq x\} .$$

where the infimum of the empty set is taken to be $+\infty$. Set $N = N_0$.

When $\mu_1 > 0$, the random walk is drifting to $+\infty$ and asymptotically the conditioning $N > n$ plays no role: the random walk does not feel the barrier at the origin. In fact, we have

$$\lim_{n\to\infty} P\left\{\frac{S_n}{\alpha\sqrt{n}} \leq x \mid N > n\right\} = 0 , \qquad x > 0$$

because of Theorem 2.23 and $P\{N = \infty\} > 0$ [see CHUNG (1968), proof of Theorem 8.4.4]. In this case it is more meaningful to examine, instead, the limit distribution of S_n, conditioned on $(n < N < \infty)$. From Theorem 2.50 we have, for $0 \leq t \leq 1$,

$$\lim_{n\to\infty} P\left\{\frac{S_{[nt]}}{\alpha\sqrt{n}} \leq x \mid n < N < \infty\right\} = P\{W_0^+(t) \leq x\} .$$

But $P\{W_0^+(1) \le x\} = 1$ and this result does not tell us the limiting behavior of S_n, because normalization by $n^{\frac{1}{2}}$ kills it. It suggests that we need not normalize S_n by $n^{\frac{1}{2}}$ to get convergence to a non-degenerate limit.

The main result in Section 2 is to show $(S_n \mid n < N < \infty)$ converges to a non-degenerate random variable $S^\#$, and to identify the distribution function of $S^\#$. As we have pointed out before, this problem is closely related to the asymptotic analysis of the distribution function of M_n, where $M_n = \max\{S_k: 0 \le k \le n\}$. We shall get the asymptotic result of $P\{M_n \le x\}$ in Section 3.

2. Limit Theorem For $(S_n \mid n < N < \infty)$

Throughout this chapter, we shall assume that the distribution of X_1 satisfies the following conditions:

(3.1) $0 < \mu_1 \le \infty$;

(3.2) $\theta(s) = E \exp(sX_1)$ converges for real $s \in (-a,0]$, for some $a > 0$;

(3.3) $\theta(s)$ attains its infimum at a point $-\tau$, $0 < \tau < a$, where $\theta(\tau) \equiv \gamma < 1$, and $\theta'(\tau) = 0$;

and

(3.4) if X_1 is lattice, then $P\{X_1 = 0\} > 0$.

The proof of the main result in this section requires us to introduce the so-called associated random variable to X_1; see FELLER (1971), p. 406, for a discussion of this concept.

The distribution function $\hat{F}(x)$ of the associated random variable $\hat{X}_1$ is given by

$$\hat{F}(x) = \int_{-\infty}^{x} \exp(-\kappa y)\, F(dy) .$$

Let $\{\hat{X}_2, \hat{X}_3, \ldots\}$ be a sequence of i.i.d. copies of $\hat{X}_1$. Form the associated random walk $\{\hat{S}_n : n \geq 0\}$ by setting $\hat{S}_0 = 0$ and $\hat{S}_n = \hat{X}_1 + \cdots + \hat{X}_n$, $n \geq 1$. Also, define the stopping time $\hat{N}$ accordingly. If $\hat{\theta}(s) = E \exp(s\hat{X}_1)$ is the moment generating function of $\hat{X}_1$, then $\hat{\theta}(s) = \theta(s - \kappa)$. Note that $\theta'(-\kappa) < 0$ implies that the associated random walk $\{\hat{S}_n : n \geq 0\}$ has a negative drift. Furthermore, the distribution function $\hat{F}$ of $\hat{X}_1$ satisfies conditions (1.7) - (1.10). Hence, we have from (1.14)

$$\lim_{n\to\infty} P\{\hat{S}_n \leq x \mid \hat{N} > n\} = c \cdot \hat{V}(x) , \tag{3.5}$$

where c is a constant and $\hat{V}(x)$ is a known function solely depending on x; see Ch. 1, Sec. 2, for this result.

To get the limit result for $(S_n \mid n < N < \infty)$, we first estimate $P\{S_n \leq x, n < N < \infty\}$ for $x > 0$.

$$P\{S_n \leq x,\ n < N < \infty\} = \int_{(o,x]} P\{S_n \in dy,\ N > n\} P^y\{N < \infty\} \tag{3.6}$$

$$= \int_{\Delta_n}\!\cdots\!\int I_{(0,x]}(y_1 + \cdots + y_n) F(dy_1) \cdots F(dy_n)\ P\{M^- > y_1 + \cdots + y_n\}$$

where $\Delta_n = \{(y_1, \ldots, y_n) : \sum_{i=1}^{k} y_i > 0,\ 1 \leq k \leq n\}$. Exploiting the concept of associated random variable to $\hat{X}_1$, we can write (3.6) as

$$\int_{\Delta_n}\cdots\int I_{(o,x]}(y_1 + \cdots + y_n)\, P\{M^- > y_1+\cdots+y_n\} \exp[\kappa(y_1+\cdots+y_n)]\hat{F}(dy_1)\ldots\hat{F}(dy_n)$$

$$= \int_{(o,x]} P\{M^- > y\} e^{\kappa y}\, P\{\hat{S}_n \in dy,\ \hat{N} > n\} .$$

Hence

$$P\{S_n \le x | n < N < \infty\} = \frac{P\{S_n \le x,\ n < N < \infty\}}{P\{n < N < \infty\}}$$

$$= \frac{\int_{(o,x]} P\{M^- > y\} e^{\kappa y} P\{\hat{S}_n \in dy | \hat{N} > n\}}{\int_{(o,\infty]} P\{M^- > y\} e^{\kappa y} P\{\hat{S}_n \in dy | \hat{N} > n\}} .$$

It follows from (3.6) and Lemma 2.49 that

$$(3.7)\qquad \lim_{n\to\infty} P\{S_n \le x | n < N < \infty\} = \frac{\int_{(o,x]} P\{M^- > y\} e^{\kappa y}\, \hat{V}(dy)}{\int_{(o,\infty]} P\{M^- > y\} e^{\kappa y} \hat{V}(dy)} \equiv H(x) .$$

It is easy to see that $H(x)$ is nondecreasing and right continuous with $H(0) = 0$, $H(+\infty) = 1$. Thus $H(x)$ is a distribution function.

Hence, we arrive at our main result in this section:

(3.8) THEOREM. If conditions (3.1) - (3.4) are satisfied, then as $n \to \infty$

$$(3.9)\qquad (S_n \mid n < N < \infty) \Rightarrow S^{\#} ,$$

where $S^{\#}$ is a non-degenerate random variable.

One application of this result is in terms of the waiting time of the n^{th} customer, W_n, in a general single server queue with traffic intensity $\rho > 1$. In this case, $W_n = S_n$ on the set $\{N > n\}$ and N is the number of customers served in the first busy period. Thus $S^{\#}$ can be thought of as the limiting waiting time given that the first busy period has not ended yet but will eventually terminate.

For illustration, we take as example $X_1 = v - u$, where $v(u)$ has an exponential distribution with parameter $\mu(\lambda)$. This corresponds to M/M/1 queue. If $\rho = \lambda/\mu$, we need to assume $\rho > 1$ in order to insure $Ex_1 \equiv \mu_1 > 0$. Then, it is well-known that

$$(3.10) \qquad P\{M^- > y\} = \rho^{-1} \exp(-\kappa y) \; ;$$

see, for example, FELLER (1971), p. 199. Because of (3.5), (3.7), (3.10) and $\hat{V}(\infty) = c^{-1}$, we have

$$H(x) = c\hat{V}(x) \; ,$$

which is the distribution function of $\lim_{n\to\infty}(\hat{S}_n \mid \hat{N} > n)$. One can calculate its Laplace transform and find

$$(3.11) \qquad E \exp(uS^{\#}) = \frac{(\lambda-\mu)^2(u+\lambda)}{\lambda(2u + \lambda - \mu)^2} \; ,$$

see IGLEHART (1974b), p. 750, for details.

3. The Asymptotic Analysis of the Distribution Function of M_n

Following the exposition of the last section, we shall go on to establish the following

(3.12) THEOREM. If conditions (3.1) - (3.4) are satisfied, then for all $x \geq 0$

$$(3.13) \qquad P\{M_n \leq x\} \sim \frac{\gamma^n}{n^{\frac{3}{2}}} \cdot \frac{G(x)}{(2\pi)^{\frac{1}{2}}\alpha\tau} \exp\left[\sum_{k=1}^{\infty} \frac{\gamma^{-k}}{k} P\{S_k \leq 0\}\right],$$

as $n \to \infty$, where $G(x)$ is a known function solely depending on x.

Proof. For $x \geq 0$ let $G_n(x) = P\{M_n \leq x\}$ and $u_n(x) = P\{S_n \leq x, N > n\}$, $n \geq 1$. Employing the same argument that is used in deriving (3.7), we obtain

$$u_n(x) = \int_{(o,x]} e^{\kappa y} P\{\hat{S}_n \in dy, \hat{N} > n\}$$

$$= P\{\hat{S}_n \leq x, \hat{N} > n\} e^{\kappa x} - \kappa \int_{(o,x]} P\{S_n \leq y, \hat{N} > n\} e^{\kappa y} dy.$$

The last equality is due to integration by parts. VERAVERBEKE and TEUGELS (1975) have shown that

$$\hat{P}\{\hat{S}_n \leq x, \hat{N} > n\} \sim \frac{\gamma^n}{n^{\frac{3}{2}}} \frac{\hat{V}(x)}{(2\pi)^{\frac{1}{2}}\alpha\tau}$$

as $n \to \infty$. Hence

$$(3.14) \qquad u_n(x) \sim \frac{\gamma^n}{n^{\frac{3}{2}}} \frac{V(x)}{(2\pi)^{\frac{1}{2}}\alpha\tau}$$

where $V(x) = \hat{V}(x) e^{\kappa x} - \kappa \int_{(o,x]} \hat{V}(y) e^{\kappa y} dy$. Form the generating function for $\{u_n(x) : n \geq 0\}$:

$$U(\Delta, x) = \sum_{n=1}^{\infty} u_n(x) \Delta^n .$$

Then we have the following Spitzer identity

$$(3.15)\qquad \sum_{n=1}^{\infty} G_n(x)s^n = [1 + U(s,x)] \cdot \exp\left[\sum_{n=1}^{\infty} \frac{s^n}{n} P\{S_n \le 0\}\right]$$

(see [30], p. 280). Differentiating with respect to s yields

$$\sum_{n=1}^{\infty} n\gamma^{-n}G_n(x)s^n = \exp\left[\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\}s^n\right]$$

$$(3.16)\qquad \cdot\left[\left\{\sum_{n=1}^{\infty} \gamma^{-n}P\{S_n \le 0\}s^n\right\}[1+U(\tfrac{s}{\gamma},x)] + \sum_{n=1}^{\infty} n\gamma^{-n}u_n(x)s^n\right]$$

As we had in (2.6),

$$(3.17)\qquad P\{S_n \le 0\} \sim (2\pi n)^{-\frac{1}{2}}\gamma^n(\alpha\tau)^{-1}, \quad \text{as } n \to \infty.$$

If we set

$$(3.18)\qquad a_n(x) = \left[\sum_{n=1}^{\infty} \gamma^{-n}P\{S_n \le 0\}s^n\right]\left[1 + U(\tfrac{s}{\gamma}, x)\right],$$

then, because of (3.14), (3.17) and Lemma 2.8,

$$a_n(x) \sim a(x)n^{-\frac{1}{2}}, \quad \text{as } n \to \infty,$$

where $a(x) = (2\pi)^{-\frac{1}{2}}(\alpha\tau)^{-1}[1 + U(\gamma^{-1}, x)]$. Thus

$$(3.19)\qquad a_n(x) + n\gamma^{-n}u_n(x) \sim (2\pi n)^{-\frac{1}{2}}(\alpha\tau)^{-1}[1 + U(\gamma^{-1},x) + V(x)]$$

by virtue of (3.14). Let

$$(3.20) \qquad \sum_{n=0}^{\infty} d_n s^n = \exp\left[\sum_{n=1}^{\infty} \frac{\gamma^{-n}}{n} P\{S_n \le 0\} s^n\right]$$

Then, because of (3.17) and Lemma 2.7,

$$(3.21) \qquad d_n = O(n^{-\frac{3}{2}}), \qquad \text{as } n \to \infty .$$

And it is obvious that

$$(3.22) \qquad \sum_{n=0}^{\infty} d_n < \infty .$$

Combining (3.16), (3.18), and (3.20), we have

$$\sum_{n=0}^{\infty} n\gamma^{-n} G_n(x) s^n = \left[\sum_{n=0}^{\infty} d_n s^n\right] \cdot \left\{\sum_{n=0}^{\infty} \left[a_n(x) + n\gamma^{-n} u_n(x)\right] s^n\right\} .$$

Because of (3.19) - (3.22) and Lemma 2.8, we obtain

$$n\gamma^{-n} G_n(x) \sim (2\pi n)^{-\frac{1}{2}} (\alpha\tau)^{-1} G(x) \cdot \exp\left[\sum_{k=1}^{\infty} \frac{\gamma^{-k}}{k} P\{S_k \le 0\}\right]$$

where $G(x) = 1 + U(\gamma^{-1}, x) + V(x)$. This completes our proof.

CHAPTER 4

CONDITIONED FUNCTIONAL CENTRAL LIMIT THEOREMS FOR QUEUES WITH TRAFFIC INTENSITY EQUAL TO UNITY

1. Introduction

We shall first study a general single-server queueing system with traffic intensity, ρ, equal to 1, and then extend the result to multiple-channel queues. The GI/G/1 queueing system is constructed in Section 1 of Ch. 1. We shall follow the exposition there.

Section 2 of this chapter is concerned with studying the behavior of the workload process, $W(t)$, conditioned on the fact that the first busy period has not ended by time t, as t becomes large. A functional central limit theorem in $D[0, 1]$ is our goal. In Section 3, the queue-length process, $Q(t)$, is treated.

In Section 4 we shall extend the results for the W_n and $W(t)$ processes to a variation of GI/G/1 queueing system, where there are several initial customers instead of only one at time $t = 0$. This simple generalization enables us to establish results for our multiple-channel queueing systems, which is the subject of Section 5.

2. The Workload Process $\{W(t): t \geq 0\}$

For $t \geq 0$ let $A(t) = n+1$ on $\{t_n \leq t < t_{n+1}\}$, where as previously defined in Section 1 of Ch. 1, $t_n = u_1 + \cdots + u_n$, $n \geq 1$, and $t_0 = 0$. For convenience let $A(0-) = 0$. Clearly $\{A(t): t \geq 0\}$ is a renewal process which represents the number of arrivals in the interval $[0, t]$.

Set $I(t) = v_0 + \cdots + v_{A(t)-1} - t$. The workload process, $W(t)$, is represented as

$$(4.1) \qquad W(t) = I(t) - \inf\{I(s): 0 \le s \le t\},$$

with $W(0+) = v_0 > 0$. Define the random function

$$Y_u(t) = \frac{W(ut)}{\sigma \lambda^{\frac{1}{2}} u^{\frac{1}{2}}}, \qquad 0 \le t \le 1 .$$

Next, define a stopping time $L = \inf\{t > 0: W(t) = 0\}$, where L represents the length of the first busy period. Our goal in this section is to obtain a central limit theorem for $(Y_u | L > u)$ as $u \to \infty$. The first step toward this goal is to understand the asymptotic behavior of $d_u \equiv P\{L > u\}$.

(4.2) LEMMA. *For* $u > 0$, $P\{L > u\} \sim c(\lambda u)^{-\frac{1}{2}}$ *as* $u \to \infty$, *where* c *is a constant*.

Proof:

$$\begin{aligned}(4.3) \qquad \{L > u\} &= \{W(t) > 0, \quad 0 < t \le u\} \\ &= \{W(t_1) > 0, W(t_2) > 0, \ldots, W(A(u)-1) > 0, W(u) > 0\} \\ &= \{S_1 > 0, \ldots, S_{A(u)-1} > 0, S_{A(u)} + t_{A(u)} - u > 0\},\end{aligned}$$

and

$$\{S_i > 0: 1 \le i \le A(u)\} \subset \{L < u\} \subset \{S_i > 0: 1 \le i \le A(u)-1\} . \tag{4.4}$$

It has been shown that

$$P\{S_i > 0: 1 \le i \le A(u)\} \sim c(\lambda u)^{-\frac{1}{2}}, \qquad \text{as } u \to \infty; \tag{4.5}$$

see IGLEHART (1974a), Lemma 4.2. The desired result follows from (4.4) and (4.5).

The proof of the main theorem requires a standard result in the theory of weak convergence. Let $\{X_n: n \ge 1\}$ and $\{Y_n: n \ge 1\}$ be two sequences of random elements of a separable metrix space such that X_n and Y_n have a common domain. Let $\rho(x,y) = \sup\{|x(s) - y(s)| : 0 \le s \le 1\}$ for any $x, y \in D[0, 1]$.

(4.6) THEOREM. *If* $X_n \Rightarrow X$ *and* $\rho(X_n, Y_n) \Rightarrow 0$, *then* $Y_n \Rightarrow X$.

See BILLINGSLEY (1968), Theorem 4.1, for this result.

Let

$$Z_u(t) = \frac{S_{A(ut)}}{\sigma(\lambda u)^{\frac{1}{2}}}, \qquad 0 \le t \le 1 .$$

(4.7) LEMMA. *If* $\rho = 1$, *and* $0 < \sigma^2 < \infty$, *then as* $u \to \infty$

$$(Z_u \mid L > u) \Rightarrow W^+ .$$

Proof. It follows from Lemma 4.2 and IGLEHART (1974a), Lemma 4.3, that, if $0 < t_1 < t_2 < \ldots < t_k \le 1$,

$$(Z_u(t_1),\ldots,Z_u(t_k)\,|\,L > u) \Rightarrow (W^+(t_1),\ldots,W^+(t_k)) .$$

Thus we have the convergence of f.d.d.'s of $(Z_u|L > u)$. It remains to show that the family $\{(Z_u|L > u): u \ge 0\}$ is tight. Since $Z_u(0) = 0$, it suffices to show that, for every $\epsilon > 0$,

$$\lim_{\delta\downarrow 0}\ \overline{\lim_{u\to\infty}}\ P\{\omega_{Z_u}(\delta) \ge \epsilon \mid L > u\} = 0, \tag{4.8}$$

where $\omega_x(\delta) = \sup\{|x(s)-x(t)|: s, t \in [0, 1], |s-t| < \delta\}$; see BILLINGSLEY (1968), Theorem 15.5, for this result.

We have from (4.4)

$$0 \le P\{\omega_{Z_u}(\delta) \ge \epsilon \mid L > u\}$$

$$\le \frac{P\{N > A(u)-1\}}{P\{L > u\}} P\{\omega_{Z_u}(\delta) \ge \epsilon \mid N > A(u) - 1\} .$$

IGLEHART (1974a), Lemma 4.6, has shown that

$$\lim_{\delta\downarrow 0}\ \overline{\lim_{u\to\infty}}\ P\{\omega_{Z_u}(\delta) \ge \epsilon \mid N > A(u) - 1\} = 0.$$

Hence (4.8) follows immediately. This completes the proof.

Now we are able to establish our main result.

(4.9) THEOREM. If $\rho = 1$, $0 < \sigma^2 < \infty$ and $E\{ u_1^3 \} < \infty$, then as $u \to \infty$,

$$(Y_u \mid L > u) \Rightarrow W^+ .$$

Proof. On the set $\{L > u\}$,

$$W(u) = \sum_{i=0}^{A(u)-1} v_i - u = S_{A(u)} + (t_{A(u)} - u).$$

Let $\rho(x,y) = \sup\{|x(t) - y(t)|: 0 \le t \le 1\}$ for any x, y in $D[0, 1]$.

Then

$$\begin{aligned}\rho(Y_u, Z_u) &= \frac{1}{\sigma(\lambda u)^{\frac{1}{2}}} \sup\{|t_{A(us)} - us|: 0 \le s \le 1\} \\ &\le \frac{1}{\sigma(\lambda u)^{\frac{1}{2}}} \sup\{u_{A(us)}: 0 \le s \le 1\} \\ &= \frac{1}{\sigma(\lambda u)^{\frac{1}{2}}} \sup\{u_k: 1 \le k \le A(u)\} .\end{aligned}$$

Thus $P\{\rho(Y_u, Z_u) > \epsilon \mid L > u\}$ is less than or equal to

$$d_u^{-1} P\left\{ \sup_{1 \le k \le A(u)+1} u_k > \epsilon\, \sigma(\lambda u)^{\frac{1}{2}} \right\}$$

$$\le d_u^{-1} E\left\{ \sum_{k=1}^{A(u)+1} 1_{\{u_k > \epsilon\sigma(\lambda u)^{\frac{1}{2}}\}} \right\}$$

$$= d_u^{-1} E\left\{A(u)+1\right\} \cdot P\left\{u_1 > \epsilon\, \sigma(\lambda u)^{\frac{1}{2}}\right\} .$$

The last equality holds because of Wald's identity. It follows from Lemma 4.2 that

$$(4.10) \qquad P\{\rho(Y_u, Z_u) > \epsilon \mid L > u\}$$

$$= [1 + o(u)] \ \sigma^{-1} \frac{E\{A(u)+1\}}{u} \ u^{\frac{3}{2}} P\{u_1 > \epsilon\sigma(\lambda u)^{\frac{1}{2}}\}.$$

The elementary renewal theorem says that

$$(4.11) \qquad \lim_{u\to\infty} \frac{E\{A(u)\}}{u} = \lambda .$$

Our assumption $E\{u_1^3\} < \infty$ implies that

$$(4.12) \qquad \lim_{t\to\infty} t^3 P\{u_1 > t\} = 0 .$$

Combining (4.10) - (4.12), we have

$$(4.13) \qquad \lim_{u\to\infty} P\left\{\rho(Y_u, Z_u) > \epsilon \;\middle|\; L > u\right\} = 0 .$$

Finally, the result follows immediately from (4.6), (4.7), and (4.13).

Now let us suppose that at time t there are $[Q(t) - 1]^+$ customers waiting in the queue and the server is serving some customer, say j^{th} customer, whose residual service time v' is part of v_j. Then, we can rewrite the workload process, $W(t)$, in the following way:

$$W(t) = \sum_{i=A(t)-[Q(t)-1]^+}^{A(t)} v_i + v' ,$$

i.e., the sum of $[Q(t) - 1]^+$ complete service times and a residual service time.

Let $V(t) = W(t) - v'$. Since the difference of $V(t)$ and $W(t)$ is dominated by $\max\{v_k: 0 \le k \le A(t)\}$, it is easy to see that $V(t)$, properly normalized, has the same weak limit (see the proof of Theorem 4.9). For $0 \le t \le 1$, let $V_u(t) = V(ut)/\sigma(\lambda u)^{\frac{1}{2}}$.

(4.14) THEOREM. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u_1^3+v_1^3\} < \infty$, <u>then as</u> $u \to \infty$

$$(V_u | L > u) \Rightarrow W^{+} .$$

3. The Queue-Length Process $\{Q(t): t \geq 0\}$

We now turn to the queue-length process, $\{Q(t): t \geq 0\}$, which represents the number of customers in the system at time t, including the one being served. This process is a random step function on the positive half line with jumps of ± 1. We may define $Q(t)$ formally by introducing the departure epochs $\{d_n\}$:

$$d_n = \begin{cases} t_n + v_n + W_n, & \text{if } n \geq 1, \\ 0 \qquad\qquad , & \text{if } n = 0. \end{cases}$$

The departure process $\{D(t): t \geq 0\}$, which records the number of departure in $(0, t]$, is defined as

$$D(t) = \begin{cases} \max\{n \geq 1 : d_n \leq t\}, & \text{if } d_1 \leq t, \\ 0 \qquad\qquad , & \text{if } d_1 > t. \end{cases}$$

Clearly $Q(t) = A(t) - D(t)$, where of course $A(t)$ and $D(t)$ are highly dependent. We observe that $A(t) \geq D(t)$ for all $t \geq 0$ and

$$\{N > D(t)\} \supset \{N > A(t)\}, \quad \text{for all } t \geq 0. \tag{4.15}$$

Let us consider another renewal process $\{D'(t): t \geq 0\}$.

$$D'(t) = \begin{cases} \max\{n \geq 1: v_1 + \cdots + v_n \leq t\}, & \text{if } v_1 \leq t, \\ 0 \qquad\qquad , & \text{if } v_1 > t. \end{cases}$$

Note that $D(t) \leq D'(t)$ and they are equal if there are no idle periods in $(0,t]$. We observe that the renewal process $\{D'(t): t \geq 0\}$ has the same rate as $\{A(t): t \geq 0\}$ because $\rho = 1$. Using the same argument as in IGLEHART (1973), Lemma 4.2, we have

(4.16) LEMMA. $P\{N > D'(u)\} \sim c(\mu u)^{-\frac{1}{2}}$ <u>as</u> $u \to \infty$, <u>where</u> c <u>is a constant</u>.

The precise value of c is not our concern here; see FELLER (1971), p. 415, for its value.

(4.17) COROLLARY. $P\{N > D(u)\} \sim c(\mu u)^{-\frac{1}{2}}$ <u>as</u> $u \to \infty$.

Using relation (4.15), (4.17) and the same proof employed in IGLEHART (1973), Lemma (4.2), we have the following two results.

(4.18) LEMMA. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u^3 + v^3\} < \infty$, <u>then as</u> $t \to \infty$

$$\left(\frac{V(t)}{\sigma(\lambda t)^{\frac{1}{2}}} \;\middle|\; N > D(t)\right) \Rightarrow W^{+}(1) .$$

(4.19) COROLLARY. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$, <u>and</u> $E\{u_1^3 + v_1^3\} < \infty$, <u>then for</u> $x > 0$

$$\lim_{t\to\infty} P\left\{V(t) \leq x \;\middle|\; N > D(t)\right\} = 0.$$

Corollary 4.19 enables us to show that, as time goes to infinity, the queue length will also become large, conditioned on the event that the first busy period never ends. For the unconditioned case, a similar result is first proved in WHITT (1968), pp. 149-151.

(4.20) LEMMA. For any $M > 0$,

$$\lim_{t\to\infty} P\{Q(t) \le M \mid N > A(t)\} = 0 .$$

Proof. Let $(\Lambda_t, \Lambda_t \cap \mathfrak{F}, P_t)$ be the trace of $(\Omega, \mathfrak{F}, P)$ on $\Lambda_t = \{N > D(t)\}$. We can write, for any x and M,

$$P_t\{V(t) \le x\} = \sum_{k=0}^{\infty} P_t\{V(t) \le x,\ [Q(t) - 1]^+ = k\} .$$

Let

$$E = \{V(t) \le x\} = \left\{ \sum_{i=A(t)-[Q(t)-1]^+}^{A(t)} v_i \le x \right\},$$

$$F = \{[Q(t) - 1]^+ = k\} ,$$

and let $\sigma(x_1, \ldots, x_n)$ denote the sigma-field generated by random variables $x_1, \ldots, x_n$. Then on the set $\{A(t) = i+k,\ D(t) = i\}$, for any $i,\ k > 0$,

$$E \in \sigma(v_{i+1}, \ldots, v_{i+k}) ,$$

$$F \in \sigma(v_0, \ldots, v_i, u_1, \ldots, u_{i+k}) ,$$

and

$$\Lambda_t \in \sigma(v_0, \ldots, v_{i-1}, u_1, \ldots, u_i) .$$

Therefore, $P(EF|\Lambda_t) = P(E) \cdot P(F|\Lambda_t)$, i.e.,

$$P_t\{V(t) \le x\} = \sum_{k=o}^{\infty} P\left\{\sum_{i=[A(t)-k]^+}^{A(t)} v_i \le x\right\} \cdot P_t\left\{[Q(t)-1]^+ = k\right\}$$

$$\ge \sum_{k=o}^{M} P\left\{\sum_{i=[A(t)-k]^+}^{A(t)} v_i \le x\right\} \cdot P_t\left\{[Q(t)-1]^+ = k\right\} .$$

For any M, choose x sufficiently large so that for all $k \le M$,

$$P\left\{\sum_{i=[A(t)-k]^+}^{A(t)} v_i \le x\right\} \ge P\left\{\sum_{i=[A(t)-M]^+}^{A(t)} v_i \le x\right\} > \epsilon .$$

The choice of x is independent of t, because the last inequality involves only at most m v_i's, which are independent and identically distributed. Thus we have

$$P_t\{V(t) \le x\} \ge \epsilon \, P_t\{[Q(t) -1]^+ \le M\} .$$

Since the left side goes to zero as t goes to infinity by Corollary 4.19, so does the right side. The theorem follows from the inequality

$$P\{Q(t) \le M | N > A(t)\} \le \frac{P\{N > D(t)\}}{P\{N > A(t)\}} P_t\{Q(t) \le M\}.$$

Using essentially the same proof employed in (4.20) we have the following generalization.

(4.21) COROLLARY. For any $M > 0$ and $0 < t \le 1$,

$$\lim_{u\to\infty} P\{Q(ut) \le M \mid N > A(u)\} = 0.$$

Our main object of interest is

$$Q_u(t) = \frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}}, \qquad 0 \le t \le 1 .$$

We are interested in the limiting behavior of $(Q_u | L > u)$. The next lemma is helpful in establishing the convergence of the f.d.d.'s of $(Q_u | L > u)$.

(4.22) LEMMA. For any $\varepsilon > 0$ and $0 < t \le 1$,

$$\lim_{u\to\infty} P\left\{\left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| \ge \varepsilon \mid L > u\right\} = 0.$$

Proof. Set $E(u) = \{k : |\frac{k}{u} - \lambda^{-1}| \le \varepsilon\}$ and $F(u)$ its complement. Let $P_u(\Lambda) = P(\Lambda | N > A(u))$ for $\Lambda \in \{N > A(u)\} \cap \mathcal{F}$ and $r_u = P\{N > A(u)\}$. We shall first consider

(4.23) $$r_u^{-1} P\left\{\omega : \left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| \ge \varepsilon, N > A(u)\right\}.$$

Probability (4.23) is less than or equal to

$$P_u\left\{\left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| > \varepsilon, A(ut)\in E(ut), D(ut)\in E(ut), Q(ut) \ge M\right\}$$
$$+ P_u\{Q(ut) < M\} + r_u^{-1}\Big[P\{A(ut)\in F(ut)\} + P\{D'(ut)\in F(ut)\}\Big]$$
$$= I_u + J_u + K_u .$$

As $u \to \infty$, K_u is asymptotically equal to

$$2u^{\frac{1}{2}} P\{A(ut) \in F(ut)\} \leq \frac{2E\{[A(ut) - \lambda ut]^2\}}{\epsilon^2 t^2 u^{\frac{3}{2}}} ,$$

which will converge to zero because the numerator is finite; see SMITH (1958), pp. 248-249. Because of Corollary 4.21, J_u also converges to zero. It remains to estimate I_u. Because of (4.15),

$$I_u \leq r_u^{-1} \cdot \sum_{\substack{k,\ell \in E(ut) \\ k-\ell \geq M}} P\left\{ \left| \frac{\sum_{i=\ell+2}^{k} (v_i - \lambda^{-1})}{k - \ell} \right| \geq \epsilon, \right.$$

$$\left. A(ut) = k,\ D(ut) = \ell,\ N > \ell \right\} .$$

Since

$$\left\{ A(ut) = k,\ D(ut) = \ell,\ N > \ell \right\} \in \sigma(v_0, \ldots, v_{\ell+1};\ u_1, \ldots, u_{k+1})$$

and

$$\left\{ \left| \frac{\sum_{i=\ell+2}^{k} (v_i - \lambda^{-1})}{k - \ell} \right| \geq \epsilon \right\} \in \sigma(v_{\ell+2}, \ldots, v_k) ,$$

the two events are independent. It follows

$$I_u \leq r_u^{-1} \cdot \sum_{\substack{k,\ell \in E(ut) \\ k-\ell \geq M}} P\left\{ \frac{\left| \sum_{i=\ell+2}^{k} (v_i - \lambda^{-1}) \right|}{k - \ell} \geq \epsilon \right\} P\{N > \ell,\ A(ut) = k,\ D(ut) = \ell\} .$$

By Chebyshev's inequality we have

$$I_u \le r_u^{-1} \frac{E\{(v_1-\lambda^{-1})^2\}}{\epsilon^2 \cdot M} \sum_{\substack{k,\ell \in E(ut) \\ k-\ell \ge M}} P\{N > [n(\lambda-\epsilon)],\ A(ut)=k,\ D(ut)=\ell\}$$

$$\le \frac{E\left\{(v_1-\lambda^{-1})^2\right\}}{\epsilon^2 M} \cdot \frac{P\left\{N > [n(\lambda-\epsilon)]\right\}}{r_u} \quad .$$

Putting the estimates of I_u, J_u and K_u together, and letting u go to infinity in (4.23), we have

$$\limsup_{u\to\infty} P_u\left\{\left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| \ge \epsilon\right\} \le \frac{E\{(v_1-\lambda^{-1})^2\}}{\epsilon^2 M} \cdot \left(\frac{\lambda-\epsilon}{\lambda}\right)^{\frac{1}{2}} .$$

Since M is arbitrary, we have

$$\lim_{u\to\infty} P\left\{\left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| \ge \epsilon \ \middle|\ N > A(u)\right\} = 0 \ .$$

Using the basic relation (4.4), we have the result.

(4.24) COROLLARY. <u>For any</u> $\epsilon > 0$ <u>and</u> $0 < t_1 < \cdots < t_k \le 1$,

$$\lim_{u\to\infty} P\left\{\left|\frac{V(ut_i)}{Q(ut_i)} - \lambda^{-1}\right| > \epsilon;\ 1 \le i \le k \ \middle|\ L > u\right\} = 0 \ .$$

<u>Proof</u>. Since

$$P\left\{\left|\frac{V(ut_i)}{Q(ut_i)} - \lambda^{-1}\right| > \epsilon;\quad 1 \le i \le k \mid L > u\right\}$$

$$\le \sum_{i=1}^{k} P\left\{\left|\frac{V(ut_i)}{Q(ut_i)} - \lambda^{-1}\right| > \epsilon \mid L > u\right\},$$

result follows immediately from Lemma 4.22.

Now, we are in the position to show the convergence of the f.d.d.'s of $(Q_u | L > u)$.

(4.25) THEOREM. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u_1^3 + v_1^3\} < \infty$, <u>and if</u> $0 < t_1 < \cdots < t_k \le 1$, <u>then as</u> $u \to \infty$

$$(Q_u(t_1), \ldots, Q_u(t_k) | L > u) \Rightarrow (W^+(t_1), \ldots, W^+(t_k)).$$

<u>Proof</u>. First, suppose $k = 1$. We shall show that for any $x > 0$ and $0 < t \le 1$,

$$\lim_{u\to\infty} P\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x \mid L > u\right\} = P\{W^+(t) \le x\}. \tag{4.26}$$

Let $P_u(\Lambda) = P(\Lambda | L > u)$ for any $\Lambda \in \{L > u\} \cap \mathfrak{F}$. Then

$$P_u\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x\right\} = P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le \frac{V(ut)}{Q(ut)} \cdot \lambda x\right\}$$

$$\le P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le \frac{V(ut)}{Q(ut)} \cdot \lambda x, \quad \left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| < \epsilon\right\}$$

$$+ P_u\left\{\left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| \ge \epsilon\right\}$$

$$\le P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le (1 + \lambda\epsilon)x\right\} + \epsilon,$$

because of Lemma 4.22. Since ϵ is arbitrary

(4.27) $$\limsup_{u\to\infty} P_u\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x\right\} \le \lim_{u\to\infty} P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le x\right\} = P\{W^+(t) \le x\}.$$

On the other hand,

$$P_u\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x\right\} \ge P_u\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x, \ \left|\frac{V(ut)}{Q(ut)} - \lambda^{-1}\right| > \epsilon\right\}$$

$$\ge P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le (1 - \epsilon\lambda)x\right\}.$$

Again, because ϵ is arbitrary,

(4.28) $$\liminf_{u\to\infty} P_u\left\{\frac{Q(ut)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \le x\right\} \ge \lim_{u\to\infty} P_u\left\{\frac{V(ut)}{\sigma(\lambda u)^{\frac{1}{2}}} \le x\right\}.$$

Equality in (4.26) follows from (4.27) and (4.28).

Exactly the same argument can be employed to establish the result for $k > 1$. This time we use Corollary 4.24 instead. And Theorem 4.25 follows.

In order to get the full weak convergence in $D[0, 1]$, we need to show that the family $\{(Q_u | L > u): u \ge 0\}$ is tight. To do so it suffices to show that for any $\epsilon > 0$,

(4.29) $$\lim_{\delta\downarrow 0} \overline{\lim_{u\to\infty}} P\left\{\omega_{Q_u}(\delta, 0, 1) \ge \epsilon \mid N > A(u)\right\} = 0 ;$$

see BILLINGSLEY (1968), Theorems 15.1 and 15.5 for this result.

The first step in demonstrating (4.29) is the next lemma.

(4.30) LEMMA. <u>For every</u> $\epsilon > 0$,

(4.31) $$\lim_{\tau\downarrow 0} \overline{\lim_{u\to\infty}} r_u^{-1} P\left\{\sup_{0\le\Delta\le\tau} \frac{Q(u\Delta)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \ge \epsilon,\quad N > A(u\tau)\right\} = 0 .$$

On the set $\{N > A(u\tau)\}$, $\{Q(u\Delta): 0 \le \Delta \le \tau\}$ is a stationary process with independent increment. In fact, $Q(u\Delta) = A(u\Delta) - D'(u\Delta)$, $0 \le \Delta \le \tau$, i.e., the difference of two standard renewal processes. Based on this observation, the proof of Lemma 4.30 follows the argument of

BELKIN (1972), p. 49, and will be omitted. Also see IGLEHART (1973), pp. 19-24.

With this lemma in hand it is an easy matter to show

(4.32) THFOREM. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u_1^3 + v_1^3\} < \infty$, <u>then</u> $(Q_u | L > u) \Rightarrow W^+$, <u>as</u> $u \to \infty$.

<u>Proof</u>. As remarked above, it suffices to show (4.29). For every $\tau \in (0,1]$, $\epsilon > 0$ and $0 < \delta < \tau$

(4.33) $$P\{\omega_{Q_u}(\delta,0,1) \geq \epsilon\} \leq r_u^{-1} P\left\{\omega_{Q_u}(\delta,0,1) \geq \epsilon,\ N > A(u\tau)\right\}$$

$$\leq r_u^{-1} P\left\{\omega_{Q_u}(\delta,0,1) \geq \epsilon,\ \sup_{0 \leq s \leq \tau} \frac{Q(us)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} < \epsilon,\ N > A(u\tau)\right\}$$

$$+\ r_u^{-1} P\left\{\sup_{0 \leq s \leq \tau} \frac{Q(us)}{\sigma(\lambda^3 u)^{\frac{1}{2}}} \geq \epsilon,\ N > A(u\tau)\right\}$$

$$= I_u + J_u \ .$$

Now, I_u is bounded above by

(4.34) $$r_u^{-1} P\left\{\omega_{Q_u}(\delta,\ \tau-\delta,\ 1)\ \geq \epsilon,\ N > A(u\tau)\right\}$$

$$\leq r_u^{-1} P\left\{\omega_{Q_u}(\delta,\ \tau-\delta,\ 1)\ \leq \epsilon,\ N > D(u(\tau-\delta))\right\} .$$

On the set $\{A(u(\tau-\delta)) = k,\ D(u(\tau-\delta)) = \ell\}$, $k > \ell \geq 0$, we see that two events

$$\{N > D(u(\tau-\delta))\} \in \sigma(v_0, \ldots, v_{\ell-1};\ u_1,\ \ldots,\ u_\ell)$$

$$\{\omega_{Q_u}(\delta, \tau-\delta, 1) \geq \epsilon\} \in \sigma(v_{\ell+1},\ \ldots,\ u_{k+1},\ \ldots)$$

are independent. Thus, (4.34) is further less than or equal to

$$\frac{P\{N > D(u(\tau-\delta))\}}{r_u} P\{\omega_{Q_u}(\delta, \tau-\delta, 1) \geq \epsilon\}$$

$$\leq \frac{P\{N > D(u(\tau-\delta))\}}{r_u} P\{\omega_{Q_u}(\delta, 0, 1) \geq \epsilon\}$$

Since $Q_u \Rightarrow f(B)$ and $P\{f(B) \in C\} = 1$ where B is the standard Brownian motion in $[0, 1]$, it follows

$$\text{(4.35)} \qquad \lim_{\delta\to 0} \overline{\lim_{u\to\infty}} P\{\omega_{Q_u}(\delta,\ 0,\ 1) \geq \epsilon\} = 0 ;$$

see WHITT (1968), Theorem 8.11, for this result. Combining (4.31) - (4.35), we obtain (4.29). This completes the proof.

4. A Single Server System With Several Initial Customers

In this section we shall consider a simple variation of the standard GI/G/1 queue with $\rho = 1$. We assume that at time 0, instead of one initial customer, there are m $(m > 1)$ initial customers. Thus our queueing system is defined by the two sequences $\{u_n\colon n \geq 1\}$ and $\{v_n\colon n \geq -m+1\}$, where u_n represents the interarrival times between the n^{th} and $(n+1)^{st}$ customers, and v_n the service time of n^{th} customer.

Let $X_n = v_{n-m} - u_n$, $n \geq 1$. Form the random walk $\{S_n : n \geq 0\}$ by setting $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. And define the other quantities, e.g. N, L, W_n, $W(t)$, accordingly. Then the next theorem is an immediate result of Theorems 1.6 and 4.9.

(4.36) THEOREM. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u_1^3\} < \infty$, <u>then</u>

(4.37) $$\left(\frac{W_{[n\cdot]}}{\sigma n^{\frac{1}{2}}} \;\middle|\; N > n\right) \Rightarrow W^+, \qquad \text{as } n \to \infty$$

and

(4.38) $$\left(\frac{W_{(u\cdot)}}{\sigma(\lambda u)^{\frac{1}{2}}} \;\middle|\; L > u\right) \Rightarrow W^+, \qquad \text{as } u \to \infty .$$

We observe that N represents the total number of customers being served during the first period for which there are always at least m customers in the system. The purpose of deriving (4.37) and (4.38) is that these results will be helpful in establishing conditioned weak limit theorems in multiple channel queueing systems. This is dealt with in the next section.

5. Multiple Channel Queues With Identical Servers

In this section conditioned limit theorems for queues with traffic intensity equal unity are extended to multiple channel queueing systems. The systems are defined by two basic sequences of independent and identically distributed random variables $\{u_n : n \geq 1\}$ and $\{v_n : n \geq -m+1\}$, where u_n represents the interarrival time between the n^{th} and $(n+1)^{st}$ customers, and v_n the service duration of n^{th} customer. There are m identical service channels and one initial customer for each service channel. The queueing discipline is first come first served.

Let $E\{u_1\} = \lambda^{-1}$ and $E\{v_1\} = \mu^{-1}$. We assume that $\rho \equiv \lambda/m\mu = 1$. Let $A(t)$ be the arrival process associated with $\{u_n: n \geq 1\}$. We shall define the gross input $G(t)$ and net input $I(t)$ of the systems as follows: for $t \geq 0$

$$G(t) = \sum_{n=-m+1}^{A(t)} v_n ,$$

$$I(t) = G(t) - mt .$$

Let $H(t)$ denote the total workload of the systems at time t. Define the workload at time t in the j^{th} channel, $H_j(t)$, to be the total work that is eventually done by server j in time t. So

$$H(t) = H_1(t) + \cdots + H_m(t), \quad t \geq 0.$$

If we define the virtual waiting time $W(t)$ as

$$W(t) = \min\{H_j(t): 1 \leq j \leq m\}, \quad t \geq 0,$$

then $W(t)$ represents the time that a hypothetical customer arriving at time t would have to wait before being served by the first available server. Define a stopping time

$$L = \inf\{t > 0: W(t) = 0\} .$$

So before time L all the servers are busy.

For the moment consider a single server system $\mathcal{S}$ with m initial customers. System $\mathcal{S}$ is described by the basic information $\{(v'_{n-1}, u'_n) = n \geq 1\}$, where $u'_n = u_n$, $v'_n = v_{n-m}/m$ and the traffic intensity is unity. From a workload point of view, the multiple channel systems behave exactly like system $\mathcal{S}$ on the set $\{L > u\}$. Based on this observation, we have the following result by virtue of (4.38):

$$\left(\frac{H(u\cdot)}{c(\lambda u)^{\frac{1}{2}}} \mid L > u\right) \Rightarrow W^{+} , \tag{4.39}$$

where

$$c = \left[\sigma^2(u_1) + \frac{\sigma^2(v_1)}{m}\right]^{\frac{1}{2}} .$$

On the set $\{L > u\}$ for any i, j and $0 \leq t \leq 1$, $H_i(ut)$ and $H_j(ut)$ may differ at most by the service duration of one customer who arrives before time u, that is to say, for $0 \leq t \leq 1$,

$$|H_i(ut) - H_j(ut)| \leq \max\{v_k : -m+1 \leq k \leq A(t)\}$$

Following the argument used in proving (4.13), we see that for every $\varepsilon > 0$,

$$\lim_{u\to\infty} P\left\{\rho\left(\frac{H_i(u\cdot)}{c(\lambda u)^{\frac{1}{2}}}, \frac{H_j(u\cdot)}{c(\lambda u)^{\frac{1}{2}}}\right) > \varepsilon \mid L > u\right\} = 0 . \tag{4.40}$$

Hence all processes

$$\left(\frac{H_i(u\cdot)}{c(\lambda u)^{\frac{1}{2}}} \mid L > u\right), \quad 1 \le i \le m,$$

have the same weak limit, if there is one. But $H(t) = H_1(t) + \cdots + H_m(t)$. Using the triangular inequality, we have

$$(4.41) \qquad \rho\left(\frac{H(u\cdot)}{cm(\lambda u)^{\frac{1}{2}}}, \frac{H_j(u\cdot)}{c(\lambda u)^{\frac{1}{2}}}\right) \le \frac{1}{m} \sum_{k \ne j} \rho\left(\frac{H_k(u\cdot)}{c(\lambda u)^{\frac{1}{2}}}, \frac{H_j(u\cdot)}{c(\lambda u)^{\frac{1}{2}}}\right)$$

Combining (4.39) - (4.41), we obtain

$$(4.42) \qquad \left(\frac{H_j(u\cdot)}{cm^{-1}(\lambda u)^{\frac{1}{2}}} \mid L > u\right) \Rightarrow W^+, \quad \text{as } u \to \infty$$

for any $j = 1, 2, \ldots, m$.

Since $W(t) = \min\{H_j(t): 1 \le j \le m\}$, it is easy to see that

$$(4.43) \qquad \rho\left(W(u\cdot), H_j(u\cdot)\right) \le \sum_{k=1}^{m} \rho\left(H_k(u\cdot), H_j(u\cdot)\right).$$

Combining (4.42) and (4.43) and appealing to Theorem 4.6, we establish the following.

(4.44) THEOREM. If $\rho \equiv \frac{\lambda}{m\mu} = 1$, $0 < \sigma^2 < \infty$ and $E\{u_1^3\} < \infty$, then

$$\left(\frac{W(u\cdot)}{cm^{-1}(\lambda u)^{\frac{1}{2}}} \mid L > u\right) \Rightarrow W^+, \quad \text{as } u \to \infty,$$

where

$$c = \left[\sigma^2(u_1) + \frac{\sigma^2(v_1)}{m}\right]^{\frac{1}{2}} .$$

We shall complete this section with a weak limit theorem for the process $\{W_n: n \geq 1\}$, the waiting time of n^{th} customer. Note that W_n can be obtained by the random scaling

$$(4.45) \qquad W_n = W(t_n) ,$$

where $t_n = u_1 + \cdots + u_n$. Define

$$N = \sup\{n > 0: W(s) > 0 \text{ for all } s \leq t_n\}$$

where the supremum of the empty set is taken to be $+\infty$. Then we have the following

(4.46) THEOREM. <u>If</u> $\rho = 1$, $0 < \sigma^2 < \infty$ <u>and</u> $E\{u_1^3\} < \infty$, <u>then as</u> $n \to \infty$,

$$\left(\frac{W_{[n\cdot]}}{cm^{-1}n^{\frac{1}{2}}} \,\middle|\, N > n\right) \Rightarrow W^+ .$$

<u>Proof.</u> First we define a random change of time e_n by

$$e_n(s) = \min\left(\frac{t_{[ns]}}{n}, 1\right), \qquad 0 \leq s \leq 1.$$

Then on the set $E_n = \{\frac{t_{[ns]}}{n} \leq 1, \ N > n\}$

$$W(ne_n(s)) = W_{[ns]} .$$

But $W(s) = S'_{A(s)} + (t_{A(s)} - s)$ on the set E_n, where $S'_n = \sum_{k=1}^{n} (v'_k - u'_k)$. Hence

$$(4.47) \qquad \frac{W_{[ns]}}{cm^{-1} n^{\frac{1}{2}}} = \frac{S'_{A(ne_n(s))}}{cm^{-1} n^{\frac{1}{2}}} + \epsilon_n(s)$$

$$= \frac{S'_{[ns]}}{cm^{-1} n^{\frac{1}{2}}} + \epsilon_n(s) ,$$

on the set E_n, where

$$(4.48) \qquad (\epsilon_n \mid N > n) \Rightarrow 0 .$$

We can assume $E\{u_n\} = \lambda^{-1} < 1$ (if this were not the case, a linear rescaling of time will make $\lambda^{-1} < 1$). Using Chebyshev's inequality, we can easily show that for $0 \leq s \leq 1$

$$(4.49) \qquad \lim_{n\to\infty} P\left\{\frac{t_{[ns]}}{n} > 1 \mid N > n\right\} = 0 .$$

Combining (4.37), (4.47) to (4.49), and Theorem 4.6, we prove the result.

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20. ABSTRACT

In a general single server queueing system let ξ_k's be the differences of service and interarrival times such that $\{\xi_k: k \geq 1\}$ is a sequence of independent, identically distributed random variables. Form the random walk $\{S_n: n \geq 0\}$ by setting $S_0 = 0$ and $S_n = \xi_1 + \cdots + \xi_n$, $n \geq 1$, and let N denote the hitting time of the set $(-\infty, 0]$ by the random walk: $N = \inf\{n > 0: S_n \leq 0\}$. In queueing context N represents the number of customers served in the first busy period. The natural measure of congestion for the system is the traffic intensity $\rho = \lambda/\mu$, where λ and μ are the arrival and service rates, respectively.

The basic processes we are interested in are $\{W_n: n \geq 0\}$, $\{W(t): t \geq 0\}$, and $\{Q(t): t \geq 0\}$ where W_n represents the waiting time of n^{th} customer, $W(t)$ denotes the workload of the server at time t, and $Q(t)$ is the number of customers in the system at time t. Define the random function X_n by $X_n(t) = W_{[nt]}/\alpha n^{\frac{1}{2}}$, where α is a norming constant.

When $\rho \neq 1$ and appropriate conditions on the distribution of ξ_1 hold, it is shown that the finite-dimensional distributions of X_n, conditioned on $n < N < \infty$, converge weakly to those of Brownian excursion, as n goes to infinity. When $\rho > 1$, it is shown that W_n, conditioned on $n < N < \infty$, converges weakly to a limit random variable $S^{\#}$, and the distribution function of $S^{\#}$ is exhibited.

These problems are closely related to the asymptotic analysis of the distribution of the maximum of the random walk $\{S_n: n \geq 0\}$. Various asymptotic results on the joint distribution function of the maximum and N are obtained.

If $\rho = 1$ and some mild regularity conditions hold, the workload process, $W(t)$, suitably normed and conditioned on the event that the first busy period has not terminated yet, is shown to converge weakly to Brownian meander, as time goes to infinity. The same result for the queue-length process, $Q(t)$, is also obtained. These results are extended to a variation of the general single server queueing system, where there are several initial customers instead of only one. This simple generalization enables us to establish results for certain multiple-channel queueing systems.